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CRITICAL MANEUVERS FOR AVOIDING COLLISION AT SEA

by

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CRITICAL MANEUVERS FOR AVOIDING COLLISION AT SEA

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INTRODUCTION

In a paper intended for presentation to the *Deutsche Gesellschaft für Ortung und Navigation* there seems to be no need to emphasize the importance of sustained efforts to improve the capability of avoiding collisions at sea. However, it may be of some interest for this audience to know the original motivation behind our present work. The *Sonderforschungsbereich Schiffbau* at the *Universität Hamburg* and the *Technische Universität Hannover* has been supporting for some time a project entitled "Safety of Ships against Collisions". The specific problem is to determine how exactly the frequency of collisions (and hence the economy and safety of ship operation) depends on the maneuvering capabilities of ships(among other things). In order to solve this problem it has been found necessary to develop a mathematical model of collision avoidance. A natural by-product of this effort was the determination of optimal evasive maneuvers and of limiting conditions for ship encounters beyond which collision cannot be avoided. This paper seeks to present in a descriptive way a few typical results, which should be of as much interest to navigators as to naval architects. A full account including the complete mathematical analysis is given in an institute report by Miloh (1974).

In order to arrive at a clearer formulation of the problem let us consider how an encounter with another ship in the open sea is handled by a ship master. The complex chain of decisionmaking subject to the International Rules of the Nautical Road has been ably summarized in logicflow diagrams by Luse (1972) and Kwik (1973) from whom our Fig. 1 is adapted. Whenever another ship enters the range of observation of own ship we speak of an encounter, which may be said to last until the other ship again passes out of our range of observation. In general, both ships will maintain course and speed during an encounter so that the range will steadily decrease upto the closest point of approach (CPA) and then increase. The range at CPA is called the miss distance. Under the assumption of constant speeds and courses for both ships the expected miss distance can be deduced from continuous or successive observations of range and relative bearing at the beginning of the encounter. If the predicted miss distance is less than a certain distance regarded as safe, the encounter is said to constitute a risk (or threat) of collision. Ships are allowed and required to maneuver during an encounter if and only if such a risk exists. Depending upon whether it is a meeting, crossing or overtaking situation, either both ships are burdened (i.e. required to evade) or one is burdened and the other is privileged (i.e. required to hold course and speed upto the so-called last minute, when it is also allowed and required to evade). A collision is always preceded by a last-minute situation (also called ships in extremis) and, in general, can occur only when both ships have failed to take appropriate action.

Against this somewhat simplified background of the Rules of the Nautical Road let us ask ourselves what kind of information on the maneuvering capability of ships would be really useful to the ship master in effectively handling an encounter constituting a risk of collision. (It should be obvious that the usual kind of maneuvering data obtained for instance from turning circle and zig-zag tests are not directly applicable.) Consider, as depicted in Fig. 2, the initial or steady-state phase of a two-ship encounter during which both ships hold speed and course. This is the phase of situation assessment as expertly described by Luse (1972). Let us draw an imaginary circle of radius L_m about the center of own ship 0, such that a final miss distance less than L_m would be practically tantamount to a physical collision between the two ships. The inside of this circle we call the forbidden zone or *terminal* zone for reasons to be explained later. Consider now the set of all possible encounters with fixed starting velocities but arbitrary relative positions of the two ships. In relative coordinates attached to own ship 0 the other ship A will seem to move with relative velocity V. Let us draw tangents to the terminal circle in a direction parallel but opposite to velocity V. The inside of the semi-infinite band formed by these two tangents (from D and E) we call the *risk* zone, and the outside we call the *no-risk* zone. For it is clear from Fig. 2 that a risk of collision in the above sense exists if and only if ship A is initially observed inside this band. Of course, this risk can, in general, be obviated by a timely evasive maneuver of either ship. It can be shown that there exists a curve EH_L such that if ship A is still beyond this curve, a final miss distance larger than L_m can be attained by a hard left turn of ship 0. Similarly, there is a limiting curve DH_R for a hard right turn of ship 0. Hence, if left and right turns of 0 were the only permissible maneuvers, we could call the area enclosed by the arcs DE, EF and FD the *collision* zone. If ship 0 is burdened, it must evade before A reaches the boundary EFD. For if 0 waits until A (initially inside the risk zone) has already penetrated the collision zone, the risk transforms into certainty and collision becomes imminent, even though some further time may elapse before A finally hits the terminal circle irrespective of how 0 now tries to evade. We shall therefore call a boundary such as EFD the *barrier* and the evasive maneuvers that would prevent collision (marginally) from starting positions on the barrier the *critical* maneuvers.

