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Comments to G.E. Pavlenko: ‘On the Theory of Roll with the Aspect to the Safety of the Ship in a Seaway’

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Comments to G. E. Pavlenko: 'On the Theory of Roll with the Aspect to the Safety of the Ship in a Seaway'.

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Nomenclature

a) General Symbols.

t	time
$\dot{\varphi}(t) = \frac{d}{dt}\varphi(t)$	a dot denotes the derivative with respect to time
$\bar{\varphi}$	a bar denotes the amplitude of an oscillogram
K, D	complete elliptic integrals
k^2	their argument

b) Symbols of the Swell.

λ, H	length and height of waves
T_W	period of waves
$\mathfrak{r}(t), r$	orbital vector, orbital radius
ω	orbital angular velocity
$\mathfrak{g}$	gravitational acceleration vector
$g \approx$	981 cm sec^{-2}, its length
$\mathfrak{g}'(t) \equiv \mathfrak{g} - \omega^2\mathfrak{r}(t)$	resultant acceleration vector
$\vartheta(t)$	angle between $\mathfrak{g}(t)$ and $\mathfrak{g}$
$\beta(t) \equiv \frac{1}{g}\cdot\lvert\mathfrak{g}'(t)\rvert - 1$	= dimensionless acceleration normal to wave surface.
$\vartheta_a(t), \vartheta_r(t); \beta_a(t), \beta_r(t)$	active or reactive phase components with respect to ships' motion
β_0	a constant making the mean normal acceleration. vanish.

c) Symbols of the Ship.

$m(t)$	unit vector in the direction of ship's mast
$\varphi(t)$	ship's roll angle, measured between $m(t)$ and y
$\psi(t)$	ship's roll angle, measured between $m(t)$ and $y'(t)$
$h(\psi)$, $U(\psi)$	righting arm or potential energy of heel,
ψ_R	end of stability range
T_s	ship's roll period
$T_0(\bar{\psi})$, $\nu_0(\bar{\psi})$	period or circular frequency of free roll in calm water, the plot is called "skeleton curve"
J'	ship's moment of inertia including virtual masses of water
$W(T_s)$	ship's damping coefficient
$D(T_s)$, $D(T_w)$, $D(\omega)$	dimensionless damping coefficient
P	ship's weight
B	breadth of ship
$\overline{M_0 G}$	metacentric height

d) Further Symbols.

$\tau(\bar{\varphi})$, $\phi(\bar{\varphi})$	dimensionless period or phase integral of free roll
$\eta_1(\bar{\varphi})$, $\eta_2(\bar{\varphi})$, $\eta_3(\bar{\varphi})$	dimensionless mean kinetic energy
$F(\bar{\varphi})$	form factor of roll oscillogram
$\alpha(t)$	phase of roll oscillogram
$w(\varphi, \bar{\varphi})$	its reciprocal phase velocity of it
$q(\bar{\varphi}) \equiv w(\bar{\varphi}, \bar{\varphi}) / w(0, \bar{\varphi})$	ratio of non-uniformity of phase velocity
$\gamma(t)$, γ_0	phase of energy oscillogram or phase lag
$U(\varphi)$	dimensionless potential heel energy
$N_1(t)$, $N_2(t)$	moments neither active nor reactive to the ship's motion
C	a constant denoting the stability of roll states.

Footnotes

1) Physicist, Institut für Schiffbau der Universität Hamburg, (Director Professor Dr.-Ing. G. Weinblum), Germany. -
The investigations were supported by the Office of Naval Research, Washington, D.C. The treatise in German was translated by Dr. K. Hasselmann. Copies of the authors' manuscript are available at the Institut für Schiffbau.

2) Numbers in square brackets refer to the bibliography at the end of this paper.

3) For instance, the periods of free roll can thus be approximated with an error less than 1% at amplitudes $\overline{\psi} < 0{,}7\,\psi_R$ for the following righting-arm curves:

$$h^{I}(\psi) = h_{max} \cdot \sin \frac{\pi \psi}{\psi_R}\,, \quad h^{II}(\psi) = \overline{M_0 g} \cdot \psi \cdot \left[1 - \left(\frac{\psi}{\psi_R}\right)^2\right],$$

$$h^{III}(\psi) = \overline{M_0 g} \cdot \psi \cdot \left[1 - \left(\frac{\psi}{\psi_R}\right)^4\right] \text{ among others.}$$

4) Equations Nr. 36 - 40 have been cancelled in the translated version.

5) In cases of multiple resonance (T_S/T_W = 3, 4, 5 etc.) the roll numbers satisfying condition (2) would be even greater (R = 15, 20, 25, etc.).

6) Although the simultaneous use of the symbols $D(T_W)$ and $D(\omega)$ is not correct; there is no fear of ambiguity as the formulae will be used alternatively.

7) $\mathbf{D} = \frac{1}{k^2} \cdot (\mathbf{K} - \mathbf{E})$, not to be confused with the damping function $D(T_S)$!

Figures

1) Explanation of terms used in the equation of motion of a ship in a transverse swell. r = orbital radius, ω = orbital angular velocity, g = gravitational vector, g' = appar~~t~~ent gravitational vector.

2) The amplitude function $q(\bar{\varphi})$

3a) The quarter oscillograms and periods of a ship with a sinusoidal righting-arm curve rolling at various amplitudes in calm water.

3b) The quarter oscillograms and periods of a ship with an additional stability by form in calm water.

4) The planimetered phase integrals $\oint(\bar{\varphi})$ of the roll oscillations of figs. 3a and 3b. The points were calculated according to (34).

5) The amplitude function $\eta_0(\bar{\varphi})$ of the roll oscillations of figs. 3a and 3b. The points were calculated according to (43).

6) The form factors F $(\bar{\varphi})$ of the oscillograms shown in 3a and 3b.

7) Damping moments measured for towed ship models. From Weinblum and St. Denis [9] .

8) The increase of damping with roll amplitude. From Grim [10] .

9) The construction of two points P_1 and P_2 of the response curves for roll states of the first kind from the skeleton curve and the damping. Construction (1) yields the blind excitation $\pm |\bar{\vartheta}_r|$) from the amplitude $\bar{\varphi}$ and the damping D. The corresponding frequencies are obtained from construction (2).

10a) Response curves of the roll states of the first kind for ships with righting-arm curves as shown at three constant values of the blind excitation. In the shaded region the roll states are unstable.

10b) Response curves of the roll states of the first kind for ships with righting-arm curves as shown at three constant values of the blind excitation. In the shaded regions the roll states are unstable.

11a) Response curves of the roll states of the first kind for ships with righting-arm curves as shown and damping D = 0,2 at three values of the wave steepness. In the shaded region the roll states are unstable.

11b) Response curves of the roll states of the first kind for ships with righting-arm curves as shown and damping D = 0,2 at three values of the wave steepness. In the shaded regions the roll states are unstable.

12) The construction of two points P_1 and P_2 on the limiting stability curve L for roll states of the first kind from the skeleton curve and the damping.

13) Interpretation of the transformation (120).

14a) Response curves of the roll states of the second kind for ships with righting-arm curves as shown at the blind excitation $|\bar{\beta}_r| = 0,2$. The broken regions of the curve represent unstable roll states. $\bar{\varphi}_{max}$ is the largest roll amplitude occuring with certainty, the probability of larger amplitudes occuring is small.

14b) Response curves of the roll states of the second kind for ships with righting-arm curves as shown at the blind excitation $|\bar{\beta}_r| = 0,2$. The broken regions of the curve represent unstable roll states. $\bar{\varphi}_{max}$ is the largest roll amplitude occuring with certainty, the probability of larger amplitudes occuring is small.

15) Illustrating the instable roll states of the second kind at small amplitudes.

16) Phase relation of the ship's motion in a swell of strength just exceeding the critical value for roll states of the second kind.

17) Determination of the convergence interval (125).

2. The Equation of Motion.

The roll motion of a ship in a transverse swell is described by the differential equation

$$J' \cdot \ddot{\varphi} + W(T_s) \cdot \dot{\varphi} + [1 + \beta(t)] \cdot P \cdot h(\varphi) = 0. \qquad (1)$$

The following comments are necessary regarding the derivation and the notation employed. (See fig. 1):

The wave length λ of the swell is assumed large in comparison to the ship's breadth B, at least.

$$\lambda > 4B. \qquad (2)$$

In this case the ship participates in the orbital motion of the water and rolls with constant displacement. Apart from the gravitational force, a centrifugal force of magnitude $r\omega^2$ then acts on the ship as a result of its oribital motion (r = orbital radius, ω = orbital angular velocity). The resulting vector g' (the apparent vertical) varies periodically in magnitude and direction. The static equilibrium position of the ship thus has a variable angle relative to the horizontal..In a trochoidal wave the amplitude $\bar{\vartheta}$ of $\vartheta(t)$, the effective wave steepness, is given by

$$\sin \bar{\vartheta} = \frac{r\omega^2}{g} = \frac{2\pi r}{\lambda}. \qquad (3)$$

If condition (2) is satisfied, the (righting) arm $h(\psi)$ corresponding to an angle of heel ψ relative to the position of static equilibrium is the same as in calm water. Because of the variation of the apparent vertical, , the ship's weight has the variable value

$$P'(t) = [1 + \beta(t)] \cdot P \tag{4}$$

$\beta(t)$ is positive in the wave trough and negative on the crest.

In a sinusoidal swell

$$\dot{\vartheta}(t) = \pm \omega \cdot \beta(t) \tag{5a}$$

(the sign depends on the direction of propogation) and

$$\vartheta^2(t) + \beta^2(t) = \frac{4\pi^2 r^2}{\lambda^2}, \tag{5b}$$

so that for a standard wave ($2\,r/\lambda = 1/20$):

$$|\bar{\vartheta}| = |\bar{\beta}| = \frac{\pi}{20} \approx 0{,}157. \tag{6}$$

The angle of roll relative to the horizontal is

$$\varphi(t) = \psi(t) + \vartheta(t). \tag{7}$$

We shall assume that the total moment of inertia, $\mathfrak{I}'$, of the ship is constant. The hydrodynamic component can be taken as independent of frequency if the wave generated by the ship's motion satisfies condition (2). The damping is as yet not sufficiently well known to be described accurately by an analytical expression. It depends in a hydrodynamically complicated manner on the geometry of the ship, the position of the centre of gravity, the roll period and the momentary angle and angular velocity of roll, among other factors. In the following we shall assume it to be proportional to the angular velocity $\dot{\psi}$ relative to the water surface and take into account otherwise only the dependency on the roll period T_s.

3. The Fundamental Functions for Undamped Roll in Calm Water.

As a basis for the forced roll oscillations to be analysed later we consider first the undamped roll oscillation in calm water. Setting

$$W = 0 \qquad \text{and} \qquad r = 0 \tag{8}$$

and writing $\varphi_0(t)$ for the functions $\varphi(t)$ and $\psi(t)$, which are indentical in this case, the equation of motion becomes

$$J' \cdot \ddot{\varphi}_0 + P \cdot h(\varphi_0) = 0. \tag{9}$$

As the damping is zero, the sum of the kinetic and potential energies is constant:

$$\tfrac{1}{2} J' \cdot \dot{\varphi}_0^2 + U(\varphi_0) = U(\bar{\varphi}) \tag{10}$$

where

$$U(\varphi) = P \cdot \int^{\varphi} h(\varphi) \cdot d\varphi \,, \tag{11}$$

$$U(-\varphi) = U(+\varphi) \,, \qquad U(0) = 0.$$

We now introduce three fundamental functions of the amplitude which will later also play an important role in the analysis of the forced oscillations in a swell:

a) the roll period,
b) the phase integral and, resulting from this,
c) the mean value of the kinetic energy.

a) From (10) the time differential of the free oscillation $\psi_0(t)$

$$dt(\psi_0) = \sqrt{\tfrac{1}{2} J'} \cdot \frac{d\psi}{\sqrt{U(\bar{\psi}) - U(\psi)}} \qquad (12)$$

giving the improper integral

$$T_0(\bar{\psi}) = \sqrt{\tfrac{1}{2} J'} \cdot 4 \cdot \int_0^{\bar{\psi}} \frac{d\psi}{\sqrt{U(\bar{\psi}) - U(\psi)}} \qquad (13)$$

for the roll period in calm water.

The integral (13), which is of fundamental importance for the dynamical behavior of the ship, can be expressed in a closed analytical form only in a few special cases. For instance, if the righting-arm curve is sinusoidal or a cubic parabola, it is well known that the amplitude function

$$\tau(\bar{\varphi}) \equiv \frac{T_0(\bar{\varphi})}{T_0(0)} \tag{14}$$

where

$$T_0(0) = \sqrt{\frac{J'}{P.\,\overline{hg}}} \tag{15}$$

is the roll period for small amplitudes, can be expressed in terms of the complete elliptic integral

$$K \equiv \int_0^1 \frac{dx}{\sqrt{(1-x^2)\cdot(1-k^2x^2)}} \qquad (16)$$

(16)

$$= \frac{\pi}{2}\cdot\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{264}k^6 + \dots\right).$$

For

For $$h^{I}(\psi) = h_{max} \cdot \sin \frac{\pi \psi}{\psi_R} \tag{17}$$

$$\tau^{I}(\bar{\psi}) = \frac{2}{\pi} K , \qquad k^2 = \sin^2 \frac{\pi \bar{\psi}}{2 \psi_R} \tag{18}$$

and for

and for $$h^{II}(\psi) = \overline{M_c g} \cdot \psi \cdot \left[1 - \left(\frac{\psi}{\psi_R} \right)^2 \right] \tag{19}$$

$$\tau^{II}(\bar{\psi}) = \sqrt{1 + k^2} \cdot \frac{2}{\pi} K , \text{ with } k^2 = \frac{\bar{\psi}^2}{2\psi_R^2 - \bar{\psi}^2} \tag{20}$$

The num~~b~~erical or planimetrical evaluation of (13) generally meets with some difficulty as the integrand is singular at $\psi = \bar{\varphi}$. The difficulty can be avoided by transforming to the parameter α :

$$\varphi_0(t) = \bar{\varphi} \cdot \sin \alpha(t) \tag{21}$$

representing the oscillation is then represented by a point moving non-uniformly but periodically round a circle at the angular velocity

$$\dot{\alpha} = + \sqrt{\frac{2}{J'} \cdot \frac{U(\bar{\varphi}) - U(\varphi_0)}{\bar{\varphi}^2 - \varphi_0^2}} \tag{22}$$

Equation (13) then transforms to the following equation for the amplitude function (14):

$$\tau(\bar{\varphi}) = \frac{2}{\pi} \cdot \int_0^{\pi/2} \sqrt{\frac{\frac{1}{2} P \cdot \overline{M_0 g} \cdot (\bar{\varphi}^2 - \varphi_0^2)}{U(\bar{\varphi}) - U(\varphi_0)}} \cdot d\alpha \tag{24}$$

where expression (15) has been substituted for the roll period at small amplitudes. The integrand

$$w(\varphi_0, \bar{\varphi}) \equiv \sqrt{\frac{\frac{1}{2} P \cdot \overline{M_0 G} \cdot (\bar{\varphi}^2 - \varphi_0^2)}{U(\bar{\varphi}) - U(\varphi_0)}} \tag{25}$$

remains finite within the entire range of integration. The values at the lower and upper limits of integration are

$$w(0, \bar{\varphi}) = \sqrt{\frac{\frac{1}{2} P \cdot \overline{M_0 G} \cdot \bar{\varphi}^2}{U(\bar{\varphi})}} \quad \text{and} \quad w(\bar{\varphi}, \bar{\varphi}) = \sqrt{\frac{\overline{M_0 G} \cdot \bar{\varphi}}{h(\bar{\varphi})}} \tag{26}$$

respectively.

