

# On Observability Estimates for Semigroups in Banach Spaces

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# Abstract

We consider control systems given by partial differential equations that are modeled as differential equations on infinite-dimensional state spaces using semigroups. In particular, we are interested in systems where the state space is given by a Banach space. For example, it is convenient to consider the heat equation in the non-reflexive space  $L_1$  since the norm of the states describes the total heat content. We thus consider control systems in general Banach spaces with bounded observation and control operators.

The central part of this thesis forms a general method to prove final-state observability estimates based on the Lebeau–Robbiano strategy. This approach has proven valuable for several applications and has been subsequently developed into sufficient conditions for observability in terms of an uncertainty principle and a dissipation estimate. However, the previous results on this strategy are limited to Hilbert spaces only. Therefore, we unify the respective advantages of the earlier results and generalize them to the natural framework of Banach spaces. Despite the general setting, we obtain explicit estimates of the observability constant. A novel aspect of our results is the integral expression by which we estimate the observability constant for generalized growth rates in the assumptions. Our estimates reflect the qualities of previous results and resemble the optimal time dependence in the case of the heat equation.

It is a classical fact that final-state observability can be related to different manifestations of null-controllability of the corresponding predual system. We expand this connection by elaborating a dual characterization of stabilizability in terms of a weak observability estimate in general Banach spaces. This connection enables us to study sufficient conditions for stabilizability resulting from weakening the conditions of the Lebeau–Robbiano strategy.

Applying our abstract results, we prove null-controllability properties for linear parabolic equations with constant coefficients and thick control sets in  $L_p(\mathbb{R}^d)$  for  $p \in [1, \infty)$ . Notably, our results apply to the non-reflexive limiting case  $L_1(\mathbb{R}^d)$ . Via duality, we utilize our conditions for observability, and the proof is then based on a quantitative uncertainty principle for functions with compactly supported Fourier transform. We retain the qualities of previous results for the heat equation in  $L_2(\mathbb{R}^d)$  and provide explicit estimates of the control cost in terms of the time and the thickness parameters. Furthermore, we utilize subordination techniques and prove controllability and stabilization properties of fractional order parabolic equations.

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# Introduction

This thesis deals with a general method for obtaining observability estimates for control systems in infinite-dimensional spaces. We study connections to controllability and stabilization properties and apply our results to a class of linear parabolic equations.

In the following sections, we introduce the above topics and motivate our results.

## Control systems and semigroups

It is well-known that control systems described by linear ordinary differential equations can be written in a so-called state space representation, i.e., a system

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + Bu(t) && \text{for } t \in (0, T], \\ y(t) &= Cx(t) && \text{for } t \in [0, T], \\ x(0) &= x_0, \end{aligned}$$

where  $A, B$ , and  $C$  are matrices of appropriate sizes,  $x_0$  is the initial condition, and  $T$  is the final time. The matrix  $A$  determines the equation for the state  $x$ , which can be manipulated via the control  $u$ . The observation  $y$  forms the output of the system. The two main objectives in control theory are *controllability*, i.e., the ability to steer the state function  $x$  via a control function  $u$  to a desired final state  $x(T)$ , and *observability*, i.e., the estimation of the state function  $x$  by the knowledge of the observation  $y$ . These concepts are mathematically closely related by duality results.

This thesis focuses on control systems given by partial differential equations. We take a functional analytic approach and formulate, e.g., the heat equation on the domain  $\Omega \subset \mathbb{R}^d$  as a differential equation  $\dot{x}(t) = -Ax(t)$  where  $-A$  is a realization of the Laplacian on the infinite-dimensional state space  $L_p(\Omega)$ . We consider bounded control and observation operators  $B$  and  $C$  with which we obtain a state space representation as in the case of ordinary differential equations but with an infinite-dimensional state space. Thus, these systems are called *infinite-dimensional control systems*. The well-established semigroup theory aims to describe the solution of an infinite-dimensional system as a generalization of the exponential  $e^{-At}$ . For this approach, one imposes conditions on the operator  $A$ . The natural framework we mainly focus on consists of operators  $A$  that generate a *strongly continuous semigroup*  $(S_t)_{t \geq 0}$ .

One often considers control systems in Hilbert spaces or reflexive Banach spaces, such as  $L_2(\Omega)$  or  $L_p(\Omega)$  for  $p \in (1, \infty)$ , which can simplify the theory significantly. However, there are situations in which it is more convenient to use non-reflexive state spaces. In particular, for the heat equation in  $L_1(\Omega)$ , the norm of the states describes the total heat content. The same applies to population systems in  $L_1(\Omega)$ , where the norm reflects the total population size. Therefore, we study control systems in general Banach spaces.

There are various non-equivalent notions of controllability and observability for infinite-

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dimensional control systems. While the connections between these concepts are well-established for reflexive Banach spaces, the situation is more intricate for non-reflexive spaces. In this thesis, we mostly focus on *final-state observability*, i.e., an estimate of the form

$$\|x(T)\| \leq C_{\text{obs}} \int_0^T \|y(t)\| dt,$$

where the norm of the final state  $x(T)$  can be bounded by the integral of the observation  $y$  for previous times. The quantity  $C_{\text{obs}} \geq 0$  is called *observability constant*. The concept of final-state observability can be related to different manifestations of *null-controllability* of the corresponding predual system. See, e.g., [Dou66, Emb73, DR77, Car88] for results for systems in Hilbert spaces or reflexive Banach spaces. More recently, these connections were extended to non-reflexive Banach spaces in [Vie05, YLC06].

Different control theoretic notions and their connections are presented in Chapter 1. Our main result in this chapter is a versatile duality theorem (Theorem 1.20) which states the equivalence of several controllability properties with observability estimates of the corresponding dual system in general Banach spaces. For this, we pursue an approach presented in [TWX20] in the context of Hilbert spaces for relating *stabilizability* to weak observability estimates. An elementary part of the proof is a result from [Vie05], which is included in this chapter.

## Generalized Lebeau–Robbiano strategy

The central part of this thesis forms a general method to prove observability estimates based on the *Lebeau–Robbiano strategy*. It originates in [LR95, LZ98, JL99] as a method to deduce null-controllability for the heat equation on bounded domains by verifying a *spectral inequality* for sums of eigenfunctions. This approach has proven valuable for several applications and has been further developed to meet various specifications. While the original method uses an iterative construction of a control function, Miller established in [Mil10] a more direct technique by exploiting iterative arguments in observability inequalities on spectral subspaces. Subsequently, the Lebeau–Robbiano strategy evolved into a set of sufficient conditions for observability (see, e.g., [TT11, WZ17, BP18, NTTV20a]). In the following, we briefly review the strategy along with the various improvements in recent years and explain our contribution.

An essential part of the Lebeau–Robbiano strategy forms an observability estimate for subspaces of the state space  $X$  of a control system, i.e., an inequality of the form

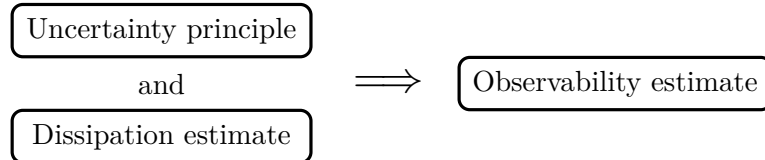
$$\forall \lambda > 0, x \in X : \quad \|P_\lambda x\| \lesssim e^\lambda \|C P_\lambda x\|, \quad (\text{UP})$$

where the linear operators  $P_\lambda$  describe the subspaces of the state space, and  $C$  is the bounded observation operator. In accordance with the phenomenon in quantum physics, this inequality is called *uncertainty principle*. The estimate (UP) is especially interesting if the operators  $P_\lambda$  are related to the state equation of the control system. This is done via a *dissipation estimate* for the semigroup  $(S_t)_{t \geq 0}$  generated by the operator  $-A$  of the control system, i.e.,

$$\forall \lambda > 0, t > 0, x \in X : \quad \|(\text{Id} - P_\lambda) S_t x\| \lesssim e^{-g(\lambda)t} \|x\|, \quad (\text{DISS})$$

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where the decay rate is specified by the positive and increasing function  $g$ . By imposing suitable conditions on the function  $g$ , the inequalities (UP) and (DISS) can be combined via iteration to deduce an observability estimate. This results in the generalized Lebeau–Robbiano strategy:



For this conclusion, it is essential to find assumptions on the asymptotic behavior of the dissipation estimate for  $\lambda \rightarrow \infty$ . Certainly, the decay in (DISS) has to be faster than the growth in (UP). While this is mainly specified by power functions  $g(\lambda) = \lambda^s$  for  $s > 1$ , the authors of [DM12] allow for functions  $g$  that grow slower by imposing an integrability condition.

For control systems in Hilbert spaces with self-adjoint operators  $A$ , the inequalities (UP) and (DISS) are often considered for spectral projections  $P_\lambda$  for the operator  $A$ . The uncertainty principle is then called spectral inequality, and the dissipation estimate is trivially satisfied. This highlights the importance of obtaining spectral inequalities in control theory. However, it is beneficial to detach the operators  $P_\lambda$  in (UP) and (DISS) from spectral theory, as in [Mil10, DM12, BP18]. This makes it possible to use well-known spectral inequalities also for a more general class of related control systems (see [BP18, BGST22b, BJP21, DSV22a] for applications).

The Lebeau–Robbiano strategy has proven helpful for obtaining explicit estimates on the observability constant, which transfer to bounds on the control cost of the related control system. The crucial part is selecting an appropriate iteration procedure to combine the uncertainty principle with the dissipation estimate. While the original approach pursued in [LR95] makes it challenging to specify the time-dependence of the control cost, a telescoping series method developed in [Mil10] yields optimal estimates for fast controls in the case of the heat equation. In [TT11], more precise estimates in the iteration were made, which, in particular, improved the large-time behavior of the control cost. This method was exploited in [NTTV20a] to provide fully explicit estimates in terms of the model parameters. However, these estimates are only available if the asymptotic behavior of the inequalities (UP) and (DISS) is described by power functions. For the more general growth rates considered in [DM12], no cost estimates are obtained.

The results on the Lebeau–Robbiano strategy discussed so far are all restricted to Hilbert spaces only. However, the theory of strongly continuous semigroups is essentially a Banach space theory, and it is natural to ask for observability estimates in this context. Therefore, we unify the respective advantages of the results in Hilbert spaces and generalize them to the framework of Banach spaces. In particular, we complement the earlier developments in the following aspects:

- (a) We consider control systems in general Banach spaces. We neither require reflexivity nor have spectral calculus available in this general framework. Furthermore, we allow for exponentially bounded semigroups and only assume a measurability assumption for the semigroup instead of strong continuity. This is crucial for obtaining controllability results in non-reflexive spaces.

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- (b) We consider the inequalities (UP) and (DISS) for general linear bounded operators  $P_\lambda$ . In particular, we do not restrict to orthogonal projections or use commutativity with the semigroup. Additionally, we require a weaker integrability condition than in [DM12] for the asymptotic behavior in (UP) and (DISS).
- (c) Despite the general setting, we obtain explicit estimates for the observability constant. A novel aspect of our results is the integral expression with which we describe the observability constant for generalized growth rates in the assumptions. Our estimates reflect the qualities of previous results and resemble the optimal time dependence in the case of the heat equation.

By means of our duality results, stabilizability of a control system can be related to weak observability estimates of the corresponding dual system. This connection enables us to provide sufficient conditions for stabilizability resulting from weakening the conditions of the Lebeau–Robbiano strategy. For stabilizability, the growth rates in the inequalities (UP) and (DISS) are not crucial. In fact, it is sufficient if both inequalities are valid for only one suitable value of the parameter  $\lambda$ .

We present our results on the Lebeau–Robbiano strategy in Chapter 2. The main results are Theorem 2.6 and Theorem 2.7 which give sufficient conditions for observability in Banach spaces. In Theorem 2.6, we obtain explicit estimates of the observability constant, provided the growth rates in the assumptions are described by power functions. Theorem 2.7 generalizes this result to growth rates satisfying an integrability condition. In the remaining part of the chapter, we study some consequences which result from subordination techniques and duality theory.

## Parabolic equations in unbounded domains

The origins of the Lebeau–Robbiano strategy lie in proving null-controllability for the heat equation with interior control in a smooth and bounded domain  $\Omega \subset \mathbb{R}^d$ . In [LR95, FI96] it is shown that the heat equation is null-controllable in  $L_2(\Omega)$  (and hence final-state observable) for any open non-empty control set. This condition was improved to sets with positive Lebesgue measure in [AEWZ14, EMZ15]. Such weak conditions on the control set are only possible for bounded domains, and they fail to be sufficient for null-controllability if the domain is unbounded (see [Mil05a]).

Here, we focus on observability estimates for linear parabolic equations in the unbounded domain  $\mathbb{R}^d$ . For instance, we study systems of the form

$$\begin{aligned} \dot{x}(t) &= \Delta_p x(t) && \text{for } t \in (0, T], \\ y(t) &= x(t)|_E && \text{for } t \in [0, T], \\ x(0) &= x_0 \in L_p(\mathbb{R}^d), \end{aligned}$$

where  $\Delta_p$  is the maximal Laplacian in  $L_p(\mathbb{R}^d)$  for  $p \in [1, \infty)$ , and  $E \subset \mathbb{R}^d$  is the observation set. Restricting to  $p = 2$ , this and related problems on unbounded domains were studied, e.g., in [CdMZ01, Mil05b, Gd07, Bar14, LM16] and the references therein. In [EV18, WWZZ19] a necessary and sufficient condition on the observation set for final-state observability of the heat equation in  $L_2(\mathbb{R}^d)$  is derived. The characterizing property

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is called *thickness*, which is a geometric condition claiming the observation set to be well-distributed in  $\mathbb{R}^d$ . This specification of observation sets is based on the Logvinenko–Sereda theorem (see [LS74, Kov01]), which associates thick sets with an uncertainty principle for functions with compactly supported Fourier transform. Following the Lebeau–Robbiano strategy, this inequality implies an observability estimate. Thereby it is possible to describe the observability constant explicitly by the parameters of the thick set (see [NTTV20a]).

The Logvinenko–Sereda uncertainty principle is valid for the whole range of  $L_p$ -spaces and, in particular, is not limited to  $p = 2$ . Hence it is natural to generalize the observability results to  $p \in [1, \infty]$ . We pursue this in the last chapter of this thesis. Besides the generalization of the state space, we replace the Laplacian with elliptic differential operators  $-A_p$  with constant coefficients and lower-order terms. We prove that the corresponding parabolic system is final-state observable for thick observation sets. Notably, this result also applies to the limiting case  $p = 1$ . For  $p = \infty$ , we obtain an observability estimate for the associated weak\*-continuous semigroup, which relates to null-controllability results in  $L_1(\mathbb{R}^d)$ .

In Chapter 3, we bring together all results needed for obtaining observability estimates for the above class of parabolic systems. This includes a complete proof of the Logvinenko–Sereda theorem. To utilize our results on the Lebeau–Robbiano strategy, we prove a dissipation estimate suitable for the frequency localization used in the Logvinenko–Sereda uncertainty principle. Our main result in this chapter is the observability estimate in Theorem 3.8 for parabolic systems with thick observation sets. The theorem retains the qualities of the Hilbert space results and provides an explicit observability constant in terms of the time and the thickness parameters. We close this chapter by studying controllability properties of fractional order parabolic equations.

## Remarks on publications

Most of the results in this thesis were developed in collaboration with Clemens Bombach, Michela Egidi, Jan Meichsner, Christian Seifert, and Martin Tautenhahn and have already been published in the articles [GST20, BGST22b] and in the preprints [EGST21, GMS22].

# 1 Control theory in Banach spaces

This chapter recalls basic concepts used in studying control systems in Banach spaces. We start by repeating some results on strongly continuous semigroups and the corresponding dual semigroups in Section 1.1. In Section 1.2, we introduce different notions of controllability, observability, and stabilizability. Since we also want to cover systems in non-reflexive spaces, we study some variants of the usual concepts. This includes cost-uniform approximate null-controllability and an observability estimate for, e.g., dual semigroups. In Section 1.3, we state and prove a versatile duality theorem that relates controllability and stabilization properties to observability estimates of the corresponding dual system.

## 1.1 Operator semigroups

Let us very briefly recall some basic properties of operator semigroups. For an extensive presentation see, e.g., [EN99]. The omitted proofs can be found in this monograph.

Throughout, we consider Banach spaces over the field  $\mathbb{R}$  or  $\mathbb{C}$ . For a Banach space  $X$  we denote the norm by  $\|\cdot\|_X$ . The space of linear bounded operators between two Banach spaces  $X$  and  $Y$  is denoted by  $\mathcal{L}(X, Y)$ . If  $X = Y$ , we just write  $\mathcal{L}(X)$ . For a densely defined closed linear operator  $A: \mathcal{D}(A) \subset X \rightarrow X$ , we consider the homogeneous abstract Cauchy problem

$$\begin{aligned}\dot{x}(t) &= -Ax(t) \quad \text{for } t \in (0, T] \\ x(0) &= x_0 \in X,\end{aligned}\tag{1.1}$$

where  $T > 0$  and  $x_0 \in X$  is the initial value for  $t = 0$ . We use  $\dot{x}$  to denote the first derivative of the function  $x$ . We call a continuously differentiable function  $x: [0, T] \rightarrow X$  a *classical solution* of (1.1) if  $x(t) \in \mathcal{D}(A)$  for all  $t \in (0, T]$  and  $x$  satisfies (1.1).

### 1.1.1 Strongly continuous semigroups

In the following, we want to restrict the class of operators  $-A$  in (1.1) to those that are the generator of a strongly continuous semigroup.

**Definition 1.1.** A family  $(S_t)_{t \geq 0}$  of bounded linear operators on a Banach space  $X$  is called a *semigroup* if  $S_0 = I$  and

$$S_{t+s} = S_t S_s \quad \text{for all } t, s \geq 0.$$

A semigroup  $(S_t)_{t \geq 0}$  is called *strongly continuous* or  *$C_0$ -semigroup* if for all  $x \in X$  the map  $t \mapsto S_t x$  is continuous from  $[0, \infty)$  to  $X$ .

By the uniform boundedness principle, it is easy to see that the norm of a  $C_0$ -semigroup grows at most exponentially.

## 1 Control theory in Banach spaces

**Proposition 1.2.** *Let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup. Then there exist  $\omega \in \mathbb{R}$  and  $M \geq 1$  such that*

$$\|S_t\|_{\mathcal{L}(X)} \leq Me^{\omega t} \quad \text{for all } t \geq 0. \quad (1.2)$$

A  $C_0$ -semigroup is called *exponentially stable* if there are  $M \geq 1$  and  $\omega < 0$  such that (1.2) holds. To link  $C_0$ -semigroups to abstract Cauchy problems with suitable operators  $-A$  in (1.1), we define the following:

**Definition 1.3.** The *generator*  $-A: \mathcal{D}(A) \subset X \rightarrow X$  of a  $C_0$ -semigroup  $(S_t)_{t \geq 0}$  on a Banach space  $X$  is the operator

$$-Ax := \lim_{t \rightarrow 0^+} \frac{1}{t} (S_t x - x)$$

defined for all  $x$  in the *domain*

$$\mathcal{D}(A) := \left\{ x \in X : \lim_{t \rightarrow 0^+} \frac{1}{t} (S_t x - x) \text{ exists} \right\}.$$

The generator of a  $C_0$ -semigroup has the following properties.

**Proposition 1.4.** *Let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup and  $-A: \mathcal{D}(A) \subset X \rightarrow X$  the corresponding generator.*

- (i)  *$-A$  is a densely defined closed linear operator which determines the  $C_0$ -semigroup uniquely.*
- (ii) *For  $x \in \mathcal{D}(A)$ , we have  $S_t x \in \mathcal{D}(A)$  for all  $t \geq 0$ . Furthermore, the map  $t \mapsto S_t x$  is continuously differentiable in  $[0, \infty)$  and*

$$\frac{d}{dt} S_t x = -A S_t x \quad \text{for all } t \geq 0.$$

- (iii) *For  $x \in X$  and  $t \geq 0$ , we have  $\int_0^t S_\tau x \, d\tau \in \mathcal{D}(A)$  and*

$$S_t x - x = -A \int_0^t S_\tau x \, d\tau.$$

Given a generator  $-A$  of a  $C_0$ -semigroup  $(S_t)_{t \geq 0}$  and an initial condition  $x_0 \in \mathcal{D}(A)$ , by Proposition 1.4 (ii), a classical solution of (1.1) is given by  $t \mapsto S_t x_0$ . However, for  $x_0 \notin \mathcal{D}(A)$  the map  $t \mapsto S_t x_0$  is not differentiable at zero from the right and hence not a classical solution. We call a function  $x: [0, T] \rightarrow X$  a *mild solution* of (1.1) if for all  $t \in [0, T]$

$$\int_0^t x(\tau) \, d\tau \in \mathcal{D}(A) \quad \text{and} \quad x(t) - x_0 = -A \int_0^t x(\tau) \, d\tau.$$

By Proposition 1.4 (iii), the function  $t \mapsto S_t x_0$  is a mild solution of (1.1) for all  $x_0 \in X$ . This solution is unique.

### 1.1.2 Dual semigroups

Let  $X'$  denote the topological dual space of the Banach space  $X$ . For a continuous linear functional  $x' \in X'$  and  $x \in X$  we write  $\langle x', x \rangle_{X', X}$  for the dual pairing.

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**Definition 1.5.** Let  $A: \mathcal{D}(A) \subset X \rightarrow Y$  be a densely defined linear operator between Banach spaces  $X$  and  $Y$ . The domain  $\mathcal{D}(A')$  of the *dual operator*  $A': \mathcal{D}(A') \subset Y' \rightarrow X'$  consists of all  $y' \in Y'$  for which there is  $x' \in X'$  such that

$$\langle x', x \rangle_{X', X} = \langle y', Ax \rangle_{Y', Y} \quad \text{for all } x \in \mathcal{D}(A).$$

The operator  $A'$  is then defined by  $A'y' = x'$  for all  $y' \in \mathcal{D}(A')$ .

Since  $\mathcal{D}(A) \subset X$  is assumed to be dense, the dual operator is well-defined. If  $A$  is closed, the domain  $\mathcal{D}(A')$  is weak\*-dense in  $X'$ . Furthermore, if  $X$  is reflexive, then  $A'$  is densely defined. If  $A \in \mathcal{L}(X, Y)$ , the dual operator is bounded as well, i.e.,  $A' \in \mathcal{L}(Y', X')$ , and we have  $\|A\|_{\mathcal{L}(X, Y)} = \|A'\|_{\mathcal{L}(Y', X')}$ .

**Definition 1.6.** Let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup. The family  $(S'_t)_{t \geq 0}$  obtained by the dual operators  $S'_t := (S_t)' \in \mathcal{L}(X')$ , for  $t \geq 0$ , is called the *dual semigroup* on  $X'$ .

Clearly, the dual semigroup  $(S'_t)_{t \geq 0}$  is actually a semigroup. It is a standard result that on reflexive Banach spaces the dual semigroup is again strongly continuous. However, for non-reflexive spaces,  $(S'_t)_{t \geq 0}$  is in general only weak\*-continuous, i.e., for all  $x' \in X'$  and  $x \in X$  the map

$$t \mapsto \langle S'_t x', x \rangle_{X', X} \tag{1.3}$$

is continuous. Indeed, for  $t \in [0, \infty)$  and  $h \in [-t, \infty)$  we have

$$|\langle S'_{t+h} x' - S'_t x', x \rangle_{X', X}| = |\langle x', S_{t+h} x - S_t x \rangle_{X', X}| \leq \|x'\|_{X'} \|S_{t+h} x - S_t x\|_X$$

for all  $x' \in X'$  and  $x \in X$ . Since  $(S_t)_{t \geq 0}$  is strongly continuous, this shows that the map in (1.3) is continuous for all  $x' \in X'$  and  $x \in X$ .

**Proposition 1.7.** Let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup with generator  $-A$ . The dual operator  $-A'$  of  $-A$  is the weak\* generator of  $(S'_t)_{t \geq 0}$ , i.e.,

$$-A'x' = \text{weak}^* \lim_{t \rightarrow 0^+} \frac{1}{t} (S'_t x' - x')$$

defined for all  $x'$  in the domain

$$\mathcal{D}(A') = \left\{ x' \in X' : \text{weak}^* \lim_{t \rightarrow 0^+} \frac{1}{t} (S'_t x' - x') \text{ exists} \right\}.$$

## 1.2 Main concepts of control theory

This section introduces the notions of controllability, observability, and stabilizability used in the later chapters. The structure of this chapter is motivated by [EN99, Chapter VI.8]. For a detailed introduction to control theory in infinite-dimensional spaces see, e.g., the monographs [CP78, Zab08, TW09, CZ12]. The novelty of this chapter starts in Section 1.2.3, where we relate stabilizability to so-called  $\alpha$ -controllability.

Let  $X, Y, U$  be Banach spaces,  $(S_t)_{t \geq 0}$  a  $C_0$ -semigroup on  $X$ , and  $-A$  the corresponding generator. For bounded linear operators  $B \in \mathcal{L}(U, X)$  and  $C \in \mathcal{L}(X, Y)$  we consider the

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inhomogeneous abstract Cauchy problem

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + Bu(t) & \text{for } t \in (0, T], \\ y(t) &= Cx(t) & \text{for } t \in [0, T], \\ x(0) &= x_0 \in X, \end{aligned} \quad \Sigma_T(A, B, C)$$

where  $T > 0$  and  $u: [0, T] \rightarrow U$  is an integrable function. The system of equations forms a *control system* in state space representation and the function  $x$  is called *state function*. The integrable *control function*  $u: [0, T] \rightarrow U$  is mapped pointwise by the *control operator*  $B$  from the *control space*  $U$  to the *state space*  $X$ . The function  $y: [0, T] \rightarrow Y$  with values in the *observation space*  $Y$  is called *observation function* and the operator  $C$  is called *observation operator*. We denote the control system by  $\Sigma_T(A, B, C)$  and write  $\Sigma_T(A, B, -)$  or  $\Sigma_T(A, -, C)$  if there is no control operator or observation operator, respectively. The block diagram in Figure 1.1 visualizes the control system  $\Sigma_T(A, B, C)$ .

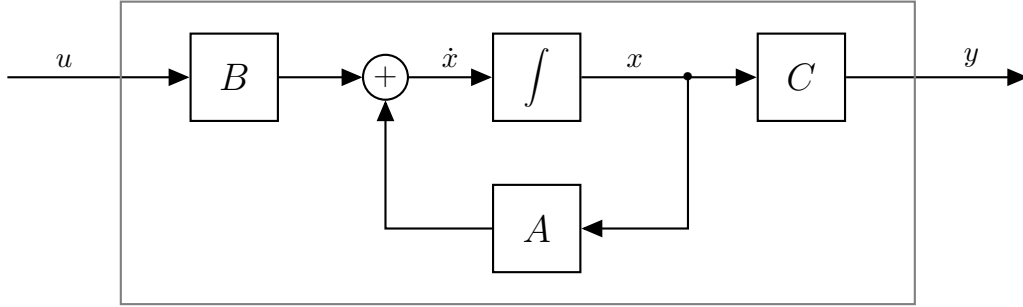


Figure 1.1: Block diagram of the system  $\Sigma_T(A, B, C)$ .

We call a continuously differentiable function  $x: [0, T] \rightarrow X$  a *classical solution* for  $\Sigma_T(A, B, C)$  if  $x(t) \in \mathcal{D}(A)$  for all  $t \in (0, T]$ , and  $x$  satisfies  $\Sigma_T(A, B, C)$ . It is a standard result that a classical solution satisfies Duhamel's formula

$$x(t) = S_t x_0 + \int_0^t S_{t-\tau} B u(\tau) \, d\tau, \quad t \in [0, T]. \quad (1.4)$$

In particular, the classical solution is unique. Clearly, the observation function  $y(t) = Cx(t)$  is uniquely determined by the solution  $x$ . For system theoretical problems one often works with the following weaker solution concept. For  $x_0 \in X$  and an integrable control function  $u: [0, T] \rightarrow U$  we call the function  $x$  given by Duhamel's formula (1.4) the *mild solution* of  $\Sigma_T(A, B, C)$ .

In the following, we define and discuss essential concepts of control theory.

### 1.2.1 Controllability

First, we consider a system without an observation operator, i.e.,

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + Bu(t) & \text{for } t \in (0, T], \\ x(0) &= x_0 \in X. \end{aligned} \quad \Sigma_T(A, B, -)$$

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Controllability is about finding a control function  $u: [0, T] \rightarrow U$  which steers the solution  $x$  of the system from a given initial state  $x_0 \in X$  to some target state  $x_T \in X$  at time  $T$ , i.e.,  $x(T) = x_T$ . Before we give a precise definition, let us mention some characteristics appearing in the infinite-dimensional setting compared to finite-dimensional systems.

A classical solution of  $\Sigma_T(A, B, -)$  satisfies  $x(t) \in \mathcal{D}(A)$  for all  $t \in (0, T]$ . Hence, it is impossible to steer the classical solution to a target state  $x_T \notin \mathcal{D}(A)$ . Note that  $\mathcal{D}(A) = X$  implies that  $-A$  is bounded. Since we are mainly interested in unbounded operators  $-A$ , we define controllability for the mild solution (1.4). For  $r \in [1, \infty]$  and  $t \in [0, T]$  we define the *controllability map*  $\mathcal{B}_t \in \mathcal{L}(L_r((0, t); U), X)$  by

$$\mathcal{B}_t u := \int_0^t S_{t-\tau} B u(\tau) \, d\tau, \quad u \in L_r((0, T); U). \quad (1.5)$$

With this map the mild solution of  $\Sigma_T(A, B, -)$  has the form

$$x(t) = S_t x_0 + \mathcal{B}_t u.$$

Hence, steering the mild solution to every target state  $x_T \in X$  is only possible if  $\text{Ran}(\mathcal{B}_T) = X$ . However, as a consequence of the open mapping theorem, this is unachievable for a compact control operator  $B \in \mathcal{L}(U, X)$  mapping to an infinite-dimensional state space  $X$  (see, e.g., [EN99, Proposition 8.5]). Since there are important applications where the control operator  $B \in \mathcal{L}(U, X)$  is compact, e.g., if the control space  $U$  is finite-dimensional, we will mainly focus on null-controllability.

**Definition 1.8.** The system  $\Sigma_T(A, B, -)$  is *null-controllable in time  $T$  w.r.t.  $L_r((0, T); U)$*  if for all initial states  $x_0 \in X$  there is a control function  $u \in L_r((0, T); U)$  such that

$$x(T) = S_T x_0 + \mathcal{B}_T u = 0.$$

Note that the system  $\Sigma_T(A, B, -)$  is null-controllable if and only if  $\text{Ran}(S_T) \subset \text{Ran}(\mathcal{B}_T)$  and hence, every target state  $x_T \in \text{Ran}(S_T)$  can be reached with a suitable control function  $u$ . In that case, we define the *control cost in time  $T$*  by

$$\mathcal{C} = \sup_{\substack{x_0 \in X \\ \|x_0\|_X \leq 1}} \inf_{u \in L_r((0, T); U)} \left\{ \|u\|_{L_r((0, T); U)} : S_T x_0 + \mathcal{B}_T u = 0 \right\}. \quad (1.6)$$

Using an open mapping theorem for convex multifunctions, it follows that for a system that is null-controllable in time  $T$ , the control cost in time  $T$  is finite (see, e.g., [Car85, Car88]). This can be related to the problem of constrained null-controllability, where one assumes that the control function has to be taken from a set of admissible control functions. If we choose

$$U_{\text{ad}} = \{u \in L_r((0, T); U) : \|u\|_{L_r((0, T); U)} \leq 1\}$$

we have the following direct connection in Proposition 1.9. For  $x_0 \in X$ , let  $B_X(x_0, \rho) \subset X$  denote the open ball in  $X$  around  $x_0$  with radius  $\rho > 0$ . Consequently, the closed ball is denoted by  $\overline{B_X(x_0, \rho)}$ . If the underlying Banach space is clear, we just write  $B(x_0, \rho)$  or  $\overline{B(x_0, \rho)}$ , respectively.

**Proposition 1.9.** *Let  $T > 0$ ,  $r \in [1, \infty]$ , and  $\rho > 0$ . For all initial states  $x_0 \in \overline{B_X(0, \rho)}$  there is a control function  $u \in U_{\text{ad}}$  such that  $x(T) = S_T x_0 + \mathcal{B}_T u = 0$  if and only if for all*

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$x_0 \in X$  there is  $u \in L_r((0, T); U)$  such that

$$\|u\|_{L_r((0, T); U)} \leq \rho^{-1} \|x_0\|_X \quad \text{and} \quad S_T x_0 + \mathcal{B}_T u = 0.$$

*Proof.* Suppose that for all  $x_0 \in \overline{B_X(0, \rho)}$  there is  $u \in U_{\text{ad}}$  with  $x(T) = S_T x_0 + \mathcal{B}_T u = 0$ . Let  $x_0 \in X$ . If  $x_0 = 0$  we choose  $u = 0$ . For  $x_0 \neq 0$ , we define

$$\tilde{x}_0 := \rho \frac{x_0}{\|x_0\|_X} \in \overline{B_X(0, \rho)}.$$

Then by assumption there exists  $\tilde{u} \in U_{\text{ad}}$  such that  $S_T \tilde{x}_0 + \mathcal{B}_T \tilde{u} = 0$ . Setting  $u = \rho^{-1} \|x_0\|_X \tilde{u}$ , we find  $\|u\|_{L_r((0, T); U)} \leq \rho^{-1} \|x_0\|_X$  and

$$S_T x_0 + \mathcal{B}_T u = \rho^{-1} \|x_0\|_X (S_T \tilde{x}_0 + \mathcal{B}_T \tilde{u}) = 0.$$

This proves the implication “ $\Rightarrow$ ”.

Suppose now that for all  $x_0 \in X$  there is  $u \in L_r((0, T); U)$  such that

$$\|u\|_{L_r((0, T); U)} \leq \rho^{-1} \|x_0\|_X \quad \text{and} \quad S_T x_0 + \mathcal{B}_T u = 0.$$

If  $\|x_0\|_X \leq \rho$ , then  $\|u\|_{L_r((0, T); U)} \leq 1$ , which proves the implication “ $\Leftarrow$ ”. □

Next, we define a weaker property than null-controllability for systems that can approximately reach the null state.

**Definition 1.10.** The system  $\Sigma_T(A, B, -)$  is *cost-uniformly approximately null-controllable in time  $T$  with respect to  $L_r((0, T); U)$*  if there exists  $C \geq 0$  such that for all  $\varepsilon > 0$  and  $x_0 \in X$  there is a control function  $u \in L_r((0, T); U)$  such that

$$\|u\|_{L_r((0, T); U)} \leq C \|x_0\|_X \quad \text{and} \quad \|x(T)\|_X = \|S_T x_0 + \mathcal{B}_T u\|_X \leq \varepsilon.$$

Clearly, cost-uniform approximate null-controllability implies  $\text{Ran}(S_T) \subset \overline{\text{Ran}(\mathcal{B}_T)}$  but the definition includes more since it guarantees the following upper bound on the control cost:

$$\mathcal{C} = \sup_{\varepsilon > 0} \sup_{\substack{x_0 \in X \\ \|x_0\|_X \leq 1}} \inf_{u \in L_r((0, T); U)} \left\{ \|u\|_{L_r((0, T); U)} : \|S_T x_0 + \mathcal{B}_T u\|_X < \varepsilon \right\} \leq C < \infty. \quad (1.7)$$

The single condition  $\text{Ran}(S_T) \subset \overline{\text{Ran}(\mathcal{B}_T)}$  is mostly called *approximate null-controllability* and is not equivalent to cost-uniform approximate null-controllability in general. See, e.g., [TWX20, Section 3.2.1] for a counterexample in Hilbert spaces. However, cost-uniform approximate null-controllability is, in many situations, the same as null-controllability. In the following proposition, we gather some interesting cases (see [Car88, Remark 2.1], [Vie05, Remark 1.2 and Remark 1.3], and [YLC06, Theorem 2.1]).

**Proposition 1.11.** *If either*

- $X$  is reflexive,
- $r \in (1, \infty]$  and  $U$  is reflexive,

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- $S_T$  is surjective, or
- for all  $\rho > 0$  the set  $\{\mathcal{B}_T u : u \in L_r((0, T); U), \|u\|_{L_r((0, T); U)} \leq \rho\}$  is closed,

then the system  $\Sigma_T(A, B, -)$  is null-controllable in time  $T$  with respect to  $L_r((0, T); U)$  if and only if it is cost-uniformly approximately null-controllable in time  $T$  with respect to  $L_r((0, T); U)$ .

Note that in each of the cases in Proposition 1.11, the definition of cost-uniform approximate null-controllability is obsolete, as it is equivalent to null-controllability. However, since we are also interested in applications where Proposition 1.11 does not apply, we will stick to the notion of cost-uniform approximate null-controllability.

### 1.2.2 Observability

We now turn to systems with observation and consider

$$\begin{aligned} \dot{x}(t) &= -Ax(t) & \text{for } t \in (0, T], \\ y(t) &= Cx(t) & \text{for } t \in [0, T], \\ x(0) &= x_0 \in X. \end{aligned} \quad \Sigma_T(A, -, C)$$

The concept of observability is about reconstruction of the initial state  $x_0$  or final state  $x(T)$  of the system via the observation function  $y(t) = Cx(t)$ ,  $t \in [0, T]$ . Given the initial state the whole state function is fixed by  $x(t) = S_t x_0$ . However, in applications it might be impossible to reconstruct the initial state of an infinite dimensional system. In fact due to duality results this is related to the restrictions for controllability (see Section 1.3). Therefore we will focus on final-state observability. For  $r \in [1, \infty]$  and  $t \in [0, T]$  we define the *observability map*  $\mathcal{C}_t \in \mathcal{L}(X, L_r((0, t); Y))$  by

$$\mathcal{C}_t x_0 := CS_{(\cdot)} x_0, \quad x_0 \in X.$$

**Definition 1.12.** The system  $\Sigma_T(A, -, C)$  is called *final-state observable in  $L_r((0, T); Y)$*  if there is  $C_{\text{obs}} \geq 0$  such that for all  $x_0 \in X$

$$\|S_T x_0\|_X \leq C_{\text{obs}} \|\mathcal{C}_T x_0\|_{L_r((0, T); Y)}. \quad (1.8)$$

By Definition 1.12, the final state of a final-state observable system is distinguishable, i.e.,  $y = \mathcal{C}_T x_0 = 0$  implies  $x(T) = S_T x_0 = 0$ . The inequality (1.8) even implies the existence of a reconstruction operator  $R_T: \text{Ran}(\mathcal{C}_T) \rightarrow X$ , such that  $R_T \mathcal{C}_T = S_T$ .

We can rewrite (1.8) as

$$\|S_T x_0\|_X \leq \begin{cases} C_{\text{obs}} \left( \int_0^T \|CS_t x_0\|_Y^r dt \right)^{1/r} & \text{if } r \in [1, \infty), \\ C_{\text{obs}} \text{ess sup}_{t \in [0, T]} \|CS_t x_0\|_Y & \text{if } r = \infty. \end{cases} \quad (1.9)$$

Note that for (1.9) the Bochner measurability of  $t \mapsto CS_t x_0$  is not needed. Hence, we can extend the above definition to semigroups  $(S_t)_{t \geq 0}$  for which the map  $t \mapsto \|CS_t x_0\|_Y$  is merely measurable.

