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Research Article

Some Properties of Weak Γ-Hyperfilters in Ordered Γ-Semihypergroups

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The main purpose of this paper is to study fundamental properties of weak Γ-hyperfilters on ordered Γ-semihypergroups that is a generalization of Γ-hyperfilters. Also, we investigate the relationship between weak Γ-hyperfilters and prime Γ-hyperideals in ordered Γ-semihypergroups. Finally, we introduce weak (m, n)-Γ-hyperfilters of ordered Γ-semihypergroups and conclude in paper with arising explicit theorems.

1. Introduction and Preliminaries

The theory of hypergroups was introduced by Marty [1] in 1934. For details about hyperstructure theory and its applications in cryptography, codes, etc, we refer to [2]. In the range of hyperstructures, Heidari and Davvaz have investigated the ordered semihypergroups [3]. In [4], Davvaz et al. provided a construction method for an ordered semigroup by using the pseudoorders. The study of ordered regular equivalence relations is one interesting topic in ordered semihypergroup theory. Gu and Tang [5] and Tang et al. [6] started the study of ordered regular equivalence relation in detail and achieved some results in this respect. The concept of hyperfilter is the most important subject in ordered semihypergroup. In [7], Mahboob and Khan introduced (m, n)-hyperfilters on ordered semihypergroups and studied its various properties and characterizations.

Many researchers have studied (ordered) Γ-semihypergroups and their related notions, for instance, Omidi et al. [8, 9], Roa et al. [10], and Tang et al. [11], also see [12, 13]. In [9], Omidi et al. investigated the notion of (intra-) regular ordered Γ-semihypergroups associated with bi-Γ-hyperideals, and in [8], Omidi and Davvaz introduced the notion of convex ordered Γ-semihypergroups. In [14], Yaqoob and Tang investigated approximations of interior hyperfilters in partially ordered LA-semihypergroups. Recently, Rao et al. [15] studied some aspects of m-k-hyperideals in ordered semihyperrings.

Fuzzy set theory has been developed in many directions. In 2015, the notion of fuzzy hyperfilters was introduced and studied by Tang et al. in [16]. In [17], Cheng and Xin investigated some types of hyperfilters on hyper-BE-algebras. In 2014, Borzooei et al. [18] studied weak filters in hyper-residuated lattices. For more results on derivations in ordered semihyperrings, one can see [15].

Recently, Rao et al. [10] investigated some properties of (m, n)-Γ-hyperfilters in ordered Γ-semihypergroups. In continuity of this paper, we study weak (m, n)-Γ-hyperfilters of ordered Γ-semihypergroups. In [19], Bouaziz and Yaqoob studied hyperfilters of ordered LA-semihypergroups in the framework of rough sets. Recently, Tang et al. defined and analyzed in [20] the weak hyperfilters of ordered semihypergroups. We know that weak hyperfilters are generalizations of hyperfilters while ordered semihypergroups generalize ordered semigroups.

In the following, some notions on ordered Γ-semihypergroups are reviewed to facilitate this study (see [10, 12], for more details and basic definitions). Let S be a nonempty set and $^{*}(S)$ be the family of all nonempty subsets of S. Then, a mapping $\gamma\colon S \times S \longrightarrow {}^{*}(S)$ is called a hyperoperation on S. A hypergroupoid is a set S together with a hyperoperation γ.

If $\varnothing \neq A, B \subseteq S$ and $x \in S$, then $A\gamma B = \cup_{a\in A, b\in B} a\gamma b$, $x\gamma A = \{x\}\gamma A$, and $B\gamma x = B\gamma\{x\}$.

We suppose that the hypergroupoid (S, γ) is equipped with the relation, $a\gamma(b\gamma c) = (a\gamma b)\gamma c$, which means that $\cup_{u\in b\gamma c} a\gamma u = \cup_{v\in a\gamma b} v\gamma c$.

For every $a, b, c \in S$, then S is called a semihypergroup.

Consider a nonempty set S and a nonempty set Γ with the following properties: for all $a, b, x, y \in S$ and all $\alpha, \beta \in \Gamma$, we have

(1) $a\gamma b \subseteq S$
(2) If $a, b, x, y \in S$ such that $a = x$ and $b = y$, then $a\gamma b = x\gamma y$
(3) $x\alpha(a\beta b) = (x\alpha a)\beta b$

Every $\gamma \in \Gamma$ denotes the hyperoperation. Such a set S is said to be a Γ-semihypergroup. The reader may see [21, 22], for detailed discussion.