As the differential ~~coefficient~~ quotient

$$\frac{dw}{d\alpha} = \frac{dw}{d\varphi_0} \cdot \frac{d\varphi_0}{d\alpha} \tag{27}$$

vanishes at the~~se~~ points $\psi_0 = 0$ and $\alpha = \pi/2$, the values are extrema. If the curvature of the righting-arm curve is of constant sign, the function w is monotonic in the range of integration and the values (26) are absolute extrema. The ratio

$$q(\bar{\varphi}) \equiv \frac{w(\bar{\varphi},\bar{\varphi})}{w(0,\bar{\varphi})} = \sqrt{\frac{2\,U(\bar{\varphi})}{\bar{\varphi}\cdot P.\,h(\bar{\varphi})}} \tag{28}$$

is shown in fig. 2 for a sinusoidal righting-arm curve and a further empirical curve representing a ship with initially increasing stiffness. $q(\bar{\varphi})$ is seen to be within 20 % of unity for a considerable range of roll amplitudes. The integrand (25) is thus limited to a relatively small interval for moderate amplitudes, so that the integral (24) can be easily evaluated numerically or by planimetration. For righting-arm curves with curvatures of constant sign the arithmetic mean of the two extremes generally gives a satisfactory approximation for moderate amplitudes. 3)

The roll period in calm water can thus be approximated by the algebraic expression

$$\tau(\bar{\varphi}) \approx \frac{1}{2}\cdot\left(\sqrt{\frac{\frac{1}{2}P.\,M_0 g\,.\,\bar{\varphi}^2}{U(\bar{\varphi})}} + \sqrt{\frac{M_0 g\,.\,\bar{\varphi}}{h(\bar{\varphi})}}\right) \tag{29}$$

i. e. by means of the ratio of the area under the initial tangent of the righting-arm curve to the area under the curve itself (both taken between the angles 0 and $\bar{\varphi}$) and by the ratio of the ordinates of the tangent and the curve at the value $\bar{\varphi}$.

The free-roll oscillograms at various amplitudes are shown in fig. 3a for the ship with a sinusoidal righting-arm curve and in fig. 3b for the ship with initially increasing stiffness. The dependence of the natural period on the roll amplitude is given by the curve passing through the maxima.

b) The normed phase integral

$$\Phi(\bar{\varphi}) \equiv \frac{1}{\pi \bar{\varphi}} \cdot \sqrt{\frac{J'}{2\,U(\bar{\varphi})}} \cdot \oint \dot{\varphi}_0 \cdot d\varphi_0 \,, \tag{30}$$

which is equal to unity for a linear righting-arm curve, can easily be evaluated by planimetration. The $\Phi(\bar{\varphi})$-curves corresponding to the ship types of fig. 3a and 3b are shown in fig. 4. It is also possible to approximate the phase integral by an algebraic expression. With the substitution (21), integral (30) transforms to

$$\Phi(\bar{\varphi}) = \sqrt{\frac{J'}{2\,U(\bar{\varphi})}} \cdot \frac{\bar{\varphi}}{\pi} \cdot \int_0^{2\pi} \cos^2\alpha \cdot \dot{\alpha} \cdot d\alpha \tag{31}$$

where $\dot{\alpha}$ is given by (22).
The extrema of $\dot{\alpha}(\alpha)$ at the values $\alpha = 0, \pi/2, \pi$ etc. are thus

$$\left.\begin{aligned} \dot{\alpha}(0) &= \dot{\alpha}(\pi) = \dot{\alpha}(2\pi) = \sqrt{\frac{2\,U(\bar{\varphi})}{J'\,\bar{\varphi}^2}} \\ \dot{\alpha}\left(\tfrac{\pi}{2}\right) &= \dot{\alpha}\left(\tfrac{3\pi}{2}\right) = \frac{1}{q(\bar{\varphi})} \cdot \sqrt{\frac{2\,U(\bar{\varphi})}{J'\,\bar{\varphi}^2}} \end{aligned}\right\} \tag{32}$$

with $q(\bar{\varphi})$ as in (28).
For righting-arm curves with curvatures of constant sign, $\dot{\alpha}(\alpha)$ is monotonic and can be approximated by

$$\dot{\alpha}(\alpha) \approx \sqrt{\frac{2\,U(\bar{\varphi})}{J'\,\bar{\varphi}^2} \cdot \frac{1}{2} \cdot \left[\left(1 + \frac{1}{q(\bar{\varphi})}\right) + \left(1 - \frac{1}{q(\bar{\varphi})}\right) \cdot \cos 2\alpha\right]} \tag{33}$$

From (31) we thus obtain the algebraic expression

$$\Phi(\bar{\varphi}) \approx \frac{1}{2}\cdot\left(1+\frac{1}{\varphi(\bar{\varphi})}\right) + \frac{1}{4}\cdot\left(1-\frac{1}{\varphi(\bar{\varphi})}\right) \tag{34}$$

$$\approx \frac{3}{4} + \frac{1}{4\varphi(\bar{\varphi})} \tag{34}$$

which is plotted in fig. 4.

c) We consider finally the normed mean value of the kinetic energy

$$\eta_0(\bar{\varphi}) \equiv \frac{2}{T_0(\bar{\varphi})}\cdot\int_0^{T_0(\bar{\varphi})}\left(1-\frac{U(\varphi)}{U(\bar{\varphi})}\right).\, dt \tag{35}$$

which will be used frequently in the following..It depends in general on the amplitude and the righting-arm curve. The expression is normed to equal unity for restoring moments proportional to the roll angle. For ships with increasing stiffness, $\eta_0(\bar{\varphi})$ is greater than one, as lower velocities occur for shorter times. For decreasing stiffness, $\eta_0(\bar{\varphi})$

is smaller than one and approaches zero as the amplitude approaches the stability limit. The upper limit for $\eta_0(\bar{\varphi})$ is two, corresponding to a body which can move freely within a certain interval and is abruptly reflected at the ends. This case is, of course, not realizable in practice for ships, and we can take as the upper limit approximately $\eta_{max} = 1{,}5$. Thus

$$0 \leq \eta_0(\bar{\varphi}) \leq \eta_{max} = \begin{cases} 1{,}0 & \text{for decreasing stiffness ,} \\ 1{,}5 & \text{for increasing stiffness ,} \end{cases} \tag{41}$$ [4]

The $\eta_0(\bar{\varphi})$ - curves for the two ships described above are shown in fig. 5. The curves were obtained from the planimetered values of $\oint(\bar{\varphi})$ and $\tau(\bar{\varphi})$ as in fig. 3a and 3b using the identity

$$\tau(\bar{\varphi}) \cdot \eta_0(\bar{\varphi}) \equiv \oint(\bar{\varphi}) \cdot \sqrt{\frac{\frac{1}{2} P \cdot \overline{MoG} \cdot \bar{\varphi}^2}{U(\bar{\varphi})}} \tag{42}$$

which can easily be deduced from the energy equation (10). For ships with monotonically increasing or decreasing stiffness we can use the approximations (34) and (29) for $\phi(\bar{\varphi})$ and $\tau(\bar{\varphi})$ respectively obtaining

$$\eta(\bar{\varphi}) \approx \frac{1}{1 + q(\bar{\varphi})} + \frac{1}{2\, q(\bar{\varphi})} \quad . \tag{43}$$

The approximate values calculated from this expression are plotted in fig. 5.

4. The Steady Roll States in a Transverse Swell.

We confine our considerations to the steady roll states of period T_s, for which

$$\psi(t+T_s) \equiv \psi(t). \tag{44}$$

Multiplying the equation of motion (1) with $\dot{\psi}(t)$ and $\psi(t)$ respectively and integrating over the roll period we obtain, using (7)

$$\int_0^{T_s} J' \cdot \ddot{\vartheta} \cdot \dot{\psi} \cdot dt + \int_0^{T_s} W(T_s) \cdot \dot{\psi}^2 \cdot dt + \int_0^{T_s} \beta \cdot P \cdot h(\psi) \cdot \dot{\psi} \cdot dt = 0, \tag{45}$$

$$\int_0^{T_s} J' \cdot \ddot{\vartheta} \cdot \psi \cdot dt + \int_0^{T_s} \left[J' \cdot \ddot{\psi} + P \cdot h(\psi)\right] \cdot \psi \cdot dt + \int_0^{T_s} \beta \cdot P \cdot h(\psi) \cdot \psi \cdot dt = 0. \tag{46}$$

From these equations certain general time-relations between the excitation and the oscillation can be deduced.

a) Relations between wave and ship periods.

The second integral in equation (45) represents the energy dissipated by the damping; it is always positive. The remaining integrals in the equation must therefore be negative, singly or in the sum, to yield the necessary energy supply. The second integral in equation (46) vanishes if $\varphi(t)$ satisfies equation (9) describing the free undamped oscillation in calm water or, in other words, if the ship rolls with its natural period. This integral is thus a measure of the detuning brought about by the remaining two integrals, which cause the ship to roll with a different period T_s. If we now let T_w be the period of the swell and $\frac{1}{n} \cdot T_s$ the period of a higher harmonic of the (generally ~~non~~ sinusoidal) oscillogram, the first integrals in equations (45) and (46) will then be unequal to zero only if

$$\frac{1}{n} \cdot T_s = T_w \quad , \quad n \text{ integer.} \tag{47}$$

On the other hand, the third integrals in (45) and (46) will be different from zero only if

$$\frac{1}{2n'} \cdot T_s = T_w \quad , \quad n' \text{ integer.} \tag{48}$$

The conditions (47) and (48) can be satisfied by an infinite number of values n and n': In the following, however, we shall confine our attention to the two cases n = 1 or n' = 1, disregarding the possibility of multiple resonances. This gives the mutually exclusive conditions

$$T_S = T_W \quad \text{or} \quad T_S = 2\,T_W, \qquad (49)\ (50)$$

b) <u>Phase ~~decompos~~ition</u>.

From equations (45) and (46) we can also determine the relative phases of the ship and wave motions. The multipliers $\dot{\varphi}(t)$ and $\varphi(t)$ are 90^0 out of phase, as

$$\int_0^{T_S} \varphi \cdot \dot{\varphi} \, dt = 0 .$$

The exciting moment can similarly be decomposed into two terms which are 90° out of phase:

$$\vartheta(t) = \vartheta_a(t) + \vartheta_r(t) \; ; \quad \beta(t) = \beta_a(t) + \beta_r(t), \qquad (51)$$

We define as the active excitation the components contributing solely to the energy supply integrals in (45). For these components then

$$\int_0^{T_s} \ddot{\vartheta}_a \cdot \varphi \cdot dt = 0 \qquad ; \qquad \int_0^{T_s} \beta_a \cdot \varphi \cdot dt = 0 , \tag{52}$$

Conversely, the components contributing solely to the detuning integrals in (46) will be termed the reactive excitation. For these

$$\int_0^{T_s} \ddot{\vartheta}_r \cdot \dot{\varphi} \cdot dt = 0 \qquad ; \qquad \int_0^{T_s} \beta_r \cdot \dot{\varphi} \cdot dt = 0 . \tag{53}$$

With condition (49) equations (45) and (46) simplify to

$$\int_0^{T_s} \left[J' \cdot \ddot{\vartheta}_a + W(T_s) \cdot \dot{\varphi}_1 \right] \cdot \dot{\varphi}_1 \cdot dt = 0 , \tag{54}$$

$$\int_0^{T_s} \left[J' \cdot \ddot{\vartheta}_r + J' \cdot \ddot{\varphi}_1 + P \cdot h(\varphi_1) \right] \cdot \varphi_1 \cdot dt = 0 . \tag{55}$$

The condition (50), on the other hand, leads to

$$\int_0^{T_s} \left[W(T_s) \cdot \dot{\psi}_2 + \beta_a \cdot P \cdot h(\psi_2) \right] \cdot \dot{\psi}_2 \cdot dt = 0, \tag{56}$$

$$\int_0^{T_s} \left[J' \cdot \ddot{\psi}_2 + (1+\beta_r) \cdot P \cdot h(\psi_2) \right] \cdot \psi_2 \cdot dt = 0. \tag{57}$$

c) Two kinds of roll states.

In a transverse swell we can thus distinguish between two kinds of roll motions, $\psi_1(t)$ and $\psi_2(t)$.

The roll states of the first kind are generated by the periodic horizontal acceleration $b_h(t) = \vartheta(t) \cdot g$ accompanying the swell; they have the same period as the swell. - The roll states of the second kind are generated by the periodic vertical acceleration $b_v(t) = \beta(t) \cdot g$ due to the swell and have a period equal to twice the swell period. In the first case the condition (2) is well satisfied. According to Kempf [5] the roll number

$$R = T_s \cdot \sqrt{\frac{g}{B}} \tag{58}$$

is not less than eight, so that for roll states of the first kind

$$\lambda > 10B. \tag{59}$$

For roll states of the second kind, condition (2) requires

$$R > 10. \tag{60}$$

We shall therefore have to exclude very stiff ships from our considerations of this case. 5)

5. The Roll States of the First Kind, $T_s = T_w$.

For roll states of the first kind, the periodic vertical acceleration $\beta(t) \cdot g$ yields no contribution to equations (45) and (46) and is thus only of secondary physical importance. We shall therefore set $\beta(t) \equiv 0$ for the first and investigate the influence of the vertical acceleration afterwards. The equation of motion now becomes

$$J' \cdot (\ddot{\vartheta} + \ddot{\varphi}_1) + W(T_s) \cdot \dot{\varphi}_1 + P \cdot h(\varphi_1) = 0 \qquad (61)$$

which by phase decomposition according to (51) reduces to the two equations

$$J' \cdot \ddot{\vartheta}_a \quad + W(T_s) \cdot \dot{\varphi}_1 \quad = + N_1(t) \qquad (62)$$

$$J' \cdot (\ddot{\vartheta}_r + \ddot{\varphi}_1) + P \cdot h(\varphi_1) \quad = - N_1(t) \qquad (63)$$

each of which contains only components of the same phase. The moment $N_1(t)$ contributes to neither the excitation nor the detuning of the ship, as from (54) and (55)

$$\int_0^{T_s} N_1 \cdot \dot{\varphi}_1 \cdot dt = 0 \quad \text{and} \quad \int_0^{T_s} N_1 \cdot \varphi_1 \cdot dt = 0 \, ; \qquad (64) \; (65)$$

$N_1(t)$ is a combination of higher harmonics which can still be disposed of within certain limits.