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**Definition 1.13.** Let  $(S_t)_{t \geq 0}$  be a semigroup on  $X$  and assume that  $[0, T] \ni t \mapsto \|CS_t x_0\|_Y$  is measurable for all  $x_0 \in X$ . We say that  $(S_t)_{t \geq 0}$  satisfies a *final-state observability estimate* if there is  $C_{\text{obs}} \geq 0$  such that (1.9) is satisfied for all  $x_0 \in X$ .

We will frequently use this notion in the connection with controllability properties of the system  $\Sigma_T(A, B, -)$ , where  $A$  is the generator of a  $C_0$ -semigroup  $(S_t)_{t \geq 0}$ , and  $B \in \mathcal{L}(U, X)$ . In this case, we consider a final-state observability estimate for the dual system, i.e.,

$$\forall x' \in X' : \quad \|S'_T x'\|_{X'} \leq \begin{cases} C_{\text{obs}} \left( \int_0^T \|B' S'_t x'\|_{U'}^{r'} dt \right)^{1/r'} & \text{if } r' \in [1, \infty), \\ C_{\text{obs}} \operatorname{ess\,sup}_{t \in [0, T]} \|B' S'_t x'\|_{U'} & \text{if } r' = \infty, \end{cases}$$

where  $B' \in \mathcal{L}(X', U')$ , and  $(S'_t)_{t \geq 0}$  is the dual semigroup of  $(S_t)_{t \geq 0}$  (see Section 1.1.2). Although the dual semigroup may fail to be strongly continuous, we abuse notation and say that *the system  $\Sigma_T(A', -, B')$  satisfies a final-state observability estimate* if  $-A'$  is the weak\* generator of  $(S'_t)_{t \geq 0}$ .

### 1.2.3 Stabilizability

For a null-controllable system  $\Sigma_T(A, B, -)$ , there exists a control function  $u$ , which steers the solution to zero at time  $T$ . In Section 1.2.1, we already started to weaken the notion of null-controllability by introducing the concept of cost-uniform approximate null-controllability, which demands the state to be only approximately zero at time  $T$ . However, it may apply that only a control function  $u$  is needed which steers the system asymptotically to zero as  $T$  goes to infinity. This leads to the concept of stabilizability. For this, we consider the following infinite time horizon problem:

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + Bu(t) & \text{for } t \in (0, \infty), \\ x(0) &= x_0 \in X, \end{aligned} \quad \Sigma_\infty(A, B, -)$$

where  $u \in L_r((0, \infty); U)$  with  $r \in [1, \infty]$ .

**Definition 1.14.** The system  $\Sigma_\infty(A, B, -)$  is *cost-uniformly open-loop stabilizable w.r.t.  $L_r((0, \infty); U)$*  if there exist  $M \geq 1$ ,  $\omega < 0$ , and  $C \geq 0$  such that for all initial conditions  $x_0 \in X$  there exists a control function  $u \in L_r((0, \infty); U)$  such that for all  $t \geq 0$

$$\|u\|_{L_r((0, \infty); U)} \leq C \|x_0\|_X \quad \text{and} \quad \|x(t)\|_X = \|S_t x_0 + \mathcal{B}_t u\|_X \leq M e^{\omega t} \|x_0\|_X. \quad (1.10)$$

As open-loop stabilizability corresponds to an exponentially fast decay of the norm of the solution, a more intuitive term for this property might be *exponential infinite-time null-controllability*. Sometimes (1.10) is replaced by the weaker condition  $x \in L_r((0, \infty); X)$ . For  $r = 2$  this is also called *optimizability* or *finite cost condition*. Recall that one says that the system  $\Sigma_\infty(A, B, -)$  is *closed-loop stabilizable* or *stabilizable by feedback* if there exists  $K \in \mathcal{L}(X, U)$  such that  $-A + BK$  generates an exponentially stable  $C_0$ -semigroup. Then  $K$  is called *state feedback operator*, and the control  $u$  given by  $u(t) = Kx(t)$  yields an exponentially stable solution  $x$ . For an open-loop stabilizable system in a Hilbert space, the existence of a state feedback operator follows from classical Riccati theory (see,

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e.g., [Zab08, Theorem IV.4.4]). For some results on the relation between open-loop and closed-loop stabilizability for systems in Banach spaces, as well as the connection to the so-called Hautus test (see [RZ98]). In this paper, also various impacts of properties of  $B$  on open-loop stabilizability are considered.

Next, we introduce a notion for systems where any decay rate can be achieved through stabilization.

**Definition 1.15.** The system  $\Sigma_\infty(A, B, -)$  is *completely cost-uniformly open-loop stabilizable w.r.t.  $L_r((0, \infty); U)$*  if for all  $\nu > 0$  the rescaled system  $\Sigma_\infty(A - \nu, B, -)$  is cost-uniformly open-loop stabilizable w.r.t.  $L_r((0, \infty); U)$ .

Equivalently, for every decay rate  $\omega < 0$  there are constants  $M \geq 1$  and  $C \geq 0$  such that for all  $x_0 \in X$  there exists  $u \in L_r((0, \infty); U)$  such that (1.10) holds. Although stabilizability is an asymptotic property, we can still relate it to a property of the finite time horizon system  $\Sigma_T(A, B, -)$ . This characterization will help us to prove a duality result for stabilizability in Section 1.3.

**Definition 1.16.** Let  $\alpha \in [0, 1)$ . The system  $\Sigma_T(A, B, -)$  is *cost-uniformly approximately  $\alpha$ -controllable in time  $T$  w.r.t.  $L_r((0, T); U)$*  if there exists  $C \geq 0$  such that for all  $\varepsilon > 0$  and  $x_0 \in X$  there exists a control function  $u \in L_r((0, T); U)$  such that

$$\|u\|_{L_r((0, T); U)} \leq C\|x_0\|_X \quad \text{and} \quad \|x(T)\|_X = \|S_T x_0 + \mathcal{B}_T u\|_X \leq (\alpha + \varepsilon)\|x_0\|_X. \quad (1.11)$$

For  $\alpha = 0$  the above concept coincides with the notion of cost-uniform approximate null-controllability given in Definition 1.10. Note that for every  $C_0$ -semigroup we have (1.11) with  $\alpha = \|S_T\|_{\mathcal{L}(X)}$  and  $C = 0$ . However, by the semigroup property we have  $\|S_T\|_{\mathcal{L}(X)} < 1$  for some  $T > 0$  if and only if  $(S_t)_{t \geq 0}$  is exponentially stable (see, e.g., [EN99, Proposition V.1.7]). It is an easy consequence that cost-uniform approximate  $\alpha$ -controllability and cost-uniform open-loop stabilizability are equivalent (see [EGST21]). With small adjustments, the proof is based on [TWX20, Lemma 31]. See also [TWX20, Theorem 26], where the same characterization was obtained in Hilbert spaces.

**Proposition 1.17.** *The system  $\Sigma_\infty(A, B, -)$  is cost-uniformly open-loop stabilizable w.r.t.  $L_r((0, \infty); U)$  if and only if there exist  $\alpha \in [0, 1)$  and  $T > 0$  such that  $\Sigma_T(A, B, -)$  is cost-uniformly approximately  $\alpha$ -controllable in time  $T$  w.r.t.  $L_r((0, T); U)$ .*

*Proof.* Assume that  $\Sigma_\infty(A, B, -)$  is cost-uniformly open-loop stabilizable, i.e., for all  $x_0 \in X$  there exists  $u \in L_r((0, \infty); U)$  such that the solution of  $\Sigma_\infty(A, B, -)$  satisfies

$$\|x(t)\|_X = \|S_t x_0 + \mathcal{B}_t u\|_X \leq M e^{\omega t} \|x_0\|_X \quad \text{for all } t > 0,$$

with uniform parameters  $M \geq 1$  and  $\omega < 0$ . Hence, for all  $\alpha \in (0, 1)$  there exists  $T > 0$  such that  $M e^{\omega T} \leq \alpha$ . Moreover, since the cost  $\|u\|_{L_r((0, \infty); U)}$  can be controlled uniformly w.r.t. the initial value  $x_0$ , the system  $\Sigma_T(A, B, -)$  is cost-uniformly approximately  $\alpha$ -controllable in time  $T$ .

We now show the converse and assume that  $\Sigma_T(A, B, -)$  is cost-uniformly approximately  $\alpha$ -controllable in time  $T$ . Since  $\alpha \in [0, 1)$ , we can choose  $\varepsilon > 0$  such that  $\alpha + \varepsilon < 1$ . By definition, for all  $x_0 \in X$  there exists a control function  $u \in L_r((0, T); U)$  such that

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$\|u\|_{L_r((0,T);U)} \leq C\|x_0\|_X$  and  $\|x(T)\|_X = \|S_T x_0 + \mathcal{B}_T u\|_X \leq (\alpha + \varepsilon)\|x_0\|_X$ . For  $k \in \mathbb{N}_0$  we recursively define  $x_{k+1} = S_T x_k + \mathcal{B}_T u_k$  and choose  $u_k \in L_r((0,T);U)$  such that

$$\|u_k\|_{L_r((0,T);U)} \leq C\|x_k\|_X \quad \text{and} \quad \|S_T x_k + \mathcal{B}_T u_k\|_X \leq (\alpha + \varepsilon)\|x_k\|_X.$$

Define  $u: [0, \infty) \rightarrow U$  as the concatenation

$$u(t) = u_k(t - kT) \quad \text{if } t \in [kT, (k+1)T).$$

Then,  $\|x_k\|_X \leq (\alpha + \varepsilon)^k \|x_0\|_X$  for all  $k \in \mathbb{N}_0$ . For  $r \in [1, \infty)$ , we have

$$\begin{aligned} \|u\|_{L_r((0,\infty);U)}^r &= \int_0^\infty \|u(\tau)\|_U^r d\tau \leq \sum_{k=0}^\infty \int_{kT}^{(k+1)T} \|u(\tau)\|_U^r d\tau \leq C^r \sum_{k=0}^\infty \|x_{k+1}\|_X^r \\ &\leq C^r \sum_{k=0}^\infty (\alpha + \varepsilon)^{rk} \|x_0\|_X^r \leq C^r \frac{1}{1 - (\alpha + \varepsilon)^r} \|x_0\|_X^r, \end{aligned}$$

and hence  $u \in L_r((0, \infty); U)$ . For  $r = \infty$ , we similarly estimate

$$\|u\|_{L_\infty((0,\infty);U)} = \sup_{k \in \mathbb{N}_0} \|u_k\|_{L_\infty((0,T);U)} \leq C \sup_{k \in \mathbb{N}_0} \|x_k\|_X \leq C \sup_{k \in \mathbb{N}_0} (\alpha + \varepsilon)^k \|x_0\|_X \leq C \|x_0\|_X,$$

and therefore also  $u \in L_\infty((0, \infty); U)$ . The control  $u$  generates a trajectory

$$x(t) = S_t x_0 + \int_0^t S_{t-\tau} B u(\tau) d\tau, \quad t > 0$$

satisfying  $x(kT) = x_k$  for all  $k \in \mathbb{N}_0$ . Let  $M_S \geq 1$  such that  $\sup_{t \in [0, T]} \|S_t\|_{\mathcal{L}(X)} \leq M_S$ . Then for all  $k \in \mathbb{N}_0$  and  $t \in [kT, (k+1)T)$ , by Hölder's inequality, we have

$$\begin{aligned} \|x(t)\|_X &= \left\| S_{t-kT} x_k + \int_0^{t-kT} S_{t-kT-\tau} B u_k(\tau - kT) d\tau \right\|_X \\ &\leq M_S \|x_k\|_X + M_S \|B\|_{\mathcal{L}(U,X)} \int_0^T \|u_k(\tau)\|_U d\tau \\ &\leq M_S \|x_k\|_X + M_S \|B\|_{\mathcal{L}(U,X)} T^{1/r'} \|u_k\|_{L_r((0,T);U)} \\ &\leq M_S (1 + \|B\|_{\mathcal{L}(U,X)} T^{1/r'} C) (\alpha + \varepsilon)^k \|x_0\|_X, \end{aligned}$$

where  $r' \in [1, \infty]$  such that  $1/r + 1/r' = 1$ , and  $1/\infty = 0$  as usual. Since  $\ln(\alpha + \varepsilon) < 0$  and  $(\alpha + \varepsilon)^{k+1} = e^{(k+1)T \frac{\ln(\alpha + \varepsilon)}{T}} \leq e^{\frac{\ln(\alpha + \varepsilon)}{T} t}$  for  $t \in [kT, (k+1)T)$ , we infer that

$$\|x(t)\|_X \leq \frac{M_S}{\alpha + \varepsilon} (1 + \|B\|_{\mathcal{L}(U,X)} T^{1/r'} C) e^{\frac{\ln(\alpha + \varepsilon)}{T} t} \|x_0\|_X.$$

Thus, we obtain the assertion with  $M = \frac{M_S}{\alpha + \varepsilon} (1 + \|B\|_{\mathcal{L}(U,X)} T^{1/r'} C) \geq 1$  and  $\omega = \ln(\alpha + \varepsilon)/T < 0$ .  $\square$

By the proof of Proposition 1.17 a system that is cost-uniformly approximately null-

controllable in time  $T$  is cost-uniformly open-loop stabilizable with a decay rate

$$\omega = \frac{\ln \varepsilon}{T} < 0,$$

for all  $\varepsilon > 0$ . For fixed  $T > 0$  we can achieve every decay rate  $\omega < 0$  by varying  $\varepsilon \in (0, 1)$ . Hence, we obtain the following sufficient condition.

**Corollary 1.18.** *Let the system  $\Sigma_T(A, B, -)$  be cost-uniformly approximately null-controllable in time  $T$  w.r.t.  $L_r((0, T); U)$ . Then the system  $\Sigma_\infty(A, B, -)$  is completely cost-uniformly open-loop stabilizable w.r.t.  $L_r((0, \infty); U)$ .*

### 1.3 Duality for control systems

It is a classical fact that null-controllability (in its various manifestations) and observability are closely related. Although this connection is an adaptation of classical duality results from functional analysis, the characteristics of infinite-dimensional Hilbert or Banach spaces accompanied by the number of different notions of controllability involved make it more intricate.

In the setting of Hilbert spaces, Douglas' lemma in [Dou66] on majorization, factorization, and range inclusion implies that null-controllability is equivalent to the final-state observability of the dual system. Unfortunately, the original version of Douglas' lemma in [Dou66] does not hold verbatim in the general framework of Banach spaces (see, e.g., [Bou78, AP91]). However, alternative formulations of this fundamental relation in Banach spaces have been studied, e.g., in [Emb73, DR77, Har78, CP78, Car88, For14]. For proving duality results for non-reflexive Banach spaces, one has to deal with dual semigroups, which may not be strongly continuous. This has been done more recently, e.g., in [Vie05, YLC06]. In particular, in [Vie05], it is shown that cost-uniform approximate null-controllability is equivalent to a final-state observability estimate for the dual system.

In this section, we follow this approach and prove a general duality theorem for  $\alpha$ -controllability derived by the author in collaboration with M. Egidi, C. Seifert, and M. Tautenhahn in [EGST21]. In addition to the duality for cost-uniform approximate null-controllability, this theorem also gives a dual characterization of cost-uniform open-loop stabilizability. By that, we generalize a result obtained in [TWX20], where the same theorem was proven in the Hilbert spaces setting.

#### 1.3.1 Duality theorem

Let us give a generalization of the final-state observability estimate in Definition 1.12. As for  $\alpha$ -controllability, we weaken the observability estimate by some  $\alpha \in [0, 1)$ . Note that here we do not assume that the semigroup  $(S_t)_{t \geq 0}$  is strongly continuous.

**Definition 1.19.** Let  $X, Y$  be Banach spaces,  $C \in \mathcal{L}(X, Y)$ , and  $(S_t)_{t \geq 0}$  a semigroup on  $X$  and assume that  $[0, T] \ni t \mapsto \|CS_t x\|_Y$  is measurable for all  $x \in X$ . We say that  $(S_t)_{t \geq 0}$  satisfies a *weak observability estimate* if there exist  $C_{\text{obs}} \geq 0$  and  $\alpha \in [0, 1)$  such that for

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all  $x \in X$  we have

$$\|S_T x\|_X \leq \begin{cases} C_{\text{obs}} \left( \int_0^T \|CS_t x\|_Y^r dt \right)^{1/r} + \alpha \|x\|_X & \text{if } r \in [1, \infty), \\ C_{\text{obs}} \operatorname{ess\,sup}_{t \in [0, T]} \|CS_t x\|_Y + \alpha \|x\|_X & \text{if } r = \infty. \end{cases} \quad (1.12)$$

Similarly as for final-state observability estimates, we say that *the system*  $\Sigma_T(A', -, B')$  *satisfies a weak observability estimate* if the above estimate holds for an operator  $B' \in \mathcal{L}(X', U')$ , and the dual semigroup  $(S'_t)_{t \geq 0}$  of a  $C_0$ -semigroup  $(S_t)_{t \geq 0}$  on  $X$ . In this case,  $-A'$  is the weak\* generator of the semigroup  $(S'_t)_{t \geq 0}$ .

For  $\alpha = 0$ , the weak observability estimate coincides with a final-state observability estimate. Note that for exponentially stable semigroups, there is  $T > 0$  such that the estimate (1.12) holds.

We state an abstract duality theorem for general Banach spaces to link the weak observability estimate to  $\alpha$ -controllability. Our result is based on [TWX20, Proposition 6], which is a Hilbert space version of the following Theorem 1.20. To deal with the framework of Banach spaces, we use a separation theorem, and a result from [Vie05] instead of the Fenchel-Rockafellar duality argument applied in [TWX20]. For the proof, we refer to Section 1.3.2 and Section 1.3.3.

**Theorem 1.20.** *Let  $X, U$  be Banach spaces,  $(S_t)_{t \geq 0}$  a  $C_0$ -semigroup on  $X$ ,  $T > 0$ ,  $r \in [1, \infty]$  and  $\mathcal{B}_T \in \mathcal{L}(L_r((0, T); U), X)$  the controllability map defined in (1.5). Let further  $C \geq 0$  and  $\alpha \geq 0$ . Then the following statements are equivalent:*

(a) *For every  $x \in X$  and  $\varepsilon > 0$  there exists  $u \in L_r((0, T); U)$  with*

$$\|u\|_{L_r((0, T); U)} \leq C \|x\|_X \quad \text{and} \quad \|S_T x + \mathcal{B}_T u\|_X < (\alpha + \varepsilon) \|x\|_X.$$

(b) *For all  $x' \in X'$  we have*

$$\|S'_T x'\|_{X'} \leq \begin{cases} C \left( \int_0^T \|B' S'_t x'\|_{U'}^{r'} dt \right)^{1/r'} + \alpha \|x'\|_{X'} & \text{if } r' \in [1, \infty), \\ C \operatorname{ess\,sup}_{t \in [0, T]} \|B' S'_t x'\|_{U'} + \alpha \|x'\|_{X'} & \text{if } r' = \infty, \end{cases}$$

where  $r' \in [1, \infty]$  with  $1/r + 1/r' = 1$ .

Theorem 1.20 can be rephrased as: cost-uniform approximate  $\alpha$ -controllability for  $\Sigma_T(A, B, -)$  is equivalent to a weak observability estimate of the corresponding dual system  $\Sigma_T(A', -, B')$ . Note that for  $\alpha = 0$ , the above theorem gives the well-known duality between cost-uniform approximate null-controllability and a final-state observability estimate. The diagram in Figure 1.2 shows the relation between the different notions introduced so far.

### 1.3.2 Separation of convex sets

To prove Theorem 1.20, we first recall some elementary results on the separation of convex sets in Banach spaces. With these results at hand, Theorem 1.20 is a direct consequence.

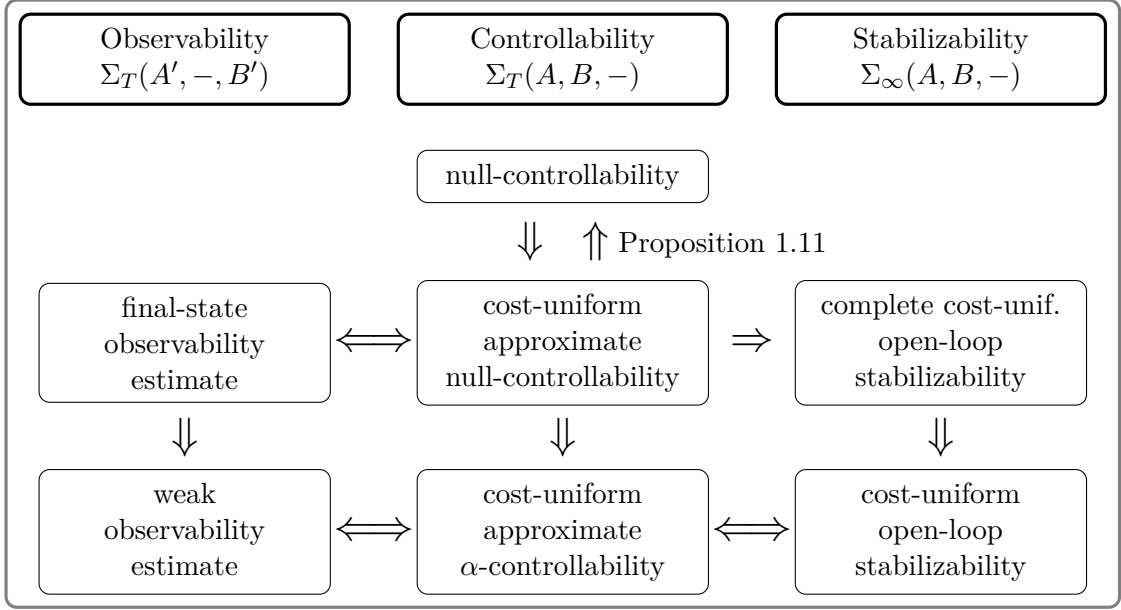


Figure 1.2: Relation between different system-theoretic notions introduced in Section 1.2.

The only technical part is to find a formula for the norm of the dual controllability map. This is a result of Vieru [Vie05], which we state and prove in Section 1.3.3.

**Definition 1.21.** Let  $X$  be a Banach space. A subset  $A \subset X$  is called *convex* if for all  $x, y \in A$  and  $\lambda \in [0, 1]$  the convex combination satisfies

$$\lambda x + (1 - \lambda)y \in A.$$

The following well-known separation theorem is a geometric version of the Hahn-Banach extension theorem. A proof can be found in most textbooks of functional analysis (see, e.g., [Rud91, Theorem 3.4]).

**Theorem 1.22** (Basic separation theorem). *Let  $X$  be a Banach space,  $A \subset X$  closed and convex, and  $x_0 \in X \setminus A$ . Then there is  $x' \in X'$  and  $c > 0$  such that for all  $x \in A$*

$$\operatorname{Re}\langle x', x_0 \rangle_{X', X} > c > \operatorname{Re}\langle x', x \rangle_{X', X}. \quad (1.13)$$

Since the set  $A$  is not only convex but also closed, one can find a convex neighborhood of  $x_0$ , which does not intersect  $A$ . Hence, one can prove the strict inequality in (1.13). Figure 1.3.2 shows an illustration of the conclusion of Theorem 1.22 in the plane. Note that in the two-dimensional setting, the set  $\{x \in X : \operatorname{Re}\langle x', x \rangle_{X', X} = c\}$  can be considered as a line in the plane.

For the proof of the duality theorem in Section 1.3.1, we need the following easy consequence of Theorem 1.22.

**Corollary 1.23.** *Let  $A, B$  be convex sets in a Banach space  $X$ . Then  $A \subset \overline{B}$  if and only if*

$$\sup_{x \in A} \operatorname{Re}\langle x', x \rangle_{X', X} \leq \sup_{x \in B} \operatorname{Re}\langle x', x \rangle_{X', X} \quad \text{for all } x' \in X'. \quad (1.14)$$

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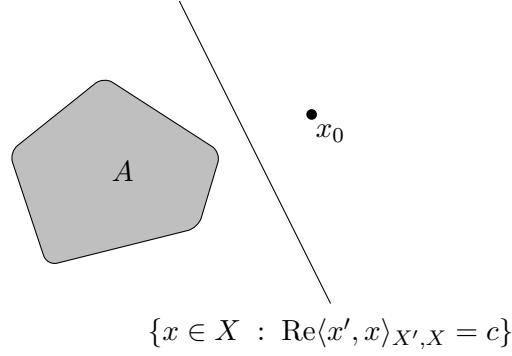


Figure 1.3: Illustration of the conclusion of Theorem 1.22 in the plane.

*Proof.* Assume that  $A \subset \overline{B}$ . Then for all  $x' \in X'$ , we have

$$\sup_{x \in A} \operatorname{Re} \langle x', x \rangle_{X', X} \leq \sup_{x \in \overline{B}} \operatorname{Re} \langle x', x \rangle_{X', X} = \sup_{x \in B} \operatorname{Re} \langle x', x \rangle_{X', X},$$

this is (1.14). For the converse implication, assume there is  $x_0 \in A$  with  $x_0 \notin \overline{B}$ . Since  $\overline{B}$  is closed and convex, by Theorem 1.22 there is  $x' \in X'$  such that

$$\operatorname{Re} \langle x', x_0 \rangle_{X', X} > \sup_{x \in \overline{B}} \operatorname{Re} \langle x', x \rangle_{X', X},$$

which contradicts (1.14). □

The duality theorem in Section 1.3.1 is an application of Corollary 1.23.

*Proof of Theorem 1.20.* Let  $C \geq 0$  and  $\alpha \geq 0$ . We consider the convex sets

$$A = \{S_T x : \|x\|_X \leq 1\} \quad \text{and} \quad B = \{\mathcal{B}_T u + \alpha x : \|u\|_{L_r((0, T); U)} \leq C, \|x\|_X \leq 1\}.$$

We observe that the following three statements are equivalent:

- (i)  $A \subset \overline{B}$
- (ii) for all  $\varepsilon > 0$  and  $x_1 \in X$  with  $\|x_1\|_X \leq 1$ , there exist  $u \in L_r((0, T); U)$  with  $\|u\|_{L_r((0, T); U)} \leq C$  and  $x_2 \in X$  with  $\|x_2\|_X \leq 1$  such that

$$\|S_T x_1 + \mathcal{B}_T u + \alpha x_2\|_X < \varepsilon.$$

- (iii) for all  $\varepsilon > 0$  and  $x_1 \in X$  with  $\|x_1\|_X \leq 1$ , there exists  $u \in L_r((0, T); U)$  with  $\|u\|_{L_r((0, T); U)} \leq C$  such that

$$\|S_T x_1 + \mathcal{B}_T u\|_X < \alpha + \varepsilon.$$

While (i)  $\Leftrightarrow$  (ii) and (ii)  $\Rightarrow$  (iii) are obvious, we note that (ii) follows from (iii) by choosing

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$x_2 = -(S_T x_1 + \mathcal{B}_T u)/(\alpha + \varepsilon)$ . Since

$$\|S_T x/\|x\| + \mathcal{B}_T u\|_X = \frac{1}{\|x\|} \|S_T x + \mathcal{B}_T \|x\|u\|_X$$

for all  $x \in X \setminus \{0\}$ , we find that (iii) (and thus also (i) and (ii)) is equivalent to statement (a) of Theorem 1.20. Next, for  $x' \in X'$  we compute

$$\sup_{x \in A} \operatorname{Re} \langle x', x \rangle_{X', X} = \sup_{\|x\|_X \leq 1} \operatorname{Re} \langle x', S_T x \rangle_{X', X} = \sup_{\|x\|_X \leq 1} \operatorname{Re} \langle S_T' x', x \rangle_{X', X} = \|S_T' x'\|_{X'}$$

where for the last equality we used that the norm of a complex functional is equal to the norm of its real part. Furthermore, we compute

$$\begin{aligned} \sup_{x \in B} \operatorname{Re} \langle x', x \rangle_{X', X} &= \sup_{\substack{\|u\|_{L_r((0, T); U)} \leq C, \\ \|x\|_X \leq 1}} \operatorname{Re} \langle x', \mathcal{B}_T u + \alpha x \rangle_{X', X} \\ &= \sup_{\|u\|_{L_r((0, T); U)} \leq C} \operatorname{Re} \langle x', \mathcal{B}_T u \rangle_{X', X} + \sup_{\|x\|_X \leq 1} \alpha \operatorname{Re} \langle x', x \rangle_{X', X} \\ &= C \|\mathcal{B}_T' x'\|_{(L_r((0, T); U))'} + \alpha \|x'\|_{X'}. \end{aligned}$$

It remains to show that

$$\|\mathcal{B}_T' x'\|_{(L_r((0, T); U))'} = \begin{cases} \left( \int_0^T \|B' S_t' x'\|_{U'}^{r'} dt \right)^{1/r'} & \text{if } r' \in [1, \infty), \\ \operatorname{ess\,sup}_{t \in [0, T]} \|B' S_t' x'\|_{U'} & \text{if } r' = \infty, \end{cases} \quad (1.15)$$

where  $r' \in [1, \infty]$  such that  $1/r + 1/r' = 1$ . In this generality, this is a result of Vieu (see [Vie05, Theorem 2.1]). For completeness, we give a proof in the following Section 1.3.3. Hence,  $\sup_{x \in A} \operatorname{Re} \langle x, x' \rangle_{X, X'} \leq \sup_{x \in B} \operatorname{Re} \langle x, x' \rangle_{X, X'}$  is equivalent to statement (b) of the theorem which is in turn equivalent to (i) from above by Corollary 1.23.  $\square$

### 1.3.3 Norm of the dual controllability map

In this section, we give a proof of (1.15) to complete the proof of Theorem 1.20. First, let us assume that the state space  $X$  and the control space  $U$  are reflexive. Then the dual semigroup  $(S_t')_{t \geq 0}$  is strongly continuous on  $X'$ . Furthermore, for  $r \in [1, \infty)$  it holds that  $(L_r((0, T); U))' \cong L_{r'}((0, T); U')$  where  $r' \in (1, \infty]$  such that  $1/r + 1/r' = 1$ . Here,  $g \in L_{r'}((0, T); U')$  is identified with the functional  $\phi_g \in (L_r((0, T); U))'$  defined by

$$\phi_g(f) = \int_0^T \langle g(t), f(t) \rangle_{U', U} dt, \quad f \in L_r((0, T); U).$$

With this, we can calculate a representation of the dual controllability map  $\mathcal{B}_T'$ . For  $u \in L_r((0, T); U)$  and  $x' \in X'$  we have

$$\begin{aligned} \langle x', \mathcal{B}_T u \rangle_{X', X} &= \left\langle x', \int_0^T S_{T-t} B u(t) dt \right\rangle_{X', X} \\ &= \int_0^T \langle x', S_{T-t} B u(t) \rangle_{X', X} dt \end{aligned}$$

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$$= \int_0^T \langle B' S'_{T-t} x', u(t) \rangle_{U', U} dt.$$

Hence, we can identify

$$\mathcal{B}'_T x' = B' S'_{T-(\cdot)} x', \quad \text{for all } x' \in X'. \quad (1.16)$$

This corresponds, up to time-reflection, to the observability map of the dual observation system  $\Sigma_T(A', -, B')$ . By (1.16), we obtain (1.15), i.e.,

$$\|\mathcal{B}'_T x'\|_{(L_r((0,T);U))'} = \|B' S'_{(\cdot)} x'\|_{L_{r'}((0,T);U')}.$$

If we turn to non-reflexive Banach spaces, the above reasoning may fail. For a non-reflexive state space  $X$ , the dual semigroup is not necessarily strongly continuous, and  $t \mapsto B' S'_t x'$  may fail to be Bochner measurable. However, assuming the strong continuity of  $(S_t)_{t \geq 0}$ , we see that the map  $t \mapsto \|B' S'_t x'\|_{U'}$ , as the supremum of continuous functions, is lower semicontinuous and hence measurable. This is outlined in the following lemma.

**Lemma 1.24.** *Let  $X, U$  be Banach spaces,  $(S_t)_{t \geq 0}$  a  $C_0$ -semigroup on  $X$ , and  $B \in \mathcal{L}(U, X)$ . Then the map*

$$[0, \infty) \ni t \mapsto \|B' S'_t x'\|_{U'}$$

*is measurable for all  $x' \in X'$ .*

*Proof.* Let  $x' \in X'$ . First, note that for all  $t \geq 0$  we have

$$\|B' S'_t x'\|_{U'} = \sup_{\|u\|_U \leq 1} |\langle B' S'_t x', u \rangle_{U', U}| = \sup_{\|u\|_U \leq 1} |\langle x', S_t B u \rangle_{X', X}|.$$

Hence, for all  $c \in \mathbb{R}$ , we have

$$\{t \in [0, \infty) : \|B' S'_t x'\|_{U'} > c\} = \bigcup_{\|u\|_U \leq 1} \{t \in [0, \infty) : |\langle x', S_t B u \rangle_{X', X}| > c\}. \quad (1.17)$$

Since  $(S_t)_{t \geq 0}$  is strongly continuous, the dual semigroup  $(S'_t)_{t \geq 0}$  is weak\*-continuous, and hence, the map

$$t \mapsto |\langle B' S'_t x', u \rangle_{U', U}| = |\langle x', S_t B u \rangle_{X', X}|$$

is continuous for all  $x' \in X$  and  $u \in U$ . Therefore, the sets on the right-hand side of (1.17) are open. Thus, as a union of open sets, the left-hand side of (1.17) is open as well, and hence a Borel set. Since this holds for all  $c \in \mathbb{R}$ , the map  $t \mapsto \|B' S'_t x'\|_{U'}$  is measurable.  $\square$

Although the identification  $\mathcal{B}'_T x' = B' S'_{T-(\cdot)} x'$  does not hold for arbitrary Banach spaces, the following Proposition states that  $\|\mathcal{B}'_T x'\|_{(L_r((0,T);U))'}$  is equal to the  $L_{r'}((0, T); \mathbb{R})$ -norm of the function  $t \mapsto \|B' S'_t x'\|_{U'}$ . In this generality, this was shown in [Vie05, Theorem 2.1]. However, for more specific cases, as reflexive Banach spaces or Hilbert spaces, this statement can be found, e.g., in [Dou66, Emb73, DR77, CP78, Car88]. We give a full proof of this result in the remaining part of this section. Note that we give a proof only for sake of completeness and that our presentation is very close to [Vie05].

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**Proposition 1.25.** *Let  $X, U$  be Banach spaces,  $(S_t)_{t \geq 0}$  a  $C_0$ -semigroup on  $X$ ,  $T > 0$ ,  $r \in [1, \infty]$ , and  $\mathcal{B}_T \in \mathcal{L}(L_r((0, T); U), X)$  the controllability map defined in (1.5). Then for all  $x' \in X'$*

$$\|\mathcal{B}'_T x'\|_{(L_r((0, T); U))'} = \begin{cases} \left( \int_0^T \|B' S'_t x'\|_{U'}^{r'} dt \right)^{1/r'} & \text{if } r' \in [1, \infty), \\ \text{ess sup}_{t \in [0, T]} \|B' S'_t x'\|_{U'} & \text{if } r' = \infty, \end{cases}$$

where  $r' \in [1, \infty]$  such that  $1/r + 1/r' = 1$ .

*Proof.* Let  $x' \in X'$  and  $r \in [1, \infty]$ . First, we observe that for  $u \in L_r((0, T); U)$  we have

$$\begin{aligned} \langle \mathcal{B}'_T x', u \rangle_{(L_r((0, T); U))', L_r((0, T); U)} &= \left\langle x', \int_0^T S_{T-t} B u(t) dt \right\rangle_{X', X} \\ &= \int_0^T \langle x', S_{T-t} B u(t) \rangle_{X', X} dt \\ &= \int_0^T \langle B' S'_{T-t} x', u(t) \rangle_{U', U} dt. \end{aligned}$$

Since the norm of a complex functional is equal to the norm of its real part, we obtain that

$$\begin{aligned} \|\mathcal{B}'_T x'\|_{(L_r((0, T); U))'} &= \sup_{\|u\|_{L_r((0, T); U)} \leq 1} \text{Re} \langle \mathcal{B}'_T x', u \rangle_{(L_r((0, T); U))', L_r((0, T); U)} \\ &= \sup_{\|u\|_{L_r((0, T); U)} \leq 1} \int_0^T \text{Re} \langle B' S'_t x', u(T-t) \rangle_{U', U} dt. \end{aligned} \quad (1.18)$$

We can estimate (1.18) by Hölder's inequality to get the upper bound

$$\begin{aligned} \|\mathcal{B}'_T x'\|_{(L_r((0, T); U))'} &\leq \sup_{\|u\|_{L_r((0, T); U)} \leq 1} \int_0^T \|B' S'_t x'\|_{U'} \|u(T-t)\|_U dt \\ &\leq \begin{cases} \left( \int_0^T \|B' S'_t x'\|_{U'}^{r'} dt \right)^{1/r'} & \text{if } r' \in [1, \infty), \\ \text{ess sup}_{t \in [0, T]} \|B' S'_t x'\|_{U'} & \text{if } r' = \infty. \end{cases} \end{aligned}$$

This proves one inequality of the proposition. For the reverse inequality, we first consider the case  $r = 1$ . Then

$$\begin{aligned} \text{ess sup}_{t \in [0, T]} \|B' S'_t x'\|_{U'} &= \sup_{t \in [0, T]} \|B' S'_t x'\|_{U'} = \sup_{t \in [0, T]} \sup_{\|u\|_U \leq 1} \text{Re} \langle B' S'_t x', u \rangle_{U', U} \\ &= \sup_{\|u\|_U \leq 1} \sup_{t \in [0, T]} \text{Re} \langle x', S_t B u \rangle_{X', X}. \end{aligned} \quad (1.19)$$

Note that the first equality holds, since  $t \mapsto \|B' S'_t x'\|_{U'}$ , as the supremum of continuous functions, is lower semicontinuous. Now, we use that  $L_1((0, T); \mathbb{R})'$  is isometrically

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isomorphic to  $L_\infty((0, T); \mathbb{R})$ , and find for  $u \in U$  that

$$\begin{aligned}
\sup_{t \in [0, T]} \operatorname{Re} \langle x', S_t B u \rangle_{X', X} &= \sup_{\|g\|_{L_1((0, T); \mathbb{R})} \leq 1} \int_0^T \operatorname{Re} \langle x', S_t B u \rangle_{X', X} \cdot g(t) \, dt \\
&= \sup_{\|g\|_{L_1((0, T); \mathbb{R})} \leq 1} \int_0^T \operatorname{Re} \langle B' S'_t x', u \rangle_{U', U} \cdot g(t) \, dt \\
&= \sup_{\|g\|_{L_1((0, T); \mathbb{R})} \leq 1} \int_0^T \operatorname{Re} \langle B' S'_t x', g(t) u \rangle_{U', U} \, dt. \tag{1.20}
\end{aligned}$$

For  $g \in L_1((0, T); \mathbb{R})$  with  $\|g\|_{L_1((0, T); \mathbb{R})} \leq 1$ , and  $u \in U$  with  $\|u\|_U \leq 1$ , we have

$$g \cdot u \in L_1((0, T); U) \quad \text{and} \quad \|g \cdot u\|_{L_1((0, T); U)} \leq 1.$$

Hence, we get by combining (1.19) and (1.20) that

$$\operatorname{ess\,sup}_{t \in [0, T]} \|B' S'_t x'\|_{U'} \leq \sup_{\|u\|_{L_1((0, T); U)} \leq 1} \int_0^T \operatorname{Re} \langle B' S'_t x', u(t) \rangle_{U', U} \, dt = \|B'_T x'\|_{(L_1((0, T); U))'},$$

where for the last equality we used (1.18). This finishes the proof in the case  $r = 1$ .

