

## A MODAL APPROACH FOR THE GYROSCOPIC QUADRATIC EIGENVALUE PROBLEM

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**Abstract.** *The Automated Multi-Level Substructuring (AMLS) has been developed to reduce the computational demands of frequency response analysis. AMLS automatically divides a large finite element model into many substructures on a number of levels based on the sparsity structure of the system matrices. Assuming that the interior degrees of freedom depend quasistatically on the interface degrees of freedom, and modeling the deviation from quasistatic dependence in terms of a small number of selected substructure eigenmodes the size of the finite element model is reduced substantially. In this paper we consider conservative gyroscopic eigenvalue problems. The original AMLS method neglects the gyroscopic effects. We generalize the AMLS approach taking advantage of the fact that for gyroscopic problems there exists a basis of eigenvectors which can be used when modeling the deviation from quasistatic behaviour. In both cases the resulting quadratic eigenproblem is still very large. We suggest to solve it by the nonlinear Arnoldi method taking advantage of the minmax characterization of its eigenvalues.*

## 1 INTRODUCTION

Acoustic simulation is increasingly becoming an important part of the automotive design process. The most costly part of the effort of optimizing noise and vibration performance is the computation of frequency response for very large finite element models because a very large number of modes may be needed for obtaining satisfactory accuracy over the frequency range of interest. Over the last few years, a new method for frequency response analysis of complex structures has been developed by Bennighof and co-authors [2], [3], [4] known as Automatic Multi-Level Substructuring (AMLS). Here the large finite element model is divided adaptively into very many substructures on several levels, and the frequency response is represented in terms of substructure eigenvectors, which are less expensive to compute than global ones because they are obtained from much smaller eigenvalue problems which have lower modal density.

In this paper we consider the conservative gyroscopic eigenvalue problem

$$Kx + i\lambda Gx - \lambda^2 Mx = 0 \quad (1)$$

where  $K$  is the stiffness matrix modified by the presence of centrifugal forces,  $M$  is the mass matrix, and  $G$  is the gyroscopic matrix.  $K$  and  $M$  are assumed to be symmetric and positive definite, and  $G$  is skew-symmetric.

This problem arises for instance from simulation of sound radiation of rolling tires which is the major source of traffic noise at speed above 40 km/h for passenger cars, while for trucks this limit is approximately 60 km/h (cf. [12], [13]). Simulating the structural dynamics of the rolling tire by an Arbitrary Lagrangian Eulerian (ALE) approach developed by Nackenhorst [12] one ends up with problem (1), where a large number of eigenfrequencies in the range from 500 Hz to 2000 Hz have to be computed since this is an interval of particular perception of the human ear. For rotating wheels the ALE method may be interpreted as introducing an observer who is riding on the axis of the wheel, not rotating with the structure.

Mathematically, AMLS for problem (1) is nothing else but a projection method where the large problem under consideration is projected to a space spanned by a small number of eigenvectors of clamped substructures on several levels where only the local stiffness and mass matrices are taken into account, whereas the gyroscopic part is neglected.

Restricting the gyroscopic problem (1) to the interior degrees of freedom of a substructure one obviously obtains a problem of the same structure. It is well known that the quadratic problem (1) of dimension  $n$  has  $2n$  real eigenvalues which are symmetric with respect to the origin, and that both sets, the eigenvectors corresponding to the positive and to the negative eigenvalues of (1) form a basis of  $\mathbb{C}^n$  [8]. This suggests to consider for gyroscopic problems an AMLS method where the original problem is projected to a small number of local eigenmodes corresponding to the smallest positive eigenvalues of the restrictions of the gyroscopic problem to the substructures on the various levels. Considering the same substructuring this approach should have better approximation properties

than the original AMLS method. On the other hand, neglecting the gyroscopic part the eigenvalue and eigenvectors of all local problems are real, and the projection can be done in real arithmetic. Only the solution of the projected gyroscopic problem requires complex arithmetic, whereas all eigenvectors of the restricted gyroscopic problems are complex, and using them in a generalized AMLS method the whole algorithm requires complex arithmetic.

Our paper is organized as follows: Section 2 summarizes briefly the AMLS approach for linear eigenproblems, and in Section 3 we discuss the application of AMLS in two ways, using modal information of the linear problem  $Kx - \lambda^2 Mx = 0$  only, and of the restriction of the gyroscopic problem to substructures. Section 4 reviews three numerical methods for gyroscopic problem, namely linearization for very small problems, safeguarded iteration for slightly larger problem, and Arnoldi's method for large and sparse problems. Section 5 reports on our numerical experiments. It turns out that the approach taking advantage of restricted gyroscopic problems indeed has better approximation properties, but overall this improvement does not pay since this case requires complex arithmetic even in the reduction phase, whereas in the linear approach the reduction of the model is done in real arithmetic, and only solving the reduced model requires complex arithmetic. The paper closes with conclusions.

## 2 AMLS FOR LINEAR EIGENPROBLEMS

In this section we summarize the *Automated Multi-Level Substructuring* (AMLS) method for the linear eigenvalue problem

$$Kx = \lambda Mx \tag{2}$$

which was developed by Bennighof and co-workers over the last few years [2], [4], [3], [11], who applied it to solving frequency response problems involving large and complex models. Here,  $K$  is the stiffness and  $M$  mass matrix of a finite element model of a structure. The method is not restricted to this type of problems but it applies to the more general problem (2) where  $K \in \mathbb{R}^{n \times n}$  and  $M \in \mathbb{R}^{n \times n}$  are large and sparse symmetric and positive definite matrices. Because of its pictographic nomenclature, however, we will use the frequency response problem to explain the AMLS method.