We now consider the swell to be characterized by its period T_W and steepness $\bar{\vartheta}$. We shall not take the exact shape of the wave profile into account but rather prescribe the wave-slope function $\vartheta(t)$ in such a manner that $N_1(t)$ vanishes, thus enabling easy integration of equations (62) and (63).

a) <u>The roll amplitudes</u>.

If we integrate (62) twice under the side condition

$$\int_0^{T_W} \vartheta_a(t).dt = 0 \tag{66}$$

we obtain the active excitation

$$\vartheta_a(t) = -\frac{W(T_s)}{\mathcal{J}'} \cdot \int^t \varphi_1(t).dt \,. \tag{67}$$

Introducing the form factor

$$F(\bar{\varphi}) = \frac{2\pi}{\bar{\varphi}\cdot T_s} \cdot \int_0^{\frac{1}{4}T_s} \varphi_1(t).dt \tag{68}$$

$[\varphi_1(0) = 0;\ \varphi_1(\frac{1}{4}T_s) = \bar{\varphi}.]$ (F(Fis plotted in fig. 6 over the amplitude for the oscillograms given in figs. 3a and 3b)) the amplitudes of the active excitation becomes

$$\bar{\vartheta}_a = F(\bar{\varphi}) \cdot D(T_s) \cdot \bar{\varphi} \tag{70}$$

where

$$D(T_s) = \frac{T_s \cdot W(T_s)}{2\pi \, J'} \tag{71}$$

As can be seen from fig. 6, F can practically be taken equal to one - at least until more precise information is available on the damping D. Examples of D are shown in figs. 7 and 8. To determine the reactive excitation $\vartheta_r(t)$ we now make the assumption that its oscillogram is similar in shape to the roll oscillogram $\varphi_1(t)$:

$$\ddot{\vartheta}_r(t) = \frac{\bar{\vartheta}_r}{\bar{\varphi}} \cdot \ddot{\varphi}_1(t) \tag{72}$$

Equation (63) then yields the differential equation

$$J'_0\left(1 + \frac{\bar{\vartheta}_r}{\bar{\varphi}}\right) \cdot \ddot{\varphi}_1 + P \cdot h(\varphi_1) = 0 \tag{73}$$

which is the same as (9) except for the change in the moment of inertia. The detuning factor relative to the natural roll period in calm water is thus

$$\frac{T_s}{T_0(\bar{\varphi})} = \sqrt{1 + \frac{\bar{\vartheta}_r}{\bar{\varphi}}} \tag{74}$$

The factor can be greater or smaller than one, depending on the phase angle between $\vartheta_r(t)$ and $\psi(t)$. For the amplitude of the reactive excitation we find, with $T_s = T_w$,

$$\bar{\vartheta}_r = \left(\frac{T_w^2}{T_0'^2(\bar{\psi})} - 1\right) \cdot \bar{\psi} \tag{75}$$

so that

$$\vartheta_r(t) = \left(\frac{T_w^2}{T_0'^2(\bar{\psi})} - 1\right) \cdot \psi_1(t). \tag{76}$$

b) The wave profile.

From equation (67) and (76) we can now determine the wave profile corresponding to the exact solutions found above. The oscillogram of the reactive excitation component is the same as the undamped roll oscillogram in calm water, apart from the scale factor (74). From (46) it can be shown that this component dominates outside the resonance interval. The amplitude here, however, is small, so that there is only a relatively small contribution from the higher harmonics. The active excitation component, on the other hand, is the integral of $\psi_0(t)$, so that here the n^{th} harmonic is reduced by a factor 1/n. However, the active excitation becomes important near the resonance point where the larger roll amplitudes entail a larger proportion of the higher harmonics (figs. 3a and 3b).

The wave profile itself is finally obtained by integrating the wave slope $\vartheta(t)$, which further reduces the higher harmonic contribution. Outside the resonance interval the profile is approximately equal to the integral of the basic oscillogram $\varphi_0(t)$ for small amplitudes, whereas near the resonance peak it is approximately equal to the second integral of the basic oscillogram for large amplitudes. The difference between a sinusoidal profile and the profile obtained in our case is thus acceptable in view of the inherent difficulty of defining the profile of an actual swell at sea.

c) <u>The response curves.</u>

Having determined the fundamental function $T_0(\bar{\varphi}) = 2\pi/\nu_0(\bar{\varphi})$ it is now a simple matter to evaluate graphically the response curves $\bar{\varphi}(\omega^2, \bar{\vartheta})$, i. e. the roll amplitude $\bar{\varphi}$ as a function of the square of the exciting frequency $\omega = 2\pi/T_W$, with the effective wave steepness $\bar{\vartheta}$ as a parameter. If the skeleton curve $\bar{\varphi}(\nu_0^2)$ is plotted in the $(\omega^2, \bar{\varphi})$-plane, the value ω^2 corresponding to a pair of values $(\bar{\varphi}, \bar{\vartheta}_r)$ can be obtained by projection, using

$$\nu_0^2(\bar{\varphi}) : \omega^2 = (\bar{\varphi} + \bar{\vartheta}_r) : \bar{\varphi} \qquad (77)$$

(which is indentical with (75)).

If the amplitude of the reactive excitation is known we can thus construct as many points $(\omega_1^2, \bar{\varphi})$ on the response curve as required. The method is illustrated in region 2 of fig. 9.

In the idealized case of a ship without damping, $\bar{\vartheta}_r$ equals $\pm\bar{\vartheta}$. The signs correspond to the two possible phase angles between $\varphi_1(t)$ and $\vartheta_r(t)$ and must both be considered. In this case the response curves can be constructed immediately and are plotted in figs. 10a and 10b for the fundamental functions shown in figs. 3a and 3b.

In the general case of finite damping the reactive excitation has first to be determined. If the oscillograms $\vartheta_a(t)$ and $\vartheta_r(t)$ are sinusoidal,

$$\bar{\vartheta}_r = \pm\sqrt{\bar{\vartheta}^2 - \bar{\vartheta}_a^2} \; . \tag{78}$$

In applying this relation to the general case of non-sinusoidal oscillograms, an appreciable error arises only if both amplitudes are of comparable magnitude. This is neither the case close to the resonance peak, where the active excitation dominates, nor further away, from it where the reverse is the case. We shall accept this small error in the following in order to facilitate the representation of the response curves. The curves calculated using (78) are shown in figs. 11a and 11b, with $D(T_s) = 0{,}2$ and $F(\bar{\varphi}) = 1$. The reactive excitation amplitudes were determined as shown in region 1 of fig. 9. For constant D and F = 1, equation (70) becomes

$$\bar{\vartheta}_a = D \cdot \bar{\varphi} . \tag{79}$$

In the resonance case $\bar{\vartheta}_r = 0$ therefore

$$\bar{\vartheta} = D \cdot \bar{\varphi}_{res} \tag{80}$$

so that (78) becomes

$$\bar{\vartheta}_r = \pm D \cdot \sqrt{\bar{\varphi}_{res}^2 - \bar{\varphi}^2} . \tag{81}$$

The length $\bar{\vartheta}_r / D$ for a given roll amplitude $\bar{\varphi} < \bar{\varphi}_{res}$ is obtained from the quarter circle of radius $\bar{\varphi}_{res} = \bar{\vartheta}/D$ (fig.9 - the radius can also be larger than φ_R) by Pythagoras (shaded triangle). The reactive excitation amplitudes themselves can then be obtained from the two straight lines of slope $\pm$ D:1.

The response curves for the roll states of the first kind can be obtained analytically from (70), (75) and (78):

$$\bar{\vartheta}^2 = \left[\left(\frac{T_w^2}{T_0'^2(\bar{\psi})} - 1\right)^2 + F^2(\bar{\psi}) \cdot D^2(T_w)\right] \cdot \bar{\psi}^2 \tag{82a}$$

or, in terms of the angular frequencies [6])

$$\bar{\vartheta}^2 = \left[\left(\frac{\nu_0^2(\bar{\psi})}{\omega^2} - 1\right)^2 + F^2(\bar{\psi}) \cdot D^2(\omega)\right] \cdot \bar{\psi}^2 \tag{82b}$$

d) Stable roll states.

The roll states derived above all satisfy the equation of motion (1). It remains to be investigated, however, if they can also be realized physically. This question is of particular interest in view of the existance of multiple-valued regions of the response curves.

If we confine our attention to steady roll states, the question of general dynamic stability cannot be immediately answered. We shall therefore formulate a stability criterion which is applicable to our response curves derived for the steady roll states as follows:

We shall regard a roll state as stable if for a given period an increment in the magnitude of the exciting amplitude leads to an increment of the same sign in the magnitude of the oscillation amplitude. Analytically,

$$C \equiv \frac{\partial \bar{\vartheta}^2}{\partial \bar{\varphi}^2} > 0. \tag{83}$$

If we now consider the response curve tangent for constant D,

$$\frac{d\bar{\varphi}}{dT_W^2} = -\frac{\frac{\partial \bar{\vartheta}^2}{\partial T_W^2}}{2\bar{\varphi}\cdot\frac{\partial \bar{\vartheta}^2}{\partial \bar{\varphi}^2}} = -\frac{\bar{\varphi}}{C\cdot T_0^2(\bar{\varphi})}\cdot\left(\frac{T_W^2}{T_0^2(\bar{\varphi})}-1\right) \tag{84a}$$

or

$$\frac{d\bar{\varphi}}{d\omega^2} = -\frac{\frac{\partial \bar{\vartheta}^2}{\partial \omega^2}}{2\bar{\varphi}\cdot\frac{\partial \bar{\vartheta}^2}{\partial \bar{\varphi}^2}} = -\frac{\bar{\varphi}\cdot \nu_0^2(\bar{\varphi})}{C\cdot\omega^4}\cdot\left(\frac{\nu_0^2(\bar{\varphi})}{\omega^2}-1\right), \tag{84b}$$

we find that in both cases for stable roll states ($C > 0$) the tangent is positive to the left of the skeleton curve and negative to the right. The tangents become infinite on the boundary curve separating the stable and unstable regions, for which $C = 0$. From the amplitude relation (82b) and with $F = 1$ the limiting curve of the stability domain can be obtained in form of the equation

(85)

$$C \equiv \left(\frac{\nu_0^2(\bar{\psi})}{\omega^2} - 1\right)^2 + D^2(\omega) + \left(\frac{\nu_0^2(\bar{\psi})}{\omega^2} - 1\right) \cdot \frac{\bar{\psi}}{\omega^2} \cdot \frac{d\,\nu_0^2(\bar{\psi})}{d\bar{\psi}} = 0 \qquad (85)$$

$$\equiv \left(\frac{\nu_0^2(\bar{\psi})}{\omega^2} - 1\right) \cdot \left(\frac{\nu_0^2(\bar{\psi})}{\omega^2} + \frac{\bar{\psi}}{\omega^2} \cdot \frac{d\,\nu_0^2(\bar{\psi})}{d\bar{\psi}} - 1\right) + D^2(\omega) = 0. \qquad (85)$$

From this equation the boundary curve can easily be constructed. Neglecting the damping for the first, we obtain the two solutions (the bracket-terms in (85) being zero):

$$\omega^2 = \nu_0^2(\bar{\psi}) \qquad \text{and} \qquad \omega^2 = \nu_*^2(\bar{\psi}). \qquad (86)$$

The first solution is the skeleton curve; the second is related to this curve by the equation

$$\nu_*^2(\bar{\psi}) = \nu_0^2(\bar{\psi}) + \bar{\psi} \cdot \frac{d\,\nu_0^2(\bar{\psi})}{d\bar{\psi}}. \qquad (87)$$

In the construction of the limiting curve $\nu_0^2(\bar{\psi})$, as shown in fig. 12, each point of the skeleton curve has been simply translated laterally by the subtangent length s = $\left|\bar{\psi} \cdot \frac{d\,\nu_0^2(\bar{\psi})}{d\bar{\psi}}\right|$ .Introducing $\nu_*^2(\bar{\psi})$ from (87) equation (85) becomes

(88)

$$C \equiv \left(\frac{\nu_0^2(\bar\varphi)}{\omega^2} - 1\right)\cdot\left(\frac{\nu_*^2(\bar\varphi)}{\omega^2} - 1\right) + D^2(\omega) = 0 \qquad (88)$$

For D = 0, C is negative in the region between the two limiting curves found above, so that this region is unstable. For finite damping, the unstable region becomes smaller, as can be seen from (88). To construct the limiting curve L in the case of finite damping we rewrite (88) in the form

$$\left(\nu_0^2(\bar\varphi) - \omega^2\right)\cdot\left(\nu_*^2(\bar\varphi) - \omega^2\right) + \left(\omega^2\cdot D(\omega)\right)^2 = 0. \qquad (89)$$

and apply the second law. The damping function $\mathcal{D}(\omega)$ can be arbitrary. If $\omega^2\cdot D(\omega)$ is drawn below the ω^2-axis, the abscissae ω_1^2, ω_2^2 of the limiting ~~curves~~ curve for a given $\bar\varphi$ are obtained as the intercepts of the curve with the semicircle drawn below the ω^2-axis through the points $\nu_0^2(\bar\varphi)$, $\nu_*^2(\bar\varphi)$ on the axis. The diameter of the semicircle is intersected normally in the points $\omega_1{}^2$ and $\omega_2{}^2$ by a secant of half length $\omega^2 \cdot D(\omega)$, giving equation (89). The method is illustrated in fig. 12 for constant D = 0,2. The shaded regions of instability in figs. 10a - 11b were determined in this way.

e) Probable roll states.

Having determined which regions of the response curves correspond to physically realizable states, the question arises which of the different possible states for a given period will actually occur in practice. This is a statistical problem, depending on the initial condition of the ship and the random moments which could bring about a transition from one state to another. These effects lie beyond the scope of our considerations, which are confined only to steady roll states. However, the following two comments may serve for a general orientation: Firstly, we shall expect roll states of larger amplitude to be less probable than those of smaller amplitude, as their higher energy content has to be imparted to the ship initially. Secondly, in experiments with a navipendulum corresponding to a sinusoidal righting-arm curve, the author always found that the higher energy roll states could be realized only with great care, very small disturbance sufficing to cause transition to the lower-energy state. The phenomenon became more marked the higher the energy of the state. Energy considerations and experiment thus suggest a greater probability for the roll states of smaller amplitude. The random disturbances which in practice are always superimposed on the swell will seldom give the ship just the impulse required for transition to the higher energy state, though they will generally be strong enough to disturb this state on account of its smaller stability.

f) The influence of the vertical acceleration.