Evidently, it would be of much help to the ship master, if the relevant part of the barrier and the associated critical maneuver with respect to any target could be calculated by a computer and displayed on a screen. The main content of this paper are sample calculations of just such barriers and critical maneuvers for various conditions obtained by the application of a recently developed new technique called the Theory of Differential Games.

FORMULATION AS A DIFFERENTIAL GAME

The name *Differential Game* was coined by Rufus Isaacs about 1951 while working for the Rand Corporation in the U.S.A. on military problems of guided pursuit. He also founded and developed most of the modern theory of differential games and published the first book on the subject, see Isaacs (1965). Very briefly, it can be described as an extension of the now well-known theory of (discrete) mathematical games, founded by von Neumann and Morgenstern (1944), to games developing continuously in time. It is an ingenious synthesis of game theory with the classical calculus of variations. Although the original impetus was to solve games of pursuit such as the pursuit of a ship by a torpedo, of a submarine by a destroyer, of a bomber by a rocket etc., the analytical techniques of the theory can be profitably applied also to problems of collision avoidance, as pointed out explicitly by Isaacs (1965) himself in the preface to his book.

The basic element of a differential game is a dynamic system controlled by two or more players. State variables characterize the current state of the system. The dynamic behavior is expressed by kinematic equations relating the time rate of change of state variables to the state variables themselves and to control variables, which are at the volition of the players. State variables whose time rate of change is zero (i.e. which do not change during a particular match according to the rules of the game, but which we might wish to vary from match to match) may be conveniently called parameters. The end of the game is marked by predefined terminal conditions of the system. The game is characterized by a payoff as a function of the terminal conditions and/or the path along which these are attained. The objective of the individual player is to maximize (or minimize) the payoff. A function relating a control variable to the state variables is called a strategy. All players are assumed to have complete information on the current state of the system and the range of controls available to all other players. If all players play their mutually optimal strategies, there exists for every starting state of the system a conceptually predetermined payoff called the Value. The theoretical solution of the game comprises the optimal strategies, the optimal paths leading to the terminal conditions and the Value as a function of starting states of the system. We shall try to illustrate all these concepts by formulating our problem of collision avoidance between ships as a differential game.

Consider the dynamic system consisting of two ships O and A maneuvering in the open sea. For the sake of simplicity, let us assume that the ships have fixed forward speeds V_o, V_a and fixed minimum turning radii R_o, R_a so that these four quantities can be considered as parameters of the system. Let us further assume that the only available controls are normalized rates of turn $-1 \leqslant \phi \leqslant 1$ for ship O and $-1 \leqslant \psi \leqslant 1$ for ship A, such that the current radius of curvature of the trajectory of O is R_o/ϕ and that of A is R_a/ψ. Then the state of the system is completely characterized by three state variables: the range r, the bearing α, and the course angle θ of ship A relative to ship O (see also List of Symbols at the end of this paper). Instead of relative polar coordinates r, α attached to ship 0 we shall also use relative Cartesian coordinates x, y whenever convenient. It is not difficult to derive the following kinematic equations (KE):

$$\frac{\partial x}{\partial t} = V_o y \phi / R_o - V_o + V_a \cos\theta \tag{1}$$

$$\frac{\partial y}{\partial t} = - V_o x \phi / R_o + V_a \sin\theta \tag{2}$$

$$\frac{\partial \theta}{\partial t} = V_a \psi / R_a - V_o \phi / R_o \tag{3}$$

Here positive values of control variables ϕ or ψ imply right turns and negative values imply left turns.