Let $\varnothing \neq A, B \subseteq S$. We define $A\gamma B = \cup\{a\gamma b | a \in A, b \in B\}$ and

$$A\Gamma B = \bigcup_{\gamma\in\Gamma} A\gamma B. \tag{1}$$

Definition 1. (see [11]). An ordered Γ-semihypergroup is a triple $(S, \Gamma, \leq)$ that

(1) (S, Γ) is a Γ-semihypergroup
(2) $(S, \leq)$ is a (partially) ordered set
(3) For all $u, v, x \in S$ and all $\gamma \in \Gamma$, $u \leq v$ implies $u\gamma x \prec v\gamma x$ and $x\gamma u \prec x\gamma v$

If $\varnothing \neq A, B \subseteq S$, then $A \prec B \Leftrightarrow \forall a \in A, \exists b \in B; a \leq b$.

Let $\varnothing \neq I \subseteq S$. $(I]$ is defined as follows: $(I] := \{x \in S | x \leq z \text{ for some } z \in I\}$.

For convenience, given $a \in I$, we write $(\{a\}] = (a]$.

By a sub-Γ-semihypergroup of an ordered Γ-semihypergroup S, we mean a nonempty subset A of S such that $A\Gamma A \subseteq A$, i.e., $a\gamma b \subseteq A$, for every $a, b \in A$ and $\gamma \in \Gamma$.

Definition 2. (see [12]). A nonempty subset I of an ordered Γ-semihypergroup S is called a Γ-hyperideal of S if

(1) $S\Gamma I \subseteq I$ and $I\Gamma S \subseteq I$
(2) $(I] = I$

For any $\varnothing \neq H \subseteq S$, we define $[H)$ by $\{x \in S | h \leq x \text{ for some } h \in H\}$. For $H = \{h\}$, we write $[h)$ instead of $[\{h\})$.

The concept of an (m, n)-Γ-hyperfilter is a generalization of the concept of a Γ-hyperfilter of S. Also, see [10], for an overview.

Definition 3. (see [10]). If $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup, then a left m-Γ-hyperfilter F is a sub-Γ-semihypergroup equipped with $[F) \subseteq F$ and, in addition, for all $a, b \in S$ and $\gamma \in \Gamma, a\gamma b \cap F \neq \varnothing$ implies $a^m \subseteq F$. Right n-Γ-hyperfilters can be defined similarly. Here, m and n are positive integers. If F is both a left m-Γ-hyperfilter and a right n-Γ-hyperfilter of S, then F is called an (m, n)-Γ-hyperfilter of S. For $m = n = 1$, F is a Γ-hyperfilter of S.

2. Main Results

In this section, we develop several definitions and results on an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Some remarkable properties associated with weak Γ-hyperfilters were investigated. Also, we examine the relationship between weak Γ-hyperfilters and prime Γ-hyperideals by mentioning a theorem.

Definition 4. Let $F_w \neq \varnothing$ be a subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then, F_w is called a left (right) weak Γ-hyperfilter of S if

(1) $(x\gamma y) \cap F_w \neq \varnothing$ for all $x, y \in F_w$ and all $\gamma \in \Gamma$
(2) For all $x, y \in S$ and all $\gamma \in \Gamma, (x\gamma y) \cap F_w \neq \varnothing \Longrightarrow x \in F_w$ $(y \in F_w)$
(3) For all $x \in F_w$ and $z \in S, x \leq z \Longrightarrow z \in F_w$, i.e., $[F_w) \subseteq F_w$

Note that if F_w is both a left weak Γ-hyperfilter and a right weak Γ-hyperfilter of S, then F_w is called a weak Γ-hyperfilter of S.

Clearly, every Γ-hyperfilter (see Definition 3) of an ordered Γ-semihypergroup S is a weak Γ-hyperfilter of S. The converse is not true, in general, that is, a weak Γ-hyperfilter may not be a Γ-hyperfilter of S.

Example 1. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\gamma, \beta\}$. The hyperoperations γ and β are given in Table 1 and Table 2. Then, (S, Γ) is a Γ-semihypergroup [23].