The roll states of the first kind as derived above will be slightly modified by the periodical vertical acceleration which we have so far neglected. To obtain an estimate of this influence we substitute the variable factor 1 + ß (t) in the restoring moment (see equation (1)) by a suitable mean value $1 \pm \varepsilon . |\bar{\beta}|$ during the appropiate semicycle ($0 < \varepsilon < 1$, - the sign is positive in the wave trough and negative on the crest). It is then seen immediately that the time required for a semicycle is reduced in the wave trough and increased on the wave crest by the vertical acceleration. The total period, however, remains the same, as for roll states of the first kind

$$\int_0^{T_s} \beta . h(\varphi) . \varphi . dt = 0 \tag{90}$$

i. e. the detuning equation (46) remains unchanged.

A further result due to the vertical acceleration is an unsymmetry of the oscillation. This is particularly marked in the resonance case $\bar{\vartheta}_r = 0$. The roll oscillogram $\psi_1(t)$ here lags a quarter of a period behind the wave slope $\vartheta(t) = \vartheta_\alpha(t)$ (equation (67)). On a wave crest the ship therefore has a maximum angle of roll, the direction being towards the latest wave slope, i.e. to lee; in a trough the ship has its maximum windward roll angle. Outside of the resonance intervals the maximum roll angles occur earlier or later. As the restoring moment is smaller on the crest than in the trough, the lee roll angle will be greater than the windward roll angle. This results in a mean heel to lee ψ_- of the roll oscillogram

$$\psi_1(t) = \psi_- + \psi_\sim(t). \tag{91}$$

To estimate the magnitude of ψ_- we equate the potential energies of the two extreme positions:

$$[1 - \varepsilon \cdot |\beta|] \cdot U(\psi_- + \bar{\psi}) = [1 + \varepsilon \cdot |\beta|] \cdot U(\psi_- - \bar{\psi}). \tag{92}$$

Expanding U in powers of ψ_- and making use of the symmetry of U we obtain in the first approximation:

$$U(\psi_- + \bar{\psi}) \pm U(\psi_- - \bar{\psi}) \approx \begin{cases} 2\,U(\bar{\psi}) \\ 2\psi_- \cdot P \cdot h(\bar{\psi}) \end{cases} \tag{93}$$

and thus finally

$$\frac{|\psi_-|}{|\bar{\psi}|} \approx \frac{\varepsilon \cdot U(\bar{\psi}) \cdot |\bar{\beta}|}{\bar{\psi} \cdot P \cdot h(\bar{\psi})} < \frac{1}{2} q^2(\bar{\psi}) \cdot |\bar{\beta}| \tag{94}$$

where $q(\bar{\psi})$ is the function defined in (28) and plotted in fig. 2. The finite mean heel angle does not invalidate our original approach, as ψ_- has no influence on the angular acceleration. However, we must emphasize subsequently that the amplitude $\bar{\psi}$ should be taken as the amplitude of $\psi_\sim(t)$.

6. The Roll States of the Second Kind, $T_s = 2\ T_w$.

In the roll states of the second kind the periodic variation of the apparent vertical, $\vartheta(t)$, yields no contribution to equations (45) and (46). We shall therefore set

$$\varphi(t) \equiv \psi(t) = \varphi_2(t) \tag{95}$$

for the first and consider the influence of the variation in the apparent vertical later. The equation of motion then becomes

$$J' \cdot \ddot{\varphi}_2 + W(T_s) \cdot \dot{\varphi}_2 + [1 + \beta(t)] \cdot P \cdot h(\varphi_2) = 0 \tag{96}$$

which can be decomposed again into the two equations

$$W(T_s) \cdot \dot{\varphi}_2 + \beta_a \cdot P \cdot h(\varphi_2) = + N_2(t) \tag{97}$$

$$J' \cdot \ddot{\varphi}_2 + [1 + \beta_r] \cdot P \cdot h(\varphi_2) = - N_2(t) \tag{98}$$

containing only components of the same phase. The moment $N_2(t)$ again has no influence on the excitation or detuning of the ship, as from (56) and (57) (98)

$$\int_0^{T_s} N_2 \cdot \dot{\varphi}_2 \cdot dt = 0 \quad \text{and} \quad \int_0^{T_s} N_2 \cdot \varphi_2 \cdot dt = 0 . \qquad (99)\ (100)$$

$N_2(t)$ is a combination of higher harmonics which can be disposed of within certain limits. It cannot vanish identically, as the terms on the left of (97) have different periods. It contains essentially a component of period $\frac{1}{3}T_s$. We shall postpone the discussion of this term until later, however.

If we again disregard the precise shape of the wave profile we can prescribe the periodical vertical acceleration in a manner enabling simplification of the analysis. We shall thus relate the active and reactive exciting moments (which must satisfy the conditions (52) and (53)) as simply as possible to the potential energy (11):

$$\beta_a(t) = -\bar{\beta}_a \cdot 2\sqrt{u(\psi_2)\cdot\left(1-u(\psi_2)\right)}\ \mathrm{sg}\,\psi_2\cdot\dot{\psi}_2 \tag{101}$$

$$\beta_r(t) = \bar{\beta}_r \cdot \left(1-2u(\psi_2)\right) + \beta_0, \quad \psi_2(t), \tag{102}$$

where

$$0 \leq u(\psi) \equiv \frac{U(\psi)}{U(\bar{\psi})} \leq 1, \quad \psi_2(t). \tag{103}$$

The active excitation amplitude $\bar{\beta}_a$ is always positive, whereas the reactive excitation amplitude $\bar{\beta}_r$ can also become negative, depending on the phase angle. β_o is a constant chosen to make the mean acceleration vanish:

$$\int_0^{T_w} \beta(t).dt = 0. \tag{104}$$

The equations can also be written in the form

$$\begin{aligned} \beta(t) &= -\bar{\beta} \cdot \sin\left(2\gamma(t) - \gamma_0\right) + \beta_0 \\ &= \bar{\beta}_r \cdot \cos 2\gamma(t) - \bar{\beta}_a \cdot \sin 2\gamma(t) + \beta_0 \end{aligned} \tag{105}$$

where

$$\bar{\beta}_r = \bar{\beta} \cdot \sin\gamma_0, \qquad \bar{\beta}_a = \bar{\beta} \cdot \cos\gamma_0 \tag{106}$$

Here γ_0 is a phase constant and $\gamma(t)$ the (non-uniformly) varying phase angle. The amplitude of the total excitation is given by the exact expression

$$\bar{\beta}^2 = \bar{\beta}_r^2 + \bar{\beta}_a^2. \tag{107}$$

As the active excitation component (101) is symmetrical, it yields no contribution to the integral (104), so that β_o depends only on the oscillogram of the reactive excitation component. From (102) then

$$\frac{\beta_o}{\bar{\beta}_r} = \frac{2}{T_W} \cdot \int_0^{T_W} \mathcal{U}(\varphi_2) \cdot dt - 1 \qquad (108)$$

$$= 1 - \eta_2(\bar{\varphi})$$

(108)

where T_W can also be replaced by a multiple, e.g. by $T_s = 2\,T_W$. The function $\eta_2(\bar{\varphi})$ differs from the fundamental function $\eta_0(\bar{\varphi})$ given in equation (35) as the oscillogram $\varphi_2(t)$ of the forced oscillation is not the same as the oscillogram $\varphi_0(t)$ of the free oscillation. It will be shown later that to a first approximation

$$\eta_2(\bar{\varphi}) = \eta_0(\bar{\varphi}) - \frac{1}{8}\bar{\beta}_r \cdot \left(2\eta_0(\bar{\varphi}) - \eta_0^2(\bar{\varphi}) + \delta\right) \qquad (109)$$

(109)

where

$$|\delta| \leq \eta_0(\bar{\varphi}) \cdot \left(2 - \eta_0(\bar{\varphi})\right) \leq 1 .$$

a) <u>The roll amplitudes.</u>

The amplitude of the active excitation component can be obtained by substituting (101) in (97) and evaluating the integral (99). With (11) we find

$$\bar{\beta}_\alpha \cdot 2\,\mathcal{U}(\bar{\varphi}) \cdot \oint_{T_s = 2T_W} \sqrt{\mathcal{U}(\varphi_2) \cdot (1 - \mathcal{U}(\varphi_2))} \cdot d\,\mathcal{U}(\varphi_2) = W(T_s) \cdot \int_0^{T_s} \dot{\varphi}_2^2 \cdot dt \, , \qquad (110)$$

where the root has the same sign as $\psi_2 \cdot \dot{\psi}_2$.

The plot of the integrand on the left side over $U(\psi)$ is a circle of radius 1/2. As $U(\psi_2)$ completes two oscillations in the time T_s, the left side of (110) is equal to $\bar{\beta}_a \cdot U(\bar{\psi}) \cdot \pi$, so that the equation becomes

$$\bar{\beta}_a = \frac{T_s \cdot W(T_s)}{\pi \cdot J'} \cdot \frac{2}{T_s} \cdot \int_0^{T_s} \frac{\frac{1}{2} J' \cdot \dot{\psi}_2^2}{U(\bar{\psi})} \cdot dt \tag{111}$$

The first factor is twice the dimensionless damping coefficient of equation (71). If we allow for the modification of the $\psi_2(t)$-oscillogram by the reactive excitation $\beta_r(t)$, the integral can be reduced to $\eta_0(\bar{\psi})$ defined in (35). It will be shown later that to a first approximation ~~it equals~~

$$\frac{2}{T_s} \int_0^{T_s} \frac{\frac{1}{2} J' \dot{\psi}_2^2}{U(\bar{\psi})} dt = \eta_0(\bar{\psi}) + \frac{1}{8} \bar{\beta}_r \cdot \left(2\eta_0(\bar{\psi}) - 5\eta_0^2(\bar{\psi}) + \delta\right) \equiv \eta_\beta(\bar{\psi}) \tag{112}$$

where

$$|\delta| \leq \eta_0(\bar{\psi}) \cdot \left(2 - \eta_0(\bar{\psi})\right) \leq 1 .$$

(112)

The amplitude of the active excitation can thus be expressed as the product

$$\bar{\beta}_a = 2D(T_s)\cdot\eta_\beta(\bar{\varphi}). \tag{113}$$

As we have neglected the depedence of the damping on the roll amplitude we can with the same consequence also neglect also the dependence of the active excitation on the amplitude. This can be further justified by the fact that experience indicates an increase of D with the amplitude (fig. 8), whereas the mean value $\eta_\beta(\bar{\varphi})$ decreases (fig. 5), the two effects thus tending to cancel. Within the limits of these uncertainties we shall thus use the simplified formula

$$\bar{\beta}_a \approx 2D(T_s) \approx \text{const.} \tag{114}$$

In contrast to the roll states of the first kind, where the active excitation was essentially proportional to the roll amplitudes, the active excitation in this case is practically independent of the roll amplitude.

This means that roll oscillations of the second kind can be generated only if the swell exceeds the critical value

$$H:\lambda \geq 2D(T_s):\pi \tag{115}$$

The amplitude of the reactive excitation then is determined by (107), and is also practically constant for a given swell strength $\bar{\beta}$.

If we substitute the reactive excitation (102) in (98) and neglect the higher harmonic combination $N_2(t)$ for the first, we obtain the equation of motion for a ship rolling in calm water with a modified restoring moment given by

$$h_\beta(\psi) = [1+\beta_r(\psi)]\cdot h(\psi) \tag{116}$$

$$= [1+\beta_0+\bar{\beta}_r(1-2u(\psi))]\cdot h(\psi). \quad (116)$$

We can then derive the following expressions analogous to equations (11) - (13); for the potential heel energy

$$U_\beta(\psi) = P\cdot\int h_\beta(\psi)\cdot d\psi \tag{117}$$

$$= U(\bar{\psi})\cdot\left[(1+\beta_0)u(\psi) + \beta_r\cdot u(\psi)\cdot(1-u(\psi))\right], \quad (117)$$

for the total energy

$$\tfrac{1}{2}J'\cdot\dot{\psi}^2 + U_\beta(\psi) = U_\beta(\bar{\psi}), \tag{118}$$

and for the time differential of the forced oscillation $\varphi_2(t)$

$$dt(\varphi_2) = \sqrt{\tfrac{1}{2} J'} \cdot \frac{d\varphi}{\sqrt{U_\beta(\bar{\varphi}) - U_\beta(\varphi)}} \tag{119}$$

$$= \sqrt{\tfrac{1}{2} J'} \cdot \frac{d\varphi}{\sqrt{U(\bar{\varphi}) - U(\bar{\varphi})} \cdot \sqrt{1 + \beta_0 - \bar{\beta}_r \cdot U(\varphi)}} \quad (119)$$

Introducing the time differential for the free oscillation $\varphi_0(t)$ given by (12), equation (119) becomes (fig. (13))

$$dt(\varphi_2) = \frac{dt(\varphi_0)}{\sqrt{1 + \beta_0 - \bar{\beta}_r \cdot U(\varphi_0)}} \tag{120}$$

and the period of the forced oscillation

$$T_S = \int_0^{T_0(\bar{\varphi})} \frac{dt}{\sqrt{1 + \beta_0 - \bar{\beta}_r \cdot U(\varphi_0)}} , \quad \varphi_0(t). \tag{121}$$

(121)

The detuning factor

$$\frac{T_S}{T_0(\bar{\varphi})} = \frac{1}{T_0(\bar{\varphi})} \cdot \int_0^{T_0(\bar{\varphi})} \left[1 + \beta_0 - \bar{\beta}_r \cdot U(\varphi_0)\right]^{-1/2} \cdot dt \tag{122}$$

resulting from the periodic vertical acceleration is thus the mean value of the expression

$$\left[1+\beta_0 - \bar{\beta}_r \cdot u(\psi_0)\right]^{-1/2}$$

taken over the natural period.

The exact evaluation of this integral is generally very difficult. We shall therefore limit ourselves to an approximation of the first order in $\bar{\beta}_r$ which is valid for all $\bar{\psi}$ (see also (108)):

$$\left[1+\beta_0 - \bar{\beta}_r \cdot u(\psi_0)\right]^{-1/2} = 1 - \tfrac{1}{2}\bar{\beta}_r \cdot \left(1 - \eta_2(\bar{\psi}) - u(\psi_0)\right) \pm \ldots \tag{123}$$

The condition for absolute convergence is

$$\left|\bar{\beta}_r \cdot \left(1-\eta_2(\bar{\psi}) - u(\psi_0)\right)\right| < 1. \tag{124}$$

As will be shown later, this leads to the conditions

$$\begin{aligned} -0{,}83 < \bar{\beta}_r < +0{,}83 \quad & \text{for decreasing stiffness,} \\ -0{,}60 < \bar{\beta}_r < +0{,}66 \quad & \text{for increasing stiffness.} \end{aligned} \tag{125}$$

With

$$\eta_2(\bar{\psi}) \approx \eta_0(\bar{\psi})$$

and - see (35) -

$$\frac{1}{T_0'(\bar{\psi})} \cdot \int_0^{T_0'(\bar{\psi})} u(\psi_0) \cdot dt = 1 - \tfrac{1}{2}\eta_0(\bar{\psi})$$

the detuning factor then becomes approximately

$$\frac{T_s}{T_0(\bar{\psi})} \approx 1 - \tfrac{1}{2}\bar{\beta}_r \cdot \left(1 - \eta_0(\bar{\psi}) - 1 + \tfrac{1}{2}\eta_0(\bar{\psi})\right) \qquad (126)$$

$$\approx 1 + \tfrac{1}{4}\bar{\beta}_r \cdot \eta_0(\bar{\psi}). \qquad (126)$$

As $T_s = 2\,T_W$ we have thus derived an implicite representation of the response curves of the second kind. A transformation to the explicite form $\bar{\psi}(T_W, \bar{\beta}_r)$ cannot be carried out, as the functions $T_0(\bar{\psi})$ and $\eta_0(\bar{\psi})$ are transcendent. However, the two branches of the response curve corresponding to a given reactive excitation amplitude $\pm|\bar{\beta}_r|$ can easily be gained from the given skeleton curve $T_0(\bar{\psi})$ using equation (126). The results are plotted in fig. 14a and 14b for the righting-arm curves shown and $|\bar{\beta}_r| = 0{,}2$.

