Chapter 15
Maximal Regularity

In this chapter, we address the issue of maximal regularity. More precisely, we provide a criterion on the 'structure' of the evolutionary equation

$$\left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right) U = F$$

in question and the right-hand side F in order to obtain $U \in \operatorname{dom}(\partial_{t,\nu} M(\partial_{t,\nu})) \cap \operatorname{dom}(A)$. If $F \in L_{2,\nu}(\mathbb{R}; H)$, $U \in \operatorname{dom}(\partial_{t,\nu} M(\partial_{t,\nu})) \cap \operatorname{dom}(A)$ is the optimal regularity one could hope for. However, one cannot expect U to be as regular since $\left(\partial_{t,\nu} M(\partial_{t,\nu}) + A\right)$ is simply not closed in general. Hence, in all the cases where $\left(\partial_{t,\nu} M(\partial_{t,\nu}) + A\right)$ is *not* closed, the desired regularity property does not hold for $F \in L_{2,\nu}(\mathbb{R}; H)$. However, note that by Picard's theorem, $F \in \operatorname{dom}(\partial_{t,\nu})$ implies the desired regularity property for U given the positive definiteness condition for the material law is satisfied and A is skew-selfadjoint. In this case, one even has $U \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(A)$, which is more regular than expected. Thus, in the general case of an unbounded, skew-selfadjoint operator A neither the condition $F \in \operatorname{dom}(\partial_{t,\nu})$ nor $F \in L_{2,\nu}(\mathbb{R}; H)$ yields precisely the regularity $U \in \operatorname{dom}(\partial_{t,\nu} M(\partial_{t,\nu})) \cap \operatorname{dom}(A)$ since

$$\operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(A) \subseteq \operatorname{dom}(\partial_{t,\nu} M(\partial_{t,\nu})) \cap \operatorname{dom}(A) \subseteq \operatorname{dom}(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}),$$

where the inclusions are proper in general. It is the aim of this chapter to provide an example case, where less regularity of F actually yields *more* regularity for U. If one focusses on time-regularity only, this improvement of regularity is in stark contrast to the general theory developed in the previous chapters. Indeed, in this regard, one can coin the (time) regularity asserted in Picard's theorem as "U is as regular as F". For a more detailed account on the usual perspective of maximal regularity (predominantly) for parabolic equations, we refer to the Comments section of this chapter.

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15.1 Guiding Examples and Non-Examples

Before we present the abstract theory, we motivate the general setting looking at a particular example. Traditionally, in the discussion of partial differential equations and their classification, people focus on regularity theory. Thus, one finds the non-exhaustive categories ‘elliptic’, ‘parabolic’, and ‘hyperbolic’. Since we do not want to dive into the intricacies of this classification much less their regularity, we only name some examples of the said subclasses. Laplace’s equation from Chap. 1 falls into the class of elliptic PDEs, the heat equation is a paradigm example of a parabolic equation and Maxwell’s equations or the transport equation are hyperbolic.

Since we predominantly treat time-dependent equations and elliptic PDEs usually are time-independent, we only look at examples for hyperbolic and parabolic equations more closely. As for the hyperbolic case, we consider the transport equation next and highlight that any ‘gain’ in regularity as hinted at in the introduction of this chapter is not possible.

Example 15.1.1 We define $\partial\colon H^1(\mathbb{R}) \subseteq L_2(\mathbb{R}) \to L_2(\mathbb{R}), \phi \mapsto \phi'$. Then, by Corollary 3.2.6, $\partial^* = -\partial$; that is, ∂ is skew-selfadjoint. We consider for $\nu > 0$ the operator

$$\partial_{t,\nu} + \partial$$

in $L_{2,\nu}(\mathbb{R}; L_2(\mathbb{R}))$. Then, by Picard’s theorem, $0 \in \rho\big(\overline{\partial_{t,\nu} + \partial}\big)$; that is, $\big(\overline{\partial_{t,\nu} + \partial}\big)^{-1} \in L(L_{2,\nu}(\mathbb{R}; L_2(\mathbb{R})))$. Next, consider the functions

$$u\colon (t,x) \mapsto \mathbb{1}_{\mathbb{R}_{\geq 0}}(t) t \mathrm{e}^{-t} h(x-t)$$
$$f\colon (t,x) \mapsto \mathbb{1}_{\mathbb{R}_{\geq 0}}(t)(1-t) \mathrm{e}^{-t} h(x-t)$$

for some $h \in L_2(\mathbb{R})$. Then it is not difficult to see that $u, f \in L_{2,\nu}(\mathbb{R}; L_2(\mathbb{R}))$. If $h \in C_c^\infty(\mathbb{R})$, then

$$u \in H_\nu^1(\mathbb{R}; H^1(\mathbb{R})) \subseteq \operatorname{dom}(\partial_{t,\nu} + \partial)$$

and

$$(\partial_{t,\nu} + \partial)u = f.$$

If $h \in L_2(\mathbb{R}) \setminus H^1(\mathbb{R})$, then one can show that $u \in \operatorname{dom}\big(\overline{\partial_{t,\nu} + \partial}\big)$, $\big(\overline{\partial_{t,\nu} + \partial}\big)u = f$ and

$$u \notin \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(\partial).$$

For this observation, we refer to Exercise 15.1. Thus, being in the domain of $\overline{\partial_{t,\nu} + \partial}$ does not necessarily imply being in the domain of either $\mathrm{dom}(\partial_{t,\nu})$ or $\mathrm{dom}(\partial)$.

The last example has shown that we cannot expect an improvement of regularity for the considered transport equation. In fact, it is possible to provide an example of a similar type for the wave equation (and similar hyperbolic type equations including Maxwell's equations). Thus, in order to have an improvement of regularity one needs to further restrict the class of evolutionary equations. We now provide a guiding example, where we discuss an abstract variant of the heat equation.