Similarly as in the component mode synthesis (CMS) the structure is partitioned into a small number of substructures based on the sparsity pattern of the system matrices, but more generally than in CMS these substructures in turn are substructured on a number of levels yielding a tree topology for the substructures. Figure 1 shows an example where each substructure has (at most) two children.

Substructures on the lowest level consist of a small number of degrees of freedom, which are partitioned into two sets: interface degrees of freedom which are shared with an adjacent substructure, and interior or local degrees of freedom which are only connected to degrees of freedom in their own substructure. Correspondingly, the substructure displacement vector  $u$  is partitioned into  $u_i$  and  $u_\ell$ . Substructure response is represented in

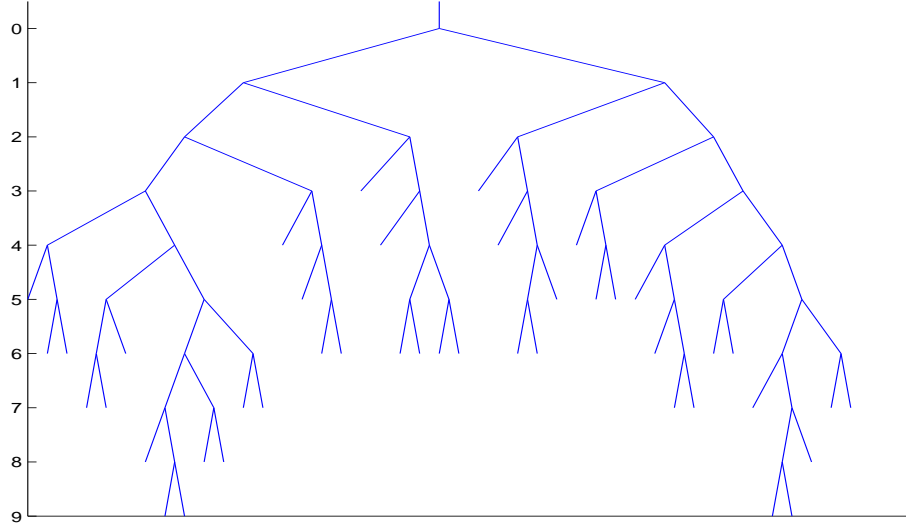


Fig. 1: Substructure tree

a Craig–Bampton form [6] as

$$\begin{pmatrix} u_i \\ u_\ell \end{pmatrix} = \begin{pmatrix} I & O \\ \Psi_\ell & \Phi_\ell \end{pmatrix} \begin{pmatrix} u_i \\ \eta \end{pmatrix} =: T_\ell \begin{pmatrix} u_i \\ \eta \end{pmatrix} \quad (3)$$

where  $\Phi_\ell$  satisfies the eigenvalue problem  $K_{\ell\ell}\Phi_\ell = M_{\ell\ell}\Phi_\ell\Lambda_\ell$ , and the matrices  $K$  and  $M$  are partitioned in the same way as the displacement vector.  $\Lambda_\ell$  is a diagonal matrix of eigenvalues,  $\eta$  is the vector of modal coordinates of the substructure, and  $\Psi_\ell = -K_{\ell\ell}^{-1}K_{\ell i}$  describes the quasistatic dependence of the local coordinates on the interface degrees of freedom.

Assuming that the eigenvectors are normalized with respect to the local mass matrix  $M_{\ell\ell}$  this transformation to a quasistatic–modal representation yields the substructure stiffness matrix

$$\tilde{K} = T_\ell^T \begin{pmatrix} K_{ii} & K_{i\ell} \\ K_{\ell i} & K_{\ell\ell} \end{pmatrix} T_\ell = \begin{pmatrix} \tilde{K}_{ii} & O \\ O & \Lambda_\ell \end{pmatrix} \quad (4)$$

where  $\tilde{K}_{ii} = K_{ii} - K_{i\ell}K_{\ell\ell}^{-1}K_{\ell i}$  is the Schur complement of  $K_{\ell\ell}$ , and the substructure mass matrix is transformed to

$$\tilde{M} = \begin{pmatrix} \tilde{M}_{ii} & \tilde{M}_{i\ell} \\ \tilde{M}_{\ell i} & I \end{pmatrix} \quad (5)$$

where  $I$  denotes the identity matrix,

$$\tilde{M}_{ii} = M_{ii} - K_{i\ell}K_{\ell\ell}^{-1}M_{\ell i} - M_{i\ell}K_{\ell\ell}^{-1}K_{\ell i} + K_{i\ell}K_{\ell\ell}^{-1}M_{\ell\ell}K_{\ell\ell}^{-1}K_{\ell i} \quad (6)$$

and

$$\tilde{M}_{i\ell} = M_{i\ell}\Phi_\ell - K_{i\ell}\Phi_\ell\Lambda_\ell^{-1} = \tilde{M}_{\ell i}^T. \quad (7)$$