We have found that the roll period of a ship is only slightly detuned by the periodical vertical acceleration. Roll states of the second kind are thus not possible for all swell periods. Because of the steeply rising response curve, however, the amplitude may become dangerously large.

b) <u>Stable roll states.</u>

From (107), (114) and (126):

$$\bar{\beta}^2 = \left(\frac{T_s}{T_0(\bar{\psi})} - 1\right)^2 \cdot \frac{16}{\eta_0^2(\bar{\psi})} + 4D^2(T_s). \qquad (127)$$

To determine the stable regions of the response curves we apply our stability criterion of section 5d. For stable roll motions we then have

$$C \equiv \frac{\partial \bar{\beta}^2}{\partial \bar{\varphi}^2} > 0. \tag{128}$$

The derivatives of the response curves $\bar{\varphi}(T_s, \bar{\beta})$ for a given wave strength $\bar{B}$ = const, are

$$\frac{d\bar{\varphi}}{dT_s} = -\frac{\partial \bar{\beta}^2 / \partial T_s}{\partial \bar{\beta}^2 / \partial \bar{\varphi}} = -\frac{\partial \bar{\beta}^2 / \partial T_s}{2\bar{\varphi} \cdot C}. \tag{129}$$

The response curves thus have infinite derivatives on the limiting curve of the stability region, C = 0, and also on the axis $\bar{\varphi} = 0$. From (127) we find further for constant D

$$\frac{d\bar{\varphi}}{dT_s} = \frac{16 \cdot (T_0(\bar{\varphi}) - T_s)}{C \cdot \bar{\varphi} \cdot T_0^2(\bar{\varphi}) \cdot \eta_0^2(\bar{\varphi})} \tag{130}$$

This yields the following rule for the response curves of the second kind (plotted over either the period or the frequency): For stable roll motions the curve tangents are positive on the left of the skeleton curve and negative on the right. This determines the sign of $\bar{\beta}_r$:

$$\bar{\beta}_r > 0 \qquad \text{for increasing stiffness,}$$
$$\bar{\beta}_r < 0 \qquad \text{for decreasing stiffness.} \tag{131}$$

The unstable parts of the response curves are shown dotted in fig. 14a and 14b. In the following it will be shown that the dotted interval on the T_s-axis, corresponding to the trivial solution $\psi_2(t) \equiv 0$ of (96), for which our stability criterion is not applicable, is also unstable.

c) <u>The critical period interval.</u>

In the limiting case of small amplitudes

$$h(\psi) = \overline{M_0 g} \cdot \psi \,, \qquad U(\psi) = \tfrac{1}{2} P \cdot \overline{M_0 g} \cdot \psi^2 \tag{132}$$

and

$$T_0(\bar{\psi}) = T_0(0) = 2\pi \cdot \sqrt{\frac{J'}{P \cdot \overline{M_0 g}}} \,. \tag{133}$$

This case is particularly useful for studying certain characteristic properties of the roll states of the second kind. Neglecting $N_2(t)$, (98) becomes a linear (Hill) differential equation

$$J' \cdot \ddot{\psi}_2 + [1 + \beta_r(t)] \cdot P \cdot \overline{M_0 g} \cdot \psi_2 = 0 \,. \tag{134}$$

With

$$u(\varphi) = \varphi^2 / \bar{\varphi}^2 \qquad (135)$$

equation (102) then leads to the non-linear differential equation

$$J' \cdot \ddot{\varphi}_2 + \left[1 + \beta_0 + \bar{\beta}_r \left(1 - 2 \cdot \frac{\varphi_2^2}{\bar{\varphi}^2}\right)\right] \cdot P \cdot \overline{M_0 g} \cdot \varphi_2 = 0 \qquad (136)$$

for the steady roll states. This is the free roll equation for a ship in calm water whose righting-arm curve is a cubic parabola:

$$h_\beta(\varphi) = \left[1 + \beta_0 + \bar{\beta}_r - 2 \bar{\beta}_r \cdot \frac{\varphi^2}{\bar{\varphi}^2}\right] \cdot \overline{M_0 g} \cdot \varphi . \qquad (137)$$

The roll oscillogram is therefore not sinusoidal, the difference increasing with the amplitude of the reactive excitation. Nevertheless, the period of oscillation is independent of the roll amplitude, as the latter is proportional to the stability limit of the righting-arm curve (137), (i. e. to the angle $\varphi = \varphi_{(R)}$ for which the expression in rectangular brackets vanishes) - the proportionality factor depending only on the reactive excitation amplitude -

$$\frac{\bar{\varphi}^2}{\varphi_{(R)}^2} = \frac{2 \bar{\beta}_r}{1 + \beta_0 + \bar{\beta}_r} , \qquad (138)$$

The response curves therefore run parallel to the ordinate axis, so that $\bar{\varphi}$ is variable independently of $\bar{B}$. The resonance states of the second kind are thus indifferent for small amplitudes, as the expression (128) vanishes.

Using (133), (135) and the substitution $x = \varphi/\bar{\varphi}$, the roll period can be obtained from (119) as an elliptic integral of the first kind:

$$T_S = \frac{T_0(0)}{\sqrt{1+\beta_0}} \cdot \frac{2}{\pi} \cdot \int_0^1 \frac{dx}{\sqrt{(1-x^2)\cdot(1-k^2x^2)}} = \frac{T_0(0)}{\sqrt{1+\beta_0}} \cdot \frac{2}{\pi} K \tag{139}$$

where

$$k^2 = \frac{\bar{\beta}_r}{1+\beta_0} . \tag{140}$$

The excitation can also be expressed as a function of the parameter k^2 by means of elliptic integrals. From (140) and (108) we obtain two simultaneous equations for the amplitude of the reactive excitation

$$\left.\begin{aligned} \bar{\beta}_r - k^2\cdot\beta_0 &= k^2 \\ [1-\eta_2(0)]\cdot\bar{\beta}_r - \beta_0 &= 0 \end{aligned}\right\} \tag{141}$$

with the solutions

$$\bar{\beta}_r = \frac{k^2}{1-k^2\cdot[1-\eta_2(0)]} \quad ; \quad \beta_0 = \frac{k^2\cdot[1-\eta_2(0)]}{1-k^2\cdot[1-\eta_2(0)]} \quad . \qquad (142)$$

Using equations (119) and (135) and the substitution $x = \varphi_2/\bar{\varphi}$ the mean value

$$\eta_2(0) = \frac{2}{T_S}\cdot\int_0^{T_S}[1-u(\varphi_2)]\cdot dt \qquad (143)$$

which depends on the oscillogram shape but not on its amplitude, can be written

$$\eta_2(0) = 2 - \frac{2}{T_S}\cdot\sqrt{\frac{J'}{2\,M_0 g\cdot(1+\beta_0)}}\cdot 4\cdot\int_0^1 \frac{x^2\cdot dx}{\sqrt{(1-x^2)\cdot(1-k^2x^2)}} \quad , \qquad (144)$$

so that from (133) and (139)

$$\eta_2(0) = 2\cdot\left(1-\frac{D}{K}\right) \quad , \qquad (146)$$

where D is the complete elliptic integral

$$D \equiv \int_0^1 \frac{x^2\cdot dx}{\sqrt{(1-x^2)\cdot(1-k^2x^2)}} \quad . \qquad ~~(146)~~ \; (145)$$

7)

With the aid of (142) we can thus determine both the reactive excitation amplitude $\bar{\beta}_r$ and the roll period (139) as functions of k^2.

To determine the active excitation we evaluate the mean value of $\eta_\beta(\bar{\varphi})$ defined by equations (111) and (113). For small amplitudes $\bar{\varphi} \to 0$, we find from (118), (117), (132) and (135), with $x = \varphi_2/\bar{\varphi}$

$$\eta_\beta(0) = \frac{2}{T_s} \cdot \sqrt{\frac{J'(1+\beta_0)}{P \cdot M_0 g}} \cdot 4 \cdot \int_0^1 \sqrt{(1-x^2)\cdot(1-k^2x^2)} \cdot dx \, . \tag{147}$$

This expression can also be reduced to the complete elliptic integrals K and D, since

$$\int_0^1 \sqrt{(1-x^2)\cdot(1-k^2x^2)} \cdot dx = \frac{3}{2} K - \frac{1}{3}(1+k^2) \cdot D \, . \tag{148}$$

Using (139) and (133), the expression (147) then becomes

$$\eta_\beta(0) = \frac{1+\beta_0}{3} \cdot \left(4 - (1+k^2) \cdot \frac{2D}{K}\right) . \tag{149}$$

$\bar{\beta}_a = 2D(T_s) \cdot \eta_\beta(0)$

In fig. 15 the active exciting amplitude $\bar{\beta}_a$ and the detuning factor $T_s/T_0(0)$ are plotted over the reactive excitation amplitude for constant D = 0,2. From the figure we can read off the magnitude $\bar{\beta}$ and phase γ_0 of the excitation (~~on~~ the equivalent components $\bar{\beta}_a$ and $\bar{\beta}_r$) required to sustain a roll state of small amplitude within a finite interval around the natural period $T_o(o)$.

The figure is also useful for explaining qualitatively the nascent state of the roll motions of the second kind and the way in which the final steady state is attained. If twice the value of the exciting period , $2\ T_w$, lies within the interval determined by the arc of radius $\bar{\beta}$ about the origine, the ship becomes unstable as soon as the periodic vertical acceleration exceeds the critical value $\bar{\beta}$ crit (which is practically equal to $2\ D(T_o)$ as the curve is only slightly inclined to the $\bar{\beta}_r$-axis). The active excitation amplitude $\bar{\beta} \cdot \cos \gamma_o$ then exceeds the damping value $2 D(T_s) \cdot \gamma_\beta(\varphi)$ and after an initial random impulse the ship begins to roll with the period $T_s = 2\ T_w$. As the initial roll amplitude can be assumed arbitrarily small for this argumentation, it is seen that within this critical period interval the trivial solution $\varphi_2(t) \equiv o$ of equation (96), to which our criterion (128) could not be applied, represents an unstable roll state of the second kind. Using (126), the critical period interval determined by the arc in fig. 15 can be written for small $|\bar{\beta}_r|$ in the approximate form

$$\left|\frac{T_s}{T_{d(0)}} - 1\right| < \frac{1}{4}\left|\bar{\beta}_r\right|. \tag{150}$$

Under the said conditions the roll amplitude increases after an initial random impulse until a point is reached where the reactive excitation corresponding to the changed natural period $T_0(\bar{\varphi})$ reduces the active excitation amplitude $\bar{\beta}_a = +\sqrt{\bar{\beta}^2 - \bar{\beta}_r^2}$ to the value $\varepsilon D(T_s) \cdot \eta_\beta(\bar{\varphi})$ just balancing the damping. As the active excitation (113) required to balance the damping scarcely varies with the amplitude, the final steady state is attained primarily by detuning, (see figs. 14a and 14b) It can readily be seen that the state is stable, ~~as~~ a larger roll amplitude would lead to a further detuning. ~~shift from the resonance peak.~~ This would necessate a larger reactive excitation, which could only be yielded by a larger total excitation - provided the damping does not decrease excessively with increasing amplitude. Our stability criterion (128) can thus also be understood from this view-point.

We have seen that roll states of the second kind will always occur in the critical interval (150). Outside this interval, the trivial solution $\varphi_2(t) \equiv 0$ is stable, so that a specific impulse is necessary to bring the ship into a non-trivial roll state of the second kind. Although all kinds of impulses are to be expected in a seaway, the probability for the impulse required is relatively small, as not only must it impart to the ship a large amount of energy, but it must also have just the right magnitude and phase.

The maximum roll amplitude $\bar{\varphi}_{max}$ which will occur with certainty is marked in figs. 14a and 14b. As the point representing this roll state lies above the limiting point of the critical period interval, which is also the limiting point of the (dotted) instability curve, we obtain the following equation for $\bar{\varphi}_{max}$, using (126):

$$\left(1 + \frac{1}{4}\bar{\beta}_T \cdot z_0(\bar{\varphi}_{max})\right) \cdot T_0(\bar{\varphi}_{max}) = \left(1 - \frac{1}{4}\bar{\beta}_T\right) \cdot T_0(0), \tag{151}$$

In solving the equation the sign rule (131) has to be observed. For normal values of $|\bar{\beta}_T|$ the steady state will be attained by a relatively small degree of detuning. The quotient $\tau(\bar{\varphi}_{max}) = T_0(\bar{\varphi}_{max}) / T_0(0)$ is then only slightly different from one so that it can be well approximated by (29). If we further set (see figs. 2 and 5)

$$g(\bar{\varphi}_{max}) \approx 1 \quad \text{and} \quad z_0(\bar{\varphi}_{max}) \approx 1 \tag{152}$$

the two expressions under the roots in (29) become approximately equal, yielding the (somewhat crude) approximate formula

$$\tau(\bar{\varphi}_{max}) \approx \sqrt{\frac{\overline{M_0 g} \cdot \bar{\varphi}_{max}}{h(\bar{\varphi}_{max})}} \qquad (153)$$

From (153) we obtain the following simple formula for the maximum amplitude occuring with certainty:

$$h(\bar{\varphi}_{max}) \approx (1 + \bar{\beta}_r) \cdot \overline{M_0 g} \cdot \bar{\varphi}_{max} \,. \qquad (154)$$

This expression can be easily evaluated by intersecting the righting-arm curve with the straight line passing through the origin whose slope is $(1+\bar{\beta}_r)$ – times the slope of the initial tangent $\overline{M_0 g} \cdot \varphi$ of the righting-arm curve. The algebraic solution of (154) for $h(\varphi) = \overline{M_0 g} \cdot \varphi \cdot (1 + a\,\varphi^2)$ is

$$\bar{\varphi}_{max} \approx \sqrt{\frac{\bar{\beta}_r}{a}} \qquad (155)$$

From (131) it follows that the solution is real.

d) Appendices.