Now, we consider the (partial) order relation,

$$\leq := \{(a, a), (b, b), (c, c), (d, a), (d, c), (d, d)\}, \tag{2}$$

on S. Then, $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup. Cover relation of S as given below:

$$\prec = \{(d, a), (d, c)\}. \tag{3}$$

The Hasse diagram of S is shown in Figure 1.

Here, $F_w = \{a, b, c\}$ is a weak Γ-hyperfilter of S. Clearly, F_w is not a Γ-hyperfilter of S. Since $F_w\Gamma F_w = S \nsubseteq F_w$, i.e., $a\Gamma b = \{b, d\} \nsubseteq F_w$, it follows that F_w is not a sub-Γ-semihypergroup of S.

Note that if $\{F_{w_i} | i \in I\}$ is a family of weak Γ-hyperfilters of an ordered Γ-semihypergroup S, for all $i \in I$, then $\cup_{i\in I} F_{w_i}$ is not a weak Γ-hyperfilter of S in general (see Example 3.3 in [20]). In the following, we show that if $\{F_{w_i} | i \in I\}$ is a chain of weak Γ-hyperfilters of S, then $\cup_{i\in I} F_{w_i}$ is a weak Γ-hyperfilter of S.

Lemma 1. *Suppose that F_{w_1} and F_{w_2} are weak Γ-hyperfilters of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then, $F_{w_1} \cup F_{w_2}$ is a weak Γ-hyperfilter of S if and only if $F_{w_1} \subseteq F_{w_2}$ or $F_{w_2} \subseteq F_{w_1}$.*

Proof. If $F_{w_1} \subseteq F_{w_2}$ or $F_{w_2} \subseteq F_{w_1}$, then it is clear that $F_{w_1} \cup F_{w_2}$ is a weak Γ-hyperfilter of S.

Table 1: Table of γ for Example 1.

γ	a	b	c	d
a	a	$\{b, d\}$	c	d
b	$\{b,d\}$	b	$\{b,d\}$	d
c	c	$\{b,d\}$	a	d
d	d	d	d	d

Table 2: Table of β for Example 1.

β	a	b	c	d
a	$\{a, c\}$	$\{b, d\}$	$\{a, c\}$	d
b	$\{b, d\}$	b	$\{b, d\}$	d
c	$\{a, c\}$	$\{b, d\}$	$\{a, c\}$	d
d	d	d	d	d

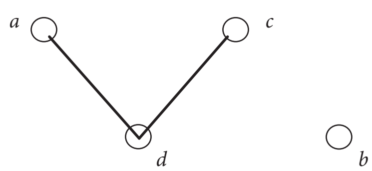

Figure 1: Figure of S for Example 1.

Assume that $F_{w_1} \cup F_{w_2}$ is a weak Γ-hyperfilter of S and $F_{w_1} \nsubseteq F_{w_2}$ and $F_{w_2} \nsubseteq F_{w_1}$. Then, there exist $x, y \in F_{w_1} \cup F_{w_2}$ such that $x \in F_{w_1}$, $x \notin F_{w_2}$, $y \in F_{w_2}$, and $y \notin F_{w_1}$. Since $F_{w_1} \cup F_{w_2}$ is a weak Γ-hyperfilter of S, we obtain $(x\gamma y) \cap (F_{w_1} \cup F_{w_2}) \neq \varnothing$, for each $\gamma \in \Gamma$. □

Case 1. $(x\gamma y) \cap F_{w_1} \neq \varnothing$. Since F_{w_1} is a weak Γ-hyperfilter of S, we get $y \in F_{w_1}$, a contradiction.

Case 2. Next, we consider the case when $(x\gamma y) \cap F_{w_2} \neq \varnothing$. Since F_{w_2} is a weak Γ-hyperfilter of S, we have $x \in F_{w_2}$, a contradiction.

From Lemma 1, we get the following result.