For the remaining part assume  $r \in (1, \infty]$  and let  $r' \in [1, \infty)$  such that  $1/r + 1/r' = 1$ . For  $t \in [0, T]$ , we abbreviate by  $f(t)$  the real part of the functional  $B' S'_t x'$ , i.e.,

$$\langle f(t), u \rangle_{U', U} = \operatorname{Re} \langle B' S'_t x', u \rangle_{U', U} \quad \text{for } u \in U.$$

With this we obtain  $\|f(t)\|_{U'} = \|B' S'_t x'\|_{U'}$ . By (1.18) and Lusin's theorem, it is enough to show that for all compact  $K \subset [0, T]$  such that  $\|f(\cdot)\|_K$  is continuous we have

$$\left( \int_K \|f(t)\|_{U'}^{r'} \, dt \right)^{1/r'} \leq \sup_{\|u\|_{L_r((0, T); U)} \leq 1} \int_K \langle f(t), u(t) \rangle_{U', U} \, dt.$$

We show that for all  $\varepsilon > 0$  there is a simple function  $g: K \rightarrow U$  with  $\|g\|_{L_r(K; U)} \leq 1$  such that

$$\left( \int_K \|f(t)\|_{U'}^{r'} \, dt \right)^{1/r'} \leq \int_K \langle f(t), g(t) \rangle_{U', U} \, dt + \varepsilon. \tag{1.21}$$

Fix  $\varepsilon > 0$  and  $K \subset [0, T]$  compact such that  $\|f(\cdot)\|_K$  is continuous. For all  $t \in K$ , there is  $u_t \in U$  with  $\|u_t\|_U \leq 1$  such that

$$\|f(t)\|_{U'} < \langle f(t), u_t \rangle_{U', U} + \varepsilon. \tag{1.22}$$

Let  $V_t$  be the set of points  $\tau \in K$  such that  $|\|f(\tau)\|_{U'} - \|f(t)\|_{U'}| < \varepsilon$  and

$$|\langle f(\tau), u_t \rangle_{U', U} - \langle f(t), u_t \rangle_{U', U}| < \varepsilon.$$

Then the family  $(V_t)_{t \in K}$  is a covering of  $K$  of open sets. Since  $K$  is compact, there is a finite cover  $(V_{t_i})_{i=1}^d$ , for some  $d \in \mathbb{N}$ , which we assume to be minimal, i.e., no subfamily of  $(V_{t_i})_{i=1}^d$  covers  $K$ . Now, we construct a disjoint partition  $(A_i)_{i=1}^d$  of  $K$  by choosing

## 1 Control theory in Banach spaces

$A_1 := V_{t_1}$  and

$$A_i := V_{t_i} \setminus (A_1 \cup \dots \cup A_{i-1}).$$

Since  $(V_{t_i})_{i=1}^d$  is minimal, we have  $\lambda(A_i) \neq 0$  for all  $i \in \{1, \dots, d\}$ , where  $\lambda(A_i)$  denotes the Lebesgue measure of the set  $A_i$ . For all  $\tau \in A_i$  we have  $\|f(\tau)\|_{U'} < \|f(t_i)\|_{U'} + \varepsilon$  and

$$\langle f(t_i), u_{t_i} \rangle_{U', U} \leq \langle f(\tau), u_{t_i} \rangle_{U', U} + \varepsilon. \quad (1.23)$$

Hence,

$$\begin{aligned} \int_K \|f(s)\|_{U'}^{r'} ds &= \sum_{i=1}^d \int_{A_i} \|f(s)\|_{U'}^{r'} ds \\ &< \sum_{i=1}^d (\|f(t_i)\|_{U'} + \varepsilon)^{r'} \lambda(A_i) \\ &< \sum_{i=1}^d (\langle f(t_i), u_{t_i} \rangle_{U', U} + 2\varepsilon)^{r'} \lambda(A_i), \end{aligned} \quad (1.24)$$

where for the last inequality we used (1.22). We consider on  $\mathbb{R}^d$  the norm

$$|v|_{A, r'} := \left( \sum_{i=1}^d |v_i|^{r'} \lambda(A_i) \right)^{1/r'}, \quad v \in \mathbb{R}^d.$$

With respect to the inner product

$$\langle \xi, v \rangle = \sum_{i=1}^d \xi_i v_i \lambda(A_i), \quad \xi, v \in \mathbb{R}^d,$$

the dual norm of  $|\cdot|_{A, r'}$  satisfies

$$\max_{|v|_{A, r'} \leq 1} \langle \xi, v \rangle = |\xi|_{A, r}, \quad \xi \in \mathbb{R}^d, \quad (1.25)$$

where  $1/r + 1/r' = 1$ , and for  $r = \infty$  we set  $|\xi|_{A, \infty} := \max_{i \in \{1, \dots, d\}} |\xi_i|$ . Let  $e = (1, \dots, 1)^\top \in \mathbb{R}^d$ . Then for  $v \in \mathbb{R}^d$  given by  $v_i = \langle f(t_i), u_{t_i} \rangle_{U', U}$ , we observe with (1.24) that

$$\|f\|_{L_{r'}(K; U')} \leq |v + 2\varepsilon e|_{A, r'} \leq |v|_{A, r'} + 2\varepsilon |e|_{A, r'}. \quad (1.26)$$

For  $\xi \in \mathbb{R}^d$  with  $|\xi|_{A, r} \leq 1$  such that  $\langle \xi, v \rangle = |v|_{A, r'}$ , we choose  $g \in L_r(K; U)$  given by

$$g = \sum_{i=1}^d \xi_i u_{t_i} \mathbf{1}_{A_i},$$

where  $\mathbf{1}_{A_i}$  denotes the indicator function of the set  $A_i$ . Then, since  $\|u_{t_i}\|_U \leq 1$ , we have  $\|g\|_{L_r(K; U)} \leq |\xi|_{A, r} \leq 1$ . Next, we show that

$$|v|_{A, r'} < \int_K \langle f(t), g(t) \rangle_{U', U} dt + \varepsilon |e|_{A, r'} \quad (1.27)$$

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and hence, together with (1.26), the claim (1.21) follows since  $\varepsilon$  is arbitrary. Therefore, note that by the choice of  $\xi \in \mathbb{R}^d$  and (1.23) we have

$$\begin{aligned}
 |v|_{A,r'} = \langle \xi, v \rangle &= \sum_{i=1}^d \xi_i \int_{A_i} \langle f(t_i), u_{t_i} \rangle_{U',U} dt \\
 &\leq \sum_{i=1}^d \xi_i \int_{A_i} (\langle f(t), u_{t_i} \rangle_{U',U} + \varepsilon) dt \\
 &= \varepsilon \sum_{i=1}^d \xi_i \lambda(A_i) + \int_{A_i} \langle f(t), \xi_i u_{t_i} \rangle_{U',U} dt \\
 &= \varepsilon |\xi|_{A,1} + \int_K \langle f(t), g(t) \rangle_{U',U} dt.
 \end{aligned}$$

The inequality (1.27) now follows since  $|\xi|_{A,1} = \langle \xi, e \rangle \leq |\xi|_{A,r} |e|_{A,r'} \leq |e|_{A,r'}$ . □

## 2 Sufficient conditions for observability

This chapter builds the central part of this thesis. We will prove sufficient conditions for observability estimates in terms of an uncertainty principle and a dissipation estimate for abstract linear systems in Banach spaces. Our approach is based on the Lebeau–Robbiano strategy, and it unifies and generalizes the respective advantages from earlier results obtained in the context of Hilbert spaces.

For classification of our results, we start in Section 2.1 by recalling the original results of Lebeau with Robbiano, Zuazua, and Jerison in [LR95, LZ98, JL99] and discuss recent developments on the Lebeau–Robbiano strategy. Our main results are presented in Section 2.2. In particular, we introduce sufficient conditions for a final-state observability estimate with an observability constant that is explicit in the time and the other model parameters. As we improved our results over the last years, we present here several results with a different degree of generality. We utilize two different iterative arguments for the proofs, one of which leads to a more accurate description of the observability constant. By exploiting subordination techniques, we show in Section 2.3 how dissipation estimates of one semigroup transfer to a family of subordinated semigroups. This enables us to prove that our sufficient conditions do not just imply observability for one system but also many related ones, such as systems induced by fractional powers. We conclude this chapter with Section 2.4 by bringing together the sufficient conditions for observability with the duality results presented in Section 1.3 to obtain conditions for null-controllability and open-loop stabilizability.

### 2.1 Lebeau–Robbiano strategy

We start this chapter by surveying the so-called Lebeau–Robbiano strategy to prove null-controllability. The strategy originates in the seminal works by Lebeau with Robbiano, Zuazua, and Jerison in [LR95, LZ98, JL99] and was further developed, e.g., in [Mil10, TT11, DM12, WZ17, BP18, NTTV20a]. We revise one of the original examples and see how spectral inequalities, i.e., quantitative uncertainty principles for spectral projectors, lead to null-controllability. We analyze further developments and introduce our main results given in [GST20, BGST22b, GMS22]. For this section, we use the references [Mil10, TT11, LL12, ENS<sup>+</sup>20].

#### 2.1.1 Original example

The original examples include the heat equation with Dirichlet boundary conditions in  $L_2(\Omega)$  for a smooth and bounded domain  $\Omega \subset \mathbb{R}^d$ , i.e.,

$$\begin{aligned} \dot{x}(t) &= \Delta_{\text{Dir}} x(t) + \mathbf{1}_E u(t) && \text{for } t \in (0, T], \\ x(0) &= x_0 \in L_2(\Omega), && \Sigma(-\Delta_{\text{Dir}}, \mathbf{1}_E, -) \end{aligned}$$

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for some open non-empty control set  $E \subset \Omega$  and  $T > 0$ . Here,  $\Delta_{\text{Dir}} : \mathcal{D}(\Delta_{\text{Dir}}) \rightarrow L_2(\Omega)$  denotes the Dirichlet Laplacian, i.e., we have  $\Delta_{\text{Dir}}x = \Delta x$  for

$$x \in \mathcal{D}(\Delta_{\text{Dir}}) := \{x \in H_0^1(\Omega) : \Delta x \in L_2(\Omega)\}.$$

The null-controllability of system  $\Sigma(-\Delta_{\text{Dir}}, \mathbf{1}_E, -)$  is well understood since the independent works by Lebeau and Robbiano [LR95] and Fursikov and Imanuvilov [FI96]. Here we focus on the approach pursued in [LR95].

The Dirichlet Laplacian  $\Delta_{\text{Dir}}$  is diagonalizable. There is an orthonormal basis  $(\varphi_k)_{k \in \mathbb{N}}$  of  $L_2(\Omega)$  consisting of eigenvectors of  $\Delta_{\text{Dir}}$  and a corresponding sequence of eigenvalues  $(-\lambda_k)_{k \in \mathbb{N}}$  of  $\Delta_{\text{Dir}}$  satisfying  $\lambda_k > 0$  for all  $k \in \mathbb{N}$  and  $\lambda_k \rightarrow \infty$  for  $k \rightarrow \infty$ . Hence,  $\Delta_{\text{Dir}}$  generates a strongly continuous contraction semigroup  $(S_t)_{t \geq 0}$  satisfying

$$S_t x = \sum_{k \in \mathbb{N}} a_k e^{-\lambda_k t} \varphi_k$$

for  $x = \sum_{k \in \mathbb{N}} a_k \varphi_k \in L_2(\Omega)$ . Based on the local elliptic Carleman estimates obtained in [LR95], in [LZ98, Theorem 3] and [JL99, Theorem 14.6] the following estimate for sums of eigenfunctions of  $\Delta_{\text{Dir}}$  was proven:

$$\int_{\Omega} \left| \sum_{\lambda_k \leq \lambda} a_k \varphi_k(\xi) \right|^2 d\xi \leq C_1 e^{c_1 \sqrt{\lambda}} \int_E \left| \sum_{\lambda_k \leq \lambda} a_k \varphi_k(\xi) \right|^2 d\xi, \quad (2.1)$$

which holds with uniform  $c_1, C_1 > 0$  for all  $\lambda > 0$  and  $(a_k)_{\lambda_k \leq \lambda}$ . The estimate (2.1) is called *spectral inequality*. One of the reasons why spectral inequalities are important in control theory is that (2.1) implies an observability estimate for initial conditions  $x = \sum_{\lambda_k \leq \lambda} a_k \varphi_k$ . Indeed, by averaging the contraction semigroup  $(S_t)_{t \geq 0}$ , we obtain

$$\begin{aligned} T \|S_T x\|_{L_2(\Omega)}^2 &\leq \int_0^T \|S_\tau x\|_{L_2(\Omega)}^2 d\tau = \int_0^T \int_{\Omega} \left| \sum_{\lambda_k \leq \lambda} a_k e^{-\lambda_k \tau} \varphi_k(\xi) \right|^2 d\xi d\tau \\ &\leq C_1 e^{c_1 \sqrt{\lambda}} \int_0^T \int_E \left| \sum_{\lambda_k \leq \lambda} a_k e^{-\lambda_k \tau} \varphi_k(\xi) \right|^2 d\xi d\tau \\ &= C_1 e^{c_1 \sqrt{\lambda}} \int_0^T \|S_\tau x\|_{L_2(E)}^2 d\tau. \end{aligned}$$

Via duality (see Theorem 1.20), this proves the null-controllability of the original equation  $\Sigma(-\Delta_{\text{Dir}}, \mathbf{1}_E, -)$  for initial conditions in the finite-dimensional spectral subspaces of  $\Delta_{\text{Dir}}$ . Although the family of spectral subspaces exhausts the whole state space  $L_2(\Omega)$  in the limit  $\lambda \rightarrow \infty$ , the corresponding control cost

$$c = \left( \frac{C_1}{T} e^{c_1 \sqrt{\lambda}} \right)^{\frac{1}{2}}$$

is unbounded for  $\lambda \rightarrow \infty$ . To obtain null-controllability for the system on the whole state space  $L_2(\Omega)$ , Lebeau and Robbiano constructed in [LR95] a control function using the null-controllability of the restricted systems for an increasing sequence of spectral subspaces. For this, it is crucial to employ the exponential decay of solutions of  $\Sigma(-\Delta_{\text{Dir}}, \mathbf{1}_E, -)$  for

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initial conditions  $x = \sum_{\lambda_k > \lambda} a_k \varphi_k$  in the complement of the spectral subspace, i.e.,

$$\|S_t x\|_{L_2(\Omega)} = \left\| \sum_{\lambda_k > \lambda} a_k e^{-\lambda_k t} \varphi_k \right\|_{L_2(\Omega)} \leq e^{-\lambda t} \left\| \sum_{\lambda_k > \lambda} a_k \varphi_k \right\|_{L_2(\Omega)} = e^{-\lambda t} \|x\|_{L_2(\Omega)}. \quad (2.2)$$

In this connection, one has to exploit that the growth rate in the spectral inequality (2.1) with respect to  $\lambda$  is smaller than the decay rate in the dissipation estimate (2.2). This method of proving null-controllability by verifying a spectral inequality is called *Lebeau–Robbiano strategy*.

### 2.1.2 Further developments

Spectral inequalities have various applications in control theory, such as observability, interpolation inequalities, and controllability properties. Based on the presentation in [ENS<sup>+</sup>20, ES21], let us recall some of the recent results on spectral inequalities of the form

$$\forall x \in L_2(\Omega) : \quad \|P_\lambda x\|_{L_2(\Omega)} \leq K_{E,\lambda} \|P_\lambda x\|_{L_2(E)},$$

where  $P_\lambda$  is the projection onto the spectral subspace below  $\lambda \in \mathbb{R}$  of some lower semi-bounded selfadjoint operator in  $L_2(\Omega)$  with domain  $\Omega \subset \mathbb{R}^d$ , and where  $E$  is a measurable subset of  $\Omega$ . In connection to control theory, it is valuable to work out explicit dependencies of the constant  $K_{E,\lambda} \geq 0$  concerning  $\lambda$  and geometric parameters of  $E$ . An example of the above estimate is the spectral inequality (2.1) for sums of eigenfunctions of the Laplacian on smooth and bounded domains  $\Omega$ . This estimate has been subsequently generalized to domains with less regularity and sets  $E$  with positive Lebesgue measure (see [AEWZ14, EMZ15]). In addition to the results for bounded domains, spectral inequalities were obtained for the Laplacian on unbounded domains in the last years. Important examples are spectral inequalities derived by complex analysis, such as the Logvinenko–Sereda theorem [LS74, Kov01] for the Laplacian on  $\Omega = \mathbb{R}^d$  and so-called thick sets  $E$ , i.e., sets  $E$  satisfying a property ensuring well-distribution in the domain. The authors in [EV18, WWZZ19] show that this spectral inequality is equivalent to the null-controllability of the heat equation for thick control sets. See also Section 3.3 and Section 3.2.3 for a discussion and a full proof of these results. The thickness property for  $E$  is also sufficient for proving spectral inequalities for the Laplacian on hyperrectangles or infinite strips (see [EV20, Egi21, ES21]). For Schrödinger operators on unbounded domains, spectral inequalities have, for instance, been proven in [NTTV18, NTTV20b] for rectangular shaped domains, equidistributed sets  $E$ , and bounded potentials, as well as in [LM19] for  $\Omega = \mathbb{R}^d$ , thick sets  $E$ , and analytic potentials. Following Kovrijkine’s proof of the Logvinenko–Sereda theorem, the authors in [BJP21] obtained spectral inequalities for the harmonic oscillator in  $\Omega = \mathbb{R}^d$  and thick sets  $E$ . This approach was abstracted in [ES21], which allows, e.g., to consider other domains for the harmonic oscillator. In [DSV22b], the fast decay of the eigenfunctions of the harmonic oscillator is exploited to prove spectral inequalities for sets  $E$  that may become sparse near infinity and have finite Lebesgue measure.

Along with the development of spectral inequalities, the Lebeau–Robbiano strategy has been extended in recent years to suit various applications. In the situation of Section 2.1.1

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we can write (2.1) concisely as

$$\forall \lambda > 0, x \in L_2(\Omega) : \quad \|P_\lambda x\|_{L_2(\Omega)} \leq C_1 e^{c_1 \sqrt{\lambda}} \|\mathbf{1}_E P_\lambda x\|_{L_2(E)}, \quad (2.3)$$

where  $P_\lambda$  is the projector on the spectral subspace of  $-\Delta_{\text{Dir}}$  below  $\lambda$ . To detach the approach from spectral theory, inequalities of the form (2.3) are considered for operators  $P_\lambda$  that are allowed to be arbitrary projections onto semigroup invariant subspaces or just linear bounded operators. The operator  $\mathbf{1}_E$  is replaced by a general linear bounded operator  $C$  between Hilbert or Banach spaces  $X$  and  $Y$ , i.e., one requires inequalities of the form

$$\forall \lambda > 0, x \in X : \quad \|P_\lambda x\|_X \leq C_1 e^{c_1 \lambda^{s_1}} \|C P_\lambda x\|_Y. \quad (\text{UP})$$

In these situations, one often calls the corresponding inequality *uncertainty principle* instead of spectral inequality. As discussed in Section 2.1.1, for proving an observability estimate of the form

$$\forall x \in X : \quad \|S_T x\|_X \leq C_{\text{obs}} \int_0^T \|C S_\tau x\|_Y d\tau, \quad (\text{OBS})$$

for the operator semigroup  $(S_t)_{t \geq 0}$  associated with the control system, it is essential to employ the exponential decay of solutions on the complement of the spectral subspace. In the case that the operators  $P_\lambda$  are not spectral projections, this is ensured by a *dissipation estimate*

$$\forall \lambda > 0, t \in (0, T], x \in X : \quad \|(\text{Id} - P_\lambda) S_t x\|_X \leq C_2 e^{-c_2 \lambda^{s_2} t} \|x\|_X \quad (\text{DISS})$$

with the condition  $s_2 > s_1$  ensuring that the decay rate in (DISS) with respect to  $\lambda$  is faster than the growth rate in (UP). In the following, we review the development in the approach of combining an uncertainty principle (UP) and a dissipation estimate (DISS) to derive a final-state observability estimate, i.e.,

$$(\text{UP}) + (\text{DISS}) \implies (\text{OBS}).$$

While the original approach developed in [LR95, LZ98, JL99] uses an iterative construction of a control function, Miller introduced in [Mil10] a more direct approach of using a telescoping series argument in observability inequalities for subspaces. Miller's conditions are formulated for strongly continuous semigroups on Hilbert spaces and projections  $P_\lambda$  on semigroup invariant subspaces. Instead of the uncertainty principle (UP), he used a related inequality by employing a reference observation operator. His method allows giving the estimate

$$C_{\text{obs}} \leq K \exp\left(\frac{K}{T^\nu}\right) \quad \text{with } \nu = \frac{s_1}{s_2 - s_1} \text{ and } K > 0 \quad (2.4)$$

on the observability constant, which reflects the optimal rate for small  $T$  for the heat equation on bounded domains (see [Güi85, Mil04]). However, the constant  $K$  in (2.4) is not explicit in the other parameters of (UP) and (DISS). Later, in [BP18] a modification of [Mil10] was given in the case of contraction semigroups on  $L_2(\Omega)$ ,  $C = \mathbf{1}_E$  with  $E \subset \Omega$  open, and a family of orthogonal projections. To allow for more general asymptotic behavior in the assumptions, the authors of [DM12] require growth rates in the uncertainty

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principle that satisfy an integrability condition. However, in this abstract setting, no estimate for the observability constant is obtained. The telescoping series method is also used for obtaining observability estimates when the system is observed for a measurable set in time (see [PW13, AEWZ14, WZ17]), and for non-autonomous systems (see [BGST22a]).

In [TT11], a different iteration approach is developed, which allows to work out more precisely the dependence of the observability constant  $C_{\text{obs}}$  with respect to the different model parameters. Additionally, this method shows that for bounded semigroups, the constant  $C_{\text{obs}}$  tends to zero with the rate  $1/T$  as  $T \rightarrow \infty$ . This behavior is optimal (see [NTTV20a] and Section 3.2.4). The conditions in [TT11] apply for strongly continuous semigroups on Hilbert spaces with a negative-selfadjoint generator having a purely discrete spectrum, the corresponding family of spectral projections, and a certain class of unbounded observation operators. In [NTTV20a] the result in [TT11] was improved by dropping the spectral assumption for the generator. Furthermore, the estimates in the iteration argument are more precise and yield a fully explicit description of  $C_{\text{obs}}$  in terms of the parameters coming from the uncertainty principle and the dissipation estimate.

So far, we have restricted the discussion to Hilbert spaces only. However, a natural setup to ask for observability estimates is the context of Banach spaces and  $C_0$ -semigroups. Let us outline what the author and his collaborators C. Bombach, J. Meichsner, C. Seifert, and M. Tautenhahn have contributed to this subject in [GST20, BGST22b, GMS22]. In [GST20], it is shown that an uncertainty principle and a dissipation estimate imply final-state observability in the general framework of Banach spaces. As in [NTTV20a], the observability constant  $C_{\text{obs}}$  is given explicitly with respect to the final time  $T$  and the other model parameters. Based on this, in [BGST22b], it is observed that, instead of the strong continuity of the semigroup  $(S_t)_{t \geq 0}$ , only a measurability assumption is needed. This makes it possible to pursue duality theory for non-reflexive Banach spaces, to obtain sufficient conditions for null-controllability. Recently, the measurability assumption was exploited to prove observability estimates for bi-continuous semigroups (see [KS22]). The admissible growth and decay rates in the assumptions are extended in [GMS22] to functions satisfying an integrability condition. This improves a similar result in [DM12]. Even in this abstract situation, we obtain explicit estimates of the observability constant.

### 2.2 Conditions for observability

In this section, we develop sufficient conditions for final-state observability estimates in terms of an uncertainty principle and a dissipation estimate in Banach spaces. Before we get to the statements, we summarize the common assumptions in the Hypothesis 2.2 below. As motivation, we start by considering the observation system

$$\begin{aligned} \dot{x}(t) &= -Ax(t) && \text{for } t \in (0, T], \\ y(t) &= Cx(t) && \text{for } t \in [0, T], \\ x(0) &= x_0 \in X, \end{aligned} \qquad \Sigma_T(A, -, C)$$

where  $X, Y$  are Banach spaces,  $A$  is a densely defined closed linear operator in  $X$  such that  $-A$  generates a  $C_0$ -semigroup  $(S_t)_{t \geq 0}$ ,  $C \in \mathcal{L}(X, Y)$ , and  $T > 0$ . We are interested in a final-state observability estimate for the system  $\Sigma_T(A, -, C)$  (cf. Definition 1.13), i.e.,

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the existence of an observability constant  $C_{\text{obs}} \geq 0$  such that

$$\forall x \in X : \quad \|S_T x\|_X \leq C_{\text{obs}} \int_0^T \|CS_\tau x\|_Y \, d\tau. \quad (\text{OBS})$$

Observe that the formulation of (OBS) just requires a semigroup with suitable measurability properties, namely measurability of  $t \mapsto \|CS_t x\|_Y$ .

**Remark 2.1.** We can use Hölder's inequality to turn (OBS) into a final-state observability estimate with respect to the  $L_r$ -norm for  $r \in (1, \infty]$ . Indeed, applying (OBS) for the rescaled semigroup  $\tilde{S}_t := e^{-\omega t} S_t$ ,  $t \geq 0$ , yields for  $x \in X$  the estimate

$$\|S_T x\|_X = e^{\omega T} \|\tilde{S}_T x\|_X \leq e^{\omega T} \tilde{C}_{\text{obs}} \int_0^T e^{-\omega \tau} \|CS_\tau x\|_Y \, d\tau, \quad (2.5)$$

for some  $\tilde{C}_{\text{obs}} \geq 0$ . Using Hölder's inequality, we obtain

$$\|S_T x\|_X \leq \begin{cases} C_{\text{obs}} \left( \int_0^T \|CS_t x\|_Y^r \, dt \right)^{1/r} & \text{if } r \in (1, \infty), \\ C_{\text{obs}} \operatorname{ess\,sup}_{t \in [0, T]} \|CS_t x\|_Y & \text{if } r = \infty, \end{cases}$$

where

$$C_{\text{obs}} = \tilde{C}_{\text{obs}} \begin{cases} \left( \frac{e^{\omega r' T} - 1}{\omega r'} \right)^{\frac{1}{r'}} & \text{for } \omega \neq 0 \\ T^{1/r'} & \text{for } \omega = 0, \end{cases}$$

with  $r' \in [1, \infty)$  such that  $1/r + 1/r' = 1$ . Further, this clearly implies the upper bound

$$C_{\text{obs}} \leq \tilde{C}_{\text{obs}} e^{\omega_+ T} T^{1/r'},$$

with  $\omega_+ := \max\{\omega, 0\}$ . ◇

For most parts of this section, we assume the hypothesis below.

**Hypothesis 2.2.** Let  $T > 0$ ,  $X$  and  $Y$  be Banach spaces, and  $C \in \mathcal{L}(X, Y)$ . Further, let  $(S_t)_{t \geq 0}$  be a semigroup on  $X$ ,  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|S_t\|_{\mathcal{L}(X)} \leq M e^{\omega t}$  for all  $t \geq 0$ , and assume that for all  $x \in X$  the mapping  $t \mapsto \|CS_t x\|_Y$  is measurable.

Consider measurable and monotonically increasing functions  $f, g, h: (0, \infty) \rightarrow (0, \infty)$ , and assume that there is a family of bounded linear operators  $(P_\lambda)_{\lambda > 0}$  in  $\mathcal{L}(X)$ , and constants  $C_1 > 0$ ,  $C_2 \geq 1$  such that

$$\forall \lambda > 0, x \in X : \quad \|P_\lambda x\|_X \leq C_1 e^{f(\lambda)} \|CP_\lambda x\|_Y, \quad (\text{UPgen})$$

and

$$\forall \lambda > 0, t \in (0, T], x \in X : \quad \|(\text{Id} - P_\lambda)S_t x\|_X \leq C_2 e^{-g(\lambda)h(t) + \omega t} \|x\|_X. \quad (\text{DISSgen})$$

The first part in Hypothesis 2.2 sets a system with observation in Banach spaces, similar to  $\Sigma_T(A, -, C)$ . However, we only assume that  $t \mapsto \|CS_t x\|_Y$  is measurable, which means

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that the semigroup  $(S_t)_{t \geq 0}$  must not be strongly continuous. The second part of Hypothesis 2.2 introduces two inequalities that are generalizations of the estimates (UP) and (DISS) considered in Section 2.1. In the next subsections we impose additional assumptions on the functions  $f, g, h$ , e.g., by considering power functions, to obtain a final-state observability estimate, i.e., we prove

$$(\text{UPgen}) + (\text{DISSgen}) \implies (\text{OBS}).$$

### 2.2.1 Weak observability estimate

The basis of the results in the next subsections is the following weak observability estimate with general  $\alpha \geq 0$ .

**Lemma 2.3.** *Assume Hypothesis 2.2. Then for all  $x \in X$ ,  $t \in (0, T]$ ,  $\lambda > 0$ , and  $q \in (0, 1)$*

$$\|S_t x\|_X \leq C_{\text{obs}}(t, \lambda) \int_{qt}^t \|C S_\tau x\|_Y \, d\tau + \alpha(t, \lambda) \|x\|_X, \quad (2.6)$$

with constants satisfying

$$C_{\text{obs}}(t, \lambda) = \frac{MC_1}{(1-q)t} e^{f(\lambda) + \max\{\omega, 0\}t}, \quad (2.7)$$

$$\alpha(t, \lambda) = M \left( C_1 \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2 e^{-g(\lambda)h(qt) + f(\lambda) + \omega t}. \quad (2.8)$$

*Proof.* W.l.o.g. we may assume that the semigroup  $(S_t)_{t \geq 0}$  is bounded, i.e.,  $\omega = 0$ . Indeed, for  $\omega \in \mathbb{R}$  the rescaled semigroup  $\tilde{S}_t := e^{-\omega t} S_t$ ,  $t \geq 0$ , is bounded and applying Lemma 2.3 for bounded semigroups yields

$$\begin{aligned} \|S_t x\|_X &= e^{\omega t} \|\tilde{S}_t x\|_X \leq e^{\omega t} \tilde{C}_{\text{obs}}(t, \lambda) \int_{qt}^t \|C \tilde{S}_\tau x\|_Y \, d\tau + e^{\omega t} \tilde{\alpha}(t, \lambda) \|x\|_X \\ &\leq e^{\max\{\omega, 0\}t} \tilde{C}_{\text{obs}}(t, \lambda) \int_{qt}^t \|C S_\tau x\|_Y \, d\tau + e^{\omega t} \tilde{\alpha}(t, \lambda) \|x\|_X, \end{aligned}$$

for  $\tilde{C}_{\text{obs}}(t, \lambda) = MC_1 e^{f(\lambda)} / (1-q)t$  and  $\tilde{\alpha}(t, \lambda) = M \left( C_1 \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2 e^{-g(\lambda)h(qt) + f(\lambda)}$ . Thus, it suffices to prove the lemma for bounded semigroups.

Let  $t \in (0, T]$ ,  $\tau \in (0, t]$ ,  $\lambda > 0$ , and  $x \in X$ . Using (UPgen) and (DISSgen), we obtain

$$\begin{aligned} \|S_\tau x\| &\leq \|P_\lambda S_\tau x\|_X + \|(\text{Id} - P_\lambda) S_\tau x\|_X \\ &\leq C_1 e^{f(\lambda)} \|C P_\lambda S_\tau x\|_Y + \|(\text{Id} - P_\lambda) S_\tau x\|_X \\ &\leq C_1 e^{f(\lambda)} \|C S_\tau x\|_Y + C_1 e^{f(\lambda)} \|C\|_{\mathcal{L}(X, Y)} \|(\text{Id} - P_\lambda) S_\tau x\|_X + \|(\text{Id} - P_\lambda) S_\tau x\|_X \\ &\leq C_1 e^{f(\lambda)} \|C S_\tau x\|_Y + \left( C_1 e^{f(\lambda)} \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2 e^{-g(\lambda)h(\tau)} \|x\|_X \\ &\leq C_1 e^{f(\lambda)} \|C S_\tau x\|_Y + \left( C_1 \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2 e^{-g(\lambda)h(\tau) + f(\lambda)} \|x\|_X. \end{aligned} \quad (2.9)$$

Since  $(S_t)_{t \geq 0}$  is a semigroup, we get for  $\tau \leq t$

$$\|S_t x\|_X = \|S_{t-\tau} S_\tau x\|_X \leq M \|S_\tau x\|_X.$$

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Note that  $\omega = 0$  by assumption. Since  $t \mapsto \|CS_t x\|_Y$  is measurable and  $h$  is monotonically increasing, we obtain by integrating (2.9) with respect to  $\tau \in [qt, t]$ , for some  $q \in (0, 1)$ , that

$$\begin{aligned} \frac{(1-q)t}{M} \|S_t x\|_X &\leq C_1 e^{f(\lambda)} \int_{qt}^t \|CS_\tau x\|_Y d\tau + (C_1 \|C\|_{\mathcal{L}(X,Y)} + 1) C_2 \int_{qt}^t e^{-g(\lambda)h(\tau)+f(\lambda)} d\tau \|x\|_X \\ &\leq C_1 e^{f(\lambda)} \int_{qt}^t \|CS_\tau x\|_Y d\tau + (C_1 \|C\|_{\mathcal{L}(X,Y)} + 1) C_2 (1-q) t e^{-g(\lambda)h(qt)+f(\lambda)} \|x\|_X. \end{aligned}$$

The claim follows by multiplying both sides by  $M/(1-q)t$ .  $\square$

Provided that  $\alpha(t, \lambda) \in [0, 1)$ , the estimate (2.6) obtained in Lemma 2.3 corresponds to a weak observability estimate (cf. Definition 1.19), when the integration is performed over the whole interval  $[0, T]$ . In the following we impose suitable assumptions on the functions  $f, g$ , and  $h$  for which (2.6) implies the final-state observability estimate

$$\forall x \in X : \quad \|S_T x\|_X \leq C_{\text{obs}} \int_0^T \|CS_\tau x\|_Y d\tau, \quad (2.10)$$

for  $T > 0$  and some constant  $C_{\text{obs}} \geq 0$ . For this, it will be important to ensure that  $\alpha(t, \lambda)$  tends to zero as  $\lambda \rightarrow \infty$ . We use an iterative argument to exploit the decay of  $\alpha(t, \lambda)$  in the proof of the boundedness of the observability constant  $C_{\text{obs}}$ .

### 2.2.2 Telescoping series method

Let us give a plain iterative argument, developed in [Mil10], which turns the weak observability estimate (2.6) into a final-state observability estimate. Due to the telescoping series argument involved, this approach is referred to as the telescoping series method. We present this well-established approach for completeness and to offer a contrast to our main results in the upcoming two subsections.

For simplicity, we restrict ourselves to power functions  $f, g, h$ . However, the telescoping series method can be performed for a wider class of functions (see [DM12, Section 6]). For improvements and further developments of this method, see [Mil13, BP18], and for an approach to use this method for time-dependent observation operators or more general non-autonomous systems, see [PW13, WZ17, BGST22a].

**Proposition 2.4.** *Assume Hypothesis 2.2. For  $s_2 > s_1 > 0$  and  $\gamma, c_1, c_2 > 0$ , let  $f(\lambda) = c_1 \lambda^{s_1}$ ,  $g(\lambda) = c_2 \lambda^{s_2}$  for  $\lambda > 0$ , and  $h(t) = t^\gamma$  for  $t > 0$ . Then for all  $T' > 0$  there are  $K_1, K_2 > 0$  such that for all  $T \in (0, T']$  and  $x \in X$*

$$\|S_T x\|_X \leq C_{\text{obs}} \int_0^T \|CS_t x\|_Y dt, \quad \text{with} \quad C_{\text{obs}} = \frac{K_1}{T} \exp\left(\frac{K_2}{T^{\frac{s_1 \gamma}{s_2 - s_1}}} + \omega_+ T\right),$$

where  $\omega_+ = \max\{\omega, 0\}$ .

*Proof.* Using the rescaling  $\lambda \mapsto (\lambda/c_1)^{1/s_1}$  we can assume  $f(\lambda) = \lambda$  for all  $\lambda > 0$  by choosing  $g(\lambda) = c\lambda^s$ ,  $\lambda > 0$ , where  $s := s_2/s_1 > 1$  and  $c := c_2/c_1^s > 0$ . Furthermore,

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w.l.o.g. we may assume that the semigroup  $(S_t)_{t \geq 0}$  is bounded, i.e.,  $\omega = 0$ . Indeed, the rescaled semigroup  $\tilde{S}_t := e^{-\omega t} S_t$ ,  $t \geq 0$ , is bounded and applying Proposition 2.4 for bounded semigroups yields

$$\forall x \in X : \quad \|\tilde{S}_T x\|_X \leq \tilde{C}_{\text{obs}} \int_0^T \|C \tilde{S}_\tau x\|_Y \, d\tau,$$

and therefore, as in (2.5), we obtain

$$\|S_T x\|_X = e^{\omega T} \|\tilde{S}_T x\|_X \leq e^{\omega T} \tilde{C}_{\text{obs}} \int_0^T e^{-\omega \tau} \|C S_\tau x\|_Y \, d\tau \leq e^{\omega T} \tilde{C}_{\text{obs}} \int_0^T \|C S_\tau x\|_Y \, d\tau.$$

Thus, it suffices to prove the theorem for bounded semigroups.