I. ~~The~~ Influences of the wave slope $\vartheta(t)$ and the higher harmonic combination $N_2(t)$.

We consider finally the influences of the periodically varying apparent vertical direction $\vartheta(t)$ and the higher harmonic combination $N_2(t)$ which is given by (97), (101) and the oscillogram $\varphi_2(t)$. We can use hereby the result found above that for the maximum amplitude occuring with certainty in roll states of the second kind the natural period $T_0(\bar{\varphi}_{max})$ is $(1-\frac{1}{2}\bar{\beta}_r)$ - times the natural period at small amplitudes - i. e. the influence of finite amplitude is in practice only of the order of a few percent. In the critical period interval the ship can thus be regarded as a quasi-linear system. Its motion can then be represented as a superposition of the three solutions $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$ of the differential equation (1) corresponding to the excitation by a)

a) the periodic variation of the apparent vertical direction, $\vartheta(t)$,

b) the periodic variation of the magnitude of the apparent vertical vector, $\beta(t) \cdot g$,

c) the moment $N_2(t)$ representing the difference between the exciting and damping moments.

The sum of the three solutions gained from the differential equations

$$\mathcal{J}' \cdot \left(\ddot{\varphi}_1 + \ddot{\vartheta}(t)\right) + W(T_s) \cdot \dot{\varphi}_1 + P \cdot h(\varphi_1) = 0 \qquad (156)$$

$$\mathcal{J}' \cdot \ddot{\varphi}_2 \qquad + \left[1 + \beta_r(t)\right] \cdot P \cdot h(\varphi_2) = 0 \qquad (157)$$

$$\mathcal{J}' \cdot \ddot{\varphi}_3 \qquad + W(T_s) \cdot \dot{\varphi}_3 + P \cdot h(\varphi_3) = -N_2(t) \qquad (158)$$

must be sufficiently small.
According to (5b) the exciting functions $\beta(t)$ and $\vartheta(t)$ have the same amplitude $\frac{2\pi r}{\lambda}$ and period T_W but are 90° out of phase. Their phase lag relative to the roll motion of the second kind $\varphi_2(t)$ of period $T_S = 2\ T_W$ depends on the damping and can be determined from (101). The situation is illustrated qualitatively in fig. 16 for the case $\bar{\beta}_r \approx 0$ where the vertical acceleration $\beta(t) \cdot g$ of the wave just exceeds the critical damping value $2D(T_s)_0 g$. As $\beta_2(t)$ (here $\approx \beta(t)$) and $\varphi_2 \cdot \dot{\varphi}_2$ are of opposite sign, the ship will be in its position either of static equilibrium or of maximum roll on the maximum wave slope, for which $\beta(t) = 0$. In the wave trough $\beta(t) > 0$ the ship rolls towards its static equilibrium position and on the wave crest $\beta(t) < 0$

in the reverse direction. In fig. 16 the successive positions of the ship are to be read from left to right. The wave is approaching from the right and the maximum roll angle $\pm|\overline{\varphi}_2|$ occurs on the rear slope of the wave.

On this roll motion we now superimpose a motion of the first kind $\varphi_1(t)$ of period T_W. In the linear approximation the differential equation (156) can be written

$$J'(\ddot{\varphi}_1 + \ddot{\vartheta}) + W(T_S)\cdot\dot{\varphi}_1 + J'\nu_1^2\cdot\varphi_1 = 0 , \tag{159}$$

$\nu_1 \approx 2\pi/T_S$ being substituted for the ship's natural frequency. As the exciting frequency $\omega_1 = 2\pi/T_W \approx 2\nu_1$ is well away from the resonance point we can neglect the damping, obtaining the approximate steady solution

$$\varphi_1(t) \approx -\tfrac{4}{3}\vartheta(t) . \tag{160}$$

The angles of roll on the rear wave slope are thus

$$\overline{\varphi}_{\substack{I\\II}} = \overline{\varphi}_1 \pm|\overline{\varphi}_2| = -\tfrac{4}{3}\overline{\vartheta} \pm|\overline{\varphi}_2| . \tag{161}$$

The maximum roll angle $\overline{\varphi}_{II}$ is directed towards the wave crest i. e. to lee. From (114) the wave steepness $|\overline{\vartheta}| = |\overline{\beta}|$ is equal to $2D(T_S)$ at the critical damping value. From (159) we can thus estimate the largest roll angle for simultaneous roll oscillations of the first and second kind:

$$\bar{\varphi}_{II} \approx \frac{8}{3} D(T_s) + \sqrt{\frac{\bar{\beta}_r}{a}} \quad . \tag{162}$$

In the higher harmonic combination $N_2(t)$ the dominating term is the third harmonic

$$\omega_3 = 3 \cdot \frac{2\pi}{T_s'} , \tag{163}$$

For the limiting case of a sinusoidal oscillogram this is, in fact, the only term. $N_2(t)$ has its maximum $\bar{N}$ value $\bar{N}$ when the angular velocity $\dot{\varphi}_2$ is a maximum. Using (118) the amplitude is approximately

$$\bar{N} \approx W(T_s) . \sqrt{\frac{2\, U(\bar{\varphi}_2)}{J'}} \tag{164}$$

for small $/\bar{\beta}_r/$. If we now substitute

$$\nu_3 \approx \frac{2\pi}{T_s} \approx \sqrt{\frac{2\, U(\bar{\varphi}_2)}{J' . \bar{\varphi}_2^{\,2}}} \tag{165}$$

for the ship's natural frequency (see equ. (32)), the approximate steady solution of (158), in which the damping can be neglected as the exciting frequency (163) is well away from the resonance value, becomes

(166)

$$\varphi_3(t) \approx \frac{N_2(t)}{8 J' \nu_3^2} \quad (166)$$

The amplitude is

$$\bar{\varphi}_3 \approx \frac{W(T_s) \cdot \nu_2}{8 J' \cdot \nu_3^2} \cdot \bar{\varphi}_2 \quad (167)$$

$$\approx \frac{1}{8} D(T_s) \cdot \bar{\varphi}_2 \; . \quad (167)$$

As $D(T_s)$ seldom exceeds 0,2 for ships the amplitude $\bar{\varphi}_3$ of the higher harmonic generated by the difference moment $N_2(t)$ is only a few percent of the total roll amplitude.

II. The mean values $\eta_2(\bar{\varphi})$ and $\eta_\beta(\bar{\varphi})$.

We next investigate the influence of the reactive excitation $\beta_r(t)$ on the mean values $\eta_2(\bar{\varphi})$ and $\eta_\beta(\bar{\varphi})$ defined in (108) or (111) and (113).respectively. We consider first the integrals

$$\eta_2(\bar{\varphi}) = \frac{2}{T_s} \cdot \int_0^{T_s} [1 - u(\varphi_2)] \cdot dt \quad (168)$$

and

$$\eta_\beta(\bar{\varphi}) = \frac{2}{T_s} \cdot \int_0^{T_s} \frac{\frac{1}{2} J' \cdot \dot{\varphi}_2^2}{U(\bar{\varphi})} \cdot dt \quad (169)$$

Transforming to the time differential of the free roll oscillation $\psi_c(t)$ according to (120) we obtain

$$\eta_2(\bar{\varphi}) = \frac{2}{T_s} \cdot \int_0^{T_0(\bar{\varphi})} [1-u(\psi_0)] \cdot [1+\beta_0-\bar{\beta}_r \cdot u(\psi_0)]^{-1/2} \cdot dt \tag{170}$$

and with (118) and (117)

$$\eta_\beta(\bar{\varphi}) = \frac{2}{T_s} \cdot \int_0^{T_0(\bar{\varphi})} [1-u(\psi_0)] \cdot [1+\beta_0-\bar{\beta}_r \cdot u(\psi_0)]^{+1/2} \cdot dt \, . \tag{171}$$

Expanding the roots and using (108) and (126) we obtain to the first order in $\bar{\beta}_r$

(172)

$$\eta_2(\bar{\varphi}) \approx \frac{2}{T_0(\bar{\varphi})} \cdot \int_0^{T_0(\bar{\varphi})} [1-u(\psi_0)] \cdot \left[1+\frac{1}{4}\bar{\beta}_r \cdot \left(\eta_0(\bar{\varphi})-2[1-u(\psi_0)]\right)\right] \cdot dt \tag{172}$$

and

(173)

$$\eta_\beta(\bar{\varphi}) \approx \frac{2}{T_0(\bar{\varphi})} \cdot \int_0^{T_0(\bar{\varphi})} [1-u(\psi_0)] \left[1-\frac{1}{4}\bar{\beta}_r \cdot \left(3\eta_0(\bar{\varphi})-2[1-u(\psi_0)]\right)\right] \cdot dt \, . \tag{173}$$

The expressions involve the mean values of the function $[1-u(\psi_0)]$ and its square.

Using equation (35) and introducing the symbol

$$\Delta \equiv \frac{1}{T_0(\bar{\varphi})} \cdot \int_0^{T_0(\bar{\varphi})} [1-u(\psi_0)]^2 \cdot dt \tag{174}$$

we can write

$$\eta_2(\bar{\varphi}) \approx \eta_0(\bar{\varphi}) + \frac{1}{4}\bar{\beta}_r \cdot \eta_0^2(\bar{\varphi}) - \bar{\beta}_r \cdot \Delta \tag{175}$$

and

$$\eta_3(\bar{\varphi}) \approx \eta_0(\bar{\varphi}) - \frac{3}{4}\bar{\beta}_r \cdot \eta_0^2(\bar{\varphi}) + \bar{\beta}_r \cdot \Delta . \tag{176}$$

Now the mean square of a value is always greater than the square of the mean value and further

$$0 \leq [1 - u(\varphi_0)]^2 \leq [1 - u(\varphi_0)] \leq 1 \tag{177}$$

so that the following bounds can be given for Δ:

$$\frac{1}{4}\eta_0^2(\bar{\varphi}) \leq \Delta \leq \frac{1}{2}\eta_0(\bar{\varphi}) . \tag{178}$$

We can equally well write

$$\Delta = \frac{1}{2} \cdot \left(\frac{1}{4}\eta_0^2(\bar{\varphi}) + \frac{1}{2}\eta_0(\bar{\varphi}) \right) + \frac{1}{8}\delta \tag{179}$$

where

$$|\delta| \leq \eta_0(\bar{\varphi}) \cdot \left(2 - \eta_0(\bar{\varphi})\right) \leq 1 . \tag{180}$$

The influence of the reactive excitation on $\eta_2(\bar{\varphi})$ and $\eta_3(\bar{\varphi})$ can thus be estimated by the formulae

$$\eta_2(\bar{\varphi}) \approx \eta_0(\bar{\varphi}) - \frac{1}{8}\bar{\beta}_r \cdot \left[\eta_0(\bar{\varphi}) \cdot (2 - \eta_0(\bar{\varphi})) + \delta\right] \tag{181}$$

and

$$\eta_3(\bar{\varphi}) \approx \eta_0(\bar{\varphi}) + \frac{1}{8}\bar{\beta}_r \cdot \left[\eta_0(\bar{\varphi}) \cdot (2 - 5\eta_0(\bar{\varphi})) + \delta\right] \tag{182}$$

III. <u>The convergence interval of the expansion (123).</u>
To determine the convergence interval of the expansion (123) we write (181) in the form

$$\eta_2(\bar{\varphi}) = \eta_0(\bar{\varphi}) - A \cdot \bar{\beta}_r \tag{183}$$

with

$$A = \frac{1}{8} \cdot \left[\eta_0(\bar{\varphi}) \cdot (2 - \eta_0(\bar{\varphi})) + \delta\right]. \tag{184}$$

The convergence condition (124) then becomes

$$\left| A \cdot \bar{\beta}_r^2 + m \cdot \bar{\beta}_r \right| < 1 \tag{185}$$

with

$$m = 1 - \eta_0(\bar{\varphi}) - u(\varphi_0). \tag{186}$$

The convergence interval

$$\beta_I < \bar{\beta}_r < \beta_{II} \tag{187}$$

can thus be determined from the intersects of the family of parabolas

$$\left.\begin{aligned} y_1 &= A \cdot \bar{\beta}_r^2 \\ 0 &\leq A \leq \tfrac{1}{4} \end{aligned}\right\} \tag{188}$$

with the two families of straight lines

$$\left.\begin{aligned} y_{2,3} &= \pm 1 - m \cdot \bar{\beta}_r \\ -\eta_{max} &\leq m \leq 1 \end{aligned}\right\} \tag{189}$$

The parameter intervals of A and m ~~follows~~ follow from (41), (103), and (180). The curve families are shown in fig.17. The abscissae of the two points on opposite sides of the y-axis which lie nearest to the axis are the limiting values $\beta_I < 0$ and $\beta_{II} > 0$ of the convergence interval, as all $\bar{\beta}_r$ lying between these limits satisfy condition (185). The intercepts and the corresponding convergence intervals are shown in the figure for $\eta_{max} = 1$ and $\eta_{max} = 1{,}5$. Having determined the limiting values of the parameters A and m from the intercepts nearest to the y-axis the convergence intervals were evaluated numerically:

$$-0{,}83 < \bar{\beta}_r < +0{,}83 \qquad \text{for} \qquad = 1{,}0 \text{ (decreasing stiffness} \tag{190}$$

$$-0{,}60 < \bar{\beta}_r < +0{,}66 \qquad \text{for} \qquad = 1{,}5 \text{ (increasing stiffnes}$$

Comments to G. E. Pavlenko: "On the Theory of Roll with the Aspect to the Safety of the Ship in a Seaway".

Iswjestija Akademii Nauk SSSR, Otdelenije Technitschesknich Nauk No 12, 1947, Pages 1589-1604.

Abstract

The treatment deals with both free rolls in calm water and steady states of forced roll in a swell, considering a nonlinear righting arm-curve and damping. As to calm water, formulae are given for the frequency of roll as well as for finding the damping moment from the decrement depending on the amplitudes. As to swell, response-curves are derived by way of harmonic approximation. A criterion for capsizing under the influence of exciting movements of constant amplitudes but different frequencies is given by the "Hyperbola of Safety". After discussing the exciting moments in a seaway it is shown as to how a "Curve of Safety" can be obtained by which it can be decided if the ship must capsize or may mot. Influences of course and unregular waves are not considered mathematically.