Theorem 1. *Suppose $\{F_{w_i} | i \in I\}$ is a family of weak Γ-hyperfilters of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$ such that $F_{w_u} \subseteq F_{w_v}$ or $F_{w_v} \subseteq F_{w_u}$, for all $u, v \in I$. Then, $\cup_{i \in I} F_{w_i}$ is a weak Γ-hyperfilter of S, where $|I| \geq 2$.*

Proof. Straightforward. □

Theorem 2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup and $\varnothing \neq F_w \subseteq S$. Then, the following statements are equivalent:*

(1) F_w is a weak Γ-hyperfilter of S

(2) $S \smallsetminus F_w = \varnothing$ or $S \smallsetminus F_w$ is a prime Γ-hyperideal of S

Proof. (1) $\Longrightarrow$ (2): assume that (1) holds and $S \smallsetminus F_w \neq \varnothing$. We first show that $S \smallsetminus F_w$ is a Γ-hyperideal of S. Let $a \in S$, $b \in S \smallsetminus F_w$, and $\gamma \in \Gamma$. If $b\gamma a \nsubseteq S \smallsetminus F_w$, then there exists $x \in b\gamma a$ such that $x \in F_w$. So, $(b\gamma a) \cap F_w \neq \varnothing$. Since F_w is a weak Γ-hyperfilter of S, we get $b \in F_w$, which is a contradiction. So, $b\gamma a \subseteq S \smallsetminus F_w$. It means that $(S \smallsetminus F_w)\Gamma S \subseteq S \smallsetminus F_w$.

Similarly, $S\Gamma (S \smallsetminus F_w) \subseteq S \smallsetminus F_w$.

Now, let $b \in S \smallsetminus F_w$, $a \in S$, and $a \leq b$. Since F_w is a weak Γ-hyperfilter of S, it follows that $[F_w) \subseteq F_w$. If $a \in F_w$, then $b \in F_w$, which is a contradiction. It implies that $a \in S \smallsetminus F_w$. Hence, $(S \smallsetminus F_w] \subseteq S \smallsetminus F_w$. Hence, by Definition 2, $S \smallsetminus F_w$ is a Γ-hyperideal of S.

Next, we show that $S \smallsetminus F_w$ is prime. Let $a, b \in S, \gamma \in \Gamma$, and $a\gamma b \subseteq S \smallsetminus F_w$. If $a \in F_w$ and $b \in F_w$, then, since F_w is a weak Γ-hyperfilter of S, we get $(a\gamma b) \cap F_w \neq \varnothing$, a contradiction. So, $a \in S \smallsetminus F_w$ or $b \in S \smallsetminus F_w$. Therefore, $S \smallsetminus F_w$ is a prime Γ-hyperideal of S.

(2) $\Longrightarrow$ (1): if $S \smallsetminus F_w = \varnothing$, then $F_w = S$, and so F_w is a weak Γ-hyperfilter of S. Now, let $S \smallsetminus F_w$ is a prime Γ-hyperideal of S. We assert that F_w is a weak Γ-hyperfilter of S. Let $a, b \in F_w$ and $\gamma \in \Gamma$. If $(a\gamma b) \cap F_w = \varnothing$, then $(a\gamma b) \subseteq (S \smallsetminus F_w)$. Since $S \smallsetminus F_w$ is prime, it follows that $a \in S \smallsetminus F_w$ or $b \in S \smallsetminus F_w$, which is a contradiction. Hence, $(a\gamma b) \cap F_w \neq \varnothing$, for all $a, b \in F_w$ and $\gamma \in \Gamma$. So, the first condition of Definition 4 is verified.

Now, let $a, b \in S$, $\gamma \in \Gamma$, and $(a\gamma b) \cap F_w \neq \varnothing$. If $a \in S \smallsetminus F_w$, then $a\gamma b \subseteq (S \smallsetminus F_w)\Gamma S$ Since $S \smallsetminus F_w$ is a Γ-hyperideal, we get $a\gamma b \subseteq S \smallsetminus F_w$. So, $a\gamma b \cap F_w \neq \varnothing$, which is a contradiction. It implies that $a \in F_w$. Similarly, $b \in F_w$. Thus, the second condition of Definition 4 is verified.