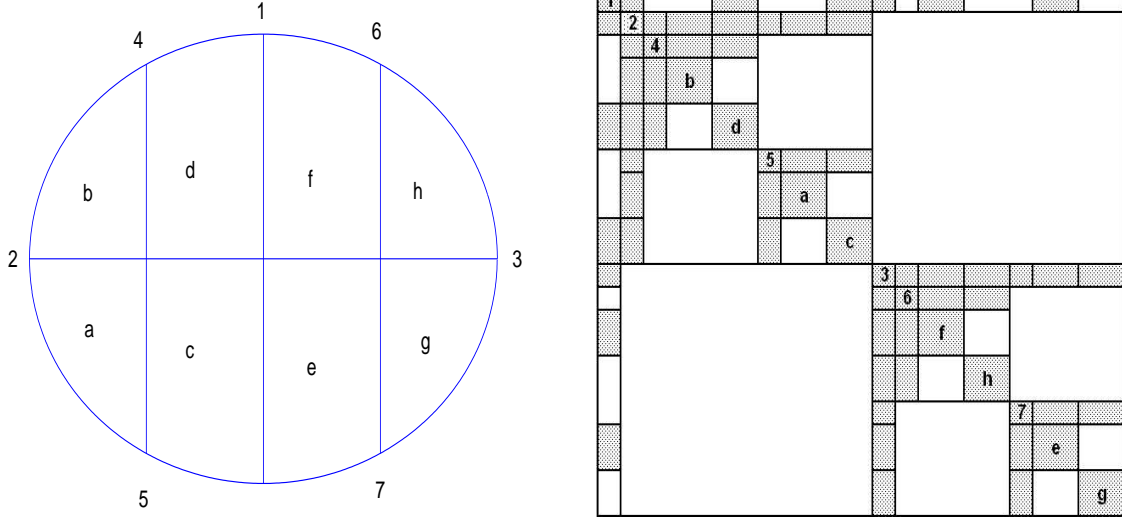


Fig. 2: Substructuring / sparsity pattern of stiffness matrix

Once substructures on the lowest level have been transformed they are assembled to substructures on the next level. Again interface and local degrees of freedom are identified, and the substructure models are transformed similarly as on the lowest level. Assembly to higher-level substructures, and the transformation to quasistatic-modal representation continues, until a model for the entire structure has been assembled, which is equivalent to the original problem and which obtains the following block form:

$$\begin{pmatrix} \tilde{K}_{II} & O \\ O & \Omega \end{pmatrix} \tilde{u} = \lambda \begin{pmatrix} \tilde{M}_{II} & \tilde{M}_{IL} \\ \tilde{M}_{LI} & \tilde{M}_{LL} \end{pmatrix} u. \quad (8)$$

Here  $\tilde{K}_{II}$  is the Schur complement of all interior degrees of freedom of the coarsest substructuring in  $K$ ,  $\Omega$  is a diagonal matrix containing all eigenvalues obtained from the transformations on the various levels, and  $\tilde{M}_{II}$ ,  $\tilde{M}_{IL}$ , and  $\tilde{M}_{LL}$  are compiled in the course of the algorithm from the contributions (6) and (7) of the substructures.

Figure 2 on the left shows a substructuring of a plain domain where the subdomains on the lowest level are labeled  $a - h$ . The only interface on the coarsest level is 1, the interfaces on the second level are 2 and 3, and on the third level are 4 – 7. The picture on the right shows the sparsity pattern of the stiffness matrix after reordering according to the partition. Only the diagonal blocks corresponding to the substructures and the interfaces, and the shaded off-diagonal blocks contain nonzero elements (actually only those rows and columns of the off-diagonal blocks which correspond to interface degrees of freedom of the corresponding substructure hold nonzero elements).

It is well known that the high frequency modes of the substructures do not influence the low frequency modes of the entire structure. Hence, similarly as in the component mode synthesis method we can reduce the dimension of the eigenvalue problem (8) considerably if we delete rows and columns in (8) corresponding to high frequencies of the substructures,

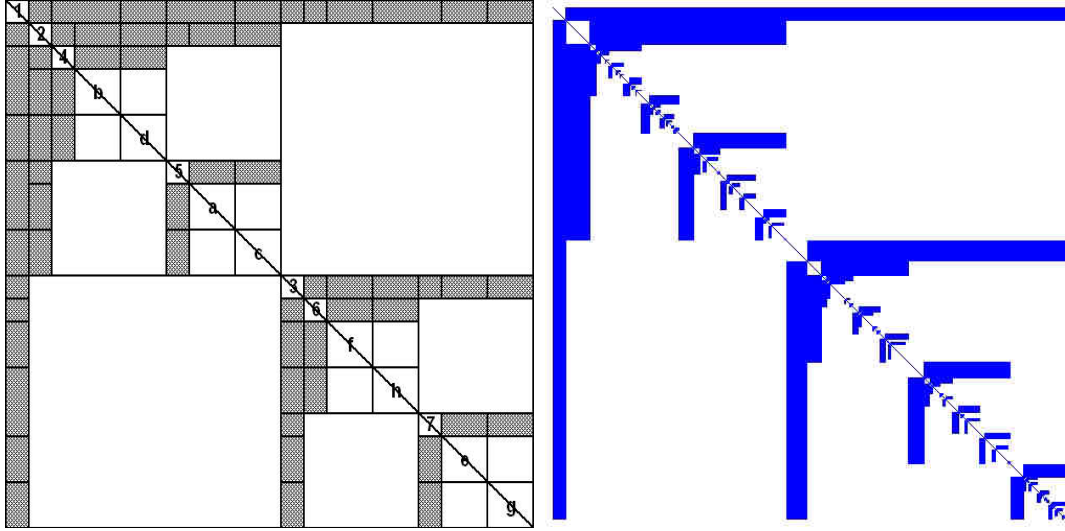


Fig. 3: Sparsity pattern of mass matrix after AMLS projection

and we do this not only for the lowest level, but for the subsequent substructures (i.e. interfaces) as well. Obviously, this is equivalent to replacing the basis transformation (3) of a substructure by the projection to the space spanned by the columns of (3) where only those columns are considered in  $\Phi_\ell$  which correspond to low frequencies of the substructure, namely those frequencies which are less than a given threshold  $\gamma$ .

The cost of performing the projection above consists of the cost of obtaining the matrices  $\Psi_\ell$  and  $\Phi_\ell$  and transforming the substructure stiffness and mass matrices  $K$  and  $M$ . Notice that for every substructure only a partial eigenproblem has to be solved, and only a small number of eigenpairs are needed. Moreover, the eigenproblems are usually very small because most of the local degrees of freedom of a substructure are local degrees of freedom of the substructures of the next lower level which form the current substructure. Hence, the part of the substructure stiffness matrix corresponding to these degrees of freedom is already diagonal, and we only have to consider those local degrees of freedom which did not have this property on the next lower level, i.e. those interface degrees of freedom of the next lower level which are not interface degrees of freedom on the current level.