Example 15.1.2 Let ℓ_2 be the space of square summable sequences indexed by $n \in \mathbb{N}$. We note that ℓ_2 is isomorphic to $L_2(\#_{\mathbb{N}})$, where $\#_{\mathbb{N}}$ is the counting measure on $\mathbb{N}$. We introduce $\mathrm{m}\colon \mathrm{dom}(\mathrm{m}) \subseteq \ell_2 \to \ell_2$ the operator of multiplying by the argument. Then, m is an unbounded, selfadjoint operator. Next, we consider the operator

$$\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{m} \\ \mathrm{m} & 0 \end{pmatrix}$$

on $L_{2,\nu}(\mathbb{R}; \ell_2)$. Then, Picard's theorem applies and we obtain

$$0 \in \rho\left(\overline{\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{m} \\ \mathrm{m} & 0 \end{pmatrix}}\right).$$

For $f \in L_{2,\nu}(\mathbb{R}; \ell_2)$ define

$$\begin{pmatrix} u \\ q \end{pmatrix} := \left(\overline{\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{m} \\ \mathrm{m} & 0 \end{pmatrix}}\right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Then $u \in \mathrm{dom}(\partial_{t,\nu}) \cap \mathrm{dom}(\mathrm{m})$ and $q \in \mathrm{dom}(\mathrm{m})$. We ask the reader to fill in the details in Exercise 15.2.

Remark 15.1.3 The last example is in fact an abstract version of the heat equation on bounded domains. We refer to [90, Section 2.2.2] for a corresponding reasoning for the Schrödinger equation.

Let us compare the two different examples, the transport equation and the abstract parabolic equation. From the perspective of evolutionary equations; that is, looking at equations of the form

$$(\partial_{t,\nu} M_0 + M_1 + A)U = F,$$

for the transport equation we have $M_0 = 1$ and $M_1 = 0$. In the case of the abstract parabolic equation, M_0 has a nontrivial kernel, which is compensated in M_1. Moreover, the decomposition of kernel and range of M_0 is comparable to the block structure of A. Thus, we may hope for an improvement of regularity as in

Example 15.1.2 if these abstract conditions are met. This observation is the starting point of parabolic evolutionary pairs to be defined in the next section.

15.2 The Maximal Regularity Theorem and Fractional Sobolev Spaces

In order to be able to formulate the main theorem of this chapter, we need the notion of fractional Sobolev spaces. For this, we recall from Example 5.3.4 and Sect. 7.2 that we already dealt with fractional powers of the time-derivative. For α, $\nu \geq 0$, we thus consistently define

$$\partial_{t,\nu}^{\alpha} := \mathcal{L}_{\nu}^{*}(\mathrm{im}+\nu)^{\alpha}\mathcal{L}_{\nu},$$

with maximal domain in $L_{2,\nu}(\mathbb{R}; H)$, where we agree with setting $\mathcal{L}_0 := \mathcal{F}$. Note that in this case, using Proposition 7.2.1, $0 \in \rho(\partial_{t,\nu}^{\alpha})$ given $\nu > 0$. Hence, the following construction yields Hilbert spaces; for this also recall that $\langle \cdot, \cdot \rangle_A$ denotes the graph inner product of a linear operator A defined in a Hilbert space.

Definition Let α, $\nu \geq 0$. Then we define

$$H_{\nu}^{\alpha}(\mathbb{R}; H) := \left(\mathrm{dom}(\partial_{t,\nu}^{\alpha}), (f, g) \mapsto \langle \partial_{t,\nu}^{\alpha} f, \partial_{t,\nu}^{\alpha} g\rangle_{L_{2,\nu}(\mathbb{R};H)}\right)$$

for $\nu > 0$ and

$$H_0^{\alpha}(\mathbb{R}; H) := \left(\{f \in L_2(\mathbb{R}; H);\ \mathcal{F}f \in \mathrm{dom}((\mathrm{im})^{\alpha})\}, (f, g) \mapsto \langle \mathcal{F}f, \mathcal{F}g\rangle_{(\mathrm{im})^{\alpha}}\right).$$

Lemma 15.2.1 *For all α, $\nu \geq 0$ the space $H_{\nu}^{\alpha}(\mathbb{R}; H)$ is a Hilbert space. Moreover, $H_{\nu}^{\alpha}(\mathbb{R}; H) \hookrightarrow L_{2,\nu}(\mathbb{R}; H)$ continuously and densely.*

Proof We only show the claim for $\nu > 0$. By Fourier–Laplace transformation, the claim follows if we show that

$$(\mathrm{im}+\nu)^{\alpha}\colon\ \mathrm{dom}((\mathrm{im}+\nu)^{\alpha}) \subseteq L_2(\mathbb{R}; H) \to L_2(\mathbb{R}; H)$$

is densely defined and continuously invertible. For this, we find $n \in \mathbb{N}$ and $\beta \in [0, 1)$ such that $\alpha = n + \beta$. It is easy to see that $(\mathrm{im}+\nu)^{\alpha} = (\mathrm{im}+\nu)^{n}(\mathrm{im}+\nu)^{\beta}$. Thus, continuous invertibility readily follows from the continuous invertibility of $(\mathrm{im}+\nu)$ and $(\mathrm{im}+\nu)^{\beta}$ (for the latter, see also Proposition 7.2.1). For the case when $H = \mathbb{K}$, it follows from Theorem 2.4.3 that $(\mathrm{im}+\nu)^{\alpha}$ is densely defined. Thus, it follows from Lemma 3.1.8 that $(\mathrm{im}+\nu)^{\alpha}$ is densely defined also for general H. □

In order to state our main theorem, we introduce the notion of parabolic pairs.

Definition Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law, $A\colon \operatorname{dom}(A) \subseteq H \to H$ and $\alpha \in (0, 1]$. We call (M, A) an *(α-)fractional parabolic pair* if the following conditions are met: there exist $\nu > \max\{0, s_b(M)\}$ and $c > 0$ such that

$$\operatorname{Re} zM(z) \geqslant c \quad (z \in \mathbb{C}_{\operatorname{Re}>\nu}),$$

and moreover, we find a closed subspace $H_0 \subseteq H$, $H_1 := H_0^{\perp}$, $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ closed and densely defined, and $M_{00} \in \mathcal{M}(H_0; \nu)$, $N \in \mathcal{M}(H; \nu)$ such that

$$M(z) = \begin{pmatrix} M_{00}(z) & 0 \\ 0 & 0 \end{pmatrix} + z^{-1}N(z), \quad A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix},$$

and

$$\operatorname{Re} z^{1-\alpha} M_{00}(z) \geqslant c' \quad (z \in \mathbb{C}_{\operatorname{Re}>\nu})$$

for some $c' > 0$, and $\mathbb{C}_{\operatorname{Re}>\nu} \ni z \mapsto z^{1-\alpha} M_{00}(z) \in L(H_0)$ is bounded. A 1-fractional parabolic pair is called *parabolic*.