Assume that $a \in F_w$ and $a \leq x$, where $x \in S$. If $x \in S \smallsetminus F_w$, then, since $S \smallsetminus F_w$ is a Γ-hyperideal of S, we get $a \in S \smallsetminus F_w$, a contradiction. So, $x \in F_w$, and hence, the third condition of Definition 4 is verified. Therefore, F_w is a weak Γ-hyperfilter of S. □

Theorem 3. *Suppose that W is a weak Γ-hyperfilter of a commutative ordered Γ-semihypergroup $(S, \Gamma, \leq)$. If $W \ll a\gamma b$ and $a \in W$, then $b \in W$ for all $a, b \in S$ and $\gamma \in \Gamma$. Here, $U \ll V$ means that there exist $u \in U$ and $v \in V$ such that $u \leq v$, for all nonempty subsets U and V of S.*

Proof. Let W be a weak Γ-hyperfilter of S, $a \in W$ and $W \ll a\gamma b$, where $b \in S$. As $W \ll a\gamma b$, there exists $u \in W$ and $v \in a\gamma b$ such that $u \leq v$. Since W is a weak Γ-hyperfilter of S, we get $v \in W$. It implies that $(a\gamma b) \cap W \neq \varnothing$, for each $\gamma \in \Gamma$. By condition (2) of Definition 4, we obtain $b \in W$. □

Theorem 4. *Let W be a weak Γ-hyperfilter of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. If $U \cap W \neq \varnothing$ and $U \prec V$, then $V \cap W \neq \varnothing$, where $\varnothing \neq U, V \subseteq S$.*

Proof. Since $U \cap W \neq \varnothing$, then there exists $x \in S$ such that $x \in W$ and $x \in U$. As $U \prec V$ and $x \in U$, there exists $y \in V$ such that $x \leq y$. Since W is a weak Γ-hyperfilter of S and $x \in W$, we have $y \in W$, by condition (3) of Definition 4. So, $V \cap W \neq \varnothing$.

A proper weak Γ-hyperfilter F_w of S is said to be maximal if there does not exist any proper weak Γ-hyperfilter of S which properly contains F_w. Indeed, let F_w be a proper weak Γ-hyperfilter of S. Then, F_w is said to be maximal if $F_w \subseteq W \subseteq S$ implies $F_w = W$ or $W = S$, for all weak Γ-hyperfilters W of S. □

Example 2. In Example 1, $F_w = \{a, b, c\}$ is a maximal weak Γ-hyperfilter of S.

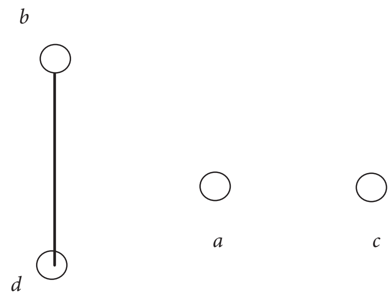

Figure 2: Figure of S for Example 3.

Theorem 5. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup with zero. If F_w is a proper weak Γ-hyperfilter of S, then there is a maximal weak Γ-hyperfilter of S containing F_w.*

Proof. Set $\mathcal{X} = \{W \subsetneq S \mid W$ as a weak Γ-hyperfilter of S, $F_w \neq S$ and $F_w \subseteq W\}$.

Since $F_w \in \mathcal{X}$, we have $\mathcal{X} \neq \varnothing$. Also, $(\mathcal{X}, \subseteq)$ is an ordered set by inclusion relation $\subseteq$. Let $\{W_i \mid i \in I\}$ be a chain in $\mathcal{X}$. By Theorem 1, $\cup_{i\in I} W_i$ is a weak Γ-hyperfilter of S. Clearly, $\cup_{i\in I} W_i$ is an upper bound of the chain $\{W_i \mid i \in I\}$. Since $0 \notin W_i$ for any $i \in I$, we get $0 \notin \cup_{i\in I} W_i$. This shows that $\cup_{i\in I} W_i \subsetneq S$. So, $\cup_{i\in I} W_i \in \mathcal{X}$. By Zorn's Lemma, there exists a maximal element M_w in $\mathcal{X}$. We assert that M_w is a maximal weak Γ-hyperfilter of S. Now, let J be a proper weak Γ-hyperfilter of S containing M_w. Then, J contains F_w and so it belongs to $\mathcal{X}$. Since M_w is maximal in $\mathcal{X}$, we get $J = M_w$. Therefore, M_w is a maximal weak Γ-hyperfilter of S.