Finally, we diagonalize the diagonal blocks corresponding to the interface on the coarsest level again neglecting eigenpairs which are larger than the threshold  $\gamma$ . Thus, we end up with a projected eigenproblem, where the stiffness matrix has become diagonal, and the mass matrix is replaced by a matrix the diagonal of which is the identity, and the only off-diagonal blocks containing non-zero elements are the ones describing the coupling of the substructures and its interfaces. Figure 3 shows the sparsity pattern of the projected mass matrix (on the left for the example in Figure 2, and on the right for the reduction of the dimension of a FE model of a container ship from 35262 to 2786). Notice, that the dimension of the projected problem is much smaller than the one of the original problem since a great deal of the eigenpairs of the substructures are neglected.

It is obvious that the AMLS method has a lot of parallelization potential. In [9] we proposed a parallelization concept.

### 3 GYROSCOPIC QUADRATIC EIGENPROBLEMS

Small vibrations of a conservative rotating system are governed by the system of linear differential equations

$$M\ddot{q} + G\dot{q} + Kq = 0, \quad (9)$$

where  $M$  is the mass matrix,  $K$  the stiffness matrix modified by the presence of centrifugal forces, and  $G = -G^T$  is the skew-symmetric gyroscopic matrix. Using the ansatz  $q = xe^{i\lambda t}$  one obtains the quadratic eigenvalue problem

$$Kx + i\lambda Gx - \lambda^2 Mx = 0. \quad (10)$$

The approach of Bennighof for quadratic eigenproblems (10) containing a linear term  $\lambda Gx$  is to neglect this term when defining the basis transformation corresponding to the substructures, i.e. to define the Craig-Bampton form as in (4) where  $\Phi_\ell$  satisfies the eigenvalue problem  $K_{\ell\ell}\Phi_\ell = M_{\ell\ell}\Phi_\ell\Lambda_\ell^2$ , and  $\Psi_\ell$  is the Schur complement of  $K_{\ell\ell}$ . Then this basis transformation (4) yields the same substructure mass matrix (5) – (7) as before, the only difference in the substructure stiffness matrix is that we have to replace  $\Lambda_\ell$  by  $\Lambda_\ell^2$ , and the gyroscopic substructure matrix is transformed to

$$T_\ell^T G T_\ell = \begin{pmatrix} \tilde{G}_{ii} & \tilde{G}_{il} \\ \tilde{G}_{li} & \Phi_\ell^T G_{\ell\ell} \Phi_\ell \end{pmatrix} \quad (11)$$

where

$$\tilde{G}_{ii} = G_{ii} - K_{il}K_{\ell\ell}^{-1}G_{\ell i} - G_{il}K_{\ell\ell}^{-1}K_{\ell i} + K_{il}K_{\ell\ell}^{-1}G_{\ell\ell}K_{\ell\ell}^{-1}K_{\ell i} \quad (12)$$

and

$$\tilde{G}_{il} = G_{il}\Phi_\ell - K_{il}K_{\ell\ell}^{-1}G_{\ell\ell}\Phi_\ell = -\tilde{G}_{li}^T. \quad (13)$$

Neglecting eigenmodes corresponding to eigenvalues exceeding a given threshold, and assembling the substructures at the consecutive levels, one gets the reduced model

$$\hat{K}\hat{u} + i\lambda\hat{G}\hat{u} - \lambda^2\hat{M}\hat{u} = 0, \quad (14)$$

where the stiffness and mass matrix have the same structure as in the linear case, and the gyroscopic matrix  $\hat{G}$  is a skew-symmetric block matrix containing diagonal blocks corresponding to the (reduced) substructures and interfaces, and only off-diagonal blocks describing the coupling of a substructure and its interface contain non-zero elements. Notice, that all projectors are real, and therefore the reduction can be performed in real arithmetic.

A disadvantage of Bennighof's approach may be the fact that the eigenvectors of the clamped substructures not taking into account the gyroscopic part are not appropriate to model the deviation from quasi-static behaviour of the substructures.

Problem (10) obviously is equivalent to the linear eigenvalue problem

$$Ay := \begin{pmatrix} iG & K \\ K & O \end{pmatrix} \begin{pmatrix} \lambda x \\ x \end{pmatrix} = \lambda \begin{pmatrix} M & O \\ O & K \end{pmatrix} \begin{pmatrix} \lambda x \\ x \end{pmatrix} =: \lambda By. \quad (15)$$

The matrices  $A$  and  $B$  are sparse, and in principle the AMLS method applies to problem (15). However, since linearization doubles the dimension of the problem we suggest a direct AMLS method for the quadratic problem (10) taking advantage of spectral properties of gyroscopic quadratic eigenproblems.

Assuming that the matrices  $K \in \mathbb{R}^{n \times n}$  and  $M \in \mathbb{R}^{n \times n}$  are symmetric and positive definite, and that  $G$  is skew-symmetric, it is obvious that the matrix  $A$  on the left of (15) is Hermitean, and  $B$  is symmetric and positive definite. Hence, problem (10) has  $2n$  real eigenvalues, and if  $(\lambda, x)$  is an eigenpair of problem (10) then it is easily seen that  $(-\lambda, \bar{x})$  is an eigenpair, as well. Hence there are  $n$  positive eigenvalues  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ , and  $n$  negative eigenvalues  $\lambda_{-n} \leq \dots \leq \lambda_{-1} < 0$ , and the spectrum is symmetric with respect to the origin.