Remark 15.2.2

(a) If (M, A) is α-fractional parabolic and β-fractional parabolic with the same decomposition $H = H_0 \oplus H_1$, then $\alpha = \beta$. Indeed, assume that $\alpha < \beta$. Then

$$z^{1-\beta} M_{00}(z) = z^{\alpha-\beta} z^{1-\alpha} M_{00}(z) \to 0 \quad (|z| \to \infty, z \in \mathbb{C}_{\operatorname{Re}>\nu})$$

contradicting the real-part condition.

(b) If (M, A) is α-fractional parabolic, then there exists $\mu > \nu$ such that for all $z \in \mathbb{C}_{\operatorname{Re}>\mu}$

$$\operatorname{Re} z^{1-\alpha} \Big(M_{00}(z) + z^{-1} N_{00}(z) \Big) \geqslant c'/2 \tag{15.1}$$

for some $c' > 0$, where $N_{00}(z) := \iota_{H_0}^* N(z) \iota_{H_0} \in L(H_0)$. Indeed, this follows from the fact that $z^{-\alpha} N_{00}(z) \to 0$ as $\operatorname{Re} z \to \infty$.

The main theorem of this chapter is the following:

Theorem 15.2.3 *Let $\alpha \in (0, 1]$ and (M, A) be α-fractional parabolic (with $H = H_0 \oplus H_1$ and C from H_0 to H_1) and assume that (15.1) holds for all $z \in \mathbb{C}_{Re>\nu}$ for some $\nu > \max\{0, s_b(M)\}$. Let $f \in L_{2,\nu}(\mathbb{R}; H_0)$ and $g \in H_\nu^{\alpha/2}(\mathbb{R}; H_1)$. Then the solution $(u, v) := \left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)^{-1} (f, g) \in L_{2,\nu}(\mathbb{R}; H)$ satisfies*

$$u \in H_\nu^{\alpha}(\mathbb{R}; H_0) \cap H_\nu^{\alpha/2}\big(\mathbb{R}; \operatorname{dom}(C)\big)$$
$$v \in H_\nu^{\alpha/2}(\mathbb{R}; H_1) \cap L_{2,\nu}\big(\mathbb{R}; \operatorname{dom}(C^*)\big).$$

More precisely,

$$\left(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\right)^{-1} : L_{2,\nu}(\mathbb{R}; H_0) \oplus H_\nu^{\alpha/2}(\mathbb{R}; H_1) \\ \to \left(H_\nu^\alpha(\mathbb{R}; H_0) \cap H_\nu^{\alpha/2}\big(\mathbb{R}; \operatorname{dom}(C)\big)\right) \oplus \left(H_\nu^{\alpha/2}(\mathbb{R}; H_1) \cap L_{2,\nu}\big(\mathbb{R}; \operatorname{dom}(C^*)\big)\right)$$

is continuous.

Example 15.2.4 (Heat Equation) Let us recall the heat equation from Theorem 6.2.4. For $\Omega \subseteq \mathbb{R}^d$ open, we let $a \in L(L_2(\Omega)^d)$ such that

$$\operatorname{Re} a \geqslant c$$

in the sense of positive definiteness. It is not difficult to see that

$$\left(z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & az^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}\right),$$

is parabolic; with the obvious orthogonal decomposition of the underlying Hilbert space. Let $f \in L_{2,\nu}(\mathbb{R}; L_2(\Omega))$. Then

$$\begin{pmatrix} \theta \\ q \end{pmatrix} := \left(\overline{\partial_{t,\nu} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div}_0 \\ \operatorname{grad} & 0 \end{pmatrix}}\right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix}$$

particularly satisfies the regularity statement

$$\theta \in H_\nu^1(\mathbb{R}; L_2(\Omega)) \cap L_{2,\nu}\big(\mathbb{R}; H^1(\Omega)\big) \text{ and } q \in L_{2,\nu}\big(\mathbb{R}; H_0(\operatorname{div}, \Omega)\big).$$

The next example deals with a parabolic variant of the equations introduced in (7.3) and (7.4) describing fractional elasticity. We modify the equations at hand by considering $\alpha \in [1, 2]$.

Example 15.2.5 (Parabolic Fractional Viscoelasticity) Let $\Omega \subseteq \mathbb{R}^d$ open and recall the differential operators Div and Grad_0 from Sect. 7.1 defined in the spaces $L_2(\Omega)^{d\times d}_{\mathrm{sym}}$ and $L_2(\Omega)^d$, respectively. Let $c > 0$ and $D \in L\big(L_2(\Omega)^{d\times d}_{\mathrm{sym}}\big)$, $\rho = \rho^* \in L(L_2(\Omega)^d)$. For $\nu > 0$ and $f \in L_{2,\nu}(\mathbb{R}; L_2(\Omega)^d)$ consider the problem of finding $u : \mathbb{R} \times \Omega \to \mathbb{R}^d$ such that

$$\partial_{t,\nu} \rho \partial_{t,\nu} u - \operatorname{Div} T = f \tag{15.2}$$

$$T = D\partial_{t,\nu}^\alpha \operatorname{Grad}_0 u, \tag{15.3}$$

for some $\alpha \in [1,2)$, where $\rho \geqslant c$ and $\operatorname{Re} D \geqslant c$ in the sense of positive definiteness. We rewrite the system just introduced by using $v := \partial_{t,\nu}^{\alpha} u$ to (formally) obtain

$$\begin{aligned} \partial_{t,\nu}\rho\partial_{t,\nu}^{1-\alpha} v - \operatorname{Div} T &= f \\ T &= D \operatorname{Grad}_0 v. \end{aligned}$$