The weak (m, n)-Γ-hyperfilter of ordered Γ-semihypergroups is defined as follows. □

Definition 5. Let $F_w \neq \varnothing$ be a subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then, F_w is said to be a left weak m-Γ-hyperfilter (right weak n-Γ-hyperfilter) of S if

(1) For all $a, b \in F_w$ and $\gamma \in \Gamma$, $(a\gamma b) \cap F_w \neq \varnothing$

(2) For all $a, b \in S$ and $\gamma \in \Gamma$, $(a\gamma b) \cap F_w \neq \varnothing \Longrightarrow a^m \subseteq F_w$ $(b^n \subseteq F_w)$

(3) For all $a \in F_w$ and $z \in S, a \leq z \Longrightarrow z \in F_w$

Here, m and n are positive integers. Note that if F_w is both a left weak m-Γ-hyperfilter and a right weak n-Γ-hyperfilter of S, then F_w is called a weak (m, n)-Γ-hyperfilter of S.

Clearly, every (m, n)-Γ-hyperfilter of S is a weak (m, n)-Γ-hyperfilter. However, the converse is not true, in general, that is, a weak (m, n)-Γ-hyperfilter may not be an (m, n)-Γ-hyperfilter of S.

Example 3. Suppose S is Γ-semihypergroup in Example 1 and put

$$\leq := \{(a, a), (b, b), (c, c), (d, b), (d, d)\}. \tag{4}$$

Then, $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup. The covering relation of S is given by

$$\prec = \{(d, b)\}. \tag{5}$$

The figure of S is shown in Figure 2.

Clearly, $F_w = \{a, b, c\}$ is a weak (m, n)-Γ-hyperfilter on S, but it is not an (m, n)-Γ-hyperfilter. Indeed, $a, b \in F_w$ and $a\Gamma b = \{b, d\} \nsubseteq F_w$.

Lemma 2. *Let $\{F_{w_i} \mid i \in I\}$ be a family of weak (m, n)-Γ-hyperfilters of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. If $\cap_{i\in I} F_{w_i} \neq \varnothing$, then $\cap_{i\in I} F_{w_i}$ is a weak (m, n)-Γ-hyperfilter of S.*

Proof. Let $x, y \in \cap_{i\in I} F_{w_i}$. Then, $x, y \in F_{w_i}$, for each $i \in I$. Since F_{w_i} is a weak (m, n)-Γ-hyperfilter of S for each $i \in I$, we get $(x\gamma y) \cap F_{w_i} \neq \varnothing$, for all $\gamma \in \Gamma$. It implies that $(x\gamma y) \cap (\cap_{i\in I} F_{w_i}) \neq \varnothing$.

Now, let $x, y \in S$, $\gamma \in \Gamma$, and $(x\gamma y) \cap (\cap_{i\in I} F_{w_i}) \neq \varnothing$. Then, there exists $u \in \cap_{i\in I} F_{w_i}$, for some $u \in x\gamma y$. Since $u \in \cap_{i\in I} F_{w_i}$, it follows that $u \in F_{w_i}$, for each $i \in I$. Since F_{w_i} is a weak (m, n)-Γ-hyperfilter of S, for each $i \in I$, we get $x^m, y^n \subseteq F_{w_i}$ for each $i \in I$. It implies that $x^m, y^n \subseteq \cap_{i\in I} F_{w_i}$.

Now, let $x \in \cap_{i\in I} F_{w_i}$ and $x \leq z \in S$. Then, $x \in F_{w_i}$ for each $i \in I$. Since F_{w_i} is a weak (m, n)-Γ-hyperfilter of S, for all $i \in I$, we have $z \in F_{w_i}$, for all $i \in I$. So, $z \in \cap_{i\in I} F_{w_i}$. Therefore, $\cap_{i\in I} F_{w_i}$ is a weak (m, n)-Γ-hyper filter of S.

Let $(S_i, \Gamma_i, \leq_i)$ be an ordered Γ_i-semihypergroup, for all $i \in \Omega$. Define $\odot$: $(\prod_{i\in I} S_i) \times (\prod_{i\in I} \Gamma_i) \times (\prod_{i\in I} S_i) \longrightarrow {}^{*}(\prod_{i\in I} S_i)$ by $(u_i)_{i\in\Omega} \odot (\gamma_i)_{i\in\Omega} \odot (v_i)_{i\in\Omega} = \{(t_i)_{i\in\Omega} \mid t_i \in u_i\gamma_i v_i\}$, for all $(u_i)_{i\in\Omega}, (v_i)_{i\in\Omega} \in \prod_{i\in\Omega} S_i$ and $(\gamma_i)_{i\in\Omega} \in \prod_{i\in\Omega} \Gamma_i$. Set $(u_i)_{i\in\Omega} \leq (v_i)_{i\in\Omega}$ if and only if, for all $i \in \Omega$, $u_i \leq_i v_i$.