Duffin [8] proved that eigenvectors corresponding to different positive (or negative) eigenvalues are linearly independent, and therefore there exists a basis of  $\mathbb{C}^n$  consisting of eigenvectors of (10) corresponding to its positive (or negative) eigenvalues. Taking advantage of this basis the AMLS method can be generalized to the quadratic eigenproblem (10) in a straightforward manner.

Assume that the underlying problem has been partitioned into substructures on a number of levels based on the sparsity pattern of the system matrices  $K$ ,  $M$  and  $G$ , and assume that for a given substructure on the lowest level the unknowns are partitioned into interface and local degrees of freedom. We represent the substructure response similarly as in the Craig–Bampton form as

$$\begin{pmatrix} u_i \\ u_\ell \end{pmatrix} = \begin{pmatrix} I & O \\ \Psi_\ell & \Phi_\ell \end{pmatrix} \begin{pmatrix} u_i \\ \eta \end{pmatrix} =: T_\ell \begin{pmatrix} u_i \\ \eta \end{pmatrix} \quad (16)$$

where  $\Psi_\ell = -K_{\ell\ell}^{-1}K_{\ell i}$  denotes the static response of the substructure to interface displacements, and  $\Phi_\ell \in \mathbb{C}^{i \times i}$  satisfies the quadratic gyroscopic eigenproblem

$$K_{\ell\ell}\Phi_\ell + iG_{\ell\ell}\Phi_\ell\Lambda_\ell - M_{\ell\ell}\Phi_\ell\Lambda_\ell^2 = 0. \quad (17)$$

Problem (17) can be linearized similarly to (15) and solved by a standard solver for linear Hermitean eigenvalue problems if the dimension of (17) is not too large, or by safeguarded iteration, which will be considered in Section 4.

With this change of the basis the substructure matrices obtain the following form

$$T_\ell^H \begin{pmatrix} K_{ii} & K_{i\ell} \\ K_{\ell i} & K_{\ell\ell} \end{pmatrix} T_\ell = \begin{pmatrix} K_{ii} - K_{i\ell}K_{\ell\ell}^{-1}K_{\ell i} & O \\ O & \Phi_\ell^H K_{\ell\ell} \Phi_\ell \end{pmatrix}, \quad (18)$$



and for  $N \in \{G, M\}$

$$T_\ell^H N T_\ell = \begin{pmatrix} \tilde{N}_{ii} & \tilde{N}_{i\ell} \\ \tilde{N}_{\ell i} & \Phi_\ell^H N_{\ell\ell} \Phi_\ell \end{pmatrix} \quad (19)$$

where

$$\tilde{N}_{ii} = N_{ii} - K_{i\ell} K_{\ell\ell}^{-1} N_{\ell i} - N_{i\ell} K_{\ell\ell}^{-1} K_{\ell i} + K_{i\ell} K_{\ell\ell}^{-1} N_{\ell\ell} K_{\ell\ell}^{-1} K_{\ell i} \quad (20)$$

and

$$\tilde{N}_{i\ell} = N_{i\ell} \Phi_\ell - K_{i\ell} K_{\ell\ell}^{-1} N_{\ell\ell} \Phi_\ell, \quad \tilde{N}_{\ell i} = \Phi_\ell^H N_{\ell i} - \Phi_\ell^H N_{\ell\ell} K_{\ell\ell}^{-1} K_{\ell i}. \quad (21)$$

Assembling substructures on the next higher level, and identifying interface and local degrees of freedom, the eigenproblem corresponding to the local unknowns again is a gyroscopic problem, and we can continue the accumulation to higher level substructures until a model of the entire structure has been assembled.

Again, we neglect eigenvectors of all substructures which correspond to eigenvalues exceeding a specified threshold, and we finally reduce the size of the quadratic problem corresponding to the interface degrees of freedom on the coarsest level. Thus we obtain a reduced model (14) of much smaller dimension where  $\hat{K}$  is a block diagonal matrix with Hermitean and positive definite blocks,  $\hat{M}$  is a Hermitean, positive definite block matrix which has the same diagonal block structure as  $\hat{K}$ , and where only off diagonal blocks describing the coupling of a substructure and its interface contain non-zero elements, and  $\hat{G}$  is a skew-Hermitean matrix having the same block structure as  $\hat{M}$ .

#### 4 ARNOLDI METHOD FOR NONLINEAR EIGENVALUE PROBLEMS

In this section we consider numerical methods for computing some of the smallest positive eigenvalues of the quadratic eigenvalue problem

$$T(\lambda)x := (\lambda^2 M - \lambda iG - K)x = 0 \quad (22)$$

where  $K, M \in \mathbb{C}^{n \times n}$  are Hermitean and positive definite and  $G \in \mathbb{C}^{n \times n}$  is skew-Hermitean. Problems of this type appear in two places in the AMLS method for gyroscopic problems: first for every substructure when reducing the model we have to determine all eigenpairs such that the eigenvalue does not exceed a threshold, and secondly the reduced model itself has this form. Depending on the number of local degrees of freedom of the substructure under consideration in the first case the dimension of problem (22) may be small or not. In the second case we assume the matrices  $K$ ,  $M$ , and  $G$  to be large and sparse, although the order of the model is already reduced by AMLS. Hence, methods for both, small and large dimensions are needed.

If the dimension of problem (22) is very small, then it is proximate to consider the linearization

$$\begin{pmatrix} iG & K \\ K & O \end{pmatrix} \begin{pmatrix} \lambda x \\ x \end{pmatrix} = \lambda \begin{pmatrix} M & O \\ O & K \end{pmatrix} \begin{pmatrix} \lambda x \\ x \end{pmatrix} \quad (23)$$

of problem (22) and to apply any dense solver. However, since the dimension is doubled this method becomes time consuming for not too large dimensions.