Note that $\gamma := 1 + (1-\alpha) \in (0,1]$. Thus, using the selfadjointness and positive definiteness of ρ as well as Proposition 7.2.1, we infer

$$\operatorname{Re}(z^{\gamma}\rho) \geqslant \nu^{\gamma} c \quad (z \in \mathbb{C}_{\operatorname{Re}\geqslant \nu}).$$

Consequently, applying Proposition 6.2.3(b) to $a = D$, we get that

$$\left(z \mapsto \begin{pmatrix} z^{\gamma-1}\rho & 0 \\ 0 & z^{-1}D^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -\operatorname{Div} \\ -\operatorname{Grad}_0 & 0 \end{pmatrix} \right)$$

is γ-fractional parabolic. In consequence, the solution (v, T) of

$$\left(\overline{\partial_{t,\nu} \begin{pmatrix} \partial_{t,\nu}^{\gamma-1}\rho & 0 \\ 0 & \partial_{t,\nu}^{-1} D^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{Div} \\ -\operatorname{Grad}_0 & 0 \end{pmatrix}} \right) \begin{pmatrix} v \\ T \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

additionally satisfies the following regularity properties

$$\begin{aligned} v &\in H_{\nu}^{\gamma}\big(\mathbb{R}; L_2(\Omega)^d\big) \cap H_{\nu}^{\gamma/2}\big(\mathbb{R}; \operatorname{dom}(\operatorname{Grad}_0)\big), \\ T &\in H_{\nu}^{\gamma/2}\big(\mathbb{R}; L_2(\Omega)_{\mathrm{sym}}^{d\times d}\big) \cap L_{2,\nu}\big(\mathbb{R}; \operatorname{dom}(\operatorname{Div})\big). \end{aligned}$$

Rephrasing this for $u = \partial_{t,\nu}^{-\alpha} v$, we even have

$$u \in H_{\nu}^{2}\big(\mathbb{R}; L_2(\Omega)^d\big) \cap H_{\nu}^{1+\alpha/2}\big(\mathbb{R}; \operatorname{dom}(\operatorname{Grad}_0)\big),$$

which, since $\alpha/2 \leqslant 1$, particularly implies that the equations (15.2) and (15.3) are equalities valid in $L_{2,\nu}\big(\mathbb{R}; L_2(\Omega)^d\big)$ and $L_{2,\nu}\big(\mathbb{R}; L_2(\Omega)_{\mathrm{sym}}^{d\times d}\big)$, respectively.

15.3 The Proof of Theorem 15.2.3

The decisive estimate in connection to the proof of Theorem 15.2.3 is contained in the following statement. For the entire rest of the section, we shall denote the norm and scalar product in $H_{\nu}^{\alpha}(\mathbb{R}; K)$, K some Hilbert space, by $\|\cdot\|_{\alpha}$ and $\langle\cdot,\cdot\rangle_{\alpha}$, respectively.

Lemma 15.3.1 *Let H_0, H_1 be Hilbert spaces, $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ densely defined and closed. Let $\alpha \in [0,1]$, $M_j\colon \operatorname{dom}(M_j) \subseteq \mathbb{C} \to L(H_j)$ material laws for $j \in \{0,1\}$, $\nu > \max\{\mathrm{s_b}(M_0), \mathrm{s_b}(M_1), 0\}$ with*

$$\mathbb{C}_{\mathrm{Re}\geqslant\nu} \ni z \mapsto z^{1-\alpha} M_0(z) \in L(H_0)$$

bounded. Assume there exists $c > 0$ such that for all $z \in \mathbb{C}_{\mathrm{Re}\geqslant\nu}$

$$\operatorname{Re} z M_0(z) \geqslant c, \quad \operatorname{Re} M_1(z) \geqslant c, \quad \operatorname{Re} z^{1-\alpha} M_0(z) \geqslant c.$$

Let $f \in L_{2,\nu}(\mathbb{R}; H_0)$, $g \in H_\nu^{\alpha/2}(\mathbb{R}; H_1)$ as well as $u \in H_\nu^1\big(\mathbb{R}; \operatorname{dom}(C)\big)$ and $v \in H_\nu^1\big(\mathbb{R}; \operatorname{dom}(C^*)\big)$. Assume the equalities

$$\begin{aligned} \partial_{t,\nu} M_0(\partial_{t,\nu}) u - C^* v &= f, \\ v + M_1(\partial_{t,\nu}) C u &= g. \end{aligned}$$

Then

$$\begin{aligned} &\|u\|_\alpha^2 + \|Cu\|_{\alpha/2}^2 + \|v\|_{\alpha/2}^2 + \left\|C^* v\right\|_0^2 \\ &\leqslant 2\left(1 + \left(m_1^2 + m_0^2 + \frac{1}{2}\right)\left(\frac{2}{c} + \frac{m_1}{c^2}\right)^2\right)\left(\|f\|_0^2 + \|g\|_{\alpha/2}^2\right) \end{aligned}$$

with $m_1 := \|M_1\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}$ and $m_0 := \left\|z \mapsto z^{1-\alpha} M_0(z)\right\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}$.

Proof We compute

$$\begin{aligned} c\,\|Cu\|_{\alpha/2}^2 &\leqslant c\,\|Cu\|_{\alpha/2}^2 + c\,\|u\|_{\alpha/2}^2 \\ &\leqslant \operatorname{Re}\big\langle M_1(\partial_{t,\nu}) Cu, Cu\big\rangle_{\alpha/2} + \operatorname{Re}\big\langle \partial_{t,\nu} M_0(\partial_{t,\nu}) u, u\big\rangle_{\alpha/2} \\ &= \operatorname{Re}\langle g - v, Cu\rangle_{\alpha/2} + \operatorname{Re}\big\langle \partial_{t,\nu} M_0(\partial_{t,\nu}) u, u\big\rangle_{\alpha/2} \\ &\leqslant \|g\|_{\alpha/2}\,\|Cu\|_{\alpha/2} + \operatorname{Re}\big\langle \partial_{t,\nu} M_0(\partial_{t,\nu}) u - C^* v, u\big\rangle_{\alpha/2} \\ &= \|g\|_{\alpha/2}\,\|Cu\|_{\alpha/2} + \operatorname{Re}\Big\langle f, \big(\partial_{t,\nu}^*\big)^{\alpha/2} \big(\partial_{t,\nu}\big)^{\alpha/2} u\Big\rangle_0 \\ &\leqslant \|g\|_{\alpha/2}\,\|Cu\|_{\alpha/2} + \|f\|_0\,\|u\|_\alpha\,, \end{aligned}$$