Let  $T' > 0$  and  $q \in (0, 1)$ . By Lemma 2.3 we have for all  $t \in (0, T']$ ,  $\lambda > 0$ , and  $x \in X$

$$\frac{1}{C_{\text{obs}}(t, \lambda)} \|S_t x\|_X - \frac{\alpha(t, \lambda)}{C_{\text{obs}}(t, \lambda)} \|x\|_X \leq \int_{qt}^t \|C S_\tau x\|_Y \, d\tau \quad (2.11)$$

with

$$C_{\text{obs}}(t, \lambda) = \frac{MC_1}{(1-q)t} e^{\lambda} \quad \text{and} \quad \alpha(t, \lambda) = M \left( C_1 \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2 e^{-c\lambda^s (qt)^\gamma + \lambda}.$$

We abbreviate  $K := M \left( C_1 \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2 \geq 1$  and define  $\lambda$  as a function of  $t$  by

$$\lambda(t) := K_{T'} t^{-\frac{\gamma}{s-1}} \quad \text{for } t \in (0, T'],$$

where  $K_{T'} > 0$  is large enough such that

$$\ln(K/q) T'^{\frac{\gamma}{s-1}} + K_{T'} q^{-\frac{\gamma}{s-1}} \leq c(K_{T'})^s q^\gamma. \quad (2.12)$$

Note that this is possible since  $s > 1$ . With this choice we have

$$\begin{aligned} c\lambda(t)^s (qt)^\gamma &= c(K_{T'})^s q^\gamma t^{-\frac{s\gamma}{s-1} + \gamma} = c(K_{T'})^s q^\gamma t^{-\frac{\gamma}{s-1}} \geq \left( \ln(K/q) T'^{\frac{\gamma}{s-1}} + K_{T'} q^{-\frac{\gamma}{s-1}} \right) t^{-\frac{\gamma}{s-1}} \\ &\geq \ln(K/q) + K_{T'} q^{-\frac{\gamma}{s-1}} t^{-\frac{\gamma}{s-1}} \\ &= \ln(K/q) + \lambda(qt), \end{aligned}$$

where for the last inequality we used that  $t \in (0, T']$ . This gives

$$\frac{\alpha(t, \lambda(t))}{C_{\text{obs}}(t, \lambda(t))} = K \frac{(1-q)t}{MC_1} e^{-c\lambda(t)^s (qt)^\gamma} \leq \frac{(1-q)qt}{MC_1} e^{-\lambda(qt)} = \frac{1}{C_{\text{obs}}(qt, \lambda(qt))}.$$

Using this estimate in (2.11), yields for all  $t \in (0, T']$

$$\frac{1}{C_{\text{obs}}(t, \lambda(t))} \|S_t x\|_X - \frac{1}{C_{\text{obs}}(qt, \lambda(qt))} \|x\|_X \leq \int_{qt}^t \|C S_\tau x\|_Y \, d\tau. \quad (2.13)$$

Let  $T \in (0, T']$ . We set  $T_0 := T$ , and for  $k \in \mathbb{N}_0$  let  $\tau_k := q^k(1-q)T$  and  $T_{k+1} := T_k - \tau_k$ . By this choice, we have  $T_k = q^k T$  and  $\sum_{k \in \mathbb{N}_0} \tau_k = T$ . For  $y \in X$  and  $k \in \mathbb{N}_0$  we apply

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(2.13) to  $x = S_{T_{k+1}}y$  and  $t = \tau_k$  to obtain

$$\frac{1}{C_{\text{obs}}(\tau_k, \lambda(\tau_k))} \|S_{T_k}y\|_X - \frac{1}{C_{\text{obs}}(\tau_{k+1}, \lambda(\tau_{k+1}))} \|S_{T_{k+1}}y\|_X \leq \int_{T_{k+1}}^{T_k} \|CS_\tau y\|_Y d\tau.$$

We add up these inequalities and use that the left-hand side is a telescopic sum to obtain

$$\frac{1}{C_{\text{obs}}(\tau_0, \lambda(\tau_0))} \|S_T y\|_X - \frac{1}{C_{\text{obs}}(\tau_k, \lambda(\tau_k))} \|S_{T_k} y\|_X \leq \int_{T_k}^T \|CS_\tau x\|_Y d\tau.$$

Note that for the right hand side we used that  $\cup_{k \in \mathbb{N}_0} (T_{k+1}, T_k]$  is a disjoint partition of  $(0, T]$ . Since the norm of the semigroup  $\|S_t x\|_X$  remains bounded for all  $t \in (0, T']$ , and using that  $C_{\text{obs}}(\tau_k, \lambda(\tau_k)) \rightarrow \infty$  as  $k \rightarrow \infty$ , we obtain by taking the limit  $k \rightarrow \infty$  that

$$\frac{1}{C_{\text{obs}}(\tau_0, \lambda(\tau_0))} \|S_T y\|_X \leq \int_0^T \|CS_\tau x\|_Y d\tau.$$

The claim now follows by observing that

$$C_{\text{obs}}(\tau_0, \lambda(\tau_0)) = \frac{MC_1}{(1-q)^2 T} \exp\left(K_{T'} \left(\frac{1}{(1-q)T}\right)^{\frac{\gamma}{s-1}}\right),$$

where  $K_{T'} > 0$  such that (2.12) holds. □

The key part to prove Proposition 2.4 is to find a suitable function  $\lambda(t)$  which satisfies the estimate

$$\forall t \in (0, T'] : \frac{\alpha(t, \lambda(t))}{C_{\text{obs}}(t, \lambda(t))} \leq \frac{1}{C_{\text{obs}}(qt, \lambda(qt))} \quad (2.14)$$

for some  $q \in (0, 1)$ . With this, one obtains the estimate (2.13), which can be used in a telescoping series. Note that our choice of  $\lambda(t)$  depends due to (2.12) on  $T'$  and the different model parameters  $s_1, s_2, \gamma, c_1, c_2, C_1, C_2$  and  $\|C\|_{\mathcal{L}(X, Y)}$ . For an explicit description of  $C_{\text{obs}}$  with respect to these parameters, we refer to the following Theorem 2.6 and Theorem 2.7. The estimate on  $C_{\text{obs}}$  given in Proposition 2.4 shows optimal behavior in the limit  $T \rightarrow 0$ . In fact, the exponential blow-up for  $T \rightarrow 0$  has to occur, e.g., for the controlled heat equation on bounded open subsets of  $\mathbb{R}^d$ . This will be discussed in Section 3.2.4. In the situation of Proposition 2.4 we obtain, by using that  $(S_t)_{t \geq 0}$  is a semigroup, for all  $T > T'$  and  $x \in X$  that

$$\|S_T x\|_X = \|S_{T-T'} S_{T'} x\|_X \leq M e^{\omega T} \|S_{T'} x\|_X.$$

This yields

$$\|S_T x\|_X \leq M e^{\omega T} \|S_{T'} x\|_X \leq M e^{\omega T} C_{\text{obs}} \int_0^T \|CS_\tau x\|_Y d\tau, \quad (2.15)$$

where  $C_{\text{obs}}$  is the observability constant for  $T = T'$ . This shows that  $C_{\text{obs}}$  does not grow more than the semigroup as  $T \rightarrow \infty$ . However, the estimation in (2.15) does not comply with the optimal behavior as  $T \rightarrow \infty$  (see also Section 3.2.4).

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### 2.2.3 Explicit estimate on the observability constant

The telescopic series method yields an observability estimate which shows optimal behavior for small times  $T$ . However, the requirements needed for the telescopic series hinder tracking the dependencies of the observability constant concerning the other model parameters. Therefore, we now switch to a different iteration approach employed in [TT11, NTTV20a, GST20]. Here we will present the results of [GST20]. For this, we again start with the estimate obtained in Lemma 2.3 and prove upper bounds of the form

$$\forall k \in \mathbb{N}_0 : \quad C_{\text{obs}}(\tau_k, \lambda_k) \leq a_1 e^{a_2 q^k} \quad \text{and} \quad \alpha(\tau_k, \lambda_k) \leq b_1 e^{-b_2 q^k},$$

for suitable constants, and sequences  $(\tau_k)_{k \in \mathbb{N}_0}$  and  $(\lambda_k)_{k \in \mathbb{N}_0}$ . These upper bounds carry over to explicit estimates for the observability constant. The foundation for this is laid down in the following lemma.

**Lemma 2.5.** *Let  $p \geq 2$ ,  $a_1 \geq 0$ ,  $b_1 > 1$ , and  $a_2, b_2 > 0$  such that  $d := \frac{b_2}{p-1} - a_2 \geq 1$ . Further, let  $(S_k)_{k \in \mathbb{N}_0}$  in  $[0, \infty)$  be a bounded sequence satisfying*

$$\forall k \in \mathbb{N}_0 : \quad S_k \leq a_1 e^{a_2 p^k} + b_1 e^{-b_2 p^k} S_{k+1}.$$

Then

$$S_0 \leq a_1 e^{a_2} + a_1 e^{\frac{b_2}{p-1}} \left( \frac{2 \ln(b_1)}{d \ln(2)e} \right)^{\frac{\ln(b_1)}{\ln(2)}}. \quad (2.16)$$

*Proof.* With the short hand notation  $A_k := a_1 e^{a_2 p^k}$  and  $B_k := b_1 e^{-b_2 p^k}$  we get by assumption  $S_k \leq A_k + B_k S_{k+1}$ . This inequality can be iterated, and we obtain after two steps

$$S_0 \leq A_0 + B_0 (A_1 + B_1 S_2) = A_0 + A_1 B_0 + B_0 B_1 S_2.$$

After  $N + 1$  steps of this type, we obtain

$$S_0 \leq A_0 + \sum_{k=1}^N A_k \prod_{l=0}^{k-1} B_l + S_{N+1} \prod_{k=0}^N B_k. \quad (2.17)$$

Let us show that the right-hand side in (2.17) converges as  $N \rightarrow \infty$ . Since  $(S_k)_{k \in \mathbb{N}_0}$  is bounded and  $B_k \rightarrow 0$  for  $k \rightarrow \infty$ , it is clear that

$$S_{N+1} \prod_{k=0}^N B_k \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Moreover, the middle summand of the right-hand side of (2.17) satisfies

$$\sum_{k=1}^N A_k \prod_{l=0}^{k-1} B_l = a_1 \sum_{k=1}^N b_1^k e^{-b_2 \left( \frac{p^k - 1}{p-1} \right) + a_2 p^k} = a_1 e^{\frac{b_2}{p-1}} \sum_{k=1}^N b_1^k e^{-d p^k}. \quad (2.18)$$

Since  $d = \frac{b_2}{p-1} - a_2 \geq 1$ , this shows that the right-hand side of (2.17) converges as  $N$  tends

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to infinity. Hence, with (2.17) and (2.18) we obtain

$$S_0 \leq a_1 e^{a_2} + a_1 e^{\frac{b_2}{p-1}} \sum_{k=1}^{\infty} b_1^k e^{-dp^k}. \quad (2.19)$$

To prove the estimate (2.16), we compute

$$\sum_{k=1}^{\infty} b_1^k e^{-dp^k} \leq \sup_{s \geq 1} b_1^s e^{-\frac{d}{2} p^s} \sum_{k=1}^{\infty} e^{-\frac{d}{2} p^k} \leq \sup_{s \geq 1} b_1^s e^{-\frac{d}{2} 2^s} \sum_{k=1}^{\infty} e^{-\frac{d}{2} p^k} = \left( \frac{2 \ln(b_1)}{d \ln(2)e} \right)^{\frac{\ln(b_1)}{\ln(2)}} \sum_{k=1}^{\infty} e^{-\frac{d}{2} p^k}, \quad (2.20)$$

where the last identity follows from elementary calculus. Since  $d \geq 1$ , we obtain by using that  $p^k \geq 2k$

$$\sum_{k=1}^{\infty} e^{-\frac{d}{2} p^k} \leq \sum_{k=1}^{\infty} e^{-k} = \frac{e^{-1}}{1 - e^{-1}} \leq 1. \quad (2.21)$$

The claim now follows by using the estimates (2.20) and (2.21) in (2.19).  $\square$

Let us now state the main theorem of [GST20], in the slight generalization presented in [BGST22b], which gives an explicit estimate on the observability constant for power functions  $f, g, h$  in the uncertainty principle (UPgen) and dissipation estimate (DISSgen).

**Theorem 2.6.** *Assume Hypothesis 2.2. For  $s_2 > s_1 > 0$  and  $\gamma, c_1, c_2 > 0$ , let  $f(\lambda) = c_1 \lambda^{s_1}$ ,  $g(\lambda) = c_2 \lambda^{s_2}$  for  $\lambda > 0$ , and  $h(t) = t^\gamma$  for  $t > 0$ . Then we have for all  $x \in X$  and  $T > 0$*

$$\|S_T x\|_X \leq C_{\text{obs}} \int_0^T \|CS_t x\|_Y dt, \quad \text{with } C_{\text{obs}} = \frac{K_1}{T} \exp\left(\frac{K_2}{T^{\frac{s_1 \gamma}{s_2 - s_1}}} + \omega_+ T\right),$$

where

$$K_1 = 8e^4 M C_1 \left( M(C_1 \|C\|_{\mathcal{L}(X,Y)} + 1) C_2 \right)^{\frac{8}{e \ln(2)}},$$

$$K_2 = 4 \left( \frac{c_1^{s_2}}{c_2^{s_1}} \right)^{\frac{1}{s_2 - s_1}} 2^{\frac{(\gamma+1)s_1}{s_2 - s_1}} \left( 1 + \frac{s_2}{s_2 - s_1} \right).$$

The proof given here is close to [GST20], with the difference that the iteration argument is shifted to Lemma 2.5.

*Proof of Theorem 2.6.* By using the rescaling  $\lambda \mapsto (\lambda/c_1)^{1/s_1}$ , we can assume  $f(\lambda) = \lambda$  for all  $\lambda > 0$  by choosing  $g(\lambda) = c\lambda^s$ ,  $\lambda > 0$ , where  $s := s_2/s_1 > 1$  and  $c := c_2/c_1^s > 0$ . Furthermore, as in the proof of Proposition 2.4, we may assume that the semigroup  $(S_t)_{t \geq 0}$  is bounded, i.e.,  $\omega = 0$ . Set  $K := 2^{\frac{8}{e}} M(C_1 \|C\|_{\mathcal{L}(X,Y)} + 1) C_2$  and let  $T > 0$ . For  $k \in \mathbb{N}_0$  we define  $T_k := 2^{-k} T$ . With this we have  $T_0 = T$  and  $\tau_k := T_k - T_{k+1} = 2^{-(k+1)} T$ . Moreover, let

$$\nu := \begin{cases} \nu_0 & \text{if } T \leq T', \\ \nu_0 \left( \frac{T}{T'} \right)^{\frac{\gamma}{s}} & \text{if } T > T', \end{cases} \quad \text{and} \quad \lambda_0 := \begin{cases} K_0 \left( \frac{T'}{T} \right)^{\frac{\gamma}{s-1}} & \text{if } T \leq T', \\ K_0 & \text{if } T > T', \end{cases}$$

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where

$$\nu_0 := 2^{\frac{\gamma+1}{s-1}}, \quad K_0 := \frac{2 \ln(K)}{e \ln(2)}, \quad T' := \left( \frac{2^{\gamma+1} \nu_0^s}{c K_0^{s-1}} \right)^{\frac{1}{\gamma}}.$$

With this notation, we define  $\lambda_k := \lambda_0 \nu^k$  for  $k \in \mathbb{N}$ . By Lemma 2.3, the estimate (2.6) holds. For  $k \in \mathbb{N}_0$  and  $y \in X$ , we apply this estimate for  $t = \tau_k$ ,  $q = \frac{1}{2}$ ,  $\lambda = \lambda_k$  and  $x = S_{T_{k+1}} y$  to obtain

$$\|S_{T_k} y\|_X \leq C_{\text{obs}}(\tau_k, \lambda_k) \int_0^T \|CS_{\tau} x\|_Y d\tau + \alpha(\tau_k, \lambda_k) \|S_{T_{k+1}} y\|_X, \quad (2.22)$$

with  $C_{\text{obs}}(\tau_k, \lambda_k)$  and  $\alpha(\tau_k, \lambda_k)$  as in (2.7) and (2.8). Note that we performed integration on  $(0, T]$  instead of  $(\tau_k/2, \tau_k]$ . Next we give estimates for  $C_{\text{obs}}(\tau_k, \lambda_k)$  and  $\alpha(\tau_k, \lambda_k)$  with which we can apply Lemma 2.5. Since  $\lambda_0 \geq 1$  and  $2\nu \leq \nu^s/2^\gamma$ , we obtain from (2.7) that

$$C_{\text{obs}}(\tau_k, \lambda_k) = \frac{4MC_1}{T} 2^k e^{\lambda_0 \nu^k} = \frac{4MC_1}{T} e^{\ln(2)k + \lambda_0 \nu^k} \leq \frac{4MC_1}{T} e^{\lambda_0 (2\nu)^k} \leq \frac{4MC_1}{T} e^{\lambda_0 \left(\frac{\nu^s}{2^\gamma}\right)^k}. \quad (2.23)$$

Using that  $\nu \leq \nu^s/2^\gamma$ , the estimate (2.8) implies

$$\alpha(\tau_k, \lambda_k) \leq K e^{-c\lambda_0^s \nu^{sk} \left(\frac{T}{2^{k+2}}\right)^\gamma + \lambda_0 \nu^k} \leq K e^{-(c\lambda_0^s \left(\frac{T}{4}\right)^\gamma - \lambda_0) \left(\frac{\nu^s}{2^\gamma}\right)^k}. \quad (2.24)$$

Using the estimates (2.23) and (2.24) in the weak observability estimate (2.22), we obtain

$$\|S_{T_k} y\|_X \leq \frac{4MC_1}{T} e^{\lambda_0 \left(\frac{\nu^s}{2^\gamma}\right)^k} \int_0^T \|CS_{\tau} x\|_Y d\tau + K e^{-(c\lambda_0^s \left(\frac{T}{4}\right)^\gamma - \lambda_0) \left(\frac{\nu^s}{2^\gamma}\right)^k} \|S_{T_{k+1}} y\|_X.$$

This inequality fits the one in Lemma 2.5 with the assignments

$$\begin{aligned} S_k &:= \|S_{T_k} y\|_X, & a_1 &:= \frac{4MC_1}{T} \int_0^T \|CS_{\tau} x\|_Y d\tau, & a_2 &:= \lambda_0, \\ p &:= \frac{\nu^s}{2^\gamma}, & b_1 &:= K, & b_2 &:= c\lambda_0^s \left(\frac{T}{4}\right)^\gamma - \lambda_0. \end{aligned} \quad (2.25)$$

Let us check, that these quantities fulfill the assumptions of Lemma 2.5. It is clear that  $S_k, a_1 \geq 0$ , and (by definition of  $K, \nu$ , and  $\lambda_0$ ) we have  $a_2 > 0$ ,  $p \geq 2$ , and  $b_1 \geq 1$ . Furthermore, in both cases  $T \leq T'$  and  $T > T'$ , the constants  $\nu$  and  $\lambda_0$  satisfy the equality

$$cT^\gamma \lambda_0^s = 2^{\gamma+1} \lambda_0 \nu^s. \quad (2.26)$$

This ensures that  $d := \frac{b_2}{p-1} - a_2 \geq 1$ . Indeed, using (2.26) and  $p = \nu^s/2^\gamma \geq 2$  we find

$$d = \frac{b_2}{p-1} - a_2 = \frac{c\lambda_0^s (T/4)^\gamma - \lambda_0 \nu^s/2^\gamma}{\nu^s/2^\gamma - 1} = \lambda_0 \frac{\nu^s/2^\gamma}{\nu^s/2^\gamma - 1} \geq \lambda_0 \geq K_0 \geq 1. \quad (2.27)$$

where the last inequality holds since  $\ln(K) \geq \frac{e \ln(2)}{2}$ . Since  $b_2 \geq d$ , we obtain by (2.27)

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that  $b_2$  is positive. Therefore, we can apply Lemma 2.5 and obtain

$$S_0 = \|S_T y\|_X \leq \underbrace{\frac{4MC_1}{T} \left( e^{a_2} + e^{\frac{b_2}{p-1}} \left( \frac{2 \ln(b_1)}{d \ln(2)e} \right)^{\frac{\ln(b_1)}{\ln(2)}} \right)}_{=: \tilde{C}_{\text{obs}}} \int_0^T \|CS_\tau y\|_Y d\tau, \quad (2.28)$$

with constants as in (2.25). By (2.27) we have  $d \geq K_0$  which gives

$$\frac{2 \ln(b_1)}{d \ln(2)e} \leq \frac{2 \ln(K)}{K_0 \ln(2)e} = 1.$$

Hence, we can estimate the constant  $\tilde{C}_{\text{obs}}$  from (2.28) by

$$\tilde{C}_{\text{obs}} \leq \frac{4MC_1}{T} \left( e^{a_2} + e^{\frac{b_2}{p-1}} \right). \quad (2.29)$$

Next we calculate, using (2.26),

$$\frac{b_2}{p-1} = \frac{c(T/4)^\gamma \lambda_0^s - \lambda_0}{\nu^s/2^\gamma - 1} \leq \frac{c(T/4)^\gamma \lambda_0^s}{\nu^s/2^\gamma - 1} = \lambda_0 \frac{2\nu^s/2^\gamma}{\nu^s/2^\gamma - 1} \leq 4\lambda_0,$$

where for the last inequality we used that  $\nu^s/2^\gamma \geq 2$ . Since  $a_2 = \lambda_0 \leq 4\lambda_0$ , we obtain from (2.29) that

$$\tilde{C}_{\text{obs}} \leq \frac{8MC_1}{T} e^{4\lambda_0}. \quad (2.30)$$

By our choice of  $\lambda_0$ , we have

$$\lambda_0 \leq K_0 + K_0 \left( \frac{T'}{T} \right)^{\frac{\gamma}{s-1}} = K_0 + \left( \frac{2^{\gamma+1} \nu_0^s}{cT^\gamma} \right)^{\frac{1}{s-1}} = \frac{2 \ln(K)}{e \ln(2)} + \left( \frac{2^{\gamma+1} \nu_0^s}{cT^\gamma} \right)^{\frac{1}{s-1}}. \quad (2.31)$$

From inequalities (2.30) and (2.31) we conclude

$$\tilde{C}_{\text{obs}} \leq \frac{8MC_1}{T} K^{\frac{s}{e \ln(2)}} \exp \left( 4 \left( \frac{2^{\gamma+1} \nu_0^s}{cT^\gamma} \right)^{\frac{1}{s-1}} \right).$$

Finally, we insert the values of  $K$  and  $\nu_0$  to obtain the assertion.  $\square$

### 2.2.4 Generalized growth and decay rates

Let us expand the class of growth and decay rates  $f, g, h$  in the uncertainty principle and dissipation estimate from power functions to functions satisfying an integrability condition. In contrast to the previous result of this kind from Duyckaerts and Miller in [DM12, Section 6], we use the iterative argument described in Lemma 2.5 instead of the telescoping series method. This enables us to allow more growth and decay rates while still obtaining a precise estimate for the observability constant. The results presented here originate from [GMS22].

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**Theorem 2.7.** *Assume Hypothesis 2.2. Let  $T > 0$ , and assume that  $f$  and  $h$  are strictly monotonically increasing and bijective onto  $(0, \infty)$ ,  $\lambda \mapsto \lambda/g(f^{-1}(\lambda))$  is monotonically decreasing, and that there exists  $\lambda_T \geq 0$  such that*

$$\int_{\lambda_T}^{\infty} h^{-1} \left( 4 \frac{\lambda}{g(f^{-1}(\lambda))} \right) \frac{1}{\lambda} d\lambda \leq \frac{\ln(2)}{4} \min\{1, T\}. \quad (2.32)$$

Then we have for all  $x \in X$

$$\|S_T x\|_X \leq C_{\text{obs}} \int_0^T \|CS_\tau x\|_Y d\tau, \quad \text{with } C_{\text{obs}} \leq \frac{K_1}{T} e^{6\lambda_T + \omega_+ T}, \quad (2.33)$$

where  $\omega_+ = \max\{\omega, 0\}$  and  $K_1 = 8e^3 M C_1 \left( M \left( C_1 \|C\|_{\mathcal{L}(X,Y)} + 1 \right) C_2 \right)^{\frac{6}{e \ln(2)}}$ .

The integrability condition (2.32) sets the class of functions  $f, g, h$  for which our strategy works. Moreover, the quantity  $\lambda_T$  describes the dependence of  $C_{\text{obs}}$  on the final time  $T$ .

**Example 2.8.** We give some examples of functions for which condition (2.32) in Theorem 2.7 applies. For simplicity we choose  $h(t) = t$  for  $T > 0$ , as well as  $f(\lambda) = \lambda$  for  $\lambda > 0$ , and only focus on the function  $g$ . Then, for given  $T \in (0, 1]$ , assumption (2.32) reads as

$$\exists \lambda_T \geq 0 : \int_{\lambda_T}^{\infty} \frac{1}{g(\lambda)} d\lambda \leq \frac{\ln(2)}{4^2} T. \quad (2.34)$$

First note that for (2.34) the function  $1/g$  has to be integrable at infinity. Hence, (2.34) excludes functions  $g$  such that  $\lambda \mapsto g(\lambda)/\lambda$  remains bounded as  $\lambda \rightarrow \infty$ , or, for example,  $g(\lambda) = \lambda \ln(1 + \lambda)$  for  $\lambda > 0$ . In Table 2.1 we give some positive examples and suitable  $\lambda_T$  such that (2.34) holds.  $\diamond$

$g(\lambda)$	$\lambda_T$
$\exp(\lambda)$	$\ln\left(\frac{1}{KT}\right)$
$\lambda^s$	$\left(\frac{1}{(s-1)KT}\right)^{\frac{1}{s-1}}$
$\lambda(\ln(1 + \lambda))^s$	$\exp\left(\left(\frac{1}{(s-1)KT}\right)^{\frac{1}{s-1}}\right)$
$\lambda(\ln(1 + \ln(1 + \lambda)))^s$	$\exp\left(\exp\left(\left(\frac{1}{(s-1)KT}\right)^{\frac{1}{s-1}}\right)\right)$

Table 2.1: Functions  $g : (0, \infty) \rightarrow (0, \infty)$  and suitable values of  $\lambda_T$  such that (2.34) holds. We abbreviate  $K := \ln(2)/4^2$  and choose  $s > 1$ .

**Example 2.9.** Let us pick up the example of anomalous diffusion on a smooth and bounded domain  $\Omega \subset \mathbb{R}^d$ , presented in [DM12, Theorem 7]. As introduced in Section 2.1.1, consider the Dirichlet Laplacian  $\Delta_{\text{Dir}}$  in  $L_2(\Omega)$  observed on an open non-empty region  $E \subset \Omega$ . Let  $(\varphi_k)_{k \in \mathbb{N}}$  be the orthonormal basis of  $L_2(\Omega)$  consisting of eigenvectors

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of  $\Delta_{\text{Dir}}$  with corresponding eigenvalues  $(-\lambda_k)_{k \in \mathbb{N}}$ . Recall that in [LR95, LZ98, JL99] the following estimate for sums of eigenfunctions of  $\Delta_{\text{Dir}}$  was proven:

$$\forall \lambda > 0, x \in L_2(\Omega) : \quad \|P_\lambda x\|_{L_2(\Omega)} \leq C_1 e^{c_1 \sqrt{\lambda}} \|\mathbf{1}_E P_\lambda x\|_{L_2(E)}, \quad (2.35)$$

with uniform  $C_1, c_1 > 0$ , and  $P_\lambda x := \sum_{\lambda_k \leq \lambda} a_k \varphi_k$  for  $x = \sum_{k \in \mathbb{N}} a_k \varphi_k$  being the projection on the spectral subspace of  $\Delta_{\text{Dir}}$  below  $\lambda$ . For  $s > 1$  let  $g: (0, \infty) \rightarrow (0, \infty)$  be any of the functions given in the table of Example 2.8 and consider the anomalous diffusion operator  $-g(\sqrt{-\Delta_{\text{Dir}}})$  defined by the functional calculus of self-adjoint operators. This yields that the  $C_0$ -semigroup  $(S_t)_{t \geq 0}$  generated by  $-g(\sqrt{-\Delta_{\text{Dir}}})$  can be represented as

$$S_t x = \sum_{k \in \mathbb{N}} a_k e^{-g(\sqrt{\lambda_k})t} \varphi_k$$

where  $x = \sum_{k \in \mathbb{N}} a_k \varphi_k$ . Hence, the semigroup  $(S_t)_{t \geq 0}$  satisfies the dissipation estimate

$$\|(\text{Id} - P_\lambda) S_t x\|_{L_2(\Omega)} = \left\| \sum_{\lambda_k > \lambda} a_k e^{-g(\sqrt{\lambda_k})t} \varphi_k \right\|_{L_2(\Omega)} \leq e^{-g(\sqrt{\lambda})t} \|x\|_{L_2(\Omega)}, \quad (2.36)$$

for all  $\lambda > 0, t > 0$ , and  $x = \sum_{k \in \mathbb{N}} a_k \varphi_k \in L_2(\Omega)$ . Therefore, the assumptions of Theorem 2.6 are satisfied with  $C \in \mathcal{L}(L_2(\Omega), L_2(E))$  being the restriction operator, and  $f(\lambda) = c_1 \sqrt{\lambda}$  for  $\lambda > 0$  and  $h(t) = t$  for  $t > 0$ . Theorem 2.6 not only gives an observability estimate for the semigroup associated to  $-g(\sqrt{-\Delta_{\text{Dir}}})$ , but also an observability constant with suitable values of  $\lambda_T$  given in the table of Example 2.8. Note that we improve the result given in [DM12, Theorem 7], since, e.g., the function  $g(\lambda) = \lambda(\ln(1 + \ln(1 + \lambda)))^s$  for  $s \in (1, 2]$  does not satisfy the integrability condition in [DM12].  $\diamond$

Our proof of Theorem 2.7 is again based on the iteration in Lemma 2.5. Remember, that in the proof of Theorem 2.6 we chose sequences  $(\tau_k)_{k \in \mathbb{N}_0}$  and  $(\lambda_k)_{k \in \mathbb{N}_0}$  by  $\tau_k := 2^{-k}T$  and  $\lambda_k := \lambda_0 \nu^k$ , where  $\nu$  depends on  $T$  as well as on the functions  $f, g, h$ . On the contrary, in the following, we choose  $\tau_k$  to be dependent on the functions  $f, g, h$ , which allows setting  $\lambda_k := \lambda_0 2^k$ .

*Proof of Theorem 2.7.* W.l.o.g. we may assume that  $f(\lambda) = \lambda$  for all  $\lambda > 0$ . Indeed, for an arbitrary function  $f: (0, \infty) \rightarrow (0, \infty)$ , satisfying the assumptions, the claim follows by replacing  $g$  by  $g \circ f^{-1}$  and considering the rescaled operators  $\tilde{P}_\lambda := P_{f(\lambda)}$ ,  $\lambda > 0$ , in (UPgen) and (DISSgen). Furthermore, as before, we may assume that the semigroup  $(S_t)_{t \geq 0}$  is bounded, i.e.,  $\omega = 0$ . Set  $K := 2^{\frac{5}{2}} M \left( C_1 \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2$  and let  $\lambda_T \geq 0$  satisfy (2.32). We define

$$\lambda_0 := \frac{2 \ln(K)}{e \ln(2)} + 2\lambda_T,$$

and for  $k \in \mathbb{N}$  we set  $\lambda_k := \lambda_0 2^k$ . Furthermore, for  $k \in \mathbb{N}_0$ , we define

$$\tau_k := \frac{T}{2} e^{-\lambda_k} + 2 \max\{1, T\} h^{-1} \left( 4 \frac{\lambda_k}{g(\lambda_k)} \right) > 0,$$

and recursively  $T_{k+1} := T_k - \tau_k$ , where  $T_0 := T$ . Next, we show that  $\tau_k, T_k \in (0, T]$  for all  $k \in \mathbb{N}_0$ . For this, note that since  $C_2 \geq 1$  by assumption, we have  $K \geq 2^{\frac{5}{2}}$  and hence

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$\ln(K) \geq \frac{e \ln(2)}{2}$ . This shows that  $\lambda_0 \geq 1$  and hence

$$\sum_{k=0}^{\infty} e^{-\lambda_k} \leq \sum_{k=0}^{\infty} e^{-2^k} \leq 1. \quad (2.37)$$

Since  $\lambda \mapsto \lambda/g(\lambda)$  is monotonically decreasing and  $h^{-1}$  is monotonically increasing, for all  $k \in \mathbb{N}_0$  we have

$$h^{-1} \left( 4 \frac{\lambda_k}{g(\lambda_k)} \right) \leq \int_{k-1}^k h^{-1} \left( 4 \frac{\lambda_0 2^s}{g(\lambda_0 2^s)} \right) ds,$$

and therefore, using (2.37),

$$\begin{aligned} \sum_{k=0}^{\infty} \tau_k &\leq \frac{T}{2} \sum_{k=0}^{\infty} e^{-\lambda_k} + 2 \max\{1, T\} \sum_{k=0}^{\infty} \int_{k-1}^k h^{-1} \left( 4 \frac{\lambda_0 2^s}{g(\lambda_0 2^s)} \right) ds \\ &\leq \frac{T}{2} + 2 \max\{1, T\} \int_{-1}^{\infty} h^{-1} \left( 4 \frac{\lambda_0 2^s}{g(\lambda_0 2^s)} \right) ds \\ &= \frac{T}{2} + \frac{2 \max\{1, T\}}{\ln(2)} \int_{\lambda_0/2}^{\infty} h^{-1} \left( 4 \frac{\lambda}{g(\lambda)} \right) \frac{1}{\lambda} d\lambda. \end{aligned}$$

Since  $\lambda_0/2 \geq \lambda_T$ , by (2.32) we observe

$$\frac{2 \max\{1, T\}}{\ln(2)} \int_{\lambda_0/2}^{\infty} h^{-1} \left( 4 \frac{\lambda}{g(\lambda)} \right) \frac{1}{\lambda} d\lambda \leq \frac{2 \max\{1, T\} \ln(2) \min\{1, T\}}{4 \ln(2)} = \frac{T}{2}.$$

Hence, we have  $\sum_{k=0}^{\infty} \tau_k \leq T$ , which gives  $T_k = T - \sum_{l=0}^{k-1} \tau_l \in (0, T]$ .

By Lemma 2.3 the weak observability estimate (2.6) holds with general  $\alpha \geq 0$ . For  $k \in \mathbb{N}_0$  and  $y \in X$ , we apply this estimate to  $t = \tau_k$ ,  $q = \frac{1}{2}$ ,  $\lambda = \lambda_k$  and  $x = S_{T_{k+1}} y$  to obtain

$$\|S_{T_k} y\|_X \leq C_{\text{obs}}(\tau_k, \lambda_k) \int_0^T \|C S_{\tau} x\|_Y d\tau + \alpha(\tau_k, \lambda_k) \|S_{T_{k+1}} y\|_X, \quad (2.38)$$

with

$$C_{\text{obs}}(\tau_k, \lambda_k) = \frac{2MC_1}{\tau_k} e^{\lambda_k} \quad \text{and} \quad \alpha(\tau_k, \lambda_k) = K e^{-g(\lambda_k)h(\tau_k/2) + \lambda_k}.$$

Note that for technical reasons the constant  $K$  in  $\alpha(\tau_k, \lambda_k)$  is by the factor  $2^{\frac{5}{2}} \geq 1$  larger compared to (2.8), and we performed integration on  $(0, T]$  instead of  $(\tau_k/2, \tau_k]$ . For  $k \in \mathbb{N}_0$ , by definition of  $\tau_k$  and since  $h$  is monotonically increasing, we obtain

$$h(\tau_k/2) \geq h \left( \max\{1, T\} h^{-1} \left( 4 \frac{\lambda_k}{g(\lambda_k)} \right) \right) \geq 4 \frac{\lambda_k}{g(\lambda_k)},$$

and therefore

$$\alpha(\tau_k, \lambda_k) \leq K e^{-g(\lambda_k) \frac{4\lambda_k}{g(\lambda_k)} + \lambda_k} \leq K e^{-3\lambda_k}. \quad (2.39)$$

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Furthermore, for  $k \in \mathbb{N}_0$  we have, by the definition of  $\tau_k$ , that  $\tau_k \geq \frac{T}{2}e^{-\lambda_k}$  and hence

$$C_{\text{obs}}(\tau_k, \lambda_k) = \frac{2MC_1}{\tau_k} e^{\lambda_k} \leq \frac{4MC_1}{T} e^{2\lambda_k}. \quad (2.40)$$

Using the estimates (2.39) and (2.40) in the weak observability estimate (2.38), we obtain

$$\|S_{T_k} y\|_X \leq \frac{4MC_1}{T} e^{2\lambda_k} \int_0^T \|CS_\tau x\|_Y d\tau + Ke^{-3\lambda_k} \|S_{T_{k+1}} y\|_X.$$

This inequality fits the one in Lemma 2.5 with the assignments

$$\begin{aligned} S_k &:= \|S_{T_k} y\|_X, & a_1 &:= \frac{4MC_1}{T} \int_0^T \|CS_\tau x\|_Y d\tau, & a_2 &:= 2\lambda_0, \\ p &:= 2, & b_1 &:= K, & b_2 &:= 3\lambda_0. \end{aligned}$$

Utilizing that  $\lambda_0 \geq 1$ , which implies that  $\frac{b_2}{p-1} - a_2 = 3\lambda_0 - 2\lambda_0 = \lambda_0 \geq 1$ , these quantities clearly fulfil the assumptions of Lemma 2.5. Therefore, we can apply Lemma 2.5 and obtain

$$\|S_T y\|_X = \|S_{T_0} y\|_X \leq \underbrace{\frac{4MC_1}{T} \left( e^{2\lambda_0} + e^{3\lambda_0} \left( \frac{2 \ln(K)}{\lambda_0 \ln(2)e} \right)^{\frac{\ln(K)}{\ln(2)}} \right)}_{=:\tilde{C}_{\text{obs}}} \int_0^T \|CS_\tau y\|_Y d\tau.$$

By definition of  $\lambda_0$ , we obtain

$$\left( \frac{2 \ln(K)}{\lambda_0 \ln(2)e} \right)^{\frac{\ln(K)}{\ln(2)}} \leq 1.$$

This yields the following estimate for the constant  $\tilde{C}_{\text{obs}}$

$$\tilde{C}_{\text{obs}} \leq \frac{4MC_1}{T} (e^{2\lambda_0} + e^{3\lambda_0}) \leq \frac{8MC_1}{T} e^{3\lambda_0}.$$

By definition of  $\lambda_0$ , we conclude

$$\tilde{C}_{\text{obs}} \leq \frac{8MC_1}{T} \exp\left(\frac{6 \ln(K)}{e \ln(2)} + 6\lambda_T\right) = \frac{8MC_1}{T} K^{\frac{6}{e \ln(2)}} e^{6\lambda_T}.$$

The claim now follows by inserting the definition of  $K$ . □

Let us relate Theorem 2.7 to the results obtained in [DM12, Section 6]. First note that Duyckaerts and Miller allow for unbounded observation operators  $C$  satisfying an admissibility condition, by using conditions involving a reference operator  $C_0$ , whereas we stick to bounded  $C$  and choose  $C_0 = \text{Id}$ . Furthermore, Duyckaerts and Miller restrict to Hilbert spaces and consider  $h(t) = t$  for  $t \in (0, T]$ . In [DM12] an integrated version of (UPgen) is considered:

$$\forall \lambda > \lambda_0, t \in (0, T], x \in \mathcal{E}_\lambda: \quad \|S_t x\|_X^2 \leq C_1 \frac{1}{t} e^{\lambda/\varphi(\lambda)} \int_0^t \|CS_\tau x\|_Y^2 d\tau, \quad (2.41)$$

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where  $\varphi$  is a positive, continuous, monotonically increasing function with  $\lambda/\varphi(\lambda) \rightarrow \infty$  for  $\lambda \rightarrow \infty$  and  $(\mathcal{E}_\lambda)_\lambda$  is a non-decreasing family of semigroup invariant spaces. On the orthogonal complement of  $\mathcal{E}_\lambda$ , Duyckaerts and Miller consider the dissipation estimate

$$\forall \lambda > \lambda_0, t \in (0, T], x \perp \mathcal{E}_\lambda : \quad \|S_t x\|_X \leq C_2 e^{-\lambda t + m\lambda/\varphi(\lambda)} \|x\|_X, \quad (2.42)$$

where  $m \geq 0$ . To turn (2.41) and (2.42) into an observability estimate, the integrability of

$$s \mapsto \frac{1}{\psi^{-1}(\varphi(q^s))} \quad (2.43)$$

at  $\infty$  is assumed, where  $q > 1$ , and  $\psi(\lambda) = \lambda \ln(\lambda)$  for  $\lambda > 1$ . The inequalities (2.41) and (2.42) can be related to (UPgen) and (DISSgen) by means of the transformation  $\varphi(\lambda) = \lambda/f(g^{-1}(\lambda))$  (assuming  $g$  is invertible). Indeed, note that if  $\lambda \mapsto \lambda/g(f^{-1}(\lambda))$  is monotonically decreasing, then  $\varphi$  is increasing. Moreover, the integrability of (2.43) at  $\infty$  yields with  $\psi^{-1}(t) \leq t$  for all  $t \geq e$ ,  $h^{-1}(t) = t$  for  $t > 0$ , and for  $R > 0$  large enough (also assuming sufficiently smooth functions):

$$\begin{aligned} & \int_{\ln(\varphi^{-1}(R))/\ln(q)}^{\infty} \frac{1}{\psi^{-1}(\varphi(q^s))} ds \geq \int_{\ln(\varphi^{-1}(R))/\ln(q)}^{\infty} \frac{1}{\varphi(q^s)} ds = \frac{1}{\ln(q)} \int_{\varphi^{-1}(R)}^{\infty} \frac{1}{\varphi(\lambda)\lambda} d\lambda \\ & = \frac{1}{\ln(q)} \int_{\varphi^{-1}(R)}^{\infty} \frac{f(g^{-1}(\lambda))}{\lambda^2} d\lambda = \frac{1}{\ln(q)} \int_{g^{-1}(\varphi^{-1}(R))}^{\infty} \frac{f(\mu)g'(\mu)}{g(\mu)^2} d\mu \\ & = \frac{1}{\ln(q)} \int_{f(g^{-1}(\varphi^{-1}(R)))}^{\infty} \frac{\xi g'(f^{-1}(\xi))}{g(f^{-1}(\xi))^2 f'(f^{-1}(\xi))} d\xi \geq \frac{1}{\ln(q)} \int_{f(g^{-1}(\varphi^{-1}(R)))}^{\infty} \frac{1}{g(f^{-1}(\xi))} d\xi, \end{aligned}$$

where we used that  $\xi g'(f^{-1}(\xi))/f'(f^{-1}(\xi)) \geq g(f^{-1}(\xi))$  for all  $\xi > 0$  since  $\xi \mapsto \xi/g(f^{-1}(\xi))$  is monotonically decreasing. Thus, (2.43) yields (2.32) in Theorem 2.7. However, (2.32) applies for more functions. Indeed, consider [DM12, Lemma 6.2] where it is shown that  $\varphi(\lambda) = (\ln(\ln(\lambda)))^s \ln(\lambda)$  satisfies condition (2.43) if and only if  $s > 2$ , whereas  $\lambda \mapsto \frac{1}{\varphi(\lambda)\lambda}$  is integrable for all  $s > 1$ . Similar as in [DM12], we can replace (DISSgen) by the dissipation estimate

$$\forall \lambda > 0, t \in (0, T], x \in X : \quad \|(\text{Id} - P_\lambda)S_t x\|_X \leq C_2 e^{-g(\lambda)h(t) + \omega_+ T + m\lambda} \|x\|_X$$

for some  $m \geq 0$ . Although in Theorem 2.7 we consider  $m = 0$ , the case  $m > 0$  can easily be implemented in our proof. In this case we get the final-state observability estimate (2.33) with  $\lambda_T \geq 0$  satisfying

$$\int_{\lambda_T}^{\infty} h^{-1} \left( (4+m) \frac{\lambda}{g(f^{-1}(\lambda))} \right) \frac{1}{\lambda} d\lambda \leq \frac{\ln(2)}{4} \min\{1, T\}.$$

Finally, let us emphasize that in [DM12] no estimate on the constant in (2.33) is obtained.