1) <u>Rolling in calm Water</u>

In calm water, the period T of roll is given by the well known integral

$$\frac{1}{4}T = \int_0^{\Theta_0} \frac{d\Theta}{\sqrt{2[\phi(\Theta_0) - \phi(\Theta)]}} , \qquad \text{(P2)} \; ^{1)}$$

where Θ denotes the angle of heel, Θ_0 its amplitude and ϕ the potential energy devided by the moment of inertia. In order to avoid the pole at $\Theta = \Theta_0$ Pavlenko, substituting $\phi = \phi(\Theta_0) . \sin^2 \xi$, uses the formula

$$\frac{1}{4}T = \sqrt{2\phi(\Theta_0)} \cdot \int_0^{\pi/2} \frac{\sin \xi . d\xi}{F} , \qquad \text{(P4)}$$

where $F(\Theta)$ denotes the righting up-moment devided by the moment of inertia. As the integrand now is always finite, this formula can be planimetered. When $\xi \to 0$ also $\Theta \to 0$ and

$$\frac{\sin \xi}{F_{\xi \to 0}} \to \left[2\phi(\Theta_0) \cdot \left(\frac{dF}{d\Theta}\right)_{\Theta \to 0}\right]^{-1/2} .$$

Fig. P1 shows how to find the integrand from the righting arm-curve; the result of the procedure has been plotted in Fig. P2 which also shows the result of the approximate formula

$$\frac{2\pi}{T} \equiv \omega \approx \sqrt{\frac{F(\Theta_0)}{\Theta_0}} . \qquad \text{(P6)}$$

1) Numbers combined with P refer to the treatment by Pavlenko.

Pavlenko assumes the damping moment of roll to be a function of the angular velocity only, f(θ').
The decrement of roll in calm water is derived by means of harmonic approximation and by treating the amplitude $\Theta_0(t)$ as a function of time:

$$-\mu \equiv \frac{\Delta\,\Theta_0(t)}{\Theta_0(t)\cdot\Delta t} = \frac{f(\omega\cdot\Theta_0)}{2\omega\cdot\Theta_0} \qquad \text{(P11) (P12)}$$

2) Rolling in regular Waves

Pavlenko describes the forced rolling motion of a ship in a beam sea by the differential equation

$$\Theta'' + f(\Theta') + F(\Theta) = R\cdot\sin\sigma t \qquad \text{(P13)}$$

By harmonic approximation again he finds the equation

$$\sigma^2 = \omega^2(\Theta_0) \pm \sqrt{\left(\frac{R}{\Theta_0}\right)^2 - 4\mu_0^2\sigma^2} \qquad \text{(P14)}$$

where μ_0 denotes the maximum value of μ, see (P11).

Fig. P3 shows the plot of Θ_0 against σ for constant R and different values of μ_0, i.e. the response curves. The most remarkable fact is that the maximum amplitude is an unsteady function of the damping. For a given exciting amplitude R, there is a critical damping and vice versa, the response curves being "limited" or not. Pavlenko concludes that in the case of an unlimited response-curve the ship must capsize. Figs. P4 and P5 illustrate the condition of safety.

In the case of resonance, the exciting moment is balanced by the damping moment. Therefore, Pavlenko finds the maximum angular velocity from the equation

$$f(\Theta_0 \cdot \omega) = R . \tag{P16}$$

In fig. P4, $f(\Theta_0 \cdot \omega)$ has been plotted over the $\Theta' = \Theta_0 \cdot \omega$ - axis and if a certain value of R is given, the corresponding value of $\Theta_0 \cdot \omega$ can easily be found. In order to determine the roll amplitude Θ_0, the product $\Theta_0 \cdot \omega$, where $\omega(\Theta_0)$ is the frequency of free roll in calm water, has been plotted over the Θ'-axis, too. When Θ_0 tends to the end of the stability range where $F(\Theta) = 0$, $\omega(\Theta_0)$ tends to zero. Thus the product $\Theta_0 \cdot \omega(\Theta_0)$ has a maximum, and to a given value of this product an amplitude Θ_0 can be found only if the value does not exceed the maximum. In this case the plot yields two amplitudes. By not explained reasons only the smaller one denotes the resonance peak. If the exciting moment is too large or the damping moment too small the value of $\Theta_0 \cdot \omega$ found from (P16) exceeds the maximum value of $\Theta_0 \cdot \omega(\Theta_0)$ calculated from (P4) and no amplitude Θ_0 can be determined. This is the case when the resonance curve is not "limited".

In fig. P5 the same condition of safety is given by the intersection of the natural frequency-curve with the "Hyperbola of Safety", $\Theta_0 \cdot \omega(\Theta_0)$ = const., which is given by (P16) for constant R.

In a seaway, R is not constant. It depends not only on the wave length but also on the route and weather condition. Then, the Hyperbola of Safety is changed into a "Curve of Safety". According to Pavlenko it is a necessary and sufficient condition of safety that there is a real intersection of the natural frequency-curve and the Curve of Safety. In order to construct it he gives the formula

$$R = \alpha_0 \cdot \sqrt{\left[\frac{D \cdot h_0}{(1+k) \cdot J} - \frac{k\sigma^2}{1+k}\right]^2 + (m_0' \cdot \sigma)^2} \tag{P20}$$

where

α_0 — the effective wave slop making allowance for the ship's dimensions

D — the boyancy

h_0 — the metacentric radius influenced by orbital motion

J — the moment of inertia

k — the coefficient of hydrodynamic inertia

m_0' — the coefficient of the linearized damping moment

As all these values are functions of the wave frequency only, so $R(\sigma)$. Fig. P9 shows how after plotting $R(\sigma)$ and $f(\theta')$ a point with the coordinates ω and θ' of the "Curve of Safety" can be found.

Comments

1))The equation of motion (P13) is not correct because there are no external moments acting on a ship in waves, but the ship is coupled by static and dynamic moments to the surrounding water as well as to the Newtonian inertial system. Therefore, the moments are functions of the relative motions resp. coordinates. Only in cases of linear coupling moments, it is possible to separate the coordinates putting e.g. the angle of ship's inclination to the inertial system on the left and the coordinates describing the motion of the surrounding water on the right of the equation, which then appears in the shape of (P13). But, as Pavlenko wants to study the ships rolling at large amplitudes even with the aspect of capsizing he ought to consider non-linear coupling moments. The hydrostatic moment then should be written $F(\Theta - \alpha)$, where in waves of sufficient length F does not differ from the function defined in calm water and α denotes the momentary apparent inclination of the wave surrounding the ship. The hydrodynamic moments of damping (esp. by wave generation) and inertia should be at least suitable functions of linear combinations of Θ' and α' resp. Θ'' and α'' with parameters depending on ship's dimensions and wave length.

2) In considering a non-linear system at large amplitudes or even capsizing one cannot expect quantitative results from a method of harmonic approximation. Especially the substitution $\omega \cdot \Theta_0$ for the maximum angular velocity is very arbitrary. With the same right one could put $\frac{1}{2}\Theta'^2 = \phi(\Theta_0)$.

Both formulae which are correct for small amplitudes do not hold for amplitudes exceeding the maximum righting-arm. - When θ_0 tends to the end of the stability range, θ_R, ω tends to zero and so does $\theta' = \omega \cdot \theta_0$; on the other hand $\phi(\theta_0)$ tends to its maximum and so does $\theta' = \sqrt{2\phi(\theta_0)}$. Thus in fig. P4 the curve denoted by $\theta_0 \cdot \omega(\theta_0)$ as well as in fig. P5, the "Hyperbola of Safety" and also the "Curve of Safety" all are arbitrary in the same extent and cannot give any reliable warning.

3) Even if the equation of motion and its evaluation are correct for a pendulum, an amplitude exceeding the range of stability would not cause capsizing for on that branch of the resonance curve (fig. P3), $\sigma > \omega(\theta_0)$ and therefore when the pendulum inclines to the left the exciting moment is directed to the right, reinforcing the righting up-moment of the pendulum. Besides, rolling states of large amplitudes have practically small stability if a state of lower amplitude is possible at the same frequency. Therefore, the upper branch of the resonance curves shown in fig. P3 is not very likely for frequencies $\sigma < \varsigma \delta$.

By the Pavlenko curve, if correctly calculated, it is not seen if the ship may capsize or not; but is is seen if her maximum amplitude of roll is determined by damping or by detuning of her natural frequency.

4) The motion of capsizing is not included in the steady states of roll. In order to find a criterion on capsizing, therefore, an investigation on transient oscillations would be necessary.

Nevertheless, we can ask for those steady states of roll, the energy of which would suffice for capsizing.

For a ship in a transverse swell of momentary effective wave inclination $\alpha(t)$ the equation of motion can be written, if damping is neglected:

$$\Theta'' + F(\psi) = 0, \tag{1}$$

where $\psi \equiv \Theta - \alpha$ is the angle of heel.

If we assume that the oscillograms $\Theta(t)$ and $\psi(t)$ are similar to one another, $\alpha(t)$ is not sinusoidal:

$$\alpha(t) = a \cdot \psi(t) \tag{2}$$

the factor a depending on the frequency of swell. Then

$$(1+a) \cdot \psi'' + F(\psi) = 0, \tag{3}$$

which means that the natural frequency $\omega(\psi_0)$ is detuned to

$$\sigma = \frac{\omega(\psi_0)}{\sqrt{1+\alpha}} . \tag{4}$$

Multiplying equation (1) by $\Theta' dt = (1+a) \cdot d\psi$ and integrating, we find

$$\tfrac{1}{2}\Theta'^2 + (1+a) \cdot \Phi(\psi) = C$$

where for $\Theta' = 0$: $\psi = \psi_0$ and so $C = (1+a) \cdot \Phi(\psi_0)$.
Therefore the total energy - which, of course, for a ship in waves is not constant - becomes

$$E \equiv \tfrac{1}{2}\Theta'^2 + \Phi(\psi) = \Phi(\psi_0) + a \cdot \left(\Phi(\psi_0) - \Phi(\psi)\right). \tag{5}$$

During one period of roll this value oscillates twice within the limits - NB (4) -

$$E_0 = (1+a)\cdot\Phi(\psi_0) = \frac{\omega^2(\psi_0)}{\sigma^2}\cdot\Phi(\psi_0) \quad \text{for } \psi = 0 \tag{6}$$

and

$$E_1 = \Phi(\psi_0) \qquad \text{for } \psi = \pm\psi_0 . \tag{7}$$

Thus, if $\sigma^2 < \omega^2(\psi_0)$, the maximum kinetic energy E_0 may exceed the value $\Phi(\psi_R)$ necessary for capsizing (ψ_R = end of the stability range) although the amplitude of roll is lower than ψ_R . The condition for those states of roll will be

$$\omega^2(\psi_0)\cdot\Phi(\psi_0) > \sigma^2\cdot\Phi(\psi_R) . \tag{8}$$

Figs. 1a and 1b show three curves each, valid for the righting arm-curves represented:

a) the skeleton curve $\omega^2(\psi_0)$

b) the (dotted) curve limiting states of roll with energy sufficient for capsizing

$$\sigma^2(\psi_0) = \frac{\Phi(\psi_0)}{\Phi(\psi_R)}\cdot\omega^2(\psi_0) \tag{9}$$

c) the curve limiting stable states of roll

$$\omega_*^2(\psi_0) = \omega^2(\psi_0) + \psi_0\cdot\frac{d\omega^2(\psi_0)}{d\psi_0} \quad ^{2)} \tag{10}$$

2) This equation has been derived in "Rollzustände grosser Amplitude in seitlicher Dünung" by Dr. H. Baumann, Schiffstechnik, Heft 10, 1955.

The vertically shaded regions represent roll states with energy sufficient for capsizing. The horizontally shaded region represent unstable states. So the double shaded regions denote dangerous roll states which may lead to capsizing. Considering steady states only, we cannot say under which circumstances a ship may come into a dangerous state of that kind, because it is no stable but a transitory state, but we can easily imagine two special processings leading momentarily into a dangerous state.

First, if by heavy roll in regular waves the freight or other weighty objects change their positions a sudden change of the ships dynamical properties occurs and per chance a transitory dangerous state may arise.

Second, if the frequency of the swell slowly rises whilst its wave slope remains constant and the point representing the roll state of the ship follows its response curve and reaches the curve $\omega_*^2(\psi_0)$ limiting stable states of roll, a transition into another state of motion is necessary. If the intersection point of response curve and $\omega_*^2(\psi_0)$ represents a dangerous state the ship may capsize although until now its amplitude of roll has been merely $\psi_0 \approx 0{,}45\,\psi_R$ and $\psi_0 \approx 0{,}61\,\psi_R$ for the response curves given in figs. 1a and 1b with $\alpha_0 \approx 0{,}6\,\psi_R$ and $0{,}5\,\psi_R$ resp. Capsizing in this way, therefore, is to be expected only if the ship's range of stability is small. A similar consideration can be made with constant frequency and slowly rising wave slope, leading to the same consequences.

5) The correspondence between the righting arm-curve shown in fig. P1 and the frequency-plot given in fig. P2 is not plausible. As the maximum slope of the righting arm-curve is at the point $\Theta = 0$ the frequency must have its maximum at $\Theta = 0$, too. The increase of frequency in the range of $0 < \Theta_0 < 15^0$ (fig. P2) cannot be obtained from the righting arm-curve given, neither by the correct formula (P4) nor by the approximate formula (P6).

By the way: If we substitute

$$\Theta = \Theta_0 \cdot \sin \alpha \tag{11}$$

we obtain

$$\tau(\Theta_0) \equiv \frac{T(\Theta_0)}{T(0)} = \frac{2}{\pi} \cdot \int_0^{\pi/2} w(\Theta, \Theta_0) \cdot d\alpha \; . \tag{12}$$

The dimensionless period equals the mean value of

$$w(\Theta, \Theta_0) \equiv \sqrt{\frac{\frac{1}{2} C (\Theta_0^2 - \Theta^2)}{\Phi(\Theta_0) - \Phi(\Theta)}} \; , \tag{13}$$

taken over the interval $0 \leq \alpha \leq \pi/2$ with $C = \left(\frac{dF(\Theta)}{d\Theta} \right)_{\Theta = 0}$.
Compared with (P4), formula (12) has the advantage that the integrand (13) varies within a smaller range than the integrand of (P4).

We find from (13)

$$q(\Theta_0) \equiv \frac{w(\Theta_0, \Theta_0)}{w(0, \Theta_0)} = \sqrt{\frac{2 \Phi(\Theta_0)}{\Theta_0 \cdot F(\Theta_0)}} \tag{14}$$

and from (P4)

$$Q(\theta_0) \equiv \left(\frac{\sin\xi}{F}\right)_{\xi=\pi/2} : \left(\frac{\sin\xi}{F}\right)_{\xi=0} \tag{15}$$

$$= \frac{\sqrt{2C\cdot\Phi(\theta_0)}}{F(\theta_0)} \quad ,$$

If we write

$$Q(\theta_0) = \frac{\Phi(\theta_0)}{\frac{1}{2}\theta_0\cdot F(\theta_0)}\cdot\sqrt{\frac{\frac{1}{2}C\theta_0^2}{\Phi(\theta_0)}} \quad , \tag{16}$$

it is seen that the first factor equals $q^2(\theta_0)$ and denotes the ratio of the areas under the righting arm-curve and under its secant, taken from $\theta = 0$ to $\theta = \theta_0$. The square of the second factor denotes the ratio of the areas under the initial tangent of the righting arm-curve and under the curve itself, taken again from $\theta = 0$ to $\theta = \theta_0$. Now if the curvature of the righting arm-curve is of constant sign both factors are either greater or smaller then unity. So the integrand of (P4) varies within a range larger than $q^2(\theta_0)$:

The bows denote the range of variation for the integrands of (13) resp. (P4).