where we used that

$$\begin{aligned}
\left\| \left(\partial_{t,\nu}^*\right)^{\alpha/2} \left(\partial_{t,\nu}\right)^{\alpha/2} u \right\|_0 &= \left\| (-\mathrm{im}+\nu)^{\alpha/2} (\mathrm{im}+\nu)^{\alpha/2} u \right\|_{L_2(\mathbb{R};H_0)} \\
&= \left\| \frac{(-\mathrm{im}+\nu)^{\alpha/2}}{(\mathrm{im}+\nu)^{\alpha/2}} (\mathrm{im}+\nu)^{\alpha} u \right\|_{L_2(\mathbb{R};H_0)} \\
&\leqslant \| (\mathrm{im}+\nu)^{\alpha} u\|_{L_2(\mathbb{R};H_0)} = \|u\|_{\alpha}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
c\,\|u\|_{\alpha}^2 &\leqslant \mathrm{Re}\left\langle \partial_{t,\nu}^{1-\alpha} M_0(\partial_{t,\nu})\partial_{t,\nu}^{\alpha} u,\, \partial_{t,\nu}^{\alpha} u\right\rangle_0 \\
&= \mathrm{Re}\left\langle \partial_{t,\nu} M_0(\partial_{t,\nu})u,\, \partial_{t,\nu}^{\alpha} u\right\rangle_0 \\
&= \mathrm{Re}\left\langle f + C^* v,\, \partial_{t,\nu}^{\alpha} u\right\rangle_0 \\
&\leqslant \|f\|_0 \|u\|_{\alpha} + \mathrm{Re}\left\langle \left(\partial_{t,\nu}^*\right)^{\alpha/2} v,\, \partial_{t,\nu}^{\alpha/2} C u\right\rangle_0 \\
&\leqslant \|f\|_0 \|u\|_{\alpha} + \|v\|_{\alpha/2} \|Cu\|_{\alpha/2} \\
&= \|f\|_0 \|u\|_{\alpha} + \left\|g - M_1(\partial_{t,\nu})Cu\right\|_{\alpha/2} \|Cu\|_{\alpha/2} \\
&\leqslant \|f\|_0 \|u\|_{\alpha} + \|g\|_{\alpha/2} \|Cu\|_{\alpha/2} + m_1 \|Cu\|_{\alpha/2}^2 \\
&\leqslant \left(1 + \frac{m_1}{c}\right)\left(\|f\|_0 \|u\|_{\alpha} + \|g\|_{\alpha/2} \|Cu\|_{\alpha/2}\right).
\end{aligned}$$

Thus, we obtain for $\varepsilon > 0$

$$\begin{aligned}
&c\left(\|u\|_{\alpha}^2 + \|Cu\|_{\alpha/2}^2\right) \\
&\leqslant \left(2 + \frac{m_1}{c}\right)\left(\|f\|_0 \|u\|_{\alpha} + \|g\|_{\alpha/2} \|Cu\|_{\alpha/2}\right) \\
&\leqslant \frac{1}{2}\left(2 + \frac{m_1}{c}\right)\left(\frac{1}{\varepsilon}\left(\|f\|_0^2 + \|g\|_{\alpha/2}^2\right) + \varepsilon\left(\|u\|_{\alpha}^2 + \|Cu\|_{\alpha/2}^2\right)\right).
\end{aligned}$$

Choosing $\varepsilon = c^2/(2c + m_1)$ and subtracting the term involving u and Cu on both sides of the inequality, we deduce

$$\begin{aligned}
\frac{c}{2}\left(\|u\|_{\alpha}^2 + \|Cu\|_{\alpha/2}^2\right) &\leqslant \frac{1}{2}\left(2 + \frac{m_1}{c}\right)\frac{1}{\varepsilon}\left(\|f\|_0^2 + \|g\|_{\alpha/2}^2\right) \\
&= \frac{1}{2c}\left(2 + \frac{m_1}{c}\right)^2\left(\|f\|_0^2 + \|g\|_{\alpha/2}^2\right)
\end{aligned}$$

and therefore

$$\left(\|u\|_\alpha^2 + \|Cu\|_{\alpha/2}^2\right) \leqslant \left(\frac{2}{c} + \frac{m_1}{c^2}\right)^2 \left(\|f\|_0^2 + \|g\|_{\alpha/2}^2\right).$$

Finally, we compute

$$\begin{aligned}\frac{1}{2}\|v\|_{\alpha/2}^2 &\leqslant \|g\|_{\alpha/2}^2 + \left\|M_1(\partial_{t,\nu})Cu\right\|_{\alpha/2}^2 \\ &\leqslant \|g\|_{\alpha/2}^2 + m_1^2\left(\frac{2}{c} + \frac{m_1}{c^2}\right)^2 \left(\|f\|_0^2 + \|g\|_{\alpha/2}^2\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\left\|C^* v\right\|_0^2 &\leqslant \left\|\partial_{t,\nu} M_0(\partial_{t,\nu})u\right\|_0^2 + \|f\|_0^2 \\ &\leqslant \left\|\partial_{t,\nu}^{1-\alpha} M_0(\partial_{t,\nu})\partial_{t,\nu}^{\alpha} u\right\|_0^2 + \|f\|_0^2 \\ &\leqslant m_0^2\|u\|_\alpha^2 + \|f\|_0^2 \\ &\leqslant m_0^2\left(\frac{2}{c} + \frac{m_1}{c^2}\right)^2 \left(\|f\|_0^2 + \|g\|_{\alpha/2}^2\right) + \|f\|_0^2.\end{aligned}$$

□

The next preliminary finding is a refinement of the surjectivity statement in Picard's theorem.