**Example 2.10.** Let us consider power functions  $f, g, h: (0, \infty) \rightarrow (0, \infty)$  in Theorem 2.7 and compare the result to Theorem 2.6. For this, let  $f(\lambda) = c_1 \lambda^{s_1}$ ,  $g(\lambda) = c_2 \lambda^{s_2}$  for  $s_2 > s_1 > 0$  and  $c_1, c_2 > 0$ , and  $h(t) = t^\gamma$  for  $\gamma > 0$ . To verify the integrability condition

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(2.32), we compute a suitable  $\lambda_T \geq 0$  by

$$\begin{aligned} \int_{\lambda_T}^{\infty} h^{-1} \left( 4 \frac{\lambda}{g(f^{-1}(\lambda))} \right) \frac{1}{\lambda} d\lambda &= \left( \frac{4c_1^{s_2/s_1}}{c_2} \right)^{1/\gamma} \int_{\lambda_T}^{\infty} \lambda^{-\frac{s_2-s_1}{s_1\gamma}-1} d\lambda \\ &= \left( \frac{4c_1^{s_2/s_1}}{c_2} \right)^{1/\gamma} \frac{s_1\gamma}{s_2-s_1} \lambda_T^{-\frac{s_2-s_1}{s_1\gamma}}. \end{aligned}$$

Hence, for  $T > 0$  condition (2.32) is satisfied for

$$\lambda_T = \frac{K_2}{\min\{1, T^{\frac{s_1\gamma}{s_2-s_1}}\}}, \quad \text{with} \quad K_2 := \left( \frac{c_1^{s_2}}{c_2^{s_1}} \right)^{\frac{1}{s_2-s_1}} \left( 4^{\frac{1}{\gamma}} \frac{s_1\gamma}{s_2-s_1} \frac{4}{\ln(2)} \right)^{\frac{s_1\gamma}{s_2-s_1}}.$$

Assuming the conditions in Theorem 2.7 with functions  $f, g$ , and  $h$  as above, we obtain the observability estimate (2.33) with the bound

$$C_{\text{obs}} \leq \frac{K_1}{T} \exp \left( \frac{6K_2}{\min\{1, T^{\frac{s_1\gamma}{s_2-s_1}}\}} + \omega_+ T \right), \quad (2.44)$$

where  $K_1$  is as in Theorem 2.7. The estimate (2.44) shares, up to numerical constants, the same behavior w.r.t.  $c_1, c_2, C_1, C_2, T$ , and  $\|C\|$  as the estimate obtained in Theorem 2.6. Only the right factor of  $K_2$  not depending on  $c_1, c_2$  differs qualitatively from Theorem 2.6.  $\diamond$

### 2.3 Observability for subordinated semigroups

In this section, we exploit subordination techniques to show that an uncertainty principle and a dissipation estimate do not just imply observability for one system but also many related ones. It turns out that only the function  $g$  in Theorem 2.7 is influenced by subordination. Namely, if  $\varphi$  is the corresponding Bernstein function and  $\varphi \circ g$  satisfies the assumptions on  $g$  instead, then the subordinated semigroup also satisfies a final-state observability estimate. This fits the intuition since the subordinated semigroup is generated by  $-\varphi(A)$ , where  $-A$  is the generator of the original semigroup.

Subordinated semigroups were first introduced by Bochner in 1949 [Boc49]. Here we will only deal with strongly continuous semigroups. However, note that subordination can also be performed in more general settings (see, e.g., [KMS21]), in particular for dual semigroups of strongly continuous semigroups. The presentation of this section follows [GMS22, Section 3] and is based on the monograph [SSV12].

**Definition 2.11.** Let  $\varphi: (0, \infty) \rightarrow [0, \infty)$ . Then  $\varphi$  is called *Bernstein function* provided  $\varphi \in C^\infty(0, \infty)$ , and  $(-1)^{k-1} \partial^k \varphi \geq 0$  for all  $k \in \mathbb{N}$ .

Standard examples for a Bernstein function are fractional powers, i.e.,  $\varphi(\lambda) = \lambda^s$  for  $s \in [0, 1]$ . An extensive collection of these functions can be found in [SSV12]. Bernstein functions can be characterized as follows.

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**Proposition 2.12** (see [SSV12, Theorem 3.2]). *Let  $\varphi: (0, \infty) \rightarrow [0, \infty)$ . The following are equivalent.*

- (a)  $\varphi$  is a Bernstein function.
- (b) There exist unique constants  $a, b \geq 0$  and a unique positive Radon measure  $\mu$  on  $(0, \infty)$  satisfying  $\int_{(0, \infty)} \min\{1, t\} \mu(dt) < \infty$  such that

$$\varphi(\lambda) = a + b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda t}) \mu(dt), \quad \lambda > 0. \quad (2.45)$$

The representation of a Bernstein function  $\varphi$  in (2.45) is called *Lévy–Khinchin representation* and the triplet  $(a, b, \mu)$  is called *Lévy triplet* of  $\varphi$ .

**Definition 2.13.** Let  $(\mu_t)_{t \geq 0}$  be a family of Radon measures on  $[0, \infty)$  and  $\mu$  a Radon measure on  $[0, \infty)$ . Then  $(\mu_t)_{t \geq 0}$  is called

- (a) a family of *sub-probability measures* if for all  $t \in [0, \infty)$  we have  $\mu_t([0, \infty)) \leq 1$ ,
- (b) a *convolution semigroup* if  $\mu_0 = \delta_0$  and for all  $s, t \in [0, \infty)$  we have  $\mu_t * \mu_s = \mu_{t+s}$ ,
- (c) *vaguely continuous* at  $s \in [0, \infty)$  with limit  $\mu$  if

$$\forall f \in C_c[0, \infty) : \lim_{t \rightarrow s} \int_{[0, \infty)} f(\lambda) \mu_t(d\lambda) = \int_{[0, \infty)} f(\lambda) \mu(d\lambda). \quad (2.46)$$

Here,  $\delta_0$  denotes the Dirac delta distribution. Note that if  $(\mu_t)_{t \geq 0}$  is a family of sub-probability measures which is vaguely continuous at 0 with limit  $\delta_0$ , then  $(\mu_t)_{t \geq 0}$  is also weakly continuous, i.e., (2.46) actually even holds for all  $f \in C_b[0, \infty)$ .

The Laplace transform relates Bernstein functions to vaguely continuous convolution semigroups of sub-probability measures on  $[0, \infty)$ .

**Proposition 2.14** (see [SSV12, Theorem 5.2]). *Let  $(\mu_t)_{t \geq 0}$  be a convolution semigroup of sub-probability measures on  $[0, \infty)$  which is vaguely continuous at 0 with limit  $\delta_0$ . Then there exists a unique Bernstein function  $\varphi: (0, \infty) \rightarrow [0, \infty)$  such that for all  $t \geq 0$  the Laplace transform of  $\mu_t$  satisfies*

$$\int_{[0, \infty)} e^{-\lambda s} \mu_t(ds) = e^{-t\varphi(\lambda)}, \quad \lambda > 0.$$

*Conversely, given any Bernstein function  $\varphi: (0, \infty) \rightarrow [0, \infty)$ , there exists a unique vaguely continuous convolution semigroup  $(\mu_t)_{t \geq 0}$  of sub-probability measures on  $[0, \infty)$  such that the above equation holds.*

**Definition 2.15.** Let  $X$  be a Banach space,  $(S_t)_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $X$ , and  $\varphi: (0, \infty) \rightarrow [0, \infty)$  a Bernstein function. Let  $(\mu_t)_{t \geq 0}$  be the vaguely continuous convolution semigroup of sub-probability measures on  $[0, \infty)$  associated with  $\varphi$ . For  $t \geq 0$  we define  $S_t^\varphi \in \mathcal{L}(X)$  by

$$S_t^\varphi x := \int_{[0, \infty)} S_s x \mu_t(ds), \quad x \in X.$$

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Then  $(S_t^\varphi)_{t \geq 0}$  is called *subordinated semigroup* to  $(S_t)_{t \geq 0}$  w.r.t.  $\varphi$ .

The first question is whether this defines a  $C_0$ -semigroup again, and the answer is affirmative.

**Proposition 2.16** (see [SSV12, Proposition 13.1]). *Let  $X$  be a Banach space,  $(S_t)_{t \geq 0}$  a bounded  $C_0$ -semigroup on  $X$ ,  $\varphi: (0, \infty) \rightarrow [0, \infty)$  a Bernstein function, and  $(S_t^\varphi)_{t \geq 0}$  the subordinated semigroup to  $(S_t)_{t \geq 0}$  w.r.t.  $\varphi$ . Then  $(S_t^\varphi)_{t \geq 0}$  is again a bounded  $C_0$ -semigroup on  $X$ .*

In view of Proposition 2.16, the generator of  $(S_t^\varphi)_{t \geq 0}$  is given by  $-\varphi(A)$ , where  $-A$  is the generator of  $(S_t)_{t \geq 0}$ , cf. [Phi52, Theorem 4.3].

The following lemma will be an essential ingredient in applying Theorem 2.7 to subordinated semigroups. It describes how the dissipation of the original semigroup transfers to subordinated semigroups.

**Lemma 2.17.** *Let  $X$  be a Banach space,  $(P_\lambda)_{\lambda > 0}$  in  $\mathcal{L}(X)$ , and  $(S_t)_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $X$ . Further, let  $T > 0$ ,  $g: (0, \infty) \rightarrow (0, \infty)$  measurable, and  $C_2 \geq 1$  such that*

$$\forall \lambda > 0, t \in (0, T], x \in X : \quad \|(\text{Id} - P_\lambda)S_t x\|_X \leq C_2 e^{-g(\lambda)t} \|x\|_X.$$

Moreover, let  $\varphi: (0, \infty) \rightarrow [0, \infty)$  be a Bernstein function and  $(S_t^\varphi)_{t \geq 0}$  be the subordinated semigroup to  $(S_t)_{t \geq 0}$  w.r.t.  $\varphi$ . Then

$$\forall \lambda > 0, t \in (0, T], x \in X : \quad \|(\text{Id} - P_\lambda)S_t^\varphi x\|_X \leq C_2 e^{-\varphi(g(\lambda))t} \|x\|_X. \quad (2.47)$$

*Proof.* Let  $(\mu_t)_{t \geq 0}$  be the vaguely continuous convolution semigroup associated with  $\varphi$ . Let  $\lambda > 0$ ,  $t \in (0, T]$ , and  $x \in X$ . Then

$$\begin{aligned} \|(\text{Id} - P_\lambda)S_t^\varphi x\|_X &= \|(\text{Id} - P_\lambda) \int_0^\infty S_s x \mu_t(ds)\|_X \\ &\leq \int_0^\infty \|(\text{Id} - P_\lambda)S_s x\|_X \mu_t(ds) \\ &\leq C_2 \int_0^\infty e^{-g(\lambda)s} \mu_t(ds) \|x\|_X. \end{aligned}$$

Note that the  $C_0$ -semigroup  $(e^{-g(\lambda)t})_{t \geq 0}$  on  $\mathbb{R}$  has the generator  $-g(\lambda)$ . Thus, the corresponding subordinated semigroup has the generator  $-\varphi(g(\lambda))$  by Proposition 2.16. Hence,  $\int_0^\infty e^{-g(\lambda)s} \mu_t(ds) = e^{-\varphi(g(\lambda))t}$  and thus

$$\|(\text{Id} - P_\lambda)S_t^\varphi x\|_X \leq C_2 e^{-\varphi(g(\lambda))t} \|x\|_X. \quad \square$$

**Theorem 2.18.** *Assume Hypothesis 2.2. Furthermore, assume that  $(S_t)_{t \geq 0}$  is a bounded  $C_0$ -semigroup on  $X$ ,  $f$  is strictly monotonically increasing and bijective, and  $h(t) = t$  for  $t > 0$ . Let  $T > 0$ ,  $\varphi: (0, \infty) \rightarrow [0, \infty)$  be a Bernstein function,  $(S_t^\varphi)_{t \geq 0}$  be the subordinated semigroup to  $(S_t)_{t \geq 0}$  w.r.t.  $\varphi$ . Assume that  $\lambda \mapsto \lambda/\varphi(g(f^{-1}(\lambda)))$  is monotonically decreasing and that  $1/(\varphi \circ g \circ f^{-1})$  integrable at  $\infty$ . Then there exists  $C_{\text{obs}} \geq 0$  such that*

$$\forall x \in X : \quad \|S_T^\varphi x\|_X \leq C_{\text{obs}} \int_0^T \|CS_\tau^\varphi x\|_Y d\tau. \quad (2.48)$$

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*Proof.* By Lemma 2.17 the dissipation estimate (2.47) for  $(S_t^\varphi)_{t \geq 0}$  is satisfied. Thus, the statement follows from Theorem 2.7.  $\square$

To verify the integrability condition in Theorem 2.18, rather explicit knowledge about the Bernstein function  $\varphi$  is needed. This can be difficult if only the Lévy triplet  $(a, b, \mu)$  of  $\varphi$  is given. We only consider the case  $a = 0$ . If  $b \neq 0$  we have  $\varphi(\lambda) \geq b\lambda$  for all  $\lambda > 0$  and thus

$$\frac{1}{\varphi(g(f^{-1}(\lambda)))} \leq \frac{1}{b} \frac{1}{g(f^{-1}(\lambda))}, \quad \lambda > 0.$$

Hence, the integrability of the function of the left-hand side is guaranteed by the same integrability condition as in Theorem 2.7. However, if  $b = 0$ , a closer examination of  $\mu$  becomes necessary. It is easy to see that for all  $\lambda > 0$  we have

$$\frac{1}{2}\lambda \int_{(0, \frac{1}{\lambda})} t \mu(dt) \leq \varphi(\lambda) \leq \lambda \int_{(0, \frac{1}{\lambda})} t \mu(dt) + 2\mu\left(\left[\frac{1}{\lambda}, \infty\right)\right), \quad (2.49)$$

since  $\frac{1}{2}x \leq 1 - e^{-x} \leq x$  for  $x \in (0, 1)$ , and  $\lambda t \in (0, 1)$  for  $t \in (0, 1/\lambda)$ . Thus, growth properties of  $\varphi$  can be studied by looking at  $\lambda \mapsto \lambda \int_{(0, \frac{1}{\lambda})} t \mu(dt)$  instead. As a standard example, let  $s \in (0, 1)$  and  $\varphi: (0, \infty) \rightarrow [0, \infty)$  be defined by the Lévy triplet  $(0, 0, \mu)$  with  $\mu(U) := \int_U t^{-1-s} dt$  for  $U \subset (0, \infty)$  measurable, i.e.,

$$\varphi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) t^{-1-s} dt, \quad \lambda > 0. \quad (2.50)$$

Then the estimate (2.49) gives the lower bound

$$\varphi(\lambda) \geq \frac{1}{2}\lambda \int_{(0, \frac{1}{\lambda})} t \mu(dt) = \frac{\lambda^s}{2(1-s)}, \quad \lambda > 0,$$

which resembles the fractional powers up to a constant.

## 2.4 Implications using duality theory

In this section, we conclude sufficient conditions for null-controllability and open-loop stabilizability by employing the conditions for observability in Section 2.2 together with duality results in Section 1.3. For this section, let  $X$  and  $U$  be Banach spaces,  $(S_t)_{t \geq 0}$  a  $C_0$ -semigroup on  $X$  with corresponding generator  $-A$ ,  $B \in \mathcal{L}(U, X)$ , and  $T > 0$ . We study the control system

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + Bu(t) & \text{for } t \in (0, T], \\ x(0) &= x_0 \in X, \end{aligned} \quad \Sigma_T(A, B, -)$$

where  $u \in L_r((0, \infty); U)$  with  $r \in [1, \infty]$ . Thanks to the duality in Theorem 1.20 we can prove controllability and stabilization properties of  $\Sigma_T(A, B, -)$  by verifying a weak observability estimate of the corresponding dual system  $\Sigma_T(A', -, B')$ , i.e., specifying  $C \geq$

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0, and  $\alpha \in [0, 1)$  such that

$$\forall x' \in X' : \quad \|S'_T x'\|_{X'} \leq \begin{cases} C \left( \int_0^T \|B' S'_t x'\|_{U'}^{r'} dt \right)^{1/r'} + \alpha \|x'\|_{X'} & \text{if } r' \in [1, \infty), \\ C \operatorname{ess\,sup}_{t \in [0, T]} \|B' S'_t x'\|_{U'} + \alpha \|x'\|_{X'} & \text{if } r' = \infty, \end{cases} \quad (2.51)$$

where  $r' \in [1, \infty]$  with  $1/r + 1/r' = 1$ . For  $\alpha = 0$ , the estimate (2.51) implies cost-uniform approximate null-controllability at time  $T$ . Whereas for  $\alpha \in (0, 1)$  the weak observability estimate is equivalent to cost-uniform open-loop stabilizability of the infinite time horizon system  $\Sigma_\infty(A, B, -)$ . In the following, we apply the sufficient conditions for observability derived in Section 2.2 to the dual system for obtaining conditions for null-controllability and stabilizability.

### 2.4.1 Null-controllability and control cost

The following theorem, stating sufficient conditions for cost-uniform approximate null-controllability, is a direct consequence of the conditions for observability in Theorem 2.7 and the duality in Theorem 1.20. Note that in many applications, the notion of cost-uniform approximate null-controllability is equivalent to null-controllability (see Proposition 1.11). Then the following Theorem gives sufficient conditions for null-controllability with a corresponding estimate of the control cost.

**Theorem 2.19.** *Let  $X$  and  $U$  be Banach spaces,  $B \in \mathcal{L}(U, X)$ , and  $T > 0$ . Further, let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$ , and  $M \geq 1$ ,  $\omega \in \mathbb{R}$  such that  $\|S_t\| \leq M e^{\omega t}$  for all  $t \geq 0$ . Consider measurable and monotonically increasing functions  $f, g, h: (0, \infty) \rightarrow (0, \infty)$  such that  $f$  and  $h$  are strictly monotonically increasing and bijective,  $\lambda \mapsto \lambda/g(f^{-1}(\lambda))$  is monotonically decreasing, and  $\lambda \mapsto h^{-1}\left(\frac{4\lambda}{g(f^{-1}(\lambda))}\right) \frac{1}{\lambda}$  is integrable at  $\infty$ . Further, assume that there are  $(P'_\lambda)_{\lambda > 0}$  in  $\mathcal{L}(X')$ , and  $C_1 > 0$ ,  $C_2 \geq 1$  such that*

$$\forall \lambda > 0, x' \in X' : \quad \|P'_\lambda x'\|_{X'} \leq C_1 e^{f(\lambda)} \|B' P'_\lambda x'\|_{U'}, \quad (2.52)$$

and

$$\forall \lambda > 0, t \in (0, T], x' \in X' : \quad \|(\operatorname{Id} - P'_\lambda) S'_t x'\|_{X'} \leq C_2 e^{-g(\lambda)h(t) + \omega t} \|x'\|_{X'}. \quad (2.53)$$

Then for all  $r \in [1, \infty]$  the system  $\Sigma_T(A, B, -)$  is cost-uniformly approximately null-controllable in time  $T$  with respect to  $L_r((0, T); U)$ . Furthermore, we have the following bound on the control cost

$$\mathcal{C} \leq \frac{8e^3 M C_1}{T^{1/r'}} \left( M \left( C_1 \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2 \right)^{\frac{6}{e \ln(2)}} e^{6\lambda_T + \max\{\omega, 0\}T},$$

where  $r' \in [1, \infty]$  such that  $1/r + 1/r' = 1$ , and  $\lambda_T \geq 0$  such that

$$\int_{\lambda_T}^{\infty} h^{-1}\left(4 \frac{\lambda}{g(f^{-1}(\lambda))}\right) \frac{1}{\lambda} d\lambda \leq \frac{\ln(2)}{4} \min\{1, T\}.$$

*Proof.* We apply Theorem 2.7 to the dual semigroup  $(S'_t)_{t \geq 0}$  on  $X'$ ,  $Y := U'$ ,  $C := B'$ ,

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and  $(P'_\lambda)_{\lambda>0}$ . Note that  $(S'_t)_{t\geq 0}$  is exponentially bounded as this applies to the semigroup  $(S_t)_{t\geq 0}$ . The measurability of the functions  $t \mapsto \|B'S'_t x'\|_{U'}$  for all  $x' \in X'$  follows from Lemma 1.24. Thus, by Theorem 2.7 we obtain that

$$\forall x' \in X' : \quad \|S'_T x'\|_{X'} \leq C_{\text{obs}} \int_0^T \|B'S'_t x'\|_{U'} dt \quad (2.54)$$

with  $C_{\text{obs}}$  as in Theorem 2.7. Let  $r' \in [1, \infty]$  such  $1/r + 1/r' = 1$ . Using Hölder's inequality, as in Remark 2.1, the estimate (2.54) implies a final-state observability estimate with respect to the  $L_{r'}$ -norm with constant

$$\tilde{C}_{\text{obs}} = \frac{K}{T^{1/r'}} e^{6\lambda T + \omega_+ T},$$

where  $K > 0$  as in Theorem 2.7, and we used that  $T^{-1}T^{1/r} = T^{-1/r'}$ . Employing the duality in Theorem 1.20 this implies that for every  $x \in X$  and  $\varepsilon > 0$  there exists  $u \in L_r((0, T); U)$  with

$$\|u\|_{L_r((0, T); U)} \leq \tilde{C}_{\text{obs}} \|x\|_X \quad \text{and} \quad \|S_T x + \mathcal{B}_T u\|_X < \varepsilon \|x\|_X,$$

i.e., cost-uniform approximate null-controllability in time  $T$  with respect to  $L_r((0, T); U)$  and control cost  $\tilde{C}_{\text{obs}}$  (cf. Definition 1.10).  $\square$

Employing abstract results of majorization and range inclusion of bounded linear operators in Banach spaces (see, e.g., [Car88]), we obtain the following equivalent formulations of (2.52) and (2.53):

$$\forall \lambda > 0 : \quad P_\lambda (B_X(0, 1)) \subset \overline{P_\lambda B (B_U(0, C_1 e^{f(\lambda)}))}, \quad (2.52')$$

and

$$\forall \lambda > 0, t \in (0, T], x \in X : \quad \|S_t(\text{Id} - P_\lambda)x\|_X \leq C_2 e^{-g(\lambda)h(t) + \omega t} \|x\|_X. \quad (2.53')$$

Recall that  $B_X(0, \rho)$  denotes the open ball in  $X$  around 0 with radius  $\rho > 0$ . There are situations where the closure in (2.52') may be waived, e.g., if the Banach space  $U$  is reflexive. Note that if  $S_t$  and  $P_\lambda$  are commuting, (2.53') is the same condition as (DISSgen).

### 2.4.2 Sufficient conditions for stabilizability

We now turn from null-controllability at some final time  $T$  to a form of infinite-time null-controllability and give sufficient conditions for cost-uniform open-loop stabilizability, similar to the results in [HWW21, LWXY22] for the Hilbert space setting. Instead of assuming an uncertainty principle and a dissipation estimate for a family  $(P_\lambda)_{\lambda>0}$  with certain dependencies of the constants on the parameter  $\lambda$ , for stabilizability, we need these assumptions to hold only for one single operator  $P$ . This analysis is based on [EGST21].

Assume that, for measurable, positive, and monotonically increasing functions  $f, g, h$  on  $(0, \infty)$ , an operator family  $(P'_\lambda)_{\lambda>0}$  in  $\mathcal{L}(X')$ , and constants  $C_1 > 0$ ,  $C_2 \geq 1$ , the dual

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observation system  $\Sigma_T(A', -, B')$  satisfies the inequalities (UPgen) and (DISSgen), i.e.,

$$\forall \lambda > 0, x' \in X' : \quad \|P'_\lambda x'\|_{X'} \leq C_1 e^{f(\lambda)} \|B' P'_\lambda x'\|_{U'}$$

and

$$\forall \lambda > 0, t \in (0, T], x' \in X' : \quad \|(\text{Id} - P'_\lambda) S'_t x'\|_{X'} \leq C_2 e^{-g(\lambda)h(t) + \omega t} \|x'\|_{X'}.$$

Then, by Lemma 2.3 and using Hölder's inequality as in Remark 2.1, we obtain for all  $q \in (0, 1)$ ,  $r' \in [1, \infty]$ ,  $\lambda > 0$ , and  $x' \in X'$  a weak observability estimate with general  $\alpha \geq 0$ , i.e.,

$$\|S'_T x'\|_{X'} \leq \begin{cases} C_{\text{obs}}(T, \lambda) \left( \int_0^T \|B' S'_t x'\|_{U'}^{r'} dt \right)^{1/r'} + \alpha(T, \lambda) \|x'\|_{X'} & \text{if } r' \in [1, \infty), \\ C_{\text{obs}}(T, \lambda) \text{ess sup}_{t \in [0, T]} \|B' S'_t x'\|_{U'} + \alpha(T, \lambda) \|x'\|_{X'} & \text{if } r' = \infty. \end{cases} \quad (2.55)$$

Note that we performed the integration in (2.55) over  $(0, T]$  instead of  $(qT, T]$ . The constants satisfy  $C_{\text{obs}}(T, \lambda) = M C_1 e^{f(\lambda) + \max\{\omega, 0\}T} / (1 - q) T^{1/r'}$  and

$$\alpha(T, \lambda) = M \left( C_1 \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2 e^{-g(\lambda)h(qT) + f(\lambda) + \omega T}, \quad (2.56)$$

where  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|S'_t\|_{\mathcal{L}(X')} = \|S_t\|_{\mathcal{L}(X)} \leq M e^{\omega t}$  for all  $t \geq 0$ . By Proposition 1.17 and Theorem 1.20 the system  $\Sigma_\infty(A, B, -)$  is cost-uniformly open-loop stabilizable w.r.t.  $L_r((0, \infty); U)$  if there are  $T > 0$  and  $\lambda > 0$  such that  $\alpha(T, \lambda) \in [0, 1)$ .

**Example 2.20.** Let us impose conditions on  $T$  and  $\lambda$  to ensure that  $\alpha(T, \lambda) \in [0, 1)$ . Therefore, we restrict to power functions  $f, g, h: (0, \infty) \rightarrow (0, \infty)$  as in Section 2.2.2 and Section 2.2.3. For  $s_1, s_2, \gamma, c_1, c_2 > 0$ , let  $f(\lambda) = c_1 \lambda^{s_1}$ ,  $g(\lambda) = c_2 \lambda^{s_2}$  for  $\lambda > 0$ , and  $h(t) = t^\gamma$  for  $t > 0$ . Abbreviate  $K := M \left( C_1 \|C\|_{\mathcal{L}(X, Y)} + 1 \right) C_2$ . We list here some interesting cases:

- (a) Assume  $s_1 > s_2$ . Let  $\gamma > 1 - s_2/s_1$ , i.e.,  $s_1 \gamma / (s_1 - s_2) > 1$ , and  $T > 0$  large enough such that

$$\ln(K) < \left( \frac{c_2}{2c_1} \right)^{\frac{s_2}{s_1 - s_2}} \frac{c_2}{2} (qT)^{\frac{s_1 \gamma}{s_1 - s_2}} - \omega T.$$

Then for  $\lambda = (c_2 (qT)^\gamma / (2c_1))^{1/(s_1 - s_2)}$  we have  $\alpha(T, \lambda) \in (0, 1)$ . Indeed one easily computes

$$\begin{aligned} \alpha(T, \lambda) &\leq K \exp \left( -c_2 \left( \frac{c_2 (qT)^\gamma}{2c_1} \right)^{\frac{s_2}{s_1 - s_2}} (qT)^\gamma + c_1 \left( \frac{c_2 (qT)^\gamma}{2c_1} \right)^{\frac{s_1}{s_1 - s_2}} + \omega T \right) \\ &= K \exp \left( - \left( \frac{c_2}{2c_1} \right)^{\frac{s_2}{s_1 - s_2}} \frac{c_2}{2} (qT)^{\frac{s_1 \gamma}{s_1 - s_2}} + \omega T \right) < 1. \end{aligned}$$

- (b) Assume  $s_1 = s_2$ . Let  $T > q(c_1/c_2)^{1/\gamma}$  and

$$\lambda > \left( \frac{\ln(K) + \omega T}{c_2 (qT)^\gamma - c_1} \right)^{\frac{1}{s_1}} > 0.$$

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Then again  $\alpha(T, \lambda) \in (0, 1)$ .

(c) Assume  $s_1 < s_2$ . For given  $T > 0$  let  $\lambda > 0$  large enough such that

$$\ln(K) + \omega T < c_2 \lambda^{s_2} (qT)^\gamma - c_1 \lambda^{s_1}.$$

Then  $\alpha(T, \lambda) \in (0, 1)$ . Apart from this, in Proposition 2.4 and Theorem 2.6 we have already determined that the inequalities (UPgen) and (DISSgen) imply the weak observability estimate (2.55) with  $\alpha = 0$ .

(d) Assume  $\omega \leq 0$ . Then for arbitrary  $\lambda, s_1, s_2, \gamma > 0$  we can achieve  $\alpha(T, \lambda) \in (0, 1)$  by choosing  $T > 0$  large enough.  $\diamond$

Note that, in contrast to the cases (a) and (b), in (c) we can ensure  $\alpha(T, \lambda) \in (0, 1)$  for every  $T > 0$  by choosing  $\lambda > 0$  appropriately. An important observation is that in all cases the inequalities (UPgen) and (DISSgen) must only hold for specific values of  $T$  and  $\lambda$ . This is different from the result obtained in [HWW21, Lemma 2.2].

By restricting to  $h(t) = t$  for all  $t > 0$ , Lemma 2.3 and the duality in Theorem 1.20 yield the following plain sufficient condition for cost-uniform open-loop stabilizability similar to the Hilbert space result on complete stabilizability in [LWXY22, Theorem 4.1].

**Proposition 2.21.** *Let  $X$  and  $U$  be Banach spaces,  $B \in \mathcal{L}(U, X)$ , and  $P \in \mathcal{L}(X)$  such that*

$$\exists R > 0 : \quad P(B_X(0, 1)) \subset \overline{PB(B_U(0, R))}. \quad (2.57)$$

Further, let  $(S_t)_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$ , and assume there exist  $M_P \geq 1$  and  $\omega_P > 0$  such that

$$\forall t > 0, x \in X : \quad \|S_t(\text{Id} - P)x\|_X \leq M_P e^{-\omega_P t} \|x\|_X. \quad (2.58)$$

Then the system  $\Sigma_\infty(A, B, -)$  is cost-uniformly open-loop stabilizable.

*Proof.* We apply Lemma 2.3 to the dual semigroup  $(S'_t)_{t \geq 0}$  on  $X'$ ,  $Y := U'$ ,  $C := B'$ , and  $P_\lambda := P'$  for all  $\lambda > 0$ . By Proposition 1.2 there are  $M \geq 1$ ,  $\omega \in \mathbb{R}$  such that  $\|S'_t\|_{\mathcal{L}(X')} = \|S_t\|_{\mathcal{L}(X)} \leq M e^{\omega t}$  for all  $t \geq 0$ . The measurability of the functions  $t \mapsto \|B'S'_t x'\|_{U'}$  for all  $x' \in X'$  follows from Lemma 1.24. It is well-known (see [Car88]), that (2.57) implies that

$$\forall x' \in X' : \quad \|P'x'\|_{X'} \leq R \|B'P'x'\|_{U'}.$$

Furthermore, (2.58) implies

$$\forall t > 0, x' \in X' : \quad \|(\text{Id} - P')S'_t x'\|_{X'} \leq M_P e^{-\omega_P t} \|x'\|_{X'} = M_P e^{-(\omega_P + \omega_+)t + \omega_+ t} \|x'\|_{X'},$$

where  $\omega_+ := \max\{\omega, 0\}$ . Thus, by Lemma 2.3 with  $C_1 = R e^{-1}$ ,  $C_2 = M_P$ ,  $f(\lambda) = 1$  and  $g(\lambda) = \omega_P + \omega_+$  for  $\lambda > 0$ ,  $h(t) = t$  for  $t > 0$ , and  $q = \frac{1}{2} + \frac{\omega_+}{2(\omega_P + \omega_+)}$ , we obtain for all  $T > 0$ ,  $r' \in [1, \infty]$ , and  $x' \in X'$

$$\|S'_T x'\|_{X'} \leq \begin{cases} C_{\text{obs}} \left( \int_0^T \|B'S'_t x'\|_{U'}^{r'} dt \right)^{1/r'} + \alpha \|x'\|_{X'} & \text{if } r' \in [1, \infty), \\ C_{\text{obs}} \text{ess sup}_{t \in [0, T]} \|B'S'_t x'\|_{U'} + \alpha \|x'\|_{X'} & \text{if } r' = \infty, \end{cases}$$

## 2 Sufficient conditions for observability

with

$$C_{\text{obs}} \leq \frac{MR e^{-1}}{(1-q)T^{1/r}} e^{1+\omega_+T} = \frac{2(\omega_P + \omega_+)MR}{\omega_P T^{1/r}} e^{\omega_+T}$$

and

$$\alpha \leq M \left( R e^{-1} \|B\|_{\mathcal{L}(U,X)} + 1 \right) M_P e^{-(\omega_P + \omega_+)qT + 1 + \omega_+T} = M \left( R \|B\|_{\mathcal{L}(U,X)} + e \right) M_P e^{-\frac{\omega_P}{2}T}.$$

Hence, for

$$T > \frac{2}{\omega_P} \ln \left( M \left( R \|B\|_{\mathcal{L}(U,X)} + e \right) M_P \right)$$

we have  $\alpha \in [0, 1)$  and the assertion follows from Theorem 1.20 and Proposition 1.17.  $\square$

Apart from the fact that the conditions (2.57) and (2.58) are required to hold for the same operator  $P$ , they are independent of each other. In applications, this means that for the uncertainty principle of the dual system, no assumption on the growth rate with respect to the spectral parameter is needed.

**Example 2.22.** An instance of a system  $\Sigma_\infty(A, B, -)$  where the growth rate in the spectral inequality varies in dependence of the control set, is the harmonic oscillator in  $L_2(\mathbb{R}^d)$ , i.e.,  $A = -\Delta + |x|^2$ , and  $B$  is the multiplication operator with a characteristic function of a measurable subset  $E \subset \mathbb{R}^d$ . Indeed, it was shown in [BJP21, HWW21, DSV22b] that a spectral inequality with  $P$  being a spectral projection associated with  $A$  is valid under different geometric assumptions on  $E$  with different growth rates of the constant with respect to the spectral parameter.  $\diamond$

By means of abstract results on majorization and range inclusion of bounded linear operators (see, e.g., [Car88]) we can replace assumption (2.57) in Proposition 2.21 by the stronger condition

$$\text{Ran}(P) \subset \text{Ran}(PB). \tag{2.57'}$$

Note that (2.57') is equivalent to (2.57) if, e.g.,  $U$  is a reflexive Banach space. However, there are counterexamples for non-reflexive Banach spaces (see, e.g., [Bou78]).