A method which only solves linear eigenproblems of the dimension of problem (22) and therefore is capable for larger (but not too large) dimensions is safeguarded iteration which takes advantage of a variational characterization of the eigenvalues of problem (22). Duffin [7], [8] proved that similarly to Hermitean linear eigenproblems the positive eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  of the gyroscopic eigenproblem (22) satisfy a minmax characterization

$$\lambda_j = \min_{\dim V=j} \max_{x \in V, x \neq 0} p(x), \quad (24)$$

where

$$p(x) = \frac{1}{2x^H M x} \left( ix^H G x + \sqrt{-(x^H G x)^2 + 4x^H K x \cdot x^H M x} \right) \quad (25)$$

denotes the (positive) Rayleigh functional of problem (22), and  $V$  denotes a subspace of  $\mathbb{C}^n$ . Notice that  $p(x)$  is the positive solution of the quadratic equation  $x^H T(\lambda)x = 0$  generalizing the Rayleigh quotient for the linear case  $T(\lambda) = \lambda M - K$ .

The minimum is attained by the invariant subspace of  $T(\lambda_j)$  corresponding to the  $j$  largest eigenvalues of the linear problem  $T(\lambda_j)y = \mu y$ ,  $\mu = 0$  is the  $j$  largest eigenvalue of  $T(\lambda_j)$ , and the maximum is attained by any eigenvector of  $T(\lambda_j)$  corresponding to  $\mu = 0$ .

These properties suggest the following method called safeguarded iteration for computing the  $j$  smallest eigenvalue  $\lambda_j$  of (22) and a corresponding eigenvector:

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**Algorithm 1** Safeguarded iteration for computing the  $j$ -smallest eigenvalue

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- 1: Start with an approximation  $\sigma_1$  to the  $j$ -th eigenvalue of (22)
  - 2: **for**  $\ell = 1, 2, \dots$  until convergence **do**
  - 3:   determine an eigenvector  $x_\ell$  corresponding to the  $j$ -largest eigenvalue of  $T(\sigma_\ell)$
  - 4:   solve  $x_\ell^H T(\sigma_{\ell+1})x_\ell = 0$  for  $\sigma_{\ell+1} = p(x_\ell)$
  - 5: **end for**
- 

Safeguarded iteration has the following convergence properties [18].

**THEOREM 1**

- (i) For  $j = 1$  the safeguarded iteration converges globally to  $\lambda_1$ .
- (ii) If  $\lambda_j$  is a simple eigenvalue of (22) then the safeguarded iteration converges locally and quadratically to  $\lambda_j$ .
- (iii) Let  $T'(\lambda) = 2\lambda M - iG$  be positive definite for  $\lambda$  in a neighbourhood of  $\lambda_j$ . If  $x_\ell$  in step 3: of Algorithm 1 is chosen to be an eigenvector corresponding to the  $j$  largest eigenvalue of the generalized eigenproblem  $T(\sigma_\ell)x = \mu T'(\sigma_\ell)x$  then the convergence is even cubic.

If problem (22) is large and sparse we consider iterative projection methods where approximations of the wanted eigenvalues and corresponding eigenvectors are obtained from projections to subspaces which are expanded in the course of the algorithm. For linear eigenproblems methods of this type like the Lanczos, Arnoldi or Jacobi–Davidson method have turned out to be very efficient [1]. Generalizations to the nonlinear eigenproblem (22) are contained in [5], [15], [16], [17].

We briefly summarize the Arnoldi approach for computing some of the smallest positive eigenvalues of problem (22). We assume that we already constructed a subspace  $\mathcal{V}$  of  $\mathbb{C}^n$ , and obtained approximations to the smallest eigenvalues  $\lambda_1 \leq \dots \leq \lambda_{\ell-1}$  of problem (22) from the projected problem

$$V^H T(\lambda) V y = \lambda^2 V^H M V - \lambda i V^H G V y - V^H K V y = 0 \quad (26)$$

where  $V$  denotes an orthonormal basis of  $\mathcal{V}$ .

To obtain approximations to further eigenvalues it is reasonable to expand the approximating space  $\mathcal{V}$  by a direction which has high approximation potential for the next wanted eigenvector. As in the linear case inverse iteration is known to converge quadratically for simple eigenvalues, and therefore,  $v = T(\lambda)^{-1} T'(\lambda) x$  is a good candidate for expanding  $\mathcal{V}$ , where  $\lambda$  and  $x$  is the current approximation to the wanted eigenvalue and eigenvector, respectively. However, solving a large linear system in each iteration step is prohibitive since it is much too expensive, and replacing  $v$  by a simplified version  $v = T(\sigma)^{-1} T'(\lambda) x$  with a fixed shift  $\sigma$  is known to lead to wrong convergence.

The following variant of inverse iteration in Algorithm 2 called residual inverse iteration and introduced by Neumaier [14] does not have these unpleasant properties.

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**Algorithm 2** Residual inverse iteration

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- 1: Start with an approximation  $x_1$  to an eigenvector of (22)
  - 2: **for**  $\ell = 1, 2, \dots$  until convergence **do**
  - 3:   evaluate the Rayleigh functional  $\mu_{\ell+1} = p(x_\ell)$
  - 4:   compute the residual  $r_\ell = T(\mu_{\ell+1}) x_\ell$
  - 5:   solve  $T(\sigma) d_\ell = r_\ell$
  - 6:   set  $x_{\ell+1} = x_\ell - d_\ell$ ,  $x_{\ell+1} = x_{\ell+1} / \|x_{\ell+1}\|$
  - 7: **end for**
- 

Theorem 2 proved in [14] describes the convergence of this method.