Proposition 15.3.2 *Let H be a Hilbert space, $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ a material law, $\nu > \mathrm{s_b}(M)$, with $\nu > 0$, and $A\colon \operatorname{dom}(A) \subseteq H \to H$ skew-selfadjoint. Assume there exists $c > 0$ such that for all $z \in \mathbb{C}_{\mathrm{Re}>\nu}$ we have*

$$\operatorname{Re} zM(z) \geqslant c.$$

Let $\beta \in [0, 1]$.

(a) *The inclusion*

$$\left(\partial_{t,\nu}M(\partial_{t,\nu}) + A\right)\left[H_\nu^2\left(\mathbb{R}; \operatorname{dom}(A)\right)\right] \subseteq H_\nu^\beta(\mathbb{R}; H)$$

is dense.

(b) *Let $H_0 \subseteq H$ be a closed subspace and $H_1 := H_0^\perp$. Then*

$$\left(\partial_{t,\nu}M(\partial_{t,\nu}) + A\right)\left[H_\nu^2\left(\mathbb{R}; \operatorname{dom}(A)\right)\right] \subseteq L_{2,\nu}(\mathbb{R}; H_0) \oplus H_\nu^\beta(\mathbb{R}; H_1)$$

is dense.

Proof

(a) Since $H_\nu^1(\mathbb{R}; H)$ is dense in $H_\nu^\beta(\mathbb{R}; H)$ (this is a consequence of Lemma 15.2.1), it suffices to show the claim for $\beta = 1$. Next, by Picard's theorem, for $f \in \operatorname{dom}(\partial_{t,\nu})$, we obtain $u = \big(\partial_{t,\nu} M(\partial_{t,\nu}) + A\big)^{-1} f \in \operatorname{dom}(\partial_{t,\nu}) \cap L_{2,\nu}\big(\mathbb{R}; \operatorname{dom}(A)\big)$. In particular, it follows that

$$\big(\partial_{t,\nu} M(\partial_{t,\nu}) + A\big)\left[H_\nu^1(\mathbb{R}; H) \cap L_{2,\nu}\big(\mathbb{R}; \operatorname{dom}(A)\big)\right] \subseteq L_{2,\nu}(\mathbb{R}; H)$$

is dense. Multiplying this inclusion by $\partial_{t,\nu}^{-1}$, we infer that

$$\big(\partial_{t,\nu} M(\partial_{t,\nu}) + A\big)\left[H_\nu^2(\mathbb{R}; H) \cap H_\nu^1\big(\mathbb{R}; \operatorname{dom}(A)\big)\right] \subseteq H_\nu^1(\mathbb{R}; H)$$

is dense. Hence, for $f \in H_\nu^1(\mathbb{R}; H)$, we find $(u_n)_n$ in $H_\nu^2(\mathbb{R}; H) \cap H_\nu^1(\mathbb{R}; \operatorname{dom}(A))$ such that $f_n := \big(\partial_{t,\nu} M(\partial_{t,\nu}) + A\big) u_n \to f$ in $H_\nu^1(\mathbb{R}; H)$ as $n \to \infty$. Next, for $\varepsilon > 0$, $(1 + \varepsilon\partial_{t,\nu})^{-1} u \in H_\nu^2(\mathbb{R}; \operatorname{dom}(A))$ given $u \in H_\nu^1(\mathbb{R}; \operatorname{dom}(A))$. Moreover, $(1 + \varepsilon\partial_{t,\nu})^{-1} f \to f$ in $H_\nu^1(\mathbb{R}; H)$ as $\varepsilon \to 0$, by Lemma 9.3.3(b) and the fact that $\partial_{t,\nu}^{-1}$ commutes with $(1 + \varepsilon\partial_{t,\nu})^{-1}$. Thus, we compute for $\varepsilon > 0$ and $n \in \mathbb{N}$

$$\begin{aligned}
&\left\| \big(\partial_{t,\nu} M(\partial_{t,\nu}) + A\big)\,(1 + \varepsilon\partial_{t,\nu})^{-1} u_n - f \right\|_1 \\
&\leqslant \left\| (1 + \varepsilon\partial_{t,\nu})^{-1} f_n - (1 + \varepsilon\partial_{t,\nu})^{-1} f \right\|_1 + \left\| (1 + \varepsilon\partial_{t,\nu})^{-1} f - f \right\|_1 \\
&\leqslant \| f_n - f \|_1 + \left\| (1 + \varepsilon\partial_{t,\nu})^{-1} f - f \right\|_1 \to 0
\end{aligned}$$

as $n \to \infty$ and $\varepsilon \to 0$, which concludes the proof of (a).

(b) By (a), it suffices to show that

$$H_\nu^\beta(\mathbb{R}; H) = H_\nu^\beta(\mathbb{R}; H_0) \oplus H_\nu^\beta(\mathbb{R}; H_1) \subseteq L_{2,\nu}(\mathbb{R}; H_0) \oplus H_\nu^\beta(\mathbb{R}; H_1)$$

is dense (note that the first equality follows from the fact that $H \ni u \mapsto (u_0, u_1) \in H_0 \oplus H_1$ is unitary). The desired density result thus follows from Lemma 15.2.1. □

Next, we shall proceed with a proof of our main theorem in this chapter.