**Remark 2.23.** Recall from Definition 1.15 that a system  $\Sigma_\infty(A, B, -)$  is completely cost-uniformly open-loop stabilizable if for all  $\nu > 0$  the system

$$\dot{x}(t) = -(A - \nu)x(t) + Bu(t), \quad t > 0, \quad x(0) = x_0 \tag{2.59}$$

is cost-uniformly open-loop stabilizable. Analogously to [LWXY22, Theorem 4.1] we obtain the following sufficient conditions: Let  $(P_\lambda)_{\lambda>0}$  in  $\mathcal{L}(X)$  satisfying (2.57) for all  $\lambda > 0$ ,  $(M_\lambda)_{\lambda>0}$  in  $[1, \infty)$ , and  $(\omega_\lambda)_{\lambda>0}$  in  $\mathbb{R}$  with  $\omega_\lambda \rightarrow \infty$  as  $\lambda \rightarrow \infty$  such that

$$\forall t > 0, \lambda > 0, x \in X : \quad \|S_t(\text{Id} - P_\lambda)x\|_X \leq M_\lambda e^{-\omega_\lambda t} \|x\|_X. \tag{2.60}$$

Then  $\Sigma_\infty(A, B, -)$  is completely cost-uniformly open-loop stabilizable. Indeed, for all  $\nu > 0$  there exists  $\lambda > 0$  such that  $\omega_\lambda > \nu$  and hence, (2.60) implies

$$\forall t > 0, x \in X : \quad \|e^{\nu t} S_t(\text{Id} - P_\lambda)x\|_X \leq M_\lambda e^{-(\omega_\lambda - \nu)t} \|x\|_X.$$

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Since  $(e^{\nu t} S_t)_{t \geq 0}$  is the semigroup generated by  $-(A - \nu)$ , Proposition 2.21 shows that the system (2.59) is cost-uniformly open-loop stabilizable.  $\diamond$

### 3 Observability estimates for parabolic equations in $L_p(\mathbb{R}^d)$

Following the strategy presented in Chapter 2, we prove observability estimates for a class of linear parabolic equations in  $L_p(\mathbb{R}^d)$  for  $p \in [1, \infty]$ . For an elliptic differential operator  $-A_p$  in  $L_p(\mathbb{R}^d)$  with constant coefficients, and a measurable observation set  $E \subset \mathbb{R}^d$  we consider the system

$$\begin{aligned} \dot{x}(t) &= -A_p x(t) && \text{for } t \in (0, T], \\ y(t) &= x(t)|_E && \text{for } t \in [0, T], \\ x(0) &= x_0 \in L_p(\mathbb{R}^d), && \Sigma_T(A_p, -, |_E) \end{aligned}$$

and aim to classify sets  $E$  such that the system  $\Sigma_T(A_p, -, |_E)$  satisfies a final-state observability estimate. In this context, we are looking for explicit bounds on the observability constant.

The most studied example of a parabolic system  $\Sigma_T(A_p, -, |_E)$  is the heat equation in  $L_2(\mathbb{R}^d)$ , i.e.,  $A_2 = -\Delta$ . For several unbounded domains  $\Omega \subset \mathbb{R}^d$  this problem has been studied, e.g., in [CdMZ01, Mil05a, Mil05b, Gd07, Bar14]. While for bounded domains it is sufficient that  $E \subset \Omega$  is open and nonempty [LR95, FI96], or even measurable with positive Lebesgue measure [AEWZ14, EMZ15], this is not true for unbounded domains (see [MZ01, Mil05a]). In [CdMZ01] observability of some parabolic equations on unbounded domains  $\Omega$  was proven if  $\Omega \setminus E$  is bounded. A sufficient geometric condition for observability of the heat equation on the whole  $\mathbb{R}^d$  was given in [LM16]. This condition was improved in [EV18, WWZZ19] to so-called thickness of the set  $E$ , which is proven to be also necessary for observability of the heat equation on  $\mathbb{R}^d$ .

Here, we maintain the class of thick observation sets  $E$  but replace the Laplacian with elliptic differential operators  $-A_p$  with constant coefficients and lower-order terms. Furthermore, we leave the Hilbert space setting and verify final-state observability estimates for the whole range of  $L_p$ -spaces with  $p \in [1, \infty]$ . For a system acting on  $L_\infty$ , we consider semigroups that are not necessarily strongly continuous.

This chapter is structured as follows. We start in Section 3.1 by introducing a class of constant-coefficient differential operators associated with strongly elliptic polynomials and prove heat kernel bounds. Our main results are presented in Section 3.2. We introduce the thickness condition on the observation set  $E$  and state that this condition is sufficient and necessary for final-state observability of this class of differential operators. Moreover, we show that the observability constant  $C_{\text{obs}}$  has optimal behavior in small and large time regimes. To prove our main result, we utilize the generalized Lebeau–Robbiano strategy presented in Chapter 2. The uncertainty principle follows directly from a Logvinenko–Sereda theorem [LS74, Kov01], which will be discussed in Section 3.3. We also give a complete proof of this result based on the work by Kovrijkine [Kov01]. The corresponding dissipation estimate is verified in Section 3.4, where we first consider powers of the Laplacian and then use a perturbation argument. With these results at hand, the proof of the

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final-state observability estimate in Section 3.5 is an application of the abstract results from Section 2.2. Finally, in Section 3.6, we consider fractional powers of the differential operators  $-A_p$  and investigate controllability and stabilization properties.

## 3.1 Linear parabolic equations in $L_p(\mathbb{R}^d)$

Let us introduce the class of differential operators considered in this chapter. We start by recalling some basic facts from Fourier analysis and fixing the notation.

### 3.1.1 Basic facts from Fourier analysis

In contrast to the common definition of the Fourier transform for functions in  $L_1(\mathbb{R}^d)$ , we first restrict to the smaller space of Schwartz functions and then extend the definition to the space of tempered distributions. Our presentation is based on the textbooks [Gra08] and [BCD11], to which we refer for further details.

For  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$  we denote the norm of  $x$  by  $|x| = (x_1^2 + \dots + x_d^2)^{1/2}$ . The  $n$ -th partial derivative of a function  $f$  on  $\mathbb{R}^d$  with respect to the  $k$ -th variable is denoted by  $\partial_k^n f$ . For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ , we abbreviate  $\partial^\alpha f := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} f$  and similarly we denote by  $x^\alpha$  the monomial  $x_1^{\alpha_1} \dots x_d^{\alpha_d}$ .

**Definition 3.1.** The *Schwartz space*, denoted by  $\mathcal{S}(\mathbb{R}^d)$ , is the space of infinitely differentiable functions  $f : \mathbb{R}^d \rightarrow \mathbb{C}$  such that for all multi-indices  $\alpha, \beta \in \mathbb{N}_0^d$

$$\rho_{\alpha, \beta}(f) := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty.$$

By definition, the derivatives of a Schwartz function decay faster than the reciprocal of any polynomial at infinity. Therefore, functions in  $\mathcal{S}(\mathbb{R}^d)$  are also called rapidly decreasing. The functions  $\rho_{\alpha, \beta}$  are a family of seminorms on  $\mathcal{S}(\mathbb{R}^d)$  with which  $\mathcal{S}(\mathbb{R}^d)$  is a Fréchet space. A sequence  $(f_k)_{k \in \mathbb{N}}$  in  $\mathcal{S}(\mathbb{R}^d)$  converges to  $f \in \mathcal{S}(\mathbb{R}^d)$  if for all  $\alpha, \beta \in \mathbb{N}_0^d$

$$\rho_{\alpha, \beta}(f_k - f) = \sup_{x \in \mathbb{R}^d} |x^\alpha (\partial^\beta (f_k - f))(x)| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We have  $\mathcal{S}(\mathbb{R}^d) \subset L_p(\mathbb{R}^d)$  for all  $p \in [1, \infty]$  and this inclusion is dense if  $p \in [1, \infty)$ . Note that the convergence in  $\mathcal{S}(\mathbb{R}^d)$  is stronger than  $L_p(\mathbb{R}^d)$ -convergence for all  $p \in [1, \infty]$ .

**Example 3.2.** For  $c > 0$  and  $m \in \mathbb{N}$  even, the function  $e^{-c|\cdot|^m} : \mathbb{R}^d \rightarrow \mathbb{R}$  is a Schwartz function. Note that for general  $m > 0$  the function  $e^{-c|\cdot|^m}$  may fail to be differentiable at  $x = 0$ . However we still have  $e^{-c|\cdot|^m} \in L_p(\mathbb{R}^d)$  for all  $p \in [1, \infty]$ .  $\diamond$

For  $x, y \in \mathbb{R}^d$ , we denote the inner product of  $x$  and  $y$  by  $x \cdot y = x_1 y_1 + \dots + x_d y_d$ .

**Definition 3.3.** For  $f \in \mathcal{S}(\mathbb{R}^d)$  the *Fourier transform*  $\mathcal{F}f : \mathbb{R}^d \rightarrow \mathbb{C}$  of  $f$  is defined by

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} dx.$$

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On the space of Schwartz functions the Fourier transform  $\mathcal{F}: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$  is bijective and continuous. Moreover, the continuous inverse of  $\mathcal{F}$  given by

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi) e^{ix \cdot \xi} d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The topological dual space of  $\mathcal{S}(\mathbb{R}^d)$ , denoted by  $\mathcal{S}'(\mathbb{R}^d)$ , is called the *space of tempered distributions*. For  $p \in [1, \infty]$ , we consider  $L_p(\mathbb{R}^d)$  as a subspace of  $\mathcal{S}'(\mathbb{R}^d)$  by identifying a function  $f \in L_p(\mathbb{R}^d)$  with the regular distribution  $u_f: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$  defined by

$$u_f(\phi) = \int_{\mathbb{R}^d} f(x)\phi(x) dx, \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

We can extend the Fourier transform to  $\mathcal{S}'(\mathbb{R}^d)$ . For a tempered distribution  $u \in \mathcal{S}'(\mathbb{R}^d)$  the Fourier transform is again denoted by  $\mathcal{F}$  and is given by

$$(\mathcal{F}u)(\phi) = u(\mathcal{F}\phi), \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

By duality, the Fourier transform is bijective on  $\mathcal{S}'(\mathbb{R}^d)$  as well and the inverse is analogously given by  $(\mathcal{F}^{-1}u)(\phi) = u(\mathcal{F}^{-1}\phi)$ .

For  $u \in \mathcal{S}'(\mathbb{R}^d)$  and  $f, \phi \in \mathcal{S}(\mathbb{R}^d)$ , we define the convolution of  $u$  and  $f$  by  $(u * f)(\phi) = u(f(-\cdot) * \phi)$  which is consistent with the usual definition if  $u \in L_p(\mathbb{R}^d)$  for some  $p \in [1, \infty]$ .

#### 3.1.2 Strongly elliptic polynomials

We now introduce the class of elliptic operators  $-A_p$  which we consider in this chapter. For the order of a multi-index  $\alpha \in \mathbb{N}_0^d$  we write  $|\alpha|_1 = \alpha_1 + \dots + \alpha_d$ . Let  $m \in \mathbb{N}$  and

$$a(\xi) = \sum_{|\alpha|_1 \leq m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^d,$$

be a polynomial of degree  $m$  with coefficients  $a_\alpha \in \mathbb{C}$ . Then

$$a_m(\xi) = \sum_{|\alpha|_1 = m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^d,$$

is called the *principal symbol* of  $a$ . We say that the polynomial  $a$  is *strongly elliptic* if there is a constant  $c > 0$  such that the principal symbol satisfies the lower bound

$$\operatorname{Re} a_m(\xi) \geq c |\xi|^m, \quad \text{for all } \xi \in \mathbb{R}^d. \quad (3.1)$$

Note that strong ellipticity implies that  $m$  is even.

Given a strongly elliptic polynomial  $a$  and  $p \in [1, \infty]$ , we define the associated semigroup  $(S_t)_{t \geq 0}$  of bounded linear operators on  $L_p(\mathbb{R}^d)$  by  $S_0 f := f$  and

$$S_t f := \mathcal{F}^{-1} e^{-ta} \mathcal{F} f = \mathcal{F}^{-1} e^{-ta} * f, \quad t > 0. \quad (3.2)$$

Note that the second equality holds since  $e^{-ta} \in \mathcal{S}(\mathbb{R}^d)$ . The convolution kernel  $\mathcal{F}^{-1} e^{-ta}$  satisfies a heat kernel estimate. See, e.g., [Fri64, Theorem 9.4.2], where this is proven in a much more general setting, and [TER96] for a sketch of the proof given here.

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**Proposition 3.4.** *Let  $a : \mathbb{R}^d \rightarrow \mathbb{C}$  be a strongly elliptic polynomial of degree  $m \in \mathbb{N}$ . There exist  $c_1, c_2 > 0$ , and  $\omega \geq 0$  such that for all  $t > 0$  and  $x \in \mathbb{R}^d$*

$$|(\mathcal{F}^{-1}e^{-ta})(x)| \leq c_1 e^{\omega t} t^{-d/m} e^{-c_2 \left(\frac{|x|^m}{t}\right)^{\frac{1}{m-1}}}. \quad (3.3)$$

*Proof.* By definition of the inverse Fourier transform, we observe

$$(\mathcal{F}^{-1}e^{-ta})(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ta(\xi)} e^{ix \cdot \xi} d\xi, \quad x \in \mathbb{R}^d.$$

Since the symbol is a polynomial, we can also extend  $a : \mathbb{C}^d \rightarrow \mathbb{C}$  and insert complex vectors. We now shift the integral by replacing  $\xi$  with  $\xi + i\eta$ , where  $\eta \in \mathbb{R}^d$ , and get for all  $x \in \mathbb{R}^d$

$$\begin{aligned} |(\mathcal{F}^{-1}e^{-ta})(x)| &= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} e^{-ta(\xi+i\eta)} e^{ix \cdot (\xi+i\eta)} d\xi \right| \\ &= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} e^{-ta(\xi+i\eta)} e^{ix \cdot \xi} e^{-x \cdot \eta} d\xi \right| \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t \operatorname{Re} a(\xi+i\eta)} e^{-x \cdot \eta} d\xi. \end{aligned}$$

Using  $a(\xi) = \sum_{|\alpha|_1 \leq m} a_\alpha \xi^\alpha$  and the binomial theorem, we write

$$\begin{aligned} \operatorname{Re} a(\xi + i\eta) &= \operatorname{Re} \sum_{|\alpha|_1 \leq m} a_\alpha (\xi + i\eta)^\alpha = \operatorname{Re} \sum_{|\alpha|_1 \leq m} a_\alpha \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} \xi^\nu (i\eta)^{\alpha-\nu} \\ &= \operatorname{Re} \sum_{|\alpha|_1 \leq m} a_\alpha \xi^\alpha + \operatorname{Re} \sum_{|\alpha|_1 \leq m} a_\alpha (i\eta)^\alpha + \operatorname{Re} a_0 + \operatorname{Re} \sum_{\substack{|\alpha|_1 \leq m \\ \alpha \neq 0}} a_\alpha \sum_{\substack{\nu \leq \alpha \\ \nu \neq 0 \\ \nu \neq \alpha}} \binom{\alpha}{\nu} \xi^\nu (i\eta)^{\alpha-\nu}. \end{aligned}$$

Since  $a$  is strongly elliptic, there is  $c > 0$  such that  $\operatorname{Re} a_m(\xi) \geq c|\xi|^m$  for all  $\xi \in \mathbb{R}^d$ . Using that one can bound a real polynomial by the term of maximal degree plus a constant, we estimate

$$\operatorname{Re} a(\xi + i\eta) \geq c|\xi|^m - k_1|\eta|^m - k_2 \left( 1 + \sum_{1 \leq r \leq m} \sum_{\substack{1 \leq s \leq r \\ s \neq m}} |\xi|^s |\eta|^{r-s} \right)$$

for some constants  $k_1, k_2 \geq 0$ . Now, using that there exist  $\sigma, \omega \geq 0$  such that

$$k_2 \left( 1 + \sum_{1 \leq r \leq m} \sum_{\substack{1 \leq s \leq r \\ s \neq m}} |\xi|^s |\eta|^{r-s} \right) \leq \frac{c}{2} |\xi|^m + (\sigma - k_1) |\eta|^m + \omega,$$

we obtain the bound

$$\operatorname{Re} a(\xi + i\eta) \geq \frac{c}{2} |\xi|^m - \sigma |\eta|^m - \omega, \quad \text{for all } \xi, \eta \in \mathbb{R}^d. \quad (3.4)$$

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With (3.4) we obtain

$$\begin{aligned} |(\mathcal{F}^{-1}e^{-ta})(x)| &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t(\frac{c}{2}|\xi|^m - \sigma|\eta|^m - \omega)} e^{-x \cdot \eta} d\xi \\ &= \frac{1}{(2\pi)^d} e^{t\sigma|\eta|^m} e^{t\omega} e^{-x \cdot \eta} \int_{\mathbb{R}^d} e^{-t\frac{c}{2}|\xi|^m} d\xi \\ &= \frac{1}{(2\pi)^d} e^{t\sigma|\eta|^m} e^{t\omega} e^{-x \cdot \eta} \left(\frac{ct}{2}\right)^{-d/m} \int_{\mathbb{R}^d} e^{-|\xi|^m} d\xi. \end{aligned}$$

In the last line we used the change of variables  $\xi \mapsto (\frac{ct}{2})^{-1/m} \xi$ . Introducing a new suitable constant  $c_1 \geq 0$ , we now have an estimate of the form

$$|(\mathcal{F}^{-1}e^{-ta})(x)| \leq c_1 e^{t\omega} t^{-d/m} e^{t\sigma|\eta|^m} e^{-x \cdot \eta}.$$

If we choose  $\eta = \left(\frac{|x|}{m\sigma t}\right)^{1/(m-1)} \frac{x}{|x|}$ , we get the desired bound by calculating

$$\begin{aligned} t\sigma|\eta|^m - x \cdot \eta &= t\sigma \left(\frac{|x|}{m\sigma t}\right)^{m/(m-1)} - \left(\frac{|x|}{m\sigma t}\right)^{1/(m-1)} |x| \\ &= \frac{1}{m} \left(\frac{|x|^m}{m\sigma t}\right)^{1/(m-1)} - \left(\frac{|x|^m}{m\sigma t}\right)^{1/(m-1)} |x| \\ &= -\frac{m-1}{m} \left(\frac{1}{m\sigma}\right)^{1/(m-1)} \left(\frac{|x|^m}{t}\right)^{1/(m-1)}. \end{aligned} \quad \square$$

It follows from the heat kernel bound (3.3) that the operator semigroup  $(S_t)_{t \geq 0}$ , defined by (3.2), is strongly continuous if  $p \in [1, \infty)$ . For  $p = \infty$  the semigroup is the dual semigroup of a strongly continuous semigroup on  $L_1(\mathbb{R}^d)$  and hence it is weak\*-continuous in general. For details we refer, e.g., to [Are02, Section 7.2]. By Young's inequality, we observe that for all  $p \in [1, \infty]$

$$\begin{aligned} \|S_t\|_{\mathcal{L}(L_p(\mathbb{R}^d))} &= \|\mathcal{F}^{-1}e^{-ta}\|_{L_1(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} c_1 e^{\omega t} t^{-d/m} e^{-c_2 \left(\frac{|x|^m}{t}\right)^{\frac{1}{m-1}}} dx \\ &= e^{\omega t} \int_{\mathbb{R}^d} c_1 e^{-c_2 |y|^{m/(m-1)}} dy. \end{aligned} \quad (3.5)$$

Hence, we have  $\|S_t\|_{L_p(\mathbb{R}^d)} \leq M e^{\omega t}$  for all  $t \geq 0$  with  $\omega \in \mathbb{R}$  as in (3.4) and

$$M = \int_{\mathbb{R}^d} c_1 e^{-c_2 |y|^{m/(m-1)}} dy. \quad (3.6)$$

Note that  $M$  is independent of  $p$ . For  $p \in [1, \infty)$ , let  $-A_p$  be the generator of the  $C_0$ -semigroup  $(S_t)_{t \geq 0}$  on  $L_p(\mathbb{R}^d)$ . Then for all  $f \in \mathcal{S}(\mathbb{R}^d)$  we have

$$A_p f = \sum_{|\alpha|_1 \leq m} a_\alpha (-i)^{|\alpha|} \partial^\alpha f.$$

**Example 3.5.** Let  $p \in [1, \infty)$ . The standard Gaussian semigroup  $(G_t)_{t \geq 0}$  on  $L_p(\mathbb{R}^d)$ ,

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given by

$$(G_t f)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy \quad t > 0, x \in \mathbb{R}^d,$$

corresponds to the strongly elliptic polynomial  $a : \mathbb{R}^d \rightarrow \mathbb{R}$  defined by  $a(\xi) = |\xi|^2$ . Indeed, for  $t > 0$  and  $x \in \mathbb{R}^d$  we have

$$(\mathcal{F}^{-1} e^{-t|\cdot|^2})(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}},$$

and hence  $G_t f = \mathcal{F}^{-1} e^{-t|\cdot|^2} * f$  for  $f \in L_p(\mathbb{R}^d)$ . The generator  $-A_p$  of  $(G_t)_{t \geq 0}$  is the maximal Laplacian in  $L_p(\mathbb{R}^d)$ , i.e.,

$$\begin{aligned} \mathcal{D}(A_p) &:= \left\{ f \in L_p(\mathbb{R}^d) : \Delta f = \sum_{k=1}^d \partial_k^2 f \in L_p(\mathbb{R}^d) \right\} \\ A_p f &:= -\Delta f \quad \text{for } f \in \mathcal{D}(A_p). \end{aligned}$$

For more details see, e.g., [EN99]. ◇

## 3.2 Geometric conditions for observability

In this section, we formulate our main theorem on final-state observability of parabolic equations corresponding to strongly elliptic polynomials. This section is based on the results obtained in [GST20, BGST22b] by the author in collaboration with C. Bombach, C. Seifert, and M. Tautenhahn.

### 3.2.1 Thick sets

To formulate our main result, we introduce the notion of thick subsets of  $\mathbb{R}^d$ . This property originates in the theory of uncertainty principles in harmonic analysis, and the term arises in [Kov01]. Earlier, e.g., in [Kac73], such sets were called relatively dense (see also [HJ12]).

**Definition 3.6.** Let  $\rho \in (0, 1]$  and  $L > 0$ . A measurable set  $E \subset \mathbb{R}^d$  is called  $(\rho, L)$ -thick if for all  $x \in \mathbb{R}^d$  we have

$$|E \cap (x + [0, L]^d)| \geq \rho L^d.$$

Here,  $|\cdot|$  denotes the Lebesgue measure in  $\mathbb{R}^d$ . Moreover,  $E \subset \mathbb{R}^d$  is called *thick* if there are  $\rho \in (0, 1]$  and  $L > 0$  such that  $E$  is  $(\rho, L)$ -thick.

By definition, the measure of the part of a  $(\rho, L)$ -thick set in a cube of length  $L$  is larger than  $\rho L^d$ . The parameter  $\rho$  can be thought of as the minimum density of  $E$ . Obviously, a thick set can not be bounded. For  $k \in \mathbb{N}$ , a  $(\rho, L)$ -thick set is also  $(\rho, kL)$ -thick. But the converse is not true if  $k \geq 2$ . Next, we give some examples of thick sets.

**Example 3.7.** Clearly,  $E = \mathbb{R}^d$  is a  $(1, L)$ -thick set in  $\mathbb{R}^d$  for every  $L > 0$  and also the set difference  $E = \mathbb{R}^d \setminus B(0, 1)$  is thick. A thick set has to be well spread in  $\mathbb{R}^d$ , such as the set  $E = \cup_{k \in \mathbb{Z}} [2k, 2k + 1]$  in  $\mathbb{R}$ . However thick sets can be more subtle, as the set

$$E = \bigcup_{k \in \mathbb{Z}} \bigcup_{i \in \mathbb{N}} \left[ k + \frac{1}{2^i}, k + \frac{1}{2^i} + \frac{1}{2^{i+1}} \right], \quad (3.7)$$

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mentioned in [WWZZ19], which is a  $(1/2, 1)$ -thick set in  $\mathbb{R}$  (see also Figure 3.1).  $\diamond$



Figure 3.1: Illustration of the thick set given in (3.7).

There is a complete description of thick sets in terms of an uncertainty principle for functions whose Fourier transform has compact support. We will discuss this in Section 3.3 in more detail.

#### 3.2.2 Observability estimate

For a strongly elliptic polynomial  $a : \mathbb{R}^d \rightarrow \mathbb{C}$ , we consider the associated semigroup  $(S_t)_{t \geq 0}$  on  $L_p(\mathbb{R}^d)$ ,  $p \in [1, \infty]$ , given by (3.2), i.e.,

$$S_0 f = f, \quad S_t f = \mathcal{F}^{-1} e^{-ta} * f, \quad t > 0.$$

In the following Theorem 3.8, we state a final-state observability estimate for the semigroup  $(S_t)_{t \geq 0}$  and thick observation sets  $E \subset \mathbb{R}^d$ . Furthermore, we formulate an upper bound of the observability constant  $C_{\text{obs}}$  with explicit dependence on the final time  $T$ , the geometry of the thick set  $E$ , and the polynomial  $a$ . Although the mapping  $t \mapsto (S_t f)|_E$  for  $f \in L_p(\mathbb{R}^d)$  may not be Bochner measurable for  $p = \infty$ , it turns out that the function  $t \mapsto \|(S_t f)|_E\|_{L_p(E)}$  is measurable for all  $p \in [1, \infty]$ . This allows us to write down the following estimate.

**Theorem 3.8.** *Let  $m \in \mathbb{N}$ ,  $a : \mathbb{R}^d \rightarrow \mathbb{C}$  be a strongly elliptic polynomial of order  $m$ ,  $c > 0$  and  $\omega \in \mathbb{R}$  as in (3.4), and  $(S_t)_{t \geq 0}$  as in (3.2). Let  $\rho \in (0, 1]$ ,  $L > 0$ ,  $E \subset \mathbb{R}^d$  a  $(\rho, L)$ -thick set,  $p, r \in [1, \infty]$ , and  $T > 0$ . Then we have for all  $f \in L_p(\mathbb{R}^d)$*

$$\|S_T f\|_{L_p(\mathbb{R}^d)} \leq \begin{cases} C_{\text{obs}} \left( \int_0^T \|(S_t f)|_E\|_{L_p(E)}^r dt \right)^{1/r} & \text{if } r \in [1, \infty), \\ C_{\text{obs}} \operatorname{ess\,sup}_{t \in [0, T]} \|(S_t f)|_E\|_{L_p(E)} & \text{if } r = \infty, \end{cases} \quad (3.8)$$

where

$$C_{\text{obs}} = \frac{K_a}{T^{1/r}} \left( \frac{K_d}{\rho} \right)^{K_d} \exp \left( \frac{K_m (L \ln(K_d/\rho))^{\frac{m}{m-1}}}{(cT)^{\frac{1}{m-1}}} + \omega_+ T \right).$$

Here,  $T^{1/r} = 1$  if  $r = \infty$ ,  $\omega_+ = \max\{\omega, 0\}$ , and  $K_a, K_d, K_m > 0$  are constants depending only on the polynomial  $a$ , on  $d$ , or on  $m$ , respectively.

In order to prove Theorem 3.8, we follow the generalized Lebeau–Robbiano strategy elaborated in Chapter 2. To this end, we define a family  $(P_\lambda)_{\lambda > 0}$  of operators in  $L_p(\mathbb{R}^d)$  such that the assumptions of Theorem 2.6, i.e., an uncertainty principle and a dissipation estimate, are satisfied. The proof is divided into several steps, each treated in a separate section. For the uncertainty principle, we shall need a so-called Logvinenko–Sereda theorem, which we prove in Section 3.3. Concerning the dissipation estimate, we first consider

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semigroups associated with powers of the Laplacian and then use a perturbation argument (see Section 3.4). Finally, in Section 3.5, we finish the proof of Theorem 3.8 by combining the previous results.

Before starting the proof, we first want to validate the quality of Theorem 3.8. In Section 3.2.3, we show that the thickness of the observation set  $E$  is not only sufficient but also necessary for final-state observability. Furthermore, in Section 3.2.4 and Section 3.2.5 we show that the constant  $C_{\text{obs}}$  behaves optimally in certain asymptotic regimes, i.e., for small and large times  $T$ , as well as, in the homogenization limit of the thick set  $E$ .

#### 3.2.3 Sharpness of the geometric conditions

By Theorem 3.8, the observability estimate (3.8) is satisfied, provided  $E$  is a thick set. In this section, we show that this geometric condition on the observation set  $E$  is sharp for  $p \in [1, \infty)$ . This full characterization was independently proven in [EV18] and [WWZZ19] for the heat equation in the Hilbert space  $L_2(\mathbb{R}^d)$ . Using the heat kernel bound in Proposition 3.4, we can extend this result for semigroups associated with strongly elliptic polynomials with lower order terms on  $L_p(\mathbb{R}^d)$  for  $p \in [1, \infty)$ .

**Theorem 3.9.** *Let  $p \in [1, \infty)$ ,  $r \in [1, \infty]$ ,  $a: \mathbb{R}^d \rightarrow \mathbb{C}$  be a strongly elliptic polynomial, and  $(S_t)_{t \geq 0}$  as in (3.2). Let  $E \subset \mathbb{R}^d$  be measurable,  $T > 0$ , and assume there exists  $C_{\text{obs}} > 0$  such that for all  $f \in L_p(\mathbb{R}^d)$  we have*

$$\|S_T f\|_{L_p(\mathbb{R}^d)} \leq C_{\text{obs}} \|(S_{(\cdot)} f)|_E\|_{L_r((0,T);L_p(E))}.$$

*Then  $E$  is a thick set.*

*Proof.* We apply the strategy developed in [EV18, WWZZ19] and show the contra position. Assume that  $E$  is not thick. Then there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^d$  such that for all  $n \in \mathbb{N}$  we have

$$|E \cap B(x_n, n)| < \frac{1}{n}, \quad (3.9)$$

where  $B(x_n, n)$  denotes the ball in  $\mathbb{R}^d$  around  $x_n$  with radius  $n$ . Recall that the semigroup  $(S_t)_{t \geq 0}$  on  $L_p(\mathbb{R}^d)$  is given as a convolution operator

$$S_t f = \mathcal{F}^{-1} e^{-ta} * f$$

with convolution kernel  $p_t := \mathcal{F}^{-1} e^{-ta} \in \mathcal{S}(\mathbb{R}^d)$  for  $t > 0$ . For  $n \in \mathbb{N}$ , we define  $f_n := p_1(\cdot - x_n)$ . As a consequence, we observe for all  $t > 0$  and  $n \in \mathbb{N}$

$$S_t f_n = p_t * f_n = p_t * p_1(\cdot - x_n) = p_{t+1}(\cdot - x_n), \quad (3.10)$$

and hence, by translation invariance of the Lebesgue measure,

$$\|S_T f_n\|_{L_p(\mathbb{R}^d)} = \|p_{T+1}(\cdot - x_n)\|_{L_p(\mathbb{R}^d)} = \|p_{T+1}\|_{L_p(\mathbb{R}^d)}. \quad (3.11)$$

For  $n \in \mathbb{N}$ , we now shift the set  $E$  by  $x_n$  and consider the set  $E - x_n = \{y \in \mathbb{R}^d: y + x_n \in E\}$ . Note that (3.9) is equivalent to  $|(E - x_n) \cap B(0, n)| < 1/n$  for all  $n \in \mathbb{N}$ . From the latter fact, (3.10), and substitution, we obtain for all  $t > 0$  and  $n \in \mathbb{N}$

$$\|(S_t f_n)|_E\|_{L_p(E)}^p = \|\mathbf{1}_E S_t f_n\|_{L_p(\mathbb{R}^d)}^p$$

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$$\begin{aligned}
&= \|\mathbf{1}_{(E-x_n) \cap B(0,n)} p_{t+1}\|_{L_p(\mathbb{R}^d)}^p + \|\mathbf{1}_{(E-x_n)}(1 - \mathbf{1}_{B(0,n)}) p_{t+1}\|_{L_p(\mathbb{R}^d)}^p \\
&\leq \|p_{t+1}\|_{L_\infty(\mathbb{R}^d)}^p |(E - x_n) \cap B(0, n)| + \|(1 - \mathbf{1}_{B(0,n)}) p_{t+1}\|_{L_p(\mathbb{R}^d)}^p \\
&< \frac{1}{n} \|p_{t+1}\|_{L_\infty(\mathbb{R}^d)}^p + \|(1 - \mathbf{1}_{B(0,n)}) p_{t+1}\|_{L_p(\mathbb{R}^d)}^p. \tag{3.12}
\end{aligned}$$

By the heat kernel bound in Proposition 3.4, there is  $M \geq 0$  such that

$$\|p_{t+1}\|_{L_\infty(\mathbb{R}^d)}^p \leq M \quad \text{for all } t \in (0, T]. \tag{3.13}$$

For the second summand we use again the heat kernel bound (Proposition 3.4) and find for all  $t > 0$

$$\begin{aligned}
\|(1 - \mathbf{1}_{B(0,n)}) p_{t+1}\|_{L_p(\mathbb{R}^d)}^p &= \int_{|x| \geq n} |p_{t+1}(x)|^p dx \\
&= \frac{c_1^p e^{p\omega(t+1)}}{(t+1)^{pd/m}} \int_{|x| \geq n} e^{-pc_2 \left(\frac{|x|^m}{t+1}\right)^{\frac{1}{m-1}}} dx \\
&\leq c_1^p e^{p\omega(T+1)} \int_{|x| \geq n} e^{-pc_2 \left(\frac{|x|^m}{T+1}\right)^{\frac{1}{m-1}}} dx \\
&\leq c_1^p e^{p\omega(T+1)} e^{-\frac{1}{2}pc_2 \left(\frac{n^m}{T+1}\right)^{1/m}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}pc_2 |x|^{\frac{m}{m-1}}} dx. \tag{3.14}
\end{aligned}$$

Since the bounds obtained in (3.13) and (3.14) are independent of  $t \in (0, T]$ , and tend to 0 as  $n$  tends to infinity, we obtain from (3.12) that

$$\|(S_{(\cdot)} f_n)|_E\|_{L_r((0,T);L_p(\mathbb{R}^d))} \rightarrow 0 \tag{3.15}$$

as  $n$  tends to infinity. From (3.11) and (3.15), we conclude that for all  $C_{\text{obs}} > 0$  there exists  $f \in L_p(\mathbb{R}^d)$  such that

$$\|S_T f\|_{L_p(\mathbb{R}^d)} > C_{\text{obs}} \|(S_{(\cdot)} f)|_E\|_{L_r((0,T);L_p(E))}.$$

This proves the contraposition of the theorem.  $\square$

Combining Theorem 3.8 and Theorem 3.9 we get the following equivalent characterization of thick sets.

**Corollary 3.10.** *Let  $p \in [1, \infty)$ ,  $r \in [1, \infty]$ ,  $T > 0$  and  $E \subset \mathbb{R}^d$  measurable. Then  $E$  is a thick set if and only if  $(S_t)_{t \geq 0}$ , given by (3.2), satisfies a final-state observability estimate in  $L_r((0, T); L_p(E))$ .*

In Section 3.3, we recall the Logvinenko–Sereda theorem (Theorem 3.13) which yields another equivalent characterization of thick sets in terms of an uncertainty principle. Together with Corollary 3.10, this yields a generalization of the main theorem in [WWZZ19, Theorem 1.1] to systems corresponding to strongly elliptic polynomials including lower order terms on  $L_p(\mathbb{R}^d)$ ,  $p \in [1, \infty)$ . Note that in [WWZZ19] additionally an equivalent characterization in terms of a Hölder-type interpolation inequality is shown.

### 3.2.4 Optimal time dependence

Let us give some comments on the time dependence of the observability constant derived in Theorem 3.8, i.e.,

$$C_{\text{obs}} = \frac{K_1}{T^{1/r}} \exp\left(\frac{K_2}{T^{\frac{1}{m-1}}} + \omega_+ T\right),$$

for some  $K_1, K_2 > 0$ . First, we discuss results that indicate that the dependence of  $C_{\text{obs}}$  on  $T$  is optimal for small  $T$ . In [Sei84] Seidman showed that for one-dimensional controlled heat systems on bounded domains the constant  $C_{\text{obs}}$  blows up at most exponentially for small  $T$ . This result was extended to arbitrary dimension by Fursikov and Imanuvilov in [FI96]. That the exponential blow-up has to occur for small  $T$ , was first shown by Güichal in [Güi85] for one-dimensional systems and by Miller in [Mil04] in arbitrary dimension. If one considers a linear control problem in  $L_2(\Omega)$  with bounded open  $\Omega \subset \mathbb{R}^d$  instead of  $\mathbb{R}^d$ , and  $(S_t)_{t \geq 0}$  being the heat-semigroup, it has been shown that the observability constant  $C_{\text{obs}}(T)$  must satisfy

$$\sup_{\overline{B(0,\rho)} \subset \Omega \setminus E} \rho^2/4 \leq \liminf_{T \rightarrow 0} T \ln C_{\text{obs}}(T) \quad (3.16)$$

(see [FCZ00, Zua01, Mil04]). This shows that the exponential blowup for  $T \rightarrow 0$  has to occur for the controlled heat equation on bounded open subsets  $\Omega \subset \mathbb{R}^d$ . In order to extend (3.16) to the case  $\Omega = \mathbb{R}^d$  it seems feasible to apply the method obtained in [SV20]. In that paper, the authors show that if the controlled heat equation on  $\Omega = \Lambda_L = (-L/2, L/2)^d$  satisfies an observability estimate with a constant independent of  $L > 0$ , then, using a continuity argument, the corresponding system on  $\Omega = \mathbb{R}^d$  satisfies an observability estimate as well with the same upper bound. By an analogous argument, the lower bound (3.16) should hold in the case  $\Omega = \mathbb{R}^d$  as well. This suggests that  $C_{\text{obs}}$  in Theorem 3.8 is optimal in the regime  $T \rightarrow 0$ .

Let us now study the large-time regime. For bounded semigroups, i.e.,  $\omega_+ = 0$ , the constant  $C_{\text{obs}}$  in Theorem 3.8 tends to zero as  $T \rightarrow \infty$ . We use the following direct estimate from [NTTV20a, Theorem 2.13] to show that the decay rate  $1/T^{1/r}$  is in fact optimal for the Gaussian semigroup.

**Proposition 3.11.** *Let  $(G_t)_{t \geq 0}$  be the Gaussian semigroup on  $L_2(\mathbb{R}^d)$  (cf. Example 3.5). For  $\omega \geq 0$  consider the rescaled semigroup  $(\tilde{G}_t)_{t \geq 0}$  given by  $\tilde{G}_t := e^{\omega t} G_t$  for all  $t \geq 0$ . Assume there are  $C_{\text{obs}} > 0$ ,  $r \in [1, \infty]$ ,  $T > 0$ , and some measurable set  $E \subset \mathbb{R}^d$  such that*

$$\forall f \in L_2(\mathbb{R}^d) : \quad \|\tilde{G}_T f\|_{L_2(\mathbb{R}^d)} \leq C_{\text{obs}} \|(\tilde{G}_{(\cdot)} f)|_E\|_{L_r((0,T);L_2(E))}.$$

Then

$$C_{\text{obs}} \geq \begin{cases} \frac{1}{T^{1/r}} & \text{for } \omega = 0, \\ \left(\frac{r\omega}{1-e^{-r\omega T}}\right)^{1/r} & \text{for } \omega > 0, \end{cases}$$

if  $r \in [1, \infty)$  and  $C_{\text{obs}} \geq 1$  if  $r = \infty$ .

For semigroups with exponential growth, i.e.,  $\omega > 0$ , Proposition 3.11 shows that the constant  $C_{\text{obs}}$  remains bounded from below as  $T \rightarrow \infty$ , whereas  $C_{\text{obs}}$  in Theorem 3.8 grows exponentially for large  $T$ .