Let us now define terminal conditions and payoff. Here we have two possibilities: (I) We can define the closest approach at first pass, characterized by

$$\frac{\partial r}{\partial t} = 0, \tag{4}$$

as the terminal condition and use the resulting terminal (also called final) range r_f as a payoff. This yields what Isaacs (1965) calls a game of *degree*. Incidentally, this is essentially the problem solved (under less general conditions than ours) recently by Merz (1973), even though he uses the terminology of control theory rather than differential games. The solution would comprise a set of optimal evasive maneuvers and maximum attainable miss-distances from all starting states. (II) We can define collision, characterized by

$$r = L_m , \tag{5}$$

as the terminal condition and use a symbolic payoff having only two discrete values, say +1 for collision avoidance and -1 for collision occurrence. This yields what Isaacs (1965) calls a game of *kind*. Here the solution would be a barrier separating the set of all possible starting states into two subsets, namely the zone of collision avoidance (Value = +1) and the zone of collision occurrence (Value = -1). The optimal strategy is defined only on the barrier and consists of the critical maneuvers which lead to a marginal collision avoidance from starting points on the barrier (by optimal paths along the barrier itself).

We prefer to formulate our problem as a game of kind rather than as a game of degree for the following reasons: First, the primary objective of a ship is not to maximize miss distance, but to carry cargo from port X to port Y. Collision avoidance is a side condition to be satisfied by ensuring a safe miss-distance in every encounter. Second, a burdened ship would, in general, not evade at first sight but await a closer approach to acquire more accurate data and then evade just in time to avoid a dangerous final approach (say one or two miles). Third, a privileged ship is in any case not free to evade before the so-called last-minute. On the other hand, it may sometimes have to evade *before* the objective last-minute, if that is the only way to achieve the absolutely minimum miss-distance corresponding to a physical collision. - Thus in most practical situations ships don't evade to maximize miss-distance but to ensure a certain preselected safe miss distance and the choice is different for burdened and privileged ships. Hence our preference for formulation as a game of kind. Fortunately, the choice of formulation is not crucial since the solutions are essentially the same in both cases. In fact, by varying the preselected miss-distance L_m (which now enters the problem as a fifth parameter) we can embed our game of kind in a game of degree.

To complete the formulation of the differential game we must now specify the objectives of our two players. This offers yet another source of variety. We obtain three essentially different versions of the game:

(a) Ship O maximizes payoff (i.e. evades optimally), while ship A is neutral (i.e. holds course and speed).
(b) Ships O and A both maximize payoff (i.e. evade optimally in collaboration).
(c) Ship O maximizes payoff (i.e. evades optimally), while ship A minimizes payoff (i.e. pursues optimally).

Note that only the last version is a genuine game involving a conflict of objectives between the players. The first two are degenerate or one-sided games, which may also be treated as ordinary problems of optimal control. Each seems to have a definite domain of relevance. Version (a) is relevant at long range when O is burdened and A is privileged. Version (b) is relevant at long range for the meeting situation and at short range (ships in extremis) for any situation. Version (c) is actually a game of pursuit, but may be relevant for collision avoidance by O when A is behaving unpredictably for some reason.Note that in order to reverse the roles of O and A in versions (a) or (c) we only need interchange their parameters in the solution. Moreover, if we substitute "minimize" for "maximize" in versions (a) or (b) we get games of rendezvous instead of collision avoidance. The case of both ships neutral is trivial.This exhausts all possible combinations.

OUTLINE OF THE SOLUTION

In conformity with our present purpose we shall skip many intricate mathematical details of the solution and indicate only its salient features. The governing equation of our differential game is

$$\mathrm{M}\left(W_x \frac{\partial x}{\partial t} + W_y \frac{\partial y}{\partial t} + W_\theta \frac{\partial \theta}{\partial t} \right) = 0 \tag{6}$$

where the symbol M stands for the operators $\max\limits_{\phi}$, $\max\limits_{\phi}\max\limits_{\psi}$, and $\max\limits_{\phi}\min\limits_{\psi}$ in versions (a), (b) and (c) of the game respectively. W stands for the unknown Value function, with the subscripts denoting partial derivatives with respect to the state variables. In the game of kind W_x, W_y, W_θ may be interpreted as the components of the outward normal vector on the surface of the barrier in the state space. Eq. (6) expresses the principle that in optimal play in a game of terminal payoff the optimal paths must lie along surfaces of constant Value. It is called the Main Equation (ME) of differential games by Isaacs (1965, p.67). Its analogue in control theory is called the Hamilton-Jacobi-Bellman Equation, see Bryson and Ho (1969, p. 135). It is a first-order partial extremal differential equation. Its solution comprises the solution of the game.