**THEOREM 2** Let  $T(\lambda)$  be twice continuously differentiable. Assume that  $\hat{\lambda}$  is a simple eigenvalue of problem (22), and let  $\hat{x}$  be a corresponding eigenvector normalized by  $\|\hat{x}\| = 1$ . Then the residual inverse iteration converges for all  $\sigma$  sufficiently close to  $\hat{\lambda}$ , and it holds

$$\frac{\|x_{\ell+1} - \hat{x}\|}{\|x_\ell - \hat{x}\|} = \mathcal{O}(|\sigma - \hat{\lambda}|), \quad \text{and} \quad |\lambda_{\ell+1} - \hat{\lambda}| = \mathcal{O}(\|x_\ell - \hat{x}\|^2).$$

Expanding the search space by residual inverse iteration (or an approximation  $v = MT(\mu_{\ell+1})x_\ell$ ,  $M \approx T(\sigma)^{-1}$  if a factorization of  $T(\sigma)$  is too expensive to obtain) one gets the projection method in Algorithm 3 which reduces to the shift-and-invert Arnoldi method in the linear case.

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**Algorithm 3** Nonlinear Arnoldi method

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1: Start with an initial shift  $\sigma$  and initial orthonormal basis  $V$ 
2: determine preconditioner  $M \approx T(\sigma)^{-1}$ 
3: while  $m \leq$  number of wanted eigenvalues do
4:   compute  $m$ -th eigenvalue  $\mu$  of projected problem  $V^T T(\lambda) V y = 0$ 
5:   determine Ritz vector  $u = Vy$  and residual  $r = T(\mu)u$ 
6:   if  $\|r\|/\|u\| < \varepsilon$  then
7:     accept eigenpair  $\lambda_m = \mu$ ,  $x_m = u$ 
8:     choose new shift  $\sigma$  and update preconditioner  $M$  if indicated
9:     restart if necessary
10:     $m=m+1$ 
11:   end if
12:    $v = Mr$ ;  $v = v - VV^T v$ ;  $v = v/\|v\|$ ;  $V = [V, v]$ ;
13:   reorthogonalize if necessary
14: end while

```

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We comment on some of the steps of the Arnoldi method in the following.

- 1: Since the safeguarded iteration converges globally, the algorithm can be started with any random vector. However, with  $v$  an approximation to an eigenvector corresponding to the maximal eigenvector of  $T(\mu)$  and  $\mu$  an approximation to the smallest eigenvalue of  $T(\cdot)$  one gets much faster convergence. On the other hand starting with a random vector the method collects information on the higher eigenvectors while iterating for the first one, which speeds up the convergence of the method for higher eigenvalues. In our numerical experiments we did not observe essential differences in the overall convergence behaviour.
- 4: The projected problem can be solved by linearization or safeguarded iteration.
- 8: Corresponding to Theorem 2 the residual inverse iteration with fixed pole  $\sigma$  converges linearly, and the contraction rate satisfies  $\mathcal{O}(|\sigma - \lambda_m|)$ . We therefore update the preconditioner if the convergence has become too slow measured by the quotient of the last two residuals before convergence. The new pole is chosen close to the eigenvalue wanted next.
- 9: As the subspaces expand in the course of the algorithm the increasing storage or the computational cost for solving the projected eigenvalue problems may make it

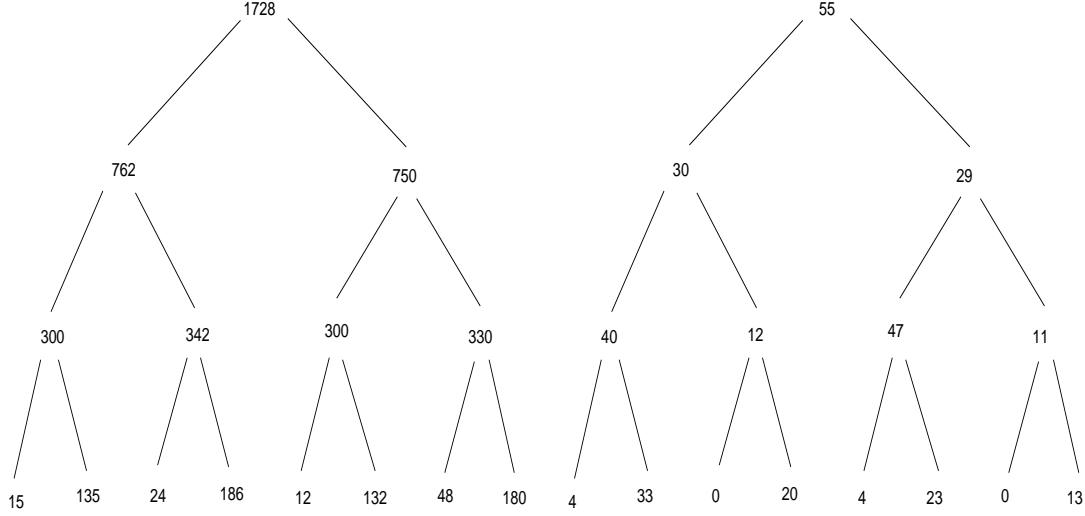


Fig. 4: Substructuring of FE model of rotating wheel

necessary to restart the algorithm and purge some of the basis vectors. Since a restart destroys information on the eigenvectors and particularly on the one the method is just aiming at we restart only if an eigenvector has converged.

By the proof of the minmax characterization the minimum is attained by the invariant subspace  $\tilde{\mathcal{V}}$  of  $T(\lambda_k)$  corresponding to the  $k$  largest eigenvalues of  $T(\lambda_k)$ . We therefore restart with  $\tilde{V} = VZ$  where  $Z$  denotes a basis of the invariant subspace of  $V^H T(\mu)V$  corresponding to the  $m$  largest eigenvalues of  $V^H T(\mu)T$ , or to retain more valuable information the invariant subspace corresponding to the  $\tilde{m}$  largest eigenvalues where  $\tilde{m}$  is slightly bigger than  $m$ .