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*Proof of Proposition 3.11.* First, we use the trivial estimate

$$\|\tilde{G}_T f\|_{L_2(\mathbb{R}^d)} \leq C_{\text{obs}} \|(\tilde{G}_{(\cdot)} f)|_E\|_{L_r((0,T);L_2(E))} \leq C_{\text{obs}} \|\tilde{G}_{(\cdot)} f\|_{L_r((0,T);L_2(\mathbb{R}^d))},$$

for  $f \in L_2(\mathbb{R}^d)$ , to obtain the lower bound

$$C_{\text{obs}} \geq \sup_{f \in L_2(\mathbb{R}^d) \setminus \{0\}} \frac{\|\tilde{G}_T f\|_{L_2(\mathbb{R}^d)}}{\|\tilde{G}_{(\cdot)} f\|_{L_r((0,T);L_2(\mathbb{R}^d))}}. \quad (3.17)$$

For  $\varepsilon > 0$ , let  $f_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$  with  $\text{supp } \mathcal{F}f_\varepsilon \subset B(0, \sqrt{\varepsilon})$ . Then, using Plancherel's theorem, we obtain

$$\begin{aligned} \|\tilde{G}_T f_\varepsilon\|_{L_2(\mathbb{R}^d)} &= e^{\omega T} \|\mathcal{F}^{-1}(e^{-T|\cdot|^2} \mathcal{F}f_\varepsilon)\|_{L_2(\mathbb{R}^d)} \\ &= e^{\omega T} (2\pi)^{-d} \|e^{-T|\cdot|^2} \mathcal{F}f_\varepsilon\|_{L_2(\mathbb{R}^d)} \\ &\geq e^{(\omega-\varepsilon)T} \|f_\varepsilon\|_{L_2(\mathbb{R}^d)}. \end{aligned} \quad (3.18)$$

On the other hand, we use that  $\|\tilde{G}_t f_\varepsilon\|_{L_2(\mathbb{R}^d)} \leq e^{\omega t} \|G_t f_\varepsilon\|_{L_2(\mathbb{R}^d)} \leq e^{\omega t} \|f_\varepsilon\|_{L_2(\mathbb{R}^d)}$  for  $t \geq 0$ , and hence

$$\|\tilde{G}_{(\cdot)} f_\varepsilon\|_{L_r((0,T);L_2(\mathbb{R}^d))} \leq \|e^{\omega(\cdot)}\|_{L_r(0,T)} \|f_\varepsilon\|_{L_2(\mathbb{R}^d)}. \quad (3.19)$$

Using inequalities (3.18) and (3.19) for the family  $(f_\varepsilon)_{\varepsilon>0}$  in (3.17), we obtain in the limit  $\varepsilon \rightarrow 0$  that

$$C_{\text{obs}} \geq \frac{e^{\omega T}}{\|e^{\omega(\cdot)}\|_{L_r(0,T)}}.$$

The claim now follows by computing

$$\|e^{\omega(\cdot)}\|_{L_r(0,T)} = \begin{cases} T^{1/r} & \text{for } \omega = 0, \\ \left(\frac{e^{r\omega T} - 1}{r\omega}\right)^{1/r} & \text{for } \omega > 0, \end{cases}$$

for  $r \in [1, \infty)$  and  $\|e^{\omega(\cdot)}\|_{L_\infty(0,T)} = e^{\omega T}$ .  $\square$

#### 3.2.5 Homogenization

The explicit description of the observability constant  $C_{\text{obs}}$  in Theorem 3.8 allows for studying homogenization properties. Here we follow the approach presented in [NTTV20a] and analyze the behavior of the observability constant as the observation set  $E$  gets more and more evenly distributed. Note that the term homogenization is also used for differential equations with oscillating coefficients (see, e.g., [Zua94, LZ02]). However, we keep the differential equation fixed and vary the observation set instead.

Recall that a thick set  $E$  can be parametrized via

$$\left| E \cap \left(x + [0, L]^d\right) \right| \geq \rho L^d, \quad \text{for all } x \in \mathbb{R}^d, \quad (3.20)$$

by two parameters, a minimum density  $\rho \in (0, 1]$  and a cube size  $L > 0$ . For a fixed density  $\rho \in (0, 1]$  and a sequence  $(L_n)_{n \in \mathbb{N}} \subset (0, \infty)$ , we call a sequence of  $(\rho, L_n)$ -thick sets

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homogenizing if  $L_n$  tends to zero for  $n \rightarrow \infty$ . This limit can be thought of as reducing the local differences in the density of a thick set, approaching a homogeneous observation set with the same density everywhere. See Figure 3.2 for a visualization.

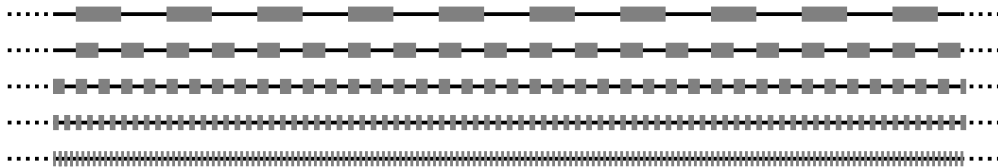


Figure 3.2: Illustration of homogenizing thick sets in  $\mathbb{R}$ .

Applying Theorem 3.8 to a  $(\rho, L_n)$ -thick set, yields the observability constant

$$C_{\text{obs}} = \frac{K_a}{T^{1/r}} \left( \frac{K_d}{\rho} \right)^{K_d} \exp \left( \frac{K_m (L_n \ln(K_d/\rho))^{\frac{m}{m-1}}}{(cT)^{\frac{1}{m-1}}} + \omega_+ T \right),$$

where  $r \in [1, \infty)$ . In the limit  $L_n \rightarrow 0$ , we observe that the exponential blow up for  $T \rightarrow 0$  vanishes and the observability constant tends to

$$\frac{K_a}{T^{1/r}} \left( \frac{K_d}{\rho} \right)^{K_d} e^{\omega_+ T}. \quad (3.21)$$

As we expect, this corresponds to a multiple of the observability constant for a system with full observation  $E = \mathbb{R}^d$ . In this case, an observability estimate is given by the trivial averages of the norm of the semigroup

$$\|S_T x\|_{L_p(\mathbb{R}^d)} \leq \frac{M}{T^{1/r}} e^{\omega_+ T} \left( \int_0^T \|S_t x\|_{L_p(\mathbb{R}^d)}^r dt \right)^{1/r}, \quad x \in L_p(\mathbb{R}^d), \quad (3.22)$$

where  $r \in [1, \infty)$ , and  $M \geq 1$  and  $\omega_+ = \max\{\omega, 0\} \geq 0$  such that  $\|S_t\|_{\mathcal{L}(L_p(\mathbb{R}^d))} \leq M e^{\omega t}$  for  $t \geq 0$ . Indeed, we obtain (3.22) by integrating both sides of the inequality

$$\|S_T x\|_{L_p(\mathbb{R}^d)} = \|S_{T-t} S_t x\|_{L_p(\mathbb{R}^d)} \leq M e^{\omega T} \|S_t x\|_{L_p(\mathbb{R}^d)}, \quad x \in L_p(\mathbb{R}^d), \quad t \in [0, T].$$

Note that, by the lower bound given in Proposition 3.11, the term  $1/T^{1/r}$  in the observability constant can not vanish.

The authors of [NTTV20a] also considered the converse problem of de-homogenization. This is the behavior of  $C_{\text{obs}}$  when for a  $(\rho, L)$ -thick set, the parameter  $L$  tends to infinity while again  $\rho$  remains constant. In this case, the observability constant must tend to infinity as  $L \rightarrow \infty$ , which can be confirmed by the estimate on  $C_{\text{obs}}$  in Theorem 3.8 again.

### 3.3 Uncertainty principle in harmonic analysis

The uncertainty principle from quantum physics states that the position and momentum of particles cannot be precisely determined simultaneously. There are various mathematical formulations of this principle in the context of harmonic analysis, for example, Heisenberg's

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uncertainty principle, the Paley–Wiener theorem, Hardy’s uncertainty principle, or the theory of annihilating pairs. A good presentation of mathematical interpretations can be found in the book [HJ12], and the survey articles [FS97, BD06].

Here, we are particularly interested in the fact that a non-zero function and its Fourier transform cannot be both compactly supported. An important result for tempered distributions with compactly supported Fourier transform is Schwartz’s Paley–Wiener theorem. We state an extract of [RS75, Theorem IX.12]. Recall that for  $u \in \mathcal{S}'(\mathbb{R}^d)$  and  $K \subset \mathbb{R}^d$  closed, we have  $\text{supp } u \subset K$  if and only if  $u(\phi) = 0$  for all  $\phi \in \mathcal{S}(\mathbb{R}^d)$  with  $\text{supp } \phi \subset \mathbb{R}^d \setminus K$ .

**Theorem 3.12** (Schwartz’s Paley–Wiener theorem). *Let  $u \in \mathcal{S}'(\mathbb{R}^d)$ . Then  $\mathcal{F}u$  has compact support if and only if  $u$  has an analytic extension to an entire analytic function on  $\mathbb{C}^d$  that satisfies*

$$|u(\xi)| \leq K(1 + |\xi|)^N e^{\lambda |\text{Im } \xi|}$$

for all  $\xi \in \mathbb{C}^d$  and some constants  $K, N, \lambda > 0$ .

Note that the proof of this theorem is the same as in [RS75, Theorem IX.12] with the only difference that we have to exchange  $\mathcal{F}$  by  $\mathcal{F}^{-1}$ . Since an entire function cannot have compact support unless it vanishes, Theorem 3.12 verifies that  $u$  and  $\mathcal{F}u$  cannot have both compact support. Schwartz’s Paley–Wiener theorem relates the compact support of  $\mathcal{F}u$  to a growth estimate for  $u$ . In this section, we focus on another quantification of the uncertainty principle, namely on characterizing subsets  $E \subset \mathbb{R}^d$  for which the following inequality is satisfied:

$$\|f\|_{L_p(\mathbb{R}^d)} \leq K_{E,\lambda} \|f\|_{L_p(E)} \quad \text{for all } f \in L_p(\mathbb{R}^d) \text{ with } \text{supp } \mathcal{F}f \subset \overline{B(0, \lambda)}, \quad (3.23)$$

and some constant  $K_{E,\lambda} > 0$  depending only on  $E \subset \mathbb{R}^d$  and  $\lambda > 0$ . By the Paley–Wiener theorem a function  $f \in L_p(\mathbb{R}^d)$  with  $\text{supp } \mathcal{F}f \subset \overline{B(0, \lambda)}$  cannot have compact support. Hence, suitable sets  $E$  have to be sufficiently large and, e.g., cannot be bounded. Since  $\|\cdot\|_{L_p(E)}$  defines a semi-norm on the space of functions in  $L_p(\mathbb{R}^d)$  with compactly supported Fourier transform, (3.23) implies that  $\|\cdot\|_{L_p(E)}$  defines a norm on this space which is equivalent to  $\|\cdot\|_{L_p(\mathbb{R}^d)}$ . If (3.23) holds, the tuple  $(\mathbb{R}^d \setminus E, \overline{B(0, \lambda)})$  is called a  $p$ -strong annihilation pair (see [BD06, HJ12]).

The problem of finding  $E \subset \mathbb{R}^d$  such that (3.23) holds for some  $K_{E,\lambda} > 0$  was first considered by Panejah in [Pan61, Pan62] for  $p = 2$ . See also [ENS<sup>+</sup>20] for a review of these results. Using a different approach, Logvinenko and Sereda [LS74] and Kacnel’son [Kac73] proved independently the following complete description of such sets.

**Theorem 3.13** (Logvinenko–Sereda, Kacnel’son). *Let  $p \in [1, \infty]$  and  $E \subset \mathbb{R}^d$ . There is  $K_{E,\lambda} > 0$  such that (3.23) holds if and only if  $E$  is a thick-set (cf. Definition 3.6).*

The constant is specified to  $K_{E,\lambda} = c_1 e^{c_2 \lambda}$  where  $c_1, c_2$  only depend on the thickness parameters  $\rho$  and  $L$ , and the dimension  $d$ . Later the constant  $K_{E,\lambda}$  was improved by Kovrijkine in [Kov00, Kov01], where he proved that  $K_{E,\lambda}$  depends polynomially on the thickness parameter  $\rho$ . This behavior is optimal as an example in [Kov00] shows.

**Theorem 3.14** (Kovrijkine). *Let  $p \in [1, \infty]$ ,  $\lambda > 0$ ,  $\rho \in (0, 1]$ ,  $L > 0$ , and  $E \subset \mathbb{R}^d$  be a  $(\rho, L)$ -thick set. Then for all  $f \in L_p(\mathbb{R}^d)$  satisfying  $\text{supp } \mathcal{F}f \subset \overline{B(0, \lambda)}$  we have*

$$\|f\|_{L_p(\mathbb{R}^d)} \leq \left(\frac{K_d}{\rho}\right)^{K_d(1+\lambda L)} \|f\|_{L_p(E)}, \quad (3.24)$$

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where  $K_d > 0$  is a constant depending only on  $d$ .

The constant in (3.24) transfers to the observability constant given in Theorem 3.8. There the explicit form is especially important since it allows, e.g., to study homogenization asymptotics (see Section 3.2.5 and [NTTV20a]). Following the ideas of Kovrijkine, Theorem 3.14 has recently been adapted to functions on hyper-rectangles or infinite strips instead of  $\mathbb{R}^d$  with suitable restrictions on the Fourier series (see [EV20, Egi21]). See also [ES21] for an abstraction of this approach.

**Example 3.15** (Spectral inequality). Let  $p = 2$  and consider the maximal Laplacian  $-\Delta$  on  $L_2(\mathbb{R}^d)$ . The Laplacian is a self-adjoint operator with spectrum  $[0, \infty)$ . The spectral projector  $P_\lambda$  of  $-\Delta$  up to  $\lambda > 0$  is given through the Fourier transform by

$$P_\lambda f = \frac{1}{(2\pi)^d} \int_{B(0, \sqrt{\lambda})} \mathcal{F}f(\xi) e^{ix \cdot \xi} d\xi.$$

Therefore the spectral subspaces satisfy  $\text{Ran } P_\lambda = \{f \in L_2(\mathbb{R}^d) : \text{supp } \mathcal{F}f \subset \overline{B(0, \sqrt{\lambda})}\}$  and, by Theorem 3.14, the following inequality holds:

$$\|f\|_{L_2(\mathbb{R}^d)} \leq \left(\frac{K_d}{\rho}\right)^{K_d(1+\sqrt{\lambda}L)} \|f\|_{L_2(E)} \quad \text{for all } f \in \text{Ran } P_\lambda.$$

If in (3.23) the condition on the support of the Fourier transform is replaced by requiring the functions to be in a spectral subspace of a self-adjoint operator, (3.23) is called *spectral inequality*. See [LR95, LZ98, JL99] for the first results on Laplace-Beltrami operators on compact manifolds and, e.g., [NTTV18, LM19, EV20, ENS<sup>+</sup>20, ES21] for an overview and recent developments.  $\diamond$

#### 3.3.1 Proof of the Logvinenko–Sereda theorem

For completeness, we give a detailed proof of Theorem 3.14 based on the work by Kovrijkine. See [Kov01] for  $d = 1$  and in [Kov00] for the higher dimensional case. In addition, we use some ideas from the complete proof given in [WWZZ19, Lemma 2.1] for the Hilbert space case  $p = 2$ . Other approaches, which lead to different constants  $K_{E,\lambda}$ , can be found in the original work by Logvinenko and Sereda [LS74] and Kacnel’son [Kac73]. See also the recent article [LM19] for a proof using Carleman estimates for functions of one complex variable. For a generalization of Kovrijkin’s proof for spectral subspaces, see [ES21].

The proof in [Kov00, Kov01] is based on two core concepts, namely, on the one hand, the analyticity of a function with compactly supported Fourier transform due to Schwartz’s Paley–Wiener theorem (Theorem 3.12) and on the other hand Bernstein’s inequality. Here, we cite a version from [BCD11, Lemma 2.1].

**Lemma 3.16** (Bernstein’s inequality). *There is  $K_B > 0$  such that for all  $\lambda > 0$ , all  $p \in [1, \infty]$ , all  $f \in L_p(\mathbb{R}^d)$  with  $\text{supp } \mathcal{F}f \subset \overline{B(0, \lambda)}$ , and all multi-indices  $\alpha \in \mathbb{N}_0^d$ , we have*

$$\|\partial^\alpha f\|_{L_p(\mathbb{R}^d)} \leq (K_B \lambda)^{|\alpha|_1} \|f\|_{L_p(\mathbb{R}^d)}. \quad (3.25)$$

For the proof of Theorem 3.14, we will use the following local estimate for analytic functions of one complex variable proved by Kovrijkine in [Kov01, Lemma 1].

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**Lemma 3.17.** *Let  $I \subset \mathbb{R}$  be an interval of length 1 such that  $0 \in I$ , and let  $J \subset I$  with positive measure  $|J| > 0$ . Let  $\Phi$  be an analytic function on  $B(0, 5) \subset \mathbb{C}$  with  $|\Phi(0)| \geq 1$ . Then there is a generic constant  $K > 0$  such that*

$$\sup_{x \in I} |\Phi(x)| \leq \left( \frac{K}{|J|} \right)^{\frac{\ln M}{\ln 2}} \sup_{x \in J} |\Phi(x)|,$$

with  $M = \sup_{|z| \leq 4} |\Phi(z)| \geq 1$ .

We are now in the position to prove Theorem 3.14.

*Proof of Theorem 3.14.* It is sufficient to prove the theorem in the case  $L = 1$ . In fact, assume we have shown the statement for the case  $L = 1$ . For  $\rho \in (0, 1]$  and  $L > 0$ , let  $E \subset \mathbb{R}^d$  be a  $(\rho, L)$ -thick set. Then the set

$$L^{-1}E := \{L^{-1}x : x \in E\}$$

is a  $(\rho, 1)$ -thick set. For a function  $f \in L_p(\mathbb{R}^d)$ ,  $p \in [1, \infty]$ , with  $\text{supp } \mathcal{F}f \subset \overline{B(0, \lambda)}$  for  $\lambda > 0$ , we define  $g \in L_p(\mathbb{R}^d)$  by

$$g(x) := f(Lx), \quad \text{for } x \in \mathbb{R}^d.$$

Then one directly computes that  $\text{supp } \mathcal{F}g \subset \overline{B(0, L\lambda)}$ . By applying Theorem 3.14 with  $L = 1$  to the set  $L^{-1}E$  and the function  $g$ , we get

$$\|g\|_{L_p(\mathbb{R}^d)} \leq \left( \frac{K_d}{\rho} \right)^{K_d(1+\lambda L)} \|g\|_{L_p(L^{-1}E)}.$$

With the change of variables  $x \mapsto Lx$ , we see that

$$\|g\|_{L_p(\mathbb{R}^d)} = L^{-d/p} \|f\|_{L_p(\mathbb{R}^d)} \quad \text{and} \quad \|g\|_{L_p(L^{-1}E)} = L^{-d/p} \|f\|_{L_p(E)},$$

where  $L^{-d/p} = 1$  if  $p = \infty$ . By dividing both sides by  $L^{-d/p}$  the statement of the theorem follows.

We now prove the theorem in the case  $L = 1$ . For each  $j \in \mathbb{Z}^d$  we consider the cube  $Q_j := j + (-1/2, 1/2)^d$  with side length 1. Then  $Q_j \cap Q_k = \emptyset$  for all  $j \neq k \in \mathbb{Z}^d$  and

$$\mathbb{R}^d = \bigcup_{j \in \mathbb{Z}^d} \overline{Q_j}.$$

Let  $\lambda > 0$ ,  $p \in [1, \infty]$ , and  $f \in L_p(\mathbb{R}^d) \setminus \{0\}$  with  $\text{supp } \mathcal{F}f \subset \overline{B(0, \lambda)}$ . Further, let  $K_B > 0$  such that by Bernstein's inequality (Lemma 3.16)

$$\|\partial^\alpha f\|_{L_p(\mathbb{R}^d)} \leq (K_B \lambda)^{|\alpha|_1} \|f\|_{L_p(\mathbb{R}^d)}, \quad \forall \alpha \in \mathbb{N}_0^d.$$

We divide the family of cubes  $(Q_j)_{j \in \mathbb{Z}^d}$  into a family of good and bad cubes. For  $j \in \mathbb{Z}^d$  the cube  $Q_j$  is called a *good cube* if for all  $\alpha \in \mathbb{N}_0^d$

$$\|\partial^\alpha f\|_{L_p(Q_j)} \leq 2^d (3K_B \lambda)^{|\alpha|_1} \|f\|_{L_p(Q_j)}. \quad (3.26)$$

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Further, we call  $Q_j$  a *bad cube* if it is not a good cube, i.e., there exists  $\alpha_j \in \mathbb{N}_0^d$ ,  $|\alpha_j|_1 > 0$  such that (3.26) does not hold. We abbreviate the disjoint union of good and bad cubes as

$$Q_{\text{good}} := \bigcup_{Q_j \text{ is good}} Q_j \quad \text{and} \quad Q_{\text{bad}} := \bigcup_{Q_j \text{ is bad}} Q_j.$$

The proof is divided into three steps.

*Step 1. Estimates on good and bad cubes.* In this step, we show that

$$\|f\|_{L_p(\mathbb{R}^d)} \leq 2\|f\|_{L_p(Q_{\text{good}})}. \quad (3.27)$$

For this, we start with an estimate on bad cubes. For a bad cube  $Q_j$ , there is  $\alpha_j \in \mathbb{N}_0^d$ ,  $|\alpha_j|_1 > 0$  such that

$$\|f\|_{L_p(Q_j)} < 2^{-d} (3K_B \lambda)^{-|\alpha_j|_1} \|\partial^{\alpha_j} f\|_{L_p(Q_j)} \leq 2^{-d} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|_1 > 0}} (3K_B \lambda)^{-|\alpha|_1} \|\partial^\alpha f\|_{L_p(Q_j)}.$$

Since  $Q_j \cap Q_k = \emptyset$  for all  $j \neq k \in \mathbb{Z}^d$ , we get for the disjoint union of bad cubes  $Q_{\text{bad}}$  that

$$\|f\|_{L_p(Q_{\text{bad}})} \leq 2^{-d} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|_1 > 0}} (3K_B \lambda)^{-|\alpha|_1} \|\partial^\alpha f\|_{L_p(Q_{\text{bad}})} \leq 2^{-d} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|_1 > 0}} (3K_B \lambda)^{-|\alpha|_1} \|\partial^\alpha f\|_{L_p(\mathbb{R}^d)}.$$

By Bernstein's inequality (3.25), this implies

$$\|f\|_{L_p(Q_{\text{bad}})} \leq 2^{-d} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|_1 > 0}} 3^{-|\alpha|_1} \|f\|_{L_p(\mathbb{R}^d)}. \quad (3.28)$$

Referring to the binomial series, we obtain

$$\sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|_1 > 0}} 3^{-|\alpha|_1} = \sum_{k=1}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|_1 = k}} 3^{-k} = \sum_{k=1}^{\infty} \binom{k+d-1}{k} 3^{-k} = \left(\frac{3}{2}\right)^d - 1 \leq \frac{1}{2} \left(\frac{3}{2}\right)^d \leq 2^{(d-1)}.$$

Together with (3.28), this implies

$$\|f\|_{L_p(Q_{\text{bad}})} \leq \frac{1}{2} \|f\|_{L_p(\mathbb{R}^d)}.$$

Since  $\|f\|_{L_p(\mathbb{R}^d)} \leq \|f\|_{L_p(Q_{\text{good}})} + \|f\|_{L_p(Q_{\text{bad}})}$ , we have

$$\|f\|_{L_p(Q_{\text{good}})} \geq \|f\|_{L_p(\mathbb{R}^d)} - \|f\|_{L_p(Q_{\text{bad}})} \geq \left(1 - \frac{1}{2}\right) \|f\|_{L_p(\mathbb{R}^d)} \geq \frac{1}{2} \|f\|_{L_p(\mathbb{R}^d)},$$

and hence (3.27).

*Step 2. Local estimate with respect to  $L_\infty$ -norm.* Let  $Q_j$  be a good cube, and let

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$\rho \in (0, 1]$  and  $E \subset \mathbb{R}^d$  such that  $|E \cap Q_j| \geq \rho$ . In this step we show the following estimate:

$$\|f\|_{L_\infty(Q_j)} \leq \left(\frac{K_d}{\rho}\right)^{K_d(1+\lambda)} \|f\|_{L_\infty(E \cap Q_j)}, \quad (3.29)$$

for a constant  $K_d > 0$  which depends only on the dimension  $d$ . Note that  $f \in L_p(\mathbb{R}^d)$  and  $Q_j$  is a good cube with respect to the  $L_p(Q_j)$ -norm. Since  $\text{supp } \mathcal{F}f \subset B(0, \lambda)$ , we obtain by Schwartz's Paley–Wiener theorem (Theorem 3.12) that  $f$  is analytic. Hence,  $|f|$  is continuous and there is  $x_j \in \overline{Q_j}$  with  $\|f\|_{L_\infty(Q_j)} = |f(x_j)|$ . Let  $\mathbb{S}^{d-1}$  be the  $(d-1)$ -dimensional unit sphere. Using spherical coordinates centred at  $x_j$ , we get by using that the diameter of the cube  $Q_j$  is  $\sqrt{d}$ , that

$$\begin{aligned} |E \cap Q_j| &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \mathbf{1}_{E \cap Q_j}(x_j + r\sigma) r^{d-1} \, d\sigma \, dr \\ &= \int_0^{\sqrt{d}} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{E \cap Q_j}(x_j + r\sigma) r^{d-1} \, d\sigma \, dr \\ &= d^{d/2} \int_0^1 \int_{\mathbb{S}^{d-1}} \mathbf{1}_{E \cap Q_j}(x_j + \sqrt{d}r\sigma) r^{d-1} \, d\sigma \, dr \\ &\leq d^{d/2} \int_0^1 \int_{\mathbb{S}^{d-1}} \mathbf{1}_{E \cap Q_j}(x_j + \sqrt{d}r\sigma) \, d\sigma \, dr \\ &= d^{d/2} \int_{\mathbb{S}^{d-1}} \int_0^1 \mathbf{1}_{I_\sigma}(r) \, dr \, d\sigma \\ &= d^{d/2} \int_{\mathbb{S}^{d-1}} |I_\sigma| \, d\sigma, \end{aligned}$$

with  $I_\sigma := \{r \in [0, 1] : x_j + \sqrt{d}r\sigma \in E \cap Q_j\}$ . Hence, there is  $\sigma_0 \in \mathbb{S}^{d-1}$  such that

$$|E \cap Q_j| \leq d^{d/2} |\mathbb{S}^{d-1}| |I_{\sigma_0}|.$$

Since  $|E \cap Q_j| \geq \rho$ , it follows that

$$|I_{\sigma_0}| \geq \frac{\rho}{d^{d/2} |\mathbb{S}^{d-1}|} > 0.$$

Note that, since  $\text{supp } \mathcal{F}f$  is compact, by Schwartz's Paley–Wiener theorem (Theorem 3.12),  $f$  is a real analytic function on  $\mathbb{R}^d$ , which has an analytic extension to  $\mathbb{C}^d$ . Hence,  $\|f\|_{L_p(Q_j)} \neq 0$  since  $f \neq 0$  by assumption, and also the function  $\phi : \mathbb{C} \rightarrow \mathbb{R}$  defined by

$$\phi(z) := \frac{f(x_j + \sqrt{d}\sigma_0 z)}{\|f\|_{L_p(Q_j)}}, \quad \text{for } z \in \mathbb{C}$$

is analytic. Then

$$|\phi(0)| = \frac{|f(x_j)|}{\|f\|_{L_p(Q_j)}} = \frac{\|f\|_{L_\infty(Q_j)}}{\|f\|_{L_p(Q_j)}} \geq 1. \quad (3.30)$$

We now apply the local estimate for analytic functions of one complex variable given in

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Lemma 3.17. With  $I = [0, 1]$ ,  $J = I_{\sigma_0}$ , and  $\Phi = \phi$  we get

$$\sup_{x \in [0, 1]} |\phi(x)| \leq \left( \frac{K}{|I_{\sigma_0}|} \right)^{\frac{\ln M}{\ln 2}} \sup_{x \in I_{\sigma_0}} |\phi(x)| \leq \left( \frac{K d^{d/2} |\mathbb{S}^{d-1}|}{\rho} \right)^{\frac{\ln M}{\ln 2}} \sup_{x \in I_{\sigma_0}} |\phi(x)|, \quad (3.31)$$

where  $M = \sup_{|z| \leq 4} |\phi(z)|$ . By definition of  $I_{\sigma_0}$  we have

$$\sup_{x \in I_{\sigma_0}} |\phi(x)| = \frac{\sup_{x \in I_{\sigma_0}} |f(x_j + \sqrt{d}\sigma_0 x)|}{\|f\|_{L_p(Q_j)}} \leq \frac{\|f\|_{L_\infty(E \cap Q_j)}}{\|f\|_{L_p(Q_j)}},$$

and by (3.30) that

$$\frac{\|f\|_{L_\infty(Q_j)}}{\|f\|_{L_p(Q_j)}} = |\phi(0)| \leq \sup_{x \in [0, 1]} |\phi(x)|.$$

Hence, (3.31) implies

$$\|f\|_{L_\infty(Q_j)} \leq \left( \frac{K d^{d/2} |\mathbb{S}^{d-1}|}{\rho} \right)^{\frac{\ln M}{\ln 2}} \|f\|_{L_\infty(E \cap Q_j)}. \quad (3.32)$$

In the remaining part of this step, we give an estimate on  $M$ . Since  $f$  is analytic on  $\mathbb{C}^d$ , we observe that for all  $z \in \mathbb{C}$

$$f(x_j + \sqrt{d}\sigma_0 z) = \sum_{\alpha \in \mathbb{N}_0^d} \frac{(\partial^\alpha f)(x_j)}{\alpha!} \sigma_0^\alpha d^{|\alpha|_1/2} z^{|\alpha|_1}.$$

Since  $|\sigma_0| = 1$ , we get

$$M = \sup_{|z| \leq 4} |\phi(z)| \leq \frac{1}{\|f\|_{L_p(Q_j)}} \sum_{\alpha \in \mathbb{N}_0^d} \frac{\|\partial^\alpha f\|_{L_\infty(Q_j)}}{\alpha!} (4\sqrt{d})^{|\alpha|_1}. \quad (3.33)$$

Using that  $Q_j$  is a good cube, for all  $\alpha \in \mathbb{N}_0^d$ , we obtain the following estimate for the Sobolev norm of  $\partial^\alpha f$

$$\begin{aligned} \|\partial^\alpha f\|_{W_p^d(Q_j)} &= \sum_{\beta \in \mathbb{N}_0^d, |\beta|_1 \leq d} \|\partial^{\alpha+\beta} f\|_{L_p(Q_j)} \\ &\leq \sum_{\beta \in \mathbb{N}_0^d, |\beta|_1 \leq d} 2^d (3K_B \lambda)^{|\alpha+\beta|_1} \|f\|_{L_p(Q_j)} \\ &\leq 2^d (1 + \lambda)^d (3K_B \lambda)^{|\alpha|_1} \sum_{\beta \in \mathbb{N}_0^d, |\beta|_1 \leq d} (3K_B)^{|\beta|_1} \|f\|_{L_p(Q_j)}. \end{aligned}$$

In the following, we denote by  $K_d$  a constant that only depends on  $d$ , and we allow the constant to change with each occurrence. Since the cubes  $Q_j$  satisfy the cone condition, by the Sobolev embedding theorem (see, e.g., [AF03, Theorem 4.12]) we have

$$\|f\|_{L_\infty(Q_j)} \leq K_d \|f\|_{W_p^d(Q_j)},$$

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and hence

$$\|\partial^\alpha f\|_{L_\infty(Q_j)} \leq K_d(1+\lambda)^d (3K_B\lambda)^{|\alpha|_1} \|f\|_{L_p(Q_j)}. \quad (3.34)$$

Now, using (3.34) in the estimate (3.33), we get

$$M \leq K_d(1+\lambda)^d \sum_{\alpha \in \mathbb{N}_0^d} \frac{(12K_B\sqrt{d}\lambda)^{|\alpha|_1}}{\alpha!}, \quad (3.35)$$

where  $\alpha! = \alpha_1! \cdots \alpha_d!$  holds. Using  $\alpha! \geq |\alpha|_1! / d^{|\alpha|_1}$ , we obtain

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^d} \frac{(12K_B\sqrt{d}\lambda)^{|\alpha|_1}}{\alpha!} &= \sum_{\alpha \in \mathbb{N}_0^d} \frac{(12K_B d^{\frac{3}{2}}\lambda)^{|\alpha|_1}}{|\alpha|_1!} = \sum_{k=0}^{\infty} \binom{k+d-1}{k} \frac{(12K_B d^{\frac{3}{2}}\lambda)^k}{k!} \\ &\leq 2^{d-1} \sum_{k=0}^{\infty} \frac{(24K_B d^{\frac{3}{2}}\lambda)^k}{k!} = 2^{d-1} e^{24K_B d^{\frac{3}{2}}\lambda}, \end{aligned}$$

where we used for the inequality that

$$\binom{k+d-1}{k} \leq \sum_{i=0}^{k+d-1} \binom{k+d-1}{i} = 2^{k+d-1}.$$

Hence, together with (3.35), we see that

$$M \leq K_d 2^{d-1} (1+\lambda)^d e^{24K_B d^{\frac{3}{2}}\lambda} \leq e^{K_d(1+\lambda)}.$$

Using this estimate on  $M$  in (3.32), we see, since  $\rho \in (0, 1]$ , that

$$\|f\|_{L_\infty(Q_j)} \leq \left( \frac{K_d d^{d/2} |\mathbb{S}^{d-1}|}{\rho} \right)^{\frac{\ln M}{\ln 2}} \|f\|_{L_\infty(E \cap Q_j)} \leq \left( \frac{K_d}{\rho} \right)^{K_d(1+\lambda)} \|f\|_{L_\infty(E \cap Q_j)}. \quad (3.36)$$

*Step 3. Recovery of the  $L_p(\mathbb{R}^d)$ -norm.* We now use the results from the previous steps to recover an estimate for the  $L_p(\mathbb{R}^d)$ -norm as in the statement of the theorem. For this, let again  $Q_j$  be a good cube,  $\rho \in (0, 1]$  and  $E$  be a  $(\rho, 1)$ -thick set. We define

$$\tilde{E} := \left\{ y \in E \cap Q_j : |f(y)| \leq \frac{2}{|E \cap Q_j|} \int_{E \cap Q_j} |f(x)| \, dx \right\}.$$

Then

$$\int_{E \cap Q_j} |f(x)| \, dx \geq \int_{(E \cap Q_j) \setminus \tilde{E}} |f(x)| \, dx \geq \frac{2|(E \cap Q_j) \setminus \tilde{E}|}{|E \cap Q_j|} \int_{E \cap Q_j} |f(x)| \, dx.$$

By using that the integrals are non-zero, since  $f \neq 0$  is analytic, we get

$$\frac{1}{2}|E \cap Q_j| \geq |(E \cap Q_j) \setminus \tilde{E}|.$$

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Since  $E$  is a thick set, we have  $|E \cap Q_j| \geq \rho$ , and thus

$$|\tilde{E}| = |E \cap Q_j| - |(E \cap Q_j) \setminus \tilde{E}| \geq \frac{1}{2}|E \cap Q_j| \geq \frac{\rho}{2} > 0.$$

Hence,  $|\tilde{E} \cap Q_j| \geq \rho/2$  and we can apply Step 2 for  $\tilde{E}$  to get the estimate

$$\|f\|_{L_\infty(Q_j)} \leq \left(\frac{2K_d}{\rho}\right)^{K_d(1+\lambda)} \|f\|_{L_\infty(\tilde{E} \cap Q_j)}.$$

By definition of  $\tilde{E}$ , we observe that

$$\begin{aligned} \|f\|_{L_\infty(Q_j)} &\leq \frac{2}{|E \cap Q_j|} \left(\frac{2K_d}{\rho}\right)^{K_d(1+\lambda)} \int_{E \cap Q_j} |f(x)| \, dx \\ &\leq \frac{2}{\rho} \left(\frac{2K_d}{\rho}\right)^{K_d(1+\lambda)} \int_{E \cap Q_j} |f(x)| \, dx \\ &\leq \left(\frac{K_d}{\rho}\right)^{K_d(1+\lambda)} \int_{E \cap Q_j} |f(x)| \, dx. \end{aligned} \quad (3.37)$$

Since  $|E \cap Q_j| \leq 1$ , we deduce from Hölder's inequality that for all  $p \in [1, \infty]$

$$\int_{E \cap Q_j} |f(x)| \, dx \leq \|f\|_{L_p(E \cap Q_j)}.$$

Hence, (3.37) implies

$$\|f\|_{L_p(Q_j)} \leq \|f\|_{L_\infty(Q_j)} \leq \left(\frac{K_d}{\rho}\right)^{K_d(1+\lambda)} \|f\|_{L_p(E \cap Q_j)}. \quad (3.38)$$

Since the cubes  $(Q_j)_{j \in \mathbb{Z}^d}$  are disjoint, (3.38) implies

$$\|f\|_{L_p(Q_{\text{good}})} \leq \left(\frac{K_d}{\rho}\right)^{K_d(1+\lambda)} \|f\|_{L_p(E \cap Q_{\text{good}})}.$$

Hence, by (3.27) we obtain,

$$\|f\|_{L_p(\mathbb{R}^d)} \leq 2\|f\|_{L_p(Q_{\text{good}})} \leq 2 \left(\frac{K_d}{\rho}\right)^{K_d(1+\lambda)} \|f\|_{L_p(E)},$$

where for the last inequality we used that  $\|f\|_{L_p(E \cap Q_{\text{good}})} \leq \|f\|_{L_p(E)}$ . This finishes the proof.  $\square$

## 3.4 Dissipation estimate

Recall that, by the Logvinenko–Sereda theorem for thick sets  $E \subset \mathbb{R}^d$  and  $\lambda > 0$ , we have the estimate

$$\|f\|_{L_p(\mathbb{R}^d)} \leq K_{E,\lambda} \|f\|_{L_p(E)}$$

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for all functions  $f \in L_p(\mathbb{R}^d)$  with low frequencies, i.e.,  $\text{supp } \mathcal{F}f \subset \overline{B(0, \lambda)}$ . The next part for proving the observability estimate in Theorem 3.8 is a dissipation estimate for functions with high frequencies. This section covers the dissipation estimate obtained in [BGST22b] by the author in collaboration with C. Bombach, C. Seifert, and M. Tautenhahn. See also [EGST21] for the modification needed to obtain Proposition 3.18.

We start with defining operators  $(P_\lambda)_{\lambda>0}$  which act as a low-pass filter on  $L_p(\mathbb{R}^d)$ . Let  $\eta \in C_c^\infty([0, \infty))$  be a smooth step function with  $0 \leq \eta \leq 1$  such that  $\eta(r) = 1$  for  $r \in [0, 1/2]$  and  $\eta(r) = 0$  for  $r \geq 1$ . See Figure 3.3 for an example.

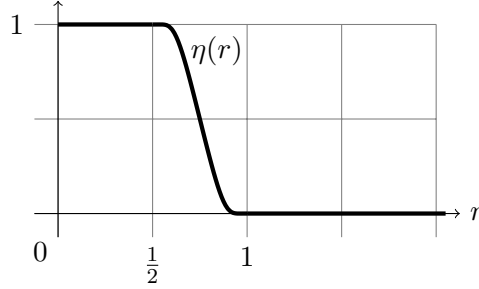


Figure 3.3: Plot of the smooth step function  $\eta(r) = 1 - \frac{f(2r-1)}{f(2r-1)+f(2-2r)}$  where  $f: \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(r) = e^{-1/r}$  for  $r > 0$  and  $f(r) = 0$  for  $r \leq 0$ . See [Tu11, Section 13] for more details of this construction.