For solving the ME we make use of an idea which is fundamental to game theory, namely the *retrogression* principle, i.e. we reverse time, start with the terminal conditions and work backward into state space. We have already defined the terminal conditions by Eq. (5). It implies that a match ends whenever the state of the system represented by a moving point in the three-dimensional r,α,θ-space reaches the cylindrical surface:

$$r = L_m , \qquad 0 \leqslant \alpha \leqslant 2\pi , \qquad 0 \leqslant \theta \leqslant 2\pi \tag{7}$$

However, not every point on the terminal surface can be reached from the outside. Theuseable part (UP) is defined by the condition of positive penetration velocity, i.e. negative range rate:

$$\frac{\partial r}{\partial t} \leqslant 0 \tag{8}$$

The boundary of the useable part (BUP) is delineated by the equality sign in Eq. (8) which again yields the condition at CPA, Eq. (4). Substitution of the KE yields a definite relation between the terminal bearing α_f and the terminal course angle θ_f on the BUP:

$$\sin\alpha_f = \pm\ (V_o - V_a\cos\theta_f)/V , \quad \cos\alpha_f = \pm\ (V_a\sin\theta_f)/V \tag{9}$$

For any given θ_f we obtain two solutions α_f^u , α_f^l corresponding to the upper and lower sign in Eq. (9) such that if we go around the terminal circle once in the clockwise sense ($0 \leqslant \alpha < 2\pi$), the UP begins at α_f^l (lower BUP) and ends at α_f^u (upper BUP).

The optimal strategies on the BUP are found to be

$$\bar{\phi} = \operatorname{sgn}(-\sin\alpha_f) = \mp \operatorname{sgn}(V_o/V_a - \cos\theta_f) \quad \text{for games (a,b,c)}$$

$$\bar{\psi} = \begin{cases} \operatorname{sgn}(\sin(\alpha_f-\theta_f)) = \mp \operatorname{sgn}(V_a/V_o - \cos\theta_f) & \text{for game (b)} \\ \operatorname{sgn}(\sin(\theta_f-\alpha_f)) = \pm \operatorname{sgn}(V_a/V_o - \cos\theta_f) & \text{for game (c)} \end{cases} \tag{10}$$

Of course, $\psi = 0$ for game (a). These strictly formal results have a simple intuitive interpretation. For optimal evasion each ship must be turning away from the other at CPA, i.e. turning left if the terminal bearing is to starboard and turning right if the terminal bearing is to port. For optimal pursuit each ship must be turning toward the other at CPA. We note parenthetically that using a more general model of the KE Miloh (1974) found equally simple rules for speed controls: For optimal evasion each ship must be accelerating away from the other at CPA, i.e. applying forward thrust if the terminal bearing is abaft of abeam and applying backward thrust if the terminal bearing is forward of abeam. Note, however, that these simple rules refer to terminal bearings at the end of the maneuver and not to initial bearings at the beginning of the maneuver. Rules for the latter are much more complicated as we shall presently see.

It can be seen from the ME (6) upon substitution of the KE (1-3) that optimal strategies are very simple functions (in fact just = ±1, i.e. the extreme values of the control variables) and at least piecewise constant along the optimal paths. This so-called optimal *bang-bang* control is characteristic of dynamic systems involving only linear controls. It enables us to obtain general equations for the optimal paths leading to the BUP by integrating the KE (1-3) in closed form assuming constant controls and using the terminal conditions (7-9) as "initial" conditions in the retrogressive sense. The final result is:

$$\bar{x} = \pm \frac{l}{\omega}\left(\eta\sin(\theta+\mu\bar{\psi}T)-\sin(\bar{\phi}T)\right) + \frac{\sin(\bar{\phi}T)}{\bar{\phi}} + \frac{\lambda}{\bar{\psi}}\left(\sin\theta - \sin(\theta+\mu\bar{\psi}T)\right) \tag{11}$$

$$\bar{y} = \pm \frac{l}{\omega}\left(\cos(\bar{\phi}T)-\eta\cos(\theta+\mu\bar{\psi}T)\right) + \left(\frac{1-\cos(\bar{\phi}T)}{\bar{\phi}}\right) + \frac{\lambda}{\bar{\psi}}\left(\cos(\theta+\mu\bar{\psi}T)-\cos\theta\right) \tag{12}$$

$$\theta = \theta_f + (\bar{\phi}-\mu\bar{\psi})T \tag{13}$$

Here we have introduced nondimensional coordinates:

$$\bar{x} = x/R_o \ , \quad \bar{y} = y/R_o \ , \tag{14}$$

a nondimensional retrograde time (i.e. time to CPA):

$$T = V_o\tau/R_o \ , \tag{15}$$

nondimensional game parameters:

$$\eta = V_a/V_o \ , \quad \lambda = R_a/R_o \ , \quad l = L_m/R_o \ , \tag{16}$$

thus reducing the number of significant parameters from five to three; and used the following abbreviations:

$$\mu = \eta/\lambda \ , \quad \omega = \left(1 + \eta^2 - 2\eta\cos\left[\theta+(\mu\bar{\psi}-\bar{\phi})T\right]\right)^{1/2} \ , \tag{17}$$

to simplify the expressions. Again the upper and lower signs in (11-12) refer to optimal paths leading to upper and lower BUP respectively. Appropriate values of optimal controls $\bar{\phi}$, $\bar{\psi}$ are to be substituted from Eq. (10). We recall that for given l the BUP consists of two curves on the terminal surface. The union of all optimal paths leading to the BUP therefore forms two curved surfaces (the upper and lower barrier) in the three-dimensional state space. In general, the upper and lower barriers intersect and enclose a finite collision zone in the state space. The intersection is a so-called *dispersal* curve separating regions of different optimal strategy (right turns on one side, left turns on the other).

Before proceeding to numerical examples we must mention two further complications in the solution. First, it is obvious from Eq. (10) that the optimal control has a constant value (either +1 or -1) along each BUP for the faster ship, but two different constant values (+1 and -1) along each BUP for the slower ship.For the latter therefore each barrier consists of two segments. On one edge the segments join to form a simple dispersal curve, on the other edge there is a void to be filled by so-called *tributaries* or secondary paths merging into a singular optimal path called a *universal curve* which in turn leads to the BUP. For a full analysis of the tributaries we refer the reader to Miloh (1974) or Isaacs (1965, Section 9.2), whose "game of two cars" is essentially the same as our game (c). However, we note that optimal maneuvers for starting points on a tributary are two-step maneuvers: first a hard right (or left) turn and then a straight course upto CPA.

NUMERICAL EXAMPLES

We are now ready to present a few numerical examples of barriers and critical maneuvers calculated by the foregoing method. We recall that the barrier is a curved surface in three-dimensional $\bar{x},\bar{y},\theta$-space. How can we conveniently display it on two-dimensional paper, or on a radar screen for that matter? The answer should be obvious from the earlier discussion of Fig. 2 in the Introduction: We display cross-sections of the barrier with individual planes θ = constant, thereby putting together the relevant information for the set of all encounters with a fixed initial course angle θ and arbitrary relative positions. Note, however, that although the barrier consists entirely of optimal paths, its cross-sections with θ = const are, in general, not optimal paths but only the starting points of optimal paths, which leave the plane θ = const on their way to a point θ_f on the BUP, see Eq. (13). The form of our path equations (11-12) has been so chosen as to be able to directly calculate and plot such cross-sections.

Consider first Fig. 3 for typical one-ship maneuvers when the evader is faster. The significance of the curves delineating the terminal zone, no-risk zone, risk zone and collision zone has already been explained in the Introduction. It remains to explain the symbols for the critical maneuvers: L denotes a hard left turn upto CPA, then a straight course (until a safe separation has been attained, at which time the original course may be resumed), R denotes a hard right turn upto CPA, then a straight course.