Then, $(\prod_{i\in\Omega} S_i = \{(u_i)_{i\in\Omega} \mid u_i \in S_i\}, \prod_{i\in\Omega} \Gamma_i, \leq)$ is an ordered $\prod_{i\in\Omega} \Gamma_i$-semihypergroup [8]. In the following, we study the behavior of weak (m, n)-Γ-hyperfilters on $\prod_{i\in\Omega} S_i$. □

Theorem 6. *Let $(S_i, \Gamma_i, \leq_i)$ be an ordered Γ_i-semihypergroup, for all $i \in \Omega$. If W_i is a weak (m, n)-Γ-hyperfilter on S_i, for all $i \in \Omega$, then $F = \prod_{i\in\Omega} W_i$ is a weak (m, n)-Γ-hyperfilter on $\prod_{i\in\Omega} S_i$.*

Proof. Let $(u_i)_{i\in\Omega}, (v_i)_{i\in\Omega} \in F = \prod_{i\in\Omega} W_i$. Then, $u_i, v_i \in W_i$, for each $i \in \Omega$. As W_i is a weak (m, n)-Γ-hyperfilter of S_i, we have $(u_i\gamma_i v_i) \cap W_i \neq \varnothing$. So, $(u_i)_{i\in\Omega} \odot (\gamma_i)_{i\in\Omega} \odot (v_i)_{i\in\Omega} = (u_i\gamma_i v_i)_{i\in\Omega} \cap F \neq \varnothing$.

Now, let $(u_i)_{i\in\Omega}, (v_i)_{i\in\Omega} \in \prod_{i\in\Omega} S_i$ and $((u_i)_{i\in\Omega} \odot (\gamma_i)_{i\in\Omega} \odot (v_i)_{i\in\Omega}) \cap F \neq \varnothing$. Then,

$$\begin{aligned}
&((u_i)_{i\in\Omega} \odot (\gamma_i)_{i\in\Omega} \odot (v_i)_{i\in\Omega}) \cap F \neq \varnothing \\
&\Longrightarrow (u_i\gamma_i v_i)_{i\in\Omega} \cap F \neq \varnothing \\
&\Longrightarrow u_i\gamma_i v_i \cap W_i \neq \varnothing, \forall i \in \Omega \\
&\Longrightarrow u_i^m \subseteq W_i \text{ and } v_i^n \subseteq W_i, \forall i \in \Omega \Longrightarrow (u_i^m)_{i\in\Omega} \subseteq F \text{ and } (v_i^n)_{i\in\Omega} \subseteq F \\
&\Longrightarrow ((u_i)_{i\in\Omega})^m \subseteq F \text{ and } ((v_i)_{i\in\Omega})^n \subseteq F.
\end{aligned} \tag{6}$$

Let $(u_i)_{i\in\Omega} \in F$ and $(a_i)_{i\in\Omega} \in \prod_{i\in\Omega} S_i$ such that $((u_i)_{i\in\Omega}, (a_i)_{i\in\Omega}) \in \leq$. Then, for all $i \in \Omega$, we have $(u_i, a_i) \in \leq_i$. Since W_i is a weak (m, n)-Γ-hyperfilter of S_i, for each $i \in \Omega$, we have $a_i \in W_i$, for each $i \in \Omega$. It implies that $(a_i)_{i\in\Omega} \in \prod_{i\in\Omega} W_i = F$. Therefore, F is a weak (m, n)-Γ-hyperfilter of $\prod_{i\in\Omega} S_i$. □

3. Conclusions

Generalization of Γ-hyperfilters in ordered Γ-semihypergroups is necessary for further study of ordered Γ-semihypergroups. In this study, we introduced the notion of weak (m, n)-Γ-hyperfilter and then obtained some related basic results. In the future, we plan to study relative weak Γ-hyperfilters, fuzzy weak Γ-hyperfilters, and rough weak Γ-hyperfilters in ordered Γ-semihypergroups. We expect further research efforts in this direction.

Question 1: under what condition a weak (m, n)-Γ-hyperfilter of S coincides with a weak Γ-hyperfilter?

Question 2: under what condition arbitrary union of weak (m, n)-Γ-hyperfilters of S is a weak (m, n)-Γ-hyperfilter?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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