In this cases the mean value (12) can be well approximated by the arithmetic mean of the extrema of the integrand which are at $\alpha = 0$ and $\alpha = \pi/2$. So

$$\tau(\Theta_0) \approx \frac{1}{2} \cdot \left(\sqrt{\frac{\frac{1}{2} C \Theta_0^2}{\Phi(\Theta_0)}} + \sqrt{\frac{C \Theta_0}{F(\Theta_0)}} \right). \tag{17}$$

In a previous treatment we have shown that for nearly every reasonable righting arm-curve the approximate formula (17) holds up to 70% of the stability range with an error less than 1%.

Hamburg, Aug. 4 '59

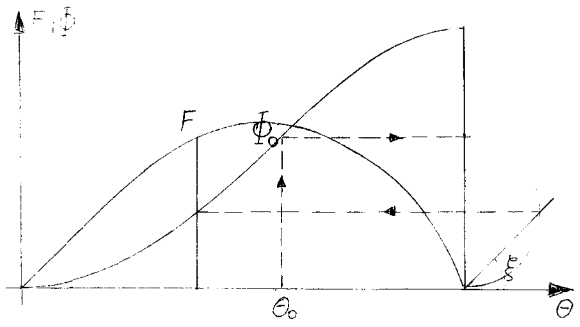


Fig. P 1

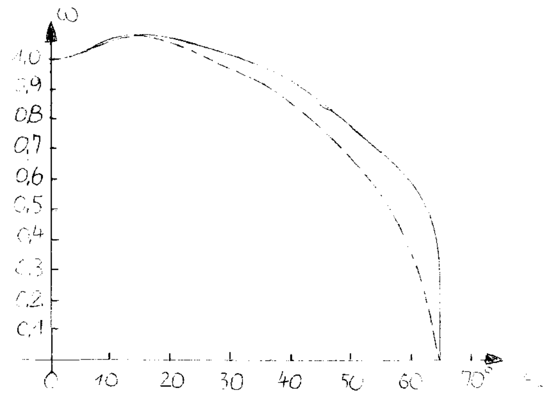


Fig. P 2

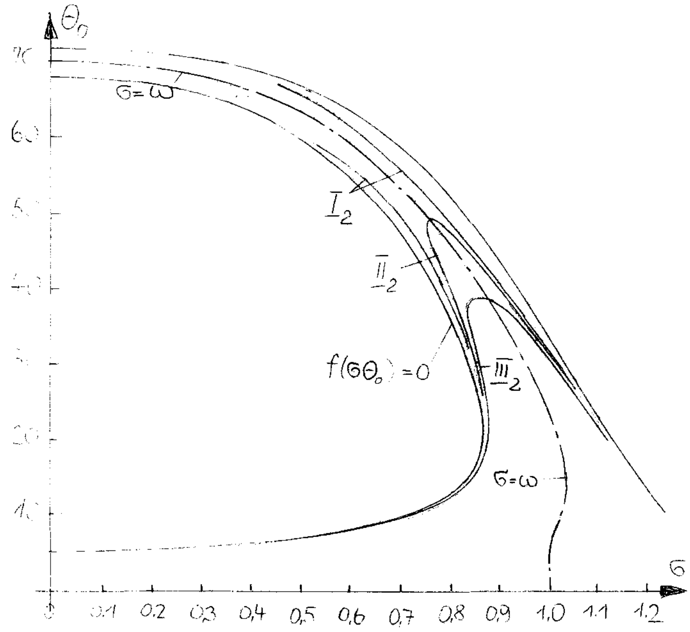


Fig. P 3

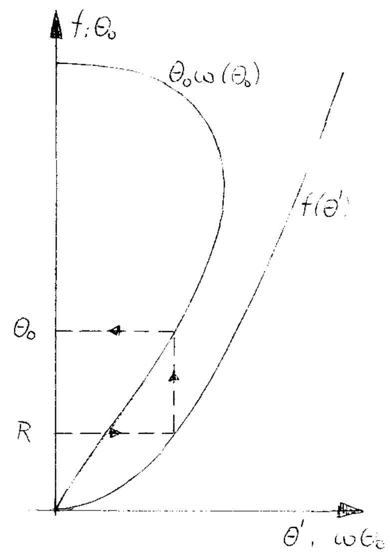


Fig. P4

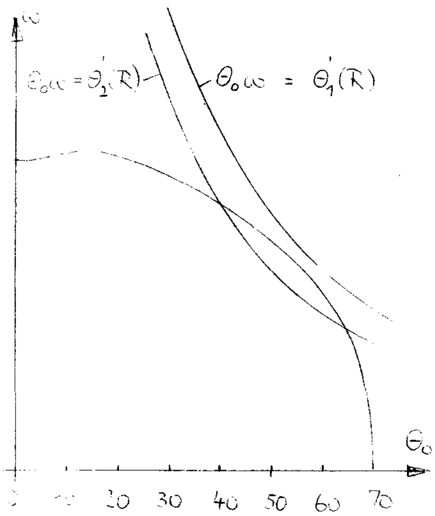


Fig. P5

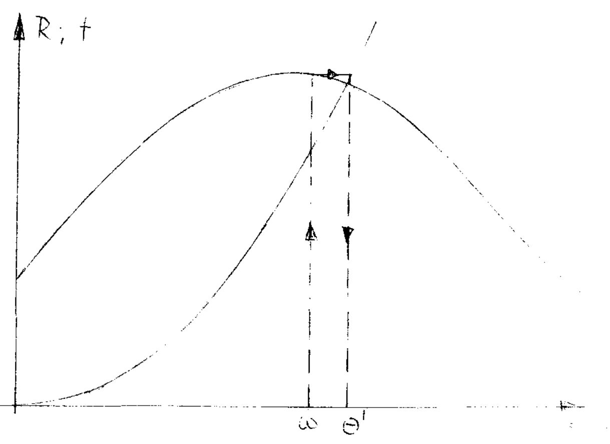
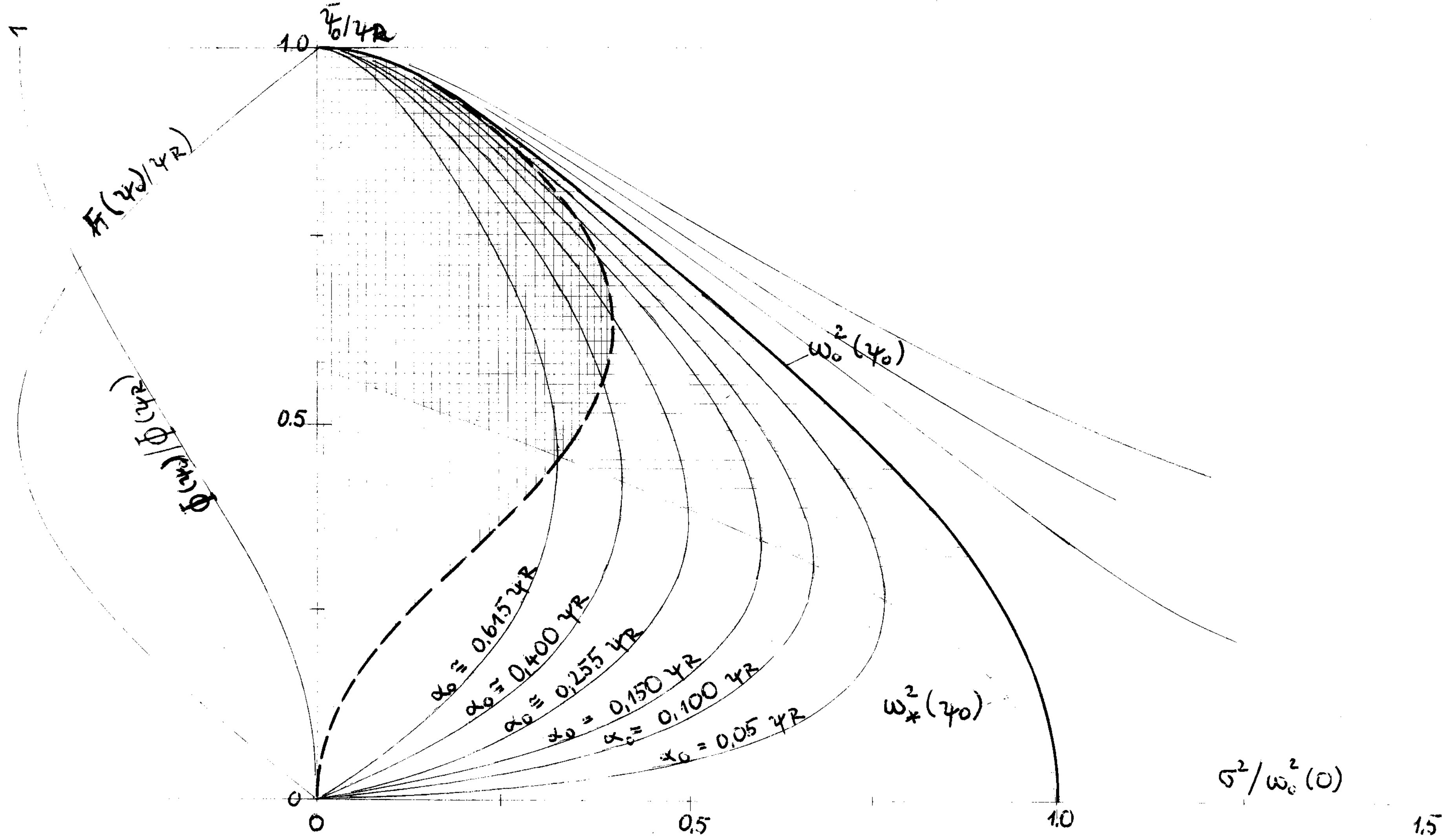
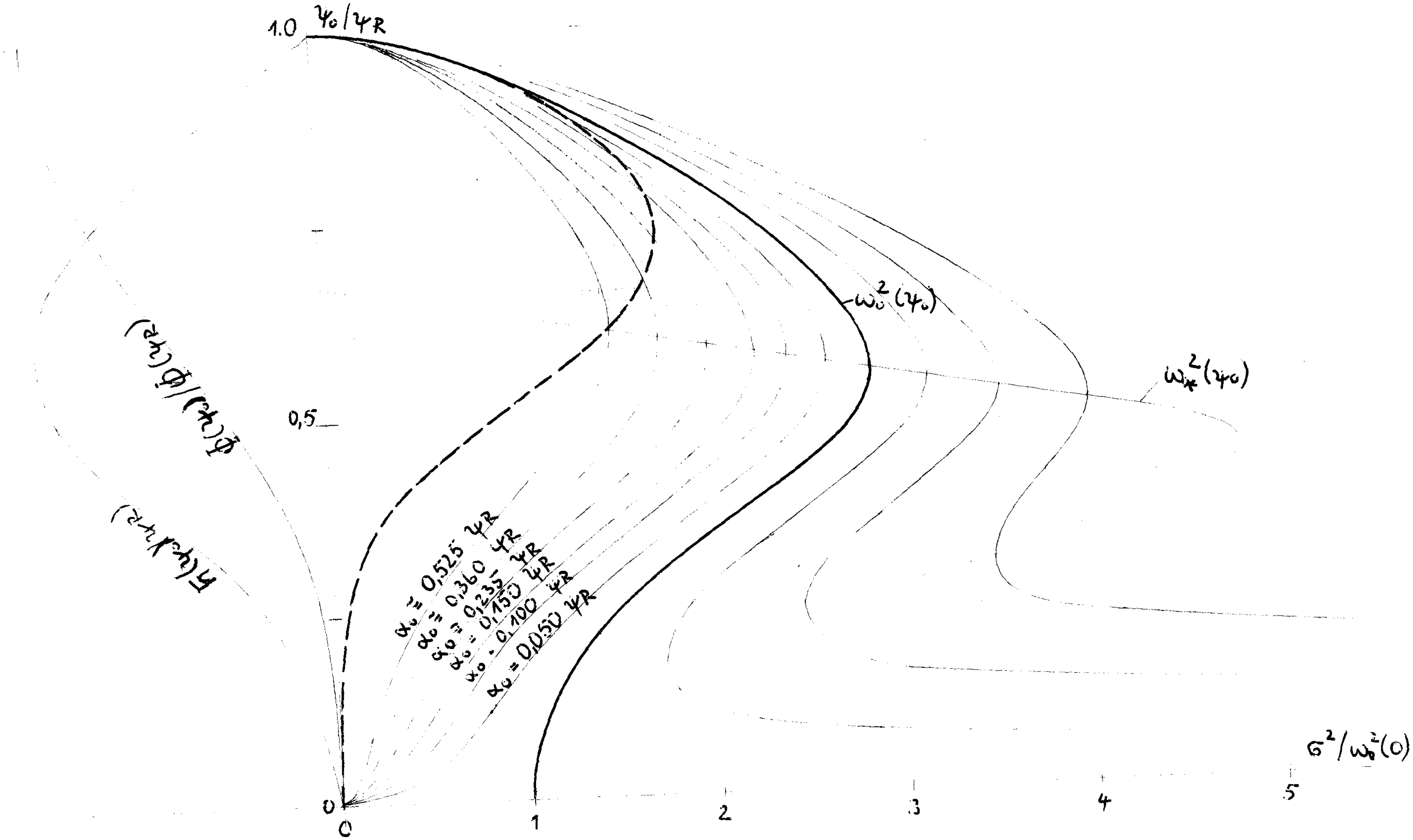


Fig. P9



1
1.0
$\bar{\psi}_0/\psi_R$
$F(\psi_0)/\psi_R$
$\Phi(\psi_0)/\Phi(\psi_R)$
0.5
0
0
0.5
1.0
1.5
$\omega_0^2(\psi_0)$
$\alpha_0 = 0.615\ \psi_R$
$\alpha_0 = 0.400\ \psi_R$
$\alpha_0 = 0.255\ \psi_R$
$\alpha_0 = 0.150\ \psi_R$
$\alpha_0 = 0.100\ \psi_R$
$\alpha_0 = 0.05\ \psi_R$
$\omega_*^2(\psi_0)$
$\sigma^2/\omega_0^2(0)$

Fig. 1a



1.0
ψ_0/ψ_R
0,5
0
$\Phi(\psi_0)/\Phi(\psi_R)$
$F(\psi_0)/F(\psi_R)$
$\alpha_0 = 0,525\ \psi_R$
$\alpha_0 = 0,360\ \psi_R$
$\alpha_0 = 0,235\ \psi_R$
$\alpha_0 = 0,150\ \psi_R$
$\alpha_0 = 0,100\ \psi_R$
$\alpha_0 = 0,050\ \psi_R$
$\omega_0^2(\psi_0)$
$\omega_*^2(\psi_0)$
0
1
2
3
4
5
$\sigma^2/\omega_0^2(0)$

Fig. 1b