There seems to be nothing surprising in Fig. 3. The picture changes when we move on to Fig. 4, which shows typical one-ship maneuvers when the evader is slower. As already announced in the previous Section here the barrier sometimes consists of several different segments including tributaries requiring two-step critical maneuvers. Consider the case of initial $\theta = 90^{o}$ for instance. Here the critical maneuvers are as follows:

Segment CE: Hard left turn upto CPA
,, EF: Hard right turn upto CPA
,, FG: Hard right turn until $\theta = \arccos(1/\eta)$, then straight to CPA
,, GH: Exactly as above
,, HD: Hard right turn upto CPA

We note that the optimal strategy of the slower vehicle is beset with all kinds of discontinuities. This is characteristic of differential games. We also note by comparison with Fig. 3 that the collision zone is relatively larger when the evader is slower.

Next, Fig. 5 shows typical results for two ships evading in collaboration. Subscripts $_o$ and $_a$ have been appended to the symbols for critical maneuvers to identify the different strategies of the ships. For the sake of variety, here ship A has been assumed to have a minimum turning radius only half as large as that of O, i.e. is more maneuverable. The collision zone is quite small.

Finally, Fig. 6 displays typical results of two ships maneuvering in conflict, with the evader O assumed faster than the pursuer to ensure a zone of collision avoidance. As expected the collision zone is larger than in Fig. 3 or Fig. 5 and the slower pursuer may be required to execute two-step maneuvers as indicated by the symbol L_a for initial $\theta = 90^{o}$.

CRITIQUE AND CONCLUSIONS

We have demonstrated how the new and powerful analytical theory of differential games can be applied to determine mathematically optimal evasive maneuvers for two-ship encounters in the open sea, using a simple but reasonable kinematical model of maneuvering. Various generalizations of this model are conceivable and much work remains to be done before the problem of maritime collision avoidance maneuvers can be regarded as completely solved.

Perhaps the most obvious criticism of our kinematical model is that it ignores two typical dynamical effects of a real maneuver: the time lag from a rudder command upto the onset of an actual turn, and the loss of speed due to increased resistance in a turn. Of course, one could use an elaborate nonlinear dynamical model for maneuvering including all significant inertial and damping effects and still apply the theory of differential games with considerably increased effort. However, there is also a simple approximate way of accounting for these effects. We have seen that almost all critical maneuvers are hard turns. For any given ship O the time lag Δt_o and average speed loss ΔV_o will be known constants. They can be simulated by pretending that the target A is at a distance $V\Delta t_o$ in advance of its true position, as indicated by the point A_1 in Fig. 2, and by using barriers calculated for a reduced speed $(V_o - \Delta V_o)$ instead of the approach speed V_o.

For lack of space we must resist the temptation of comparing our results with the classical work in the field of maritime collision avoidance, such as that of Calvert (1960), Hollingdale (1961), Morrel (1961), Jones (1971) and Cockroft (1972). Suffice it to say that apparently none of these authors used the retrogression principle which in our opinion is the key to the determination of truly optimal evasive maneuvers. A notable exception is the recent paper of Merz (1973) who, as already cited, has essentially used the same technique as ours. Although our work was carried out independently of him, we find that the results complement each other beautifully. Merz uses the approach of optimal control theory, we use differential game theory. Merz has solved a game of degree, we have solved a game of kind. Merz has calculated only examples with equal speeds ($\eta = 1$), we have presented only examples with unequal speed. However, we have in the meantime verified that our barriers for $\eta = 1$ are identical to those given graphically but not analytically by Merz (1973).