Proof of Theorem 15.2.3 For $i, j \in \{0, 1\}$ we set $N_{ij}(z) := \iota_{H_i}^* N(z) \iota_{H_j}$. Let $(f, g) \in \big(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\big)\big[H_\nu^2\big(\mathbb{R}; \operatorname{dom}(C) \oplus \operatorname{dom}(C^*)\big)\big]$. Defining

$$(u, v) := \big(\overline{\partial_{t,\nu} M(\partial_{t,\nu}) + A}\big)^{-1} (f, g) \in H_\nu^2\big(\mathbb{R}; \operatorname{dom}(C) \oplus \operatorname{dom}(C^*)\big),$$

we have

$$\begin{aligned}\partial_{t,\nu} M_{00}(\partial_{t,\nu})u + N_{00}(\partial_{t,\nu})u - C^* v &= f - N_{01}(\partial_{t,\nu})v,\\ N_{11}(\partial_{t,\nu})v + Cu &= g - N_{10}(\partial_{t,\nu})u.\end{aligned}$$

Since $\operatorname{Re} zM(z) \geqslant c$, we infer

$$\operatorname{Re} N_{11}(\partial_{t,\nu}) \geqslant c.$$

Thus, by Proposition 6.2.3(b), we deduce that $M_1(\partial_{t,\nu}) \coloneqq N_{11}(\partial_{t,\nu})^{-1}$ satisfies the real-part condition imposed on M_1 in Lemma 15.3.1. Moreover, since (M, A) is α-fractional parabolic,

$$M_0(z) \coloneqq M_{00}(z) + z^{-1} N_{00}(z)$$

fulfills the real part and boundedness assumptions in Lemma 15.3.1. Introducing

$$\begin{aligned}\widetilde{f} &\coloneqq f - N_{01}(\partial_{t,\nu})v \in H_\nu^1(\mathbb{R}; H_0) \subseteq L_{2,\nu}(\mathbb{R}; H_0)\\ \widetilde{g} &\coloneqq M_1(\partial_{t,\nu})g - M_1(\partial_{t,\nu})N_{10}(\partial_{t,\nu})u \in H_\nu^1(\mathbb{R}; H_1) \subseteq H_\nu^{\alpha/2}(\mathbb{R}; H_1)\,,\end{aligned}$$

we get

$$\begin{aligned}\partial_{t,\nu} M_0(\partial_{t,\nu})u - C^* v &= \widetilde{f},\\ v + M_1(\partial_{t,\nu})Cv &= \widetilde{g}.\end{aligned}$$

Thus, using Lemma 15.3.1, we find $\kappa \geqslant 0$ in terms of M_0, M_1 and the positivity constants such that (recall that $m_1 \coloneqq \|M_1\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}$)

$$\begin{aligned}&\|u\|_\alpha^2 + \|Cu\|_{\alpha/2}^2 + \|v\|_{\alpha/2}^2 + \left\|C^* v\right\|_0^2\\ &\leqslant \kappa\big(\,\|\widetilde{f}\|_0^2 + \|\widetilde{g}\|_{\alpha/2}^2\big)\\ &\leqslant 2\kappa\big(\,\|f\|_0^2 + \|N\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}^2 \|v\|_0^2 + m_1^2 \|g\|_{\alpha/2}^2 + m_1^2 \|N\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}^2 \|u\|_{\alpha/2}^2\big)\\ &\leqslant 2\kappa\big(\,\|f\|_0^2 + \|N\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}^2 \|v\|_0^2 + m_1^2 \|g\|_{\alpha/2}^2 + 2m_1^2 \|N\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}^2 \big(\varepsilon \|u\|_\alpha^2 + \frac{1}{\varepsilon}\|u\|_0^2\big)\big)\end{aligned}$$

for all $\varepsilon > 0$, where in the last estimate, we used

$$\|u\|_{\alpha/2}^2 = \left\langle \partial_{t,\nu}^{\alpha/2} u, \partial_{t,\nu}^{\alpha/2} u\right\rangle_0 = \left\langle u, (\partial_{t,\nu}^{\alpha/2})^* \partial_{t,\nu}^{\alpha/2} u\right\rangle_0 \leqslant \|u\|_0 \|u\|_\alpha\,.$$

Hence, choosing $\varepsilon > 0$ small enough and using that $\overline{\left(\partial_{t,\nu}M(\partial_{t,\nu}) + A\right)}^{-1}$ is continuous from $L_{2,\nu}(\mathbb{R}; H)$ into itself, we find $\kappa' \geqslant 0$ such that

$$\|u\|_{\alpha}^2 + \|Cu\|_{\alpha/2}^2 + \|v\|_{\alpha/2}^2 + \left\|C^*v\right\|_0^2 \leqslant \kappa'\left(\|f\|_0^2 + \|g\|_{\alpha/2}^2 \right),$$

which establishes the assertion (using the density result in Proposition 15.3.2(b)).

□

15.4 Comments

The issue of maximal regularity (in Hilbert spaces for simplicity) is a priori formulated for equations of the type

$$u' + Au = f,$$

where f lies in some $L_2\big((0,T); H\big)$ and A is an unbounded operator in H. The question of maximal regularity then addresses, whether a solution u to this equation exists and satisfies $u \in L_2\big((0,T); \operatorname{dom}(A)\big) \cap H^1\big((0,T); H\big)$. In Hilbert spaces, whether or not this question can be answered in the affirmative solely relies on the properties of A. Hence, one shortens this question to whether A 'has maximal regularity'. The present situation is conveniently understood: A has maximal regularity if and only if $-A$ is the generator of a holomorphic semigroup, see [33, Theorem 2.2] and [105, Lemma 3,1]. One major example class is the class of operators that are defined with the help of forms, see [5] for an introductory text. People then studied the situation of time-dependent A. It has then been shown in various contexts and under suitable conditions on the (smoothness of the) time-dependence of A, whether A has maximal regularity or not. For this, we refer to [2, 8, 30] for an account of possible conditions. The evolutionary equations case, which is addressed for the first time in [88] in the time-independent and in [123] for the non-autonomous case, is different in as much as the focus of the underlying rationale is shifted away from the spatial derivative operator towards the material law. The proof of Theorem 15.2.3 outlined above is the autonomous version of [123].

Exercises

Exercise 15.1 Consider the situation of Example 15.1.1.

(a) Show that $0 \in \rho(\overline{\partial_{t,\nu} + \partial})$ for all $\nu > 0$. Next, let u be as in Example 15.1.1 and show that $u \notin \operatorname{dom}(\partial_{t,\nu})$.
(b) Let $\nu > 0$ and show using Picard's theorem that

$$0 \in \rho\left(\overline{\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}}\right).$$

Show that there exist $f, g \in L_{2,\nu}(\mathbb{R}; L_2(\mathbb{R}))$ such that for

$$\begin{pmatrix} u_f \\ v_f \end{pmatrix} := \left(\overline{\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}}\right)^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} u_g \\ v_g \end{pmatrix} := \left(\overline{\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}}\right)^{-1} \begin{pmatrix} 0 \\ g \end{pmatrix}$$

we have $u_f, u_g \notin \operatorname{dom}(\partial_{t,\nu})$.