For  $\lambda > 0$ , we define  $\chi_\lambda: \mathbb{R}^d \rightarrow \mathbb{R}$  by  $\chi_\lambda(\xi) = \eta(|\xi|/\lambda)$ . Since  $\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$ , we have  $\mathcal{F}^{-1}\chi_\lambda \in \mathcal{S}(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$  and for all  $p \in [1, \infty]$ , we define  $P_\lambda: L_p(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$  by

$$P_\lambda f = (\mathcal{F}^{-1}\chi_\lambda) * f. \quad (3.39)$$

By Young's inequality, we have for all  $f \in L_p(\mathbb{R}^d)$

$$\|P_\lambda f\|_{L_p(\mathbb{R}^d)} = \|(\mathcal{F}^{-1}\chi_\lambda) * f\|_{L_p(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)} \|f\|_{L_p(\mathbb{R}^d)}.$$

Moreover, the norm  $\|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)}$  is independent of  $\lambda > 0$ . Indeed, by the scaling property of the Fourier transform and by change of variables we have for all  $\lambda > 0$

$$\|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)} = |\lambda|^d \|(\mathcal{F}^{-1}\chi_1)(\lambda \cdot)\|_{L_1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\chi_1\|_{L_1(\mathbb{R}^d)}. \quad (3.40)$$

Hence, for all  $\lambda > 0$  the operator  $P_\lambda$  is bounded, and the family  $(P_\lambda)_{\lambda>0}$  is uniformly bounded by  $\|\mathcal{F}^{-1}\chi_1\|_{L_1(\mathbb{R}^d)}$ .

#### 3.4.1 Powers of the Laplacian

We first prove a dissipation estimate for semigroups associated with fractional powers of the Laplacian. For this, let  $s \in (0, \infty)$  and  $a_s: \mathbb{R}^d \rightarrow \mathbb{R}$  be defined by  $a_s(\xi) = |\xi|^s$ . Note that  $a_s$  may not be differentiable at 0. However, it can be shown that for  $t > 0$  we have  $e^{-ta_s} \in L_1(\mathbb{R}^d)$  and  $\mathcal{F}^{-1}e^{-ta_s} \in L_1(\mathbb{R}^d)$ . Indeed,  $e^{-ta_s}$  decays faster than any polynomial. Thus,  $e^{-ta_s} \in L_1(\mathbb{R}^d)$  and  $\mathcal{F}^{-1}e^{-ta_s} \in C^\infty(\mathbb{R}^d)$ . Moreover, the Riemann–Lebesgue lemma yields  $\mathcal{F}^{-1}e^{-ta_s} \in C_0(\mathbb{R}^d)$ . Then by subordination techniques (see, e.g., [KMS21]), one can

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show that  $f \mapsto \mathcal{F}^{-1}e^{-tas} * f$  yields a bounded operator on  $L_1(\mathbb{R}^d)$ . Since this operator is also translation invariant,  $\mathcal{F}^{-1}e^{-tas}$  is given by a finite Borel measure (cf. [Gra08, Theorem 2.58]) and therefore  $\mathcal{F}^{-1}e^{-tas} \in L_1(\mathbb{R}^d)$ . Taking into account Young's inequality, for  $p \in [1, \infty]$  and  $t \geq 0$  we define  $G_t^s: L_p(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$  by

$$G_0^s f := f, \quad G_t^s f := \mathcal{F}^{-1}e^{-tas} * f, \quad t > 0. \quad (3.41)$$

It is easy to see that  $(G_t^s)_{t \geq 0}$  is a  $C_0$ -semigroup for  $p \in [1, \infty)$  and a weak\*-continuous exponentially bounded semigroup for  $p = \infty$ .

**Proposition 3.18.** *Let  $(P_\lambda)_{\lambda > 0}$  be given by (3.39),  $s \in (0, \infty)$ , and  $G_t^s: L_p(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$  given by (3.41). Then for all  $p \in [1, \infty]$ ,  $\lambda > 0$ ,  $t \geq 0$ , and  $f \in L_p(\mathbb{R}^d)$  we have*

$$\|(\text{Id} - P_\lambda)G_t^s f\|_{L_p(\mathbb{R}^d)} \leq K_{s,d} e^{-2^{-s-2t}\lambda^s} \|f\|_{L_p(\mathbb{R}^d)},$$

where  $K_{s,d} > 0$  is a constant depending only on  $s$  and  $d$ .

*Proof.* By definition of the operator  $P_\lambda$ , we have for all  $f \in L_p(\mathbb{R}^d)$  that

$$(\text{Id} - P_\lambda)G_t^s f = \mathcal{F}^{-1}((1 - \chi_\lambda)e^{-tas}) * f,$$

and by Young's inequality, we obtain for all  $\lambda, t > 0$  and all  $f \in L_p(\mathbb{R}^d)$

$$\|(\text{Id} - P_\lambda)G_t^s f\|_{L_p(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}((1 - \chi_\lambda)e^{-tas})\|_{L_1(\mathbb{R}^d)} \|f\|_{L_p(\mathbb{R}^d)}.$$

For  $\mu > 0$ , we define  $k_\mu: \mathbb{R}^d \rightarrow \mathbb{R}$  by  $k_\mu = \mathcal{F}^{-1}((1 - \chi_\mu)e^{-as})$ . By substitution first in Fourier space and then in direct space, we obtain, using  $|t^{1/s}\xi|^s = |t|\xi|^s$ , for all  $\lambda, t > 0$

$$\begin{aligned} & \|\mathcal{F}^{-1}((1 - \chi_\lambda)e^{-tas})\|_{L_1(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \frac{1}{t^{d/s}} \left| \int_{\mathbb{R}^d} e^{ix \cdot (t^{-1/s}\xi)} (1 - \chi_{t^{1/s}\lambda}(\xi)) e^{-|\xi|^s} d\xi \right| dx \\ &= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} e^{iy \cdot \xi} (1 - \chi_{t^{1/s}\lambda}(\xi)) e^{-|\xi|^s} d\xi \right| dy = \|k_{t^{1/s}\lambda}\|_{L_1(\mathbb{R}^d)}. \end{aligned}$$

We denote by  $K_{s,d} > 0$  a constant which depends only on  $s$  and the dimension  $d$ , and we allow this constant to change with each occurrence. By Young's inequality and (3.40), we have

$$\|\mathcal{F}^{-1}(\chi_\mu e^{-as})\|_{L_1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\chi_\mu * \mathcal{F}^{-1}e^{-as}\|_{L_1(\mathbb{R}^d)} \leq K_{s,d}.$$

Hence, we find for all  $\mu > 0$  the uniform bound

$$\|k_\mu\|_{L_1(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}e^{-as}\|_{L_1(\mathbb{R}^d)} + \|\mathcal{F}^{-1}(\chi_\mu e^{-as})\|_{L_1(\mathbb{R}^d)} \leq K_{s,d}. \quad (3.42)$$

Next, we show that the  $L_1$ -norm of  $k_\mu$  decays even exponentially as  $\mu \rightarrow \infty$ . For this purpose, let now  $\mu \geq 2$ , and  $\alpha \in \mathbb{N}_0^d$  with  $|\alpha|_1 \leq d+1$ . By differentiation properties of the Fourier transform, we have

$$x^\alpha k_\mu(x) = x^\alpha \mathcal{F}^{-1}((1 - \chi_\mu)e^{-as})(x) = i^{|\alpha|_1} \mathcal{F}^{-1}(\partial^\alpha ((1 - \chi_\mu)e^{-as}))(x), \quad x \in \mathbb{R}^d,$$

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and hence, for all  $x \in \mathbb{R}^d$

$$\begin{aligned} |x^\alpha k_\mu(x)| &= \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \partial_\xi^\alpha \left( (1 - \chi_\mu(\xi)) e^{-|\xi|^s} \right) d\xi \right| \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \partial_\xi^\alpha \left( (1 - \chi_\mu(\xi)) e^{-|\xi|^s} \right) \right| d\xi. \end{aligned} \quad (3.43)$$

To the integrand of the right-hand side for  $\xi \in \mathbb{R}^d \setminus \{0\}$ , we apply the product rule and the triangle inequality to obtain

$$\left| \partial_\xi^\alpha \left( (1 - \chi_\mu(\xi)) e^{-|\xi|^s} \right) \right| \leq \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \leq \alpha}} \binom{\alpha}{\beta} \left| \partial_\xi^{\alpha-\beta} (1 - \chi_\mu(\xi)) \right| \left| \partial_\xi^\beta e^{-|\xi|^s} \right|. \quad (3.44)$$

For all  $\beta \in \mathbb{N}_0^d$  and  $\beta \leq \alpha$ , we have

$$\left| \partial_\xi^\beta e^{-|\xi|^s} \right| \leq K_{s,d} (1 + |\xi|)^{s|\beta|_1} \left( 1 + |\xi|^{s-2|\beta|_1} \right) e^{-|\xi|^s}.$$

Since  $\xi \mapsto (1 + |\xi|)^{s|\beta|_1} (1 + |\xi|^{s-2|\beta|_1}) e^{-|\xi|^s/2}$  is bounded for  $|\xi| \geq 1$ , we observe

$$\left| \partial_\xi^\beta e^{-|\xi|^s} \right| \leq K_{s,d} e^{-|\xi|^s/2}, \quad \text{for } |\xi| \geq 1.$$

Since  $\mu \geq 2$ , for all  $\beta \in \mathbb{N}_0^d$  with  $\beta \leq \alpha$ , we have

$$\left| \partial_\xi^{\alpha-\beta} (1 - \chi_\mu(\xi)) \right| \leq \mu^{|\beta|_1 - |\alpha|_1} \left( \partial_\xi^{\alpha-\beta} \chi_1 \right) (\xi/\mu) \leq \sup_{\gamma \leq \alpha} \sup_{\xi \in \mathbb{R}^d} \left| \left( \partial_\xi^\gamma \chi_1 \right) (\xi) \right| \mathbf{1}_{\mathbb{R}^d \setminus B(0, \mu/2)}(\xi)$$

and hence, since  $\mu/2 \geq 1$ ,

$$\left| \partial_\xi^{\alpha-\beta} (1 - \chi_\mu(\xi)) \right| \left| \partial_\xi^\beta e^{-|\xi|^s} \right| \leq K_{s,d} e^{-|\xi|^s/2} \mathbf{1}_{\mathbb{R}^d \setminus B(0, \mu/2)}(\xi) \leq K_{s,d} e^{-|\xi|^s/4} e^{-\mu^s/2^{s+2}}.$$

Thus, (3.44) and  $|\alpha|_1 \leq d+1$ , imply for all  $\xi \in \mathbb{R}^d$  that

$$\left| \partial_\xi^\alpha \left( (1 - \chi_\mu(\xi)) e^{-|\xi|^s} \right) \right| \leq K_{s,d} e^{-|\xi|^s/4} e^{-\mu^s/2^{s+2}} \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \leq \alpha}} \binom{\alpha}{\beta} \leq K_{s,d} e^{-|\xi|^s/4} e^{-\mu^s/2^{s+2}}.$$

Hence, from (3.43), we obtain for all  $x \in \mathbb{R}^d$

$$|x^\alpha k_\mu(x)| \leq K_{s,d} e^{-\mu^s/2^{s+2}} \int_{\mathbb{R}^d} e^{-|\xi|^s/4} d\xi = K_{s,d} e^{-\mu^s/2^{s+2}}. \quad (3.45)$$

In particular, for  $j \in \{1, 2, \dots, d\}$  and  $\alpha_j = (d+1)e_j$ , where  $e_j$  denotes the  $j$ -th canonical unit vector in  $\mathbb{R}^d$ , we obtain  $|x_j|^{d+1} |k_\mu(x)| \leq K_{s,d} e^{-\mu^s/2^{s+2}}$ , hence  $|x|_\infty^{d+1} |k_\mu(x)| \leq K_{s,d} e^{-\mu^s/2^{s+2}}$ , and consequently for all  $x \in \mathbb{R}^d$  and all  $\mu \geq 2$  we find

$$|x|^{d+1} |k_\mu(x)| \leq K_{s,d} e^{-\mu^s/2^{s+2}}. \quad (3.46)$$

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From (3.45) with  $\alpha = 0$  and (3.46), we obtain for all  $\mu \geq 2$  that

$$\begin{aligned} \|k_\mu\|_{L_1(\mathbb{R}^d)} &\leq K_{s,d} e^{-\mu^s/2^{s+2}} \int_{B(0,1)} 1 \, dx + K_{s,d} e^{-\mu^s/2^{s+2}} \int_{\mathbb{R}^d \setminus B(0,1)} |x|^{-d-1} \, dx \\ &\leq K_{s,d} e^{-\mu^s/2^{s+2}}. \end{aligned}$$

From this inequality and (3.42), we obtain for all  $\mu > 0$  that

$$\|k_\mu\|_{L_1(\mathbb{R}^d)} \leq K_{s,d} e^{-\mu^s/2^{s+2}}. \quad \square$$

#### 3.4.2 Semigroups associated with strongly elliptic polynomials

With Proposition 3.18, we are now in the position to prove the dissipation estimate for the general case of semigroups associated with strongly elliptic polynomials.

**Proposition 3.19.** *Let  $m \in \mathbb{N}$ ,  $a: \mathbb{R}^d \rightarrow \mathbb{C}$  be a strongly elliptic polynomial of order  $m$ ,  $c > 0$  and  $\omega \in \mathbb{R}$  as in (3.4),  $(S_t)_{t \geq 0}$  as in (3.2), and  $(P_\lambda)_{\lambda > 0}$  as in (3.39). Then for all  $p \in [1, \infty]$ ,  $\lambda > 0$ ,  $t \geq 0$ , and  $f \in L_p(\mathbb{R}^d)$  we have*

$$\|(\text{Id} - P_\lambda)S_t f\|_{L_p(\mathbb{R}^d)} \leq K_a e^{-2^{-m-4}c\lambda^m t + \omega t} \|f\|_{L_p(\mathbb{R}^d)},$$

where  $K_a \geq 1$  is a constant depending only on the polynomial  $a$  (and therefore also on  $m$  and  $d$ ).

*Proof of Proposition 3.19.* We introduce  $\tilde{a}: \mathbb{R}^d \rightarrow \mathbb{C}$  given by  $\tilde{a}(\xi) = (c/4)|\xi|^m$ . Then  $\tilde{a}$  and  $(a - \tilde{a})$  are strongly elliptic polynomials of order  $m \in \mathbb{N}$ . Note that the semigroup associated to  $\tilde{a}$  is  $G_{tc/4}^m$  for all  $t \geq 0$ , where  $(G_t^m)_{t \geq 0}$  is as in (3.41). Moreover, let  $(T_t)_{t \geq 0}$  be the semigroup associated to  $a - \tilde{a}$ . Since  $a = a - \tilde{a} + \tilde{a}$ , it follows that  $S_t = T_t G_{tc/4}^m$ . One can obtain a corresponding heat kernel bound for the kernel of the semigroup  $(T_t)_{t \geq 0}$  with the same growth rate  $\omega$  as for  $(S_t)_{t \geq 0}$  as follows: Since  $m$  has to be even, by Young's inequality for products and by (3.4) there exist  $\sigma, \tilde{\sigma}, C \geq 0$  such that for all  $\xi, \eta \in \mathbb{R}^d$

$$\begin{aligned} \text{Re}(a - \tilde{a})(\xi + i\eta) &= \text{Re} a(\xi + i\eta) - (c/4)|\xi + i\eta|^m \\ &\geq (c/2)|\xi|^m - \sigma|\eta|^m - \omega - (c/4)|\xi + i\eta|^m \\ &\geq (c/2)|\xi|^m - \sigma|\eta|^m - \omega - (c/4) \sum_{k=0}^{m/2} \binom{m/2}{k} (|\xi|^2)^k (|\eta|^2)^{m/2-k} \\ &\geq (c/2)|\xi|^m - \sigma|\eta|^m - \omega - (c/4)(1 + 1/2)|\xi|^m - C|\eta|^m \\ &\geq (1/8)c|\xi|^m - \tilde{\sigma}|\eta|^m - \omega, \end{aligned}$$

with yields an estimate similar to (3.4) with  $a$  replaced by  $a - \tilde{a}$ . Then, arguing as in the proof of Proposition 3.4, one can prove the heat kernel bound

$$|(\mathcal{F}^{-1} e^{-t(a-\tilde{a})})(x)| \leq \tilde{c}_1 e^{\omega t} t^{-d/m} e^{-\tilde{c}_2 \left(\frac{|x|^m}{t}\right)^{\frac{1}{m-1}}},$$

for some  $\tilde{c}_1, \tilde{c}_2 \geq 0$  and the same constant  $\omega$  as for the polynomial  $a$ . With this, we get as in (3.5), that there exists  $\tilde{M} \geq 1$  such that  $\|T_t\| \leq \tilde{M} e^{\omega t}$  for all  $t \geq 0$ . By Proposition 3.18

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and since Fourier multipliers commute, we obtain for all  $\lambda > 0$ ,  $t \geq 0$ , and  $f \in L_p(\mathbb{R}^d)$

$$\begin{aligned} \|(\text{Id} - P_\lambda)S_t f\|_{L_p(\mathbb{R}^d)} &= \|S_t(\text{Id} - P_\lambda)f\|_{L_p(\mathbb{R}^d)} \\ &\leq \|T_t\|_{\mathcal{L}(L_p(\mathbb{R}^d))} \|G_{(c/4)t}^m(\text{Id} - P_\lambda)f\|_{L_p(\mathbb{R}^d)} \\ &\leq \widetilde{M}K_{m,d}e^{-2^{-m-4}c\lambda^m t + \omega t} \|f\|_{L_p(\mathbb{R}^d)}, \end{aligned}$$

where  $K_{m,d} > 0$  is a constant depending only on  $m$  and  $d$ . □

## 3.5 Proof of the observability estimate

For proving the observability estimate in Theorem 3.8, we follow the generalized Lebeau–Robbiano strategy in Theorem 2.6 and combine the uncertainty principle obtained in Theorem 3.14 with the dissipation estimate in Proposition 3.19.

*Proof of Theorem 3.8.* Let  $(P_\lambda)_{\lambda>0}$  be the family of operators defined by (3.39) at the beginning of Section 3.4. Since  $\text{supp } \mathcal{F}(P_\lambda f) \subset \overline{B(0, \lambda)}$  for all  $\lambda > 0$  and  $f \in L_p(\mathbb{R}^d)$ , the Logvinenko–Sereda theorem (Theorem 3.14) implies that

$$\forall \lambda > 0, f \in L_p(\mathbb{R}^d) : \quad \|P_\lambda f\|_{L_p(\mathbb{R}^d)} \leq C_1 e^{c_1 \lambda^{s_1}} \|P_\lambda f\|_{L_p(E)},$$

with  $s_1 := 1$ ,

$$C_1 := \left(\frac{K_d}{\rho}\right)^{K_d}, \quad \text{and} \quad c_1 := L \ln\left(\frac{K_d}{\rho}\right),$$

where  $K_d > 0$  is a constant depending only on  $d$ . Moreover, according to Proposition 3.19, we have

$$\forall \lambda > 0, t > 0, f \in L_p(\mathbb{R}^d) : \quad \|(\text{Id} - P_\lambda)S_t f\|_{L_p(\mathbb{R}^d)} \leq C_2 e^{-c_2 \lambda^{s_2} t^\gamma + \omega t} \|f\|_{L_p(E)},$$

where  $C_2 := K_a$  is a constant depending only on the polynomial  $a$ ,  $c_2 := 2^{-m-4}c$ ,  $s_2 := m$ , and  $\gamma := 1$ . Note that the function  $t \mapsto \|(S_t f)|_E\|_{L_p(E)}$  is measurable for all  $f \in L_p(\mathbb{R}^d)$ . Indeed, if  $p \in [1, \infty)$  the semigroup  $(S_t)_{t \geq 0}$  is strongly continuous and the measurability follows. If  $p = \infty$ , measurability is a consequence of duality and the representation of the norm in  $L_\infty(E)$  by means of the Hahn–Banach theorem (cf. Lemma 1.24). Hence, we can apply Theorem 2.6 with  $X := L_p(\mathbb{R}^d)$ ,  $Y := L_p(E)$ ,  $C: X \rightarrow Y$  given by the restriction map on  $E$ , and obtain for all  $x \in X$  and  $T > 0$

$$\|S_T f\|_{L_p(\mathbb{R}^d)} \leq \widetilde{C}_{\text{obs}} \int_0^T \|(S_t f)|_E\|_{L_p(E)} dt, \quad \text{with} \quad \widetilde{C}_{\text{obs}} = \frac{K_1}{T} \exp\left(\frac{K_2}{T^{\frac{1}{m-1}}} + \omega_+ T\right),$$

where

$$\begin{aligned} K_1 &= 8e^4 M C_1 \left(M(C_1 \|C\|_{\mathcal{L}(X,Y)} + 1)C_2\right)^{\frac{8}{6 \ln(2)}}, \\ K_2 &= 4 \left(\frac{c_1^m}{c_2}\right)^{\frac{1}{m-1}} 2^{\frac{2}{m-1}} \left(1 + \frac{m}{m-1}\right), \end{aligned}$$

with  $\omega_+ := \max\{0, \omega\}$ , and  $M$  as in (3.6). We denote by  $K_d$ ,  $K_m$ , and  $K_a$  positive

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constants depending only on the dimension  $d$ , on  $m$ , or on the polynomial  $a$ , respectively. A straightforward calculation shows that

$$K_1 \leq K_a \left( \frac{K_d}{\rho} \right)^{K_d} \quad \text{and} \quad K_2 \leq K_m c^{-\frac{1}{m-1}} \left( L \ln \left( \frac{K_d}{\rho} \right) \right)^{\frac{m}{m-1}}.$$

Thus, we obtain

$$\tilde{C}_{\text{obs}} \leq \frac{K_a}{T} \left( \frac{K_d}{\rho} \right)^{K_d} \exp \left( \frac{K_m (L \ln(K_d/\rho))^{\frac{m}{m-1}}}{(cT)^{\frac{1}{m-1}}} + \omega_+ T \right) =: C_{\text{obs}}.$$

As in Remark 2.1, we can use Hölder's inequality to turn the observability estimate w.r.t the  $L_1$ -norm into an estimate w.r.t the  $L_r$ -norm for  $r \in [1, \infty]$ . The observability constant then changes to  $C_{\text{obs}} T^{1/r'}$  for  $r' \in [1, \infty]$  with  $1/r + 1/r' = 1$ . The claim now follows by observing that  $T^{-1} T^{1/r'} = T^{-1/r}$ .  $\square$

## 3.6 Fractional powers

We now establish observability estimates of systems induced by fractional powers of strongly elliptic polynomials. Using duality results, we further deduce controllability and stabilization properties of the corresponding control systems. In contrast to the subordination technique introduced in Section 2.3, we use a direct approach which not requires the strong continuity of the semigroup. We cover the results from [EGST21] and complement recent results in the Hilbert space  $L_2(\mathbb{R}^d)$  for the fractional heat equation and more general Fourier multipliers (see [HWW21, Lis20, LWXY22, Koe20, AM22]).

### 3.6.1 Observability estimate for fractional powers

As introduced in Section 3.1.2, let  $a: \mathbb{R}^d \rightarrow \mathbb{C}$ , given by

$$a(\xi) := \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^d,$$

be a strongly elliptic polynomial of degree  $m \in \mathbb{N}$  with coefficients  $a_\alpha \in \mathbb{C}$ . For  $s \in (0, 1]$  let  $\tilde{m} \in \mathbb{N}_0$  be the largest integer less than  $sm$  and let  $b: \mathbb{R}^d \rightarrow \mathbb{C}$  be given by

$$b(\xi) := \sum_{|\alpha| \leq \tilde{m}} b_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^d,$$

with coefficients  $b_\alpha \in \mathbb{C}$ . By the strong ellipticity of  $a$ , there are  $c > 0$  and  $\omega \in \mathbb{R}$  such that for all  $\xi \in \mathbb{R}^d$

$$\operatorname{Re} a(\xi) \geq c |\xi|^m - \omega. \quad (3.47)$$

We consider the fractional powers  $a_{s,b} := (a + \omega)^s + b$ . By (3.47), there is  $\nu \in \mathbb{R}$  such that for all  $\xi \in \mathbb{R}^d$

$$\operatorname{Re} a_{s,b}(\xi) = \operatorname{Re}(a(\xi) + \omega)^s + \operatorname{Re} b(\xi) \geq c^s |\xi|^{sm} - \nu. \quad (3.48)$$

Note that  $a_{s,b}$  may not be differentiable at 0. However, with the same arguments as in Section 3.4.1, it can be shown that for  $t > 0$  we have  $e^{-ta_{s,b}} \in L_1(\mathbb{R}^d)$  and  $\mathcal{F}^{-1} e^{-ta_{s,b}} \in$

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$L_1(\mathbb{R}^d)$ . Hence, taking into account Young's inequality, for  $p \in [1, \infty]$  and  $t \geq 0$  we can define  $S_t^s: L_p(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$  by

$$S_0^s f := f, \quad S_t^s f := \mathcal{F}^{-1} e^{-ta_{s,b}} * f, \quad f \in L_p(\mathbb{R}^d). \quad (3.49)$$

It is easy to see that  $(S_t^s)_{t \geq 0}$  is a  $C_0$ -semigroup for  $p \in [1, \infty)$  and  $(S_t^s)_{t \geq 0}$  is a weak\*-continuous exponentially bounded semigroup for  $p = \infty$ . Employing the dissipation estimate in Proposition 3.18, and the perturbation argument used in the proof of Proposition 3.19, we obtain the following dissipation estimate for  $(S_t^s)_{t \geq 0}$ .

**Proposition 3.20.** *Let  $(P_\lambda)_{\lambda > 0}$  be given by (3.39),  $s \in (0, 1]$ , and  $(S_t^s)_{t \geq 0}$  be given by (3.49). Then for all  $p \in [1, \infty]$ ,  $\lambda > 0$ ,  $t \geq 0$ , and  $f \in L_p(\mathbb{R}^d)$  we have*

$$\|(\text{Id} - P_\lambda) S_t^s f\|_{L_p(\mathbb{R}^d)} \leq K_{a_{s,b}} e^{-2^{-sm-4} c^s t \lambda^{sm} + \nu t} \|f\|_{L_p(\mathbb{R}^d)},$$

where  $c > 0$  and  $\nu \in \mathbb{R}$  are as in (3.48), and  $K_{a_{s,b}} > 0$  depends only on  $a_{s,b}$ .

*Proof.* Let  $\tilde{a}(\xi) := c^s |\xi|^{sm} / 4$  and denote the corresponding semigroup by  $(G_{c^s t/4}^{sm})_{t \geq 0}$ , following the notation in (3.41). Then, by Proposition 3.18, for all  $p \in [1, \infty]$ ,  $\lambda > 0$ ,  $t \geq 0$ , and  $f \in L_p(\mathbb{R}^d)$  we have

$$\|(\text{Id} - P_\lambda) G_{c^s t/4}^{sm} f\|_{L_p(\mathbb{R}^d)} \leq K_{sm,d} e^{-2^{-sm-4} c^s t \lambda^{sm}} \|f\|_{L_p(\mathbb{R}^d)},$$

where  $K_{sm,d} > 0$  is a constant depending only on  $sm$  and  $d$ . As in the proof of Proposition 3.19, we employ a perturbation argument to prove the dissipation estimate for  $a_{s,b} = (a_{s,b} - \tilde{a}) + \tilde{a}$ . Observe that  $a_{s,b} - \tilde{a}$  satisfies an estimate similar to (3.48), so the corresponding semigroup  $(T_t)_{t \geq 0}$  obeys an exponential bound of the form

$$\|T_t\|_{\mathcal{L}(L_p(\mathbb{R}^d))} \leq M e^{\nu t}, \quad t \geq 0.$$

Thus, since  $S_t^s = T_t G_{c^s t/4}^{sm}$  and Fourier multipliers commute, we arrive at

$$\begin{aligned} \|(\text{Id} - P_\lambda) S_t^s f\|_{L_p(\mathbb{R}^d)} &= \|S_t^s (\text{Id} - P_\lambda) f\|_{L_p(\mathbb{R}^d)} \\ &\leq \|T_t\|_{\mathcal{L}(L_p(\mathbb{R}^d))} \|G_{c^s t/4}^{sm} (\text{Id} - P_\lambda) f\|_{L_p(\mathbb{R}^d)} \\ &\leq M K_{sm,d} e^{-2^{-sm-4} c^s t \lambda^{sm} + \nu t} \|f\|_{L_p(\mathbb{R}^d)}. \quad \square \end{aligned}$$

We now prove a final-state observability estimate for the semigroups  $(S_t^s)_{t \geq 0}$  by combining the dissipation estimate in Proposition 3.20 with the uncertainty principle for thick sets  $E \subset \mathbb{R}^d$  given by the Logvinenko–Sereda theorem.

**Proposition 3.21.** *Let  $p, r \in [1, \infty]$ ,  $T > 0$ ,  $E \subset \mathbb{R}^d$  be a thick set (cf. Definition 3.6), and let  $C: L_p(\mathbb{R}^d) \rightarrow L_p(E)$  be the restriction map on  $E$ . Further, let  $a: \mathbb{R}^d \rightarrow \mathbb{C}$  be a strongly elliptic polynomial of degree  $m \in \mathbb{N}$ ,  $s \in (1/m, 1]$ , and consider the semigroup  $(S_t^s)_{t \geq 0}$  in  $L_p(\mathbb{R}^d)$  given by (3.49). Then  $(S_t^s)_{t \geq 0}$  satisfies a final-state observability estimate (cf. Definition 1.13).*

*Proof.* As in the proof of Theorem 3.8 we combine the uncertainty principle in Theorem 3.14 with the dissipation estimate obtained in Proposition 3.20. Let  $(P_\lambda)_{\lambda > 0}$  be the family

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of operators defined by (3.39). By the Logvinenko–Sereda theorem (Theorem 3.14), we have

$$\forall \lambda > 0, f \in L_p(\mathbb{R}^d) : \quad \|P_\lambda f\|_{L_p(\mathbb{R}^d)} \leq C_1 e^{c_1 \lambda} \|P_\lambda f\|_{L_p(E)},$$

for some  $C_1, c_1 > 0$ . Furthermore, Proposition 3.20 implies

$$\forall \lambda > 0, t \in (0, T], f \in L_p(\mathbb{R}^d) : \quad \|(\text{Id} - P_\lambda)S_t^s f\|_{L_p(\mathbb{R}^d)} \leq C_2 e^{-c_2 \lambda^{sm} t + \nu t} \|f\|_{L_p(\mathbb{R}^d)},$$

for  $C_2 \geq 1$ ,  $c_2 > 0$ , and  $\nu \in \mathbb{R}$  as in (3.48). Since  $sm > 1$ , the decay rate with respect to  $\lambda$  of the dissipation estimate is larger than the growth rate of the uncertainty principle. Hence, Theorem 2.6 yields the desired result.  $\square$

The assumption  $s > 1/m$  is crucial for obtaining an observability estimate with our strategy. This is consistent with a counterexample obtained in [Koe20]. A. Koenig proved that the fractional Laplacian  $-(-\Delta)^s$  in  $L_2(\mathbb{R})$  does not satisfy an observability estimate for any strict open subset  $E \subset \mathbb{R}$  and  $s < 1/2$ . However, following Example 2.20 and invoking that the assumed estimates hold for all  $\lambda > 0$ , we obtain that  $(S_t^s)_{t \geq 0}$  satisfies a weak observability estimate for all  $s \in (0, 1]$ . This leads to stabilization properties of the predual system, which we discuss in the next subsection.

#### 3.6.2 Controllability and stabilization properties

Let  $p \in [1, \infty)$ ,  $s \in (0, 1]$ , and  $-A_{s,p}$  be the generator of the  $C_0$ -semigroup  $(S_t^s)_{t \geq 0}$  on  $L_p(\mathbb{R}^d)$  given by (3.49) associated with the fractional powers  $a_{s,b}$  of a strongly elliptic polynomial  $a : \mathbb{R}^d \rightarrow \mathbb{C}$  of degree  $m \in \mathbb{N}$ . We consider control systems in  $L_p(\mathbb{R}^d)$  of the form

$$\begin{aligned} \dot{x}(t) &= -A_{s,p}x(t) + \mathbf{1}_E u(t) \quad \text{for } t \in (0, T], \\ x(0) &= x_0 \in L_p(\mathbb{R}^d), \end{aligned} \quad \Sigma_T(A_{s,p}, \mathbf{1}_E, -)$$

where  $\mathbf{1}_E : L_p(E) \rightarrow L_p(\mathbb{R}^d)$  is the embedding from a thick set  $E \subset \mathbb{R}^d$  to  $\mathbb{R}^d$ ,  $T > 0$ , and  $u \in L_r((0, T); L_p(E))$  for some  $r \in [1, \infty]$ . Hence, the influence of the control function  $u$  is restricted to the subset  $E$ .

**Theorem 3.22.** *Let  $T > 0$ ,  $r \in [1, \infty]$ , and  $E \subset \mathbb{R}^d$  be a thick set (cf. Definition 3.6).*

- (a) *Let  $p \in (1, \infty)$  and  $s \in (1/m, 1]$ . Then the system  $\Sigma_T(A_{s,p}, \mathbf{1}_E, -)$  is null-controllable in time  $T$  w.r.t.  $L_r((0, T); L_p(E))$  (cf. Definition 1.8).*
- (b) *Let  $p = 1$  and  $s \in (1/m, 1]$ . Then the system  $\Sigma_T(A_{s,p}, \mathbf{1}_E, -)$  is cost-uniformly approximately null-controllable in time  $T$  w.r.t.  $L_r((0, T); L_1(E))$  (cf. Definition 1.10).*
- (c) *Let  $p \in [1, \infty)$  and  $s \in (0, 1]$ . Then the system  $\Sigma_\infty(A_{s,p}, \mathbf{1}_E, -)$  is completely cost-uniformly open-loop stabilizable w.r.t.  $L_r((0, \infty); L_p(E))$  (cf. Definition 1.15).*

*Proof.* We employ the duality in Theorem 1.20 and prove the claim via an observability estimate for the dual system. We observe that for all  $p \in [1, \infty)$  and  $s \in (0, 1]$  the dual semigroup  $((S_t^s)')_{t \geq 0}$  on  $L_{p'}(\mathbb{R}^d)$ , where  $p' \in (1, \infty]$  such that  $1/p + 1/p' = 1$ , corresponds to the symbol  $a_{s,b}(\cdot)$ . Indeed, for  $f \in L_p(\mathbb{R}^d)$  and  $g \in L_{p'}(\mathbb{R}^d)$  we compute

$$\int_{\mathbb{R}^d} (\mathcal{F}^{-1} e^{-ta_{s,b}} * f)(x) g(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\mathcal{F}^{-1} e^{-ta_{s,b}})(x - y) f(y) g(x) dy dx$$

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$$= \int_{\mathbb{R}^d} f(x) (\mathcal{F}^{-1} e^{-ta_{s,b}(\cdot)} * g)(x) dx,$$

which yields the claim. Note that the symbol  $a_{s,b}(\cdot)$  obeys the same lower bound (3.48) as  $a_{s,b}$  with identical constants.

*Proof of (b).* For  $p' \in (1, \infty]$  and  $s \in (1/m, 1]$  we apply Proposition 3.21 with  $C : L_{p'}(\mathbb{R}^d) \rightarrow L_{p'}(E)$ ,  $Cf := f|_E$  being the restriction to  $E$ , and  $((S_t^s)')_{t \geq 0}$  on  $L_{p'}(\mathbb{R}^d)$  to obtain a final-state observability estimate with respect to the  $L_{r'}$ -norm for all  $r' \in [1, \infty]$  (cf. Remark 2.1). By Theorem 1.20, this observability estimate implies for all  $p \in [1, \infty)$  cost-uniform approximate null-controllability of the system  $\Sigma_T(A_{s,p}, \mathbf{1}_E, -)$  in time  $T$  with respect to  $L_r((0, T); L_p(E))$ . So in particular statement (b) holds.

*Proof of (a).* For  $p \in (1, \infty)$  the state space  $L_p(\mathbb{R}^d)$  is reflexive and hence, by Proposition 1.11, cost-uniform approximate null-controllability implies null-controllability. Thus statement (a) follows from the above proof of (b).

*Proof of (c).* We follow the strategy explained in Remark 2.23. Let  $p \in [1, \infty)$  and consider the family  $(P_\lambda)_{\lambda > 0}$  of operators in  $\mathcal{L}(L_p(\mathbb{R}^d))$  defined by (3.39) at the beginning of Section 3.4, i.e.,  $P_\lambda f = (\mathcal{F}^{-1} \chi_\lambda) * f$  with a smooth cut-off function  $\chi_\lambda : \mathbb{R}^d \rightarrow \mathbb{R}$ . Then the dual operators  $(P'_\lambda)_{\lambda > 0} \subset \mathcal{L}(L_{p'}(\mathbb{R}^d))$ , with  $p' \in (1, \infty]$  such that  $1/p + 1/p' = 1$ , satisfy  $P'_\lambda f = (\mathcal{F}^{-1} \chi_\lambda(\cdot)) * f = (\mathcal{F}^{-1} \chi_\lambda) * f$  and hence, the Logvinenko–Sereda theorem (Theorem 3.14) implies that for all  $\lambda > 0$  there is  $K_{E,\lambda} > 0$  such that

$$\forall f \in L_{p'}(\mathbb{R}^d) : \quad \|P'_\lambda f\|_{L_{p'}(\mathbb{R}^d)} \leq K_{E,\lambda} \|(P'_\lambda f)|_E\|_{L_{p'}(E)}.$$

A standard duality argument (see, e.g., [Car88]) yields that this estimate implies the range inclusion (2.57) with  $R = K_{E,\lambda}$ , the operators  $P_\lambda$ , and the embedding  $B = \mathbf{1}_E : L_p(E) \rightarrow L_p(\mathbb{R}^d)$ . Furthermore, by the dissipation estimate obtained in Proposition 3.20 and since  $(S_t)_{t \geq 0}$  and  $(P_\lambda)_{\lambda > 0}$  are commuting, we obtain

$$\|S_t^s (\text{Id} - P_\lambda) f\|_{L_p(\mathbb{R}^d)} = \|(\text{Id} - P_\lambda) S_t^s f\|_{L_p(\mathbb{R}^d)} \leq K_{a_s,b} e^{-(2^{-s-4} c^s \lambda^{sm} - \nu)t} \|f\|_{L_p(\mathbb{R}^d)}.$$

This yields condition (2.60) with  $\omega_\lambda := 2^{-s-4} c^s \lambda^{sm} - \nu$  tending to  $\infty$  as  $\lambda \rightarrow \infty$ . By the arguments of Remark 2.23 this implies (c).  $\square$

In the situations of (a) and (b) in Theorem 3.22, we can give an upper bound on the control cost  $\mathcal{C}$  defined in (1.6) and (1.7), respectively. Indeed, as in Theorem 3.8, we can express the observability constant  $C'_{\text{obs}}$  of the dual system explicitly with respect to all model parameters. By Theorem 1.20 this description transfers to an upper bound on the control cost and we obtain  $\mathcal{C} \leq C'_{\text{obs}}$ . Note that, since  $C'_{\text{obs}}$  is the constant of the dual system, it depends on  $r'$  instead of  $r$ , where  $r' \in [1, \infty]$  such that  $1/r + 1/r' = 1$ .

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