If  $T'(\lambda)$  is positive definite then  $\tilde{\mathcal{V}}$  in the last paragraph can be replaced by the subspace spanned by the eigenvectors of  $T(\lambda_k)u = \mu T'(\lambda_k)u$  corresponding to the  $k$  largest eigenvalues. In this case we therefore restart with  $VZ$  where  $Z$  is spanned by eigenvectors of  $V^H T(\mu)Vz = \mu V^H T'(\mu)Vz$  corresponding to the  $m$  or  $\tilde{m}$  largest eigenvalues of the generalized eigenproblem.

## 5 NUMERICAL EXPERIMENTS

To test the approximation properties of both variants of the AMLS method for gyroscopic problems we determined all eigenpairs  $(\omega, x)$  of a coarse finite element model of a rotating wheel of dimension 1728 such that  $\omega \leq 12500$ , i.e. eigenpairs corresponding to eigenfrequencies up to 2000 Hz.

Using the graph partitioner CHACO [10] we determined a substructuring containing four levels. Figure 4 shows the topology of the substructuring and the number of degrees of freedom of the substructures on the left. We included all eigenvalues of substructures less than 20000 in the AMLS model, and obtained a reduced model of dimension 318

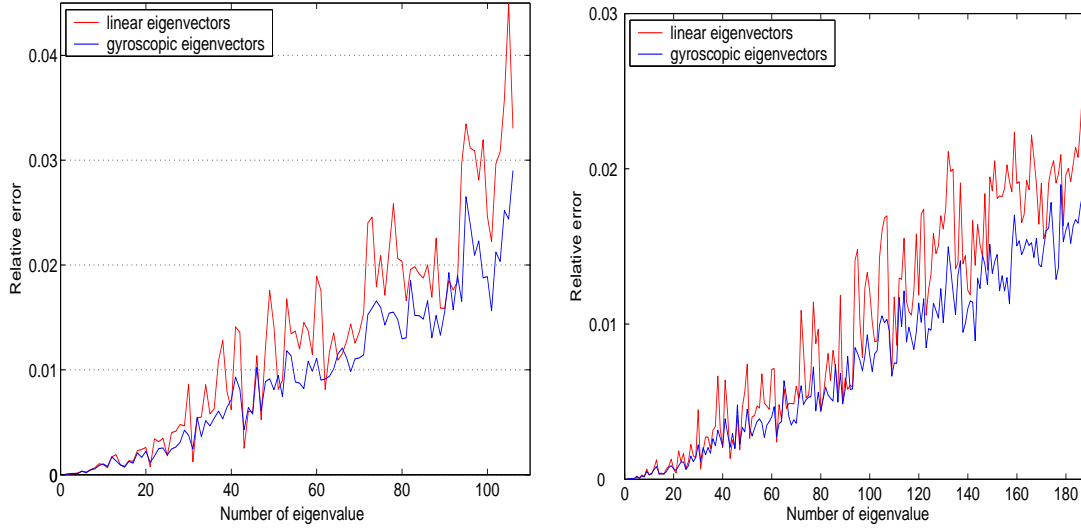


Fig. 5: Relative errors for AMLS with local linear and gyroscopic models

and 321 when taking into account eigenvalues of the local linear and quadratic eigenvalue problem, respectively. On the right, Figure 4 shows the number of eigenvalues for each substructure which were included into the model in the quadratic case. For the linear case these number are very similar differing by 1 or 2 in 6 substructures.

AMLS returned 106 and 107 eigenvalues in the interval  $[0, 12500]$ , and the maximum relative error is 0.045 and 0.029 for the reduction with linear and quadratic local eigenvalue problems, respectively. The relative errors for all eigenvalue approximations are displayed in Figure 5 on the left.

The numerical experiments were run under MATLAB 6.5 on a Pentium 4 processor with 3.0 GHz and 1 GB RAM. For substructuring and determining the reduced model the method required 9.9 seconds when using linear eigenproblems of the substructures and 12.6 seconds with gyroscopic problems, and in both cases 8.1 seconds were needed to solve the resulting gyroscopic eigenvalue problem. Solving the original gyroscopic problem by the nonlinear Arnoldi method with safeguarded iteration to solve the projected problems 72.5 second were needed to determine the smallest 107 eigenvalues and corresponding eigenvectors.

For the same model we computed eigenpair approximations in the interval  $0 \leq \omega \leq 16500$  with a threshold  $\gamma = 26000$  for the substructure eigenvalue problems. AMLS determined 188 and 190 eigenvalues with a maximum relative error 0.026 and 0.019 for local linear and gyroscopic models, respectively. The relative errors of all eigenvalue approximations are contained in Figure 5 on the right. The CPU time in this case was 10.0 seconds and 21.4 seconds for obtaining the reduced model with linear and gyroscopic substructure eigenvalue problems, respectively, and 35.3 seconds in both cases for solving the reduced gyroscopic problem. The nonlinear Arnoldi method for the original problem needed 202.1 seconds to obtain the eigenpairs corresponding to the smallest 190 eigenvalues.

## 6 CONCLUSIONS

We have presented two versions of the Automated Multi-Level Substructuring (AMLS) method for solving large and sparse conservative gyroscopic problems, where on the one hand linear eigenmodes of the substructures are in use to model the deviation from quasistatic dependence of the substructures on their boundary degrees of freedom, and on the other hand correct eigenmodes of the gyroscopic behaviour of the substructures. The improved approximation properties yield smaller relative errors in the latter case, while the necessary complex arithmetic in the reduction phase increases the total CPU time. For the relatively small problem considered here, and solved under MATLAB the improvement of the approximation properties does not pay. Further experiments with larger and more realistic models are necessary.

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