Exercise 15.2 Let u and q be defined as in Example 15.1.2. Show that $u \in \operatorname{dom}(\partial_{t,\nu})$ and $q \in \operatorname{dom}(\mathrm{m})$ by explicit computation (not using Theorem 15.2.3).
Hint: Find an ordinary differential equation satisfied by u. Use the explicit solution of this ordinary differential equation to show the claim.

Exercise 15.3 Let $\alpha \geqslant 0$ and $\nu > 0$. Show that

$$\begin{aligned} \partial_{t,\nu} \colon \operatorname{dom}(\partial_{t,\nu}^{\lceil\alpha\rceil+1}) \subseteq H_\nu^\alpha(\mathbb{R}) &\to H_\nu^\alpha(\mathbb{R}) \\ u &\mapsto \partial_{t,\nu} u \end{aligned}$$

is densely defined closable with continuous invertible closure.

Exercise 15.4 (Local Maximal Regularity) Let H_0, H_1 be Hilbert spaces, $a \in L(H_1)$ be such that $\operatorname{Re} a \geqslant c$ for some $c > 0$. Furthermore, let $C \colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be densely defined and closed. Let $T > 0$. Show that for every $f \in L_2\big((0,T); H_0\big)$ there exists a unique $u \in H^1\big((0,T); H_0\big) \cap L_2\big((0,T); \operatorname{dom}(C^*aC)\big)$ with $u(0) = 0$ such that

$$u'(t) + C^*aCu(t) = f(t) \quad (\text{a.e. } t \in (0,T)).$$

Hint: Reformulate the equation satisfied by u into an evolutionary equation, apply Theorem 15.2.3.

Exercise 15.5 Let H_0, H_1 be Hilbert spaces, $a \in L(H_1)$ be such that $\operatorname{Re} a \geqslant c$ for some $c > 0$. Furthermore, let $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be densely defined and closed. Let $T > 0$. Define $\partial_0\colon \operatorname{dom}(\partial_0) \subseteq L_2\big((0,T)\,;H_0\big) \to L_2\big((0,T)\,;H_0\big)$ with $\partial_0 u = u'$ and

$$\operatorname{dom}(\partial_0) = \Big\{u \in H^1\big((0,T)\,;H_0\big);\ u(0) = 0\Big\}.$$

Show that for $u \in H^1\big((0,T)\,;H_0\big)$ the point-evaluation $u(0) = 0$ is well-defined. Then show that $\partial_0 + C^* a C$ is continuously invertible and closed as an operator in $L_2\big((0,T)\,;H_0\big)$.
Hint: For the first part use Theorem 12.1.3. For the second part, apply the result of Exercise 15.4. Show that in the situation of the previous exercise, there exists $\kappa > 0$ independently of f and u with

$$\|u\|_{H^1\left((0,T);H_0\right)\cap L_2\left((0,T);\operatorname{dom}(C^*aC)\right)} \leqslant \kappa \|f\|_{L_2\left((0,T);H_0\right)}.$$

Exercise 15.6 Recall Maxwell's equations from Theorem 6.2.8:

$$\partial_{t,\nu}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix} + \begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -\operatorname{curl}\\ \operatorname{curl}_0 & 0\end{pmatrix}$$

in $L_{2,\nu}\left(\mathbb{R}; L_2(\Omega)^3 \times L_2(\Omega)^3\right)$ with $\varepsilon, \mu, \sigma\colon \Omega \to \mathbb{R}^{3\times 3}$ satisfying the following property: there exist $c > 0$ and $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ we have

$$\nu\varepsilon(x) + \operatorname{Re}\sigma(x) \geqslant c, \quad \mu(x) \geqslant c \quad (x \in \Omega).$$

By Theorem 6.2.8, for $\nu \geqslant \nu_0$ and $j_0 \in L_{2,\nu}(\mathbb{R}; L_2(\Omega)^3)$, there exists a unique pair $(E, H) \in L_{2,\nu}(\mathbb{R}; L_2(\Omega)^6)$ such that

$$\begin{pmatrix}E\\ H\end{pmatrix} := \left(\overline{\partial_{t,\nu}\begin{pmatrix}\varepsilon & 0\\ 0 & \mu\end{pmatrix} + \begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -\operatorname{curl}\\ \operatorname{curl}_0 & 0\end{pmatrix}}\right)^{-1}\begin{pmatrix}j_0\\ 0\end{pmatrix}.$$

Assume there exist open sets $\Omega_0, \Omega_1 \subseteq \Omega$ such that $\overline{\Omega_0} \subseteq \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega$ with $\operatorname{spt} j_0(t) \subseteq \Omega_0$ for a.e. $t \in \mathbb{R}$. Moreover, $j_0 \in H_\nu^{1/2}\left(\mathbb{R}; L_2(\Omega_1)^3\right)$. Furthermore, assume $\varepsilon = 0$ on $\overline{\Omega_1}$. Show that $t \mapsto H(t)|_{\Omega_0} \in H_\nu^1\left(\mathbb{R}; L_2(\Omega_0)^3\right)$.

Exercise 15.7 Let H_0, H_1 be Hilbert spaces, $a, b \in L(H_1)$ be such that $\operatorname{Re} b \geqslant c$ for some $c > 0$. Furthermore, let $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be densely defined and closed. Let $f \in L_2(\mathbb{R}; H_0)$ with $\inf \operatorname{spt} f > -\infty$. Show that for $\nu > 0$ large enough, there exists for a unique $u \in H_\nu^2(\mathbb{R}; H_0) \cap \operatorname{dom}\left(C^*(a + b\partial_{t,\nu})C\right)$ satisfying

$$\partial_{t,\nu}^2 u + C^*(a + b\partial_{t,\nu})Cu = f.$$

Hint: Use the substitution $w := \partial_{t,\nu}u$ and $q := -(a + b\partial_{t,\nu})Cu$ to reformulate the equation in question as an evolutionary equation. Then apply Theorem 15.2.3.

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