Finally, we must mention the remarkable work of Kenan (1972) and Webster (1974) from whom we have borrowed the ideas of critical ranges, critical maneuvers, and the different versions of the game. However, their technique of solution is entirely different. It is more realistic in that they use elaborate nonlinear equations of motion, engine setting and rudder angle as control variables and an elongated rather than circular deck contour for defining a physical collision. But it is also more restrictive in that they make a priori assumptions about permissible maneuvers and determine critical conditions by trial and error, i.e. starting from assumed initial states and verifying by numerical integration of the equations of motion whether or not collision eventually occurs. Not only is this technique extremely wasteful of computer time but also unlikely to uncover more intricate evasive maneuvers such as our two-step maneuvers on tributaries. Presumably for reasons of computing economy they have considered only starting states on perfect collision courses (zero rate of bearing), while our calculations show that this is not always the most critical starting condition! Future efforts should be directed toward combining the best features of both approaches.

REFERENCES

Bryson, A.E.; Ho, Y.-C.: Applied Optimal Control. Ginn and Company, Waltham, Mass. (1969).

Calvert, E.S.: Manoeuvres to Ensure the Avoidance of Collision. Journ. Inst. Navig. 13 (1960) 127-137.

Cockroft, A.N.: A Manoeuvring Diagram for Avoiding Collision at Sea. Journ. Inst. Navig. 25 (1972) 105-107.

Hollingdale, S.H.: The Mathematics of Collision Avoidance in Two Dimensions. Journ. Inst. Navig. 14 (1961) 243-261.

Isaacs, R.: Differential Games. John Wiley and Sons, New York (1965).

Jones, K.D.: Practical Manoeuvres to Avoid Collision at Sea. Journ. Inst. Navig. 24 (1971) 60-66.

Kenan, G.: Collision Avoidance between Surface Ships at Short Range. Soc. Nav. Arch. Mar. Eng. Northern California Section (1972).

Kwik, K.H.: Kinematik der Begegnung von Schiffen und Kollisionsverhütung mit einer Betrachtung der Fahrregeln. Institut für Schiffbau, Ham-

burg, Bericht Nr. 294 (1973).
Luse, J.D.: Collision Avoidance Systems and the Rules of the Nautical Road. Navigation 19 (1972) 80-87.
Merz, A.W.: Optimal Evasive Maneuvers in Maritime Collision Avoidance. Navigation 20 (1973) 144-152.
Miloh, T.: Determination of Critical Maneuvers for Collision Avoidance Using the Theory of Differential Games. Institut für Schiffbau, Hamburg, Bericht Nr. 319 (1974).
Morrel, J.S.: The Physics of Collisions at Sea. Journ. Inst. Navig. 14 (1961) 163-184.
Neumann, J. von; Morgenstern, O.: Theory of Games and Economic Behavior. Princeton Univ. Press, Princeton, N.J. (1944).
Webster, W.E.: When is Collision Unavoidable? Trans. 10th Naval Hydrodynamics Symposium, Boston, Mass. (1974).

LIST OF SYMBOLS

L_m	Preselected minimum miss-distance
l	Nondimensional minimum miss-distance, see Eq. (16)
Oxy	Relative coordinate system attached to ship O
R_a	Minimum turning radius of ship A
R_o	Minimum turning radius of ship O
r	Range of ship A from ship O
r_f	Final (or terminal) range at CPA
T	Nondimensional retrograde time, see Eq. (15)
t	Physical time
Δt_o	Time lag from rudder command to beginning of actual turn
x	Distance of A from O measured along current velocity of O
$\overline{x}$	Nondimensional coordinate, see Eq. (14)
y	Distance of A from O normal to velocity of O (positive to starb.)
$\overline{y}$	Nondimensional coordinate, see Eq. (14)
V	Velocity of A relative to O
V_a	Speed of ship A
V_o	Speed of ship O
W	Value function
α	Bearing of A from O taken clockwise from velocity of O
α_f	Final (or terminal) relative bearing α at CPA
η	Nondimensional speed ratio, see Eq. (16)
θ	Relative course angle of A taken clockwise from velocity of O
θ_f	Final (or terminal) value of relative course angle θ at CPA
λ	Nondimensional ratio of turning radii, see Eq. (16)
μ	Abbreviation for η/λ, see Eq. (17)
τ	Retrograde time, i.e. time to CPA
ϕ	Control variable of ship O (normalized rate of turn)
$\overline{\phi}$	Optimal value of control ϕ
ψ	Control variable of ship A (normalized rate of turn)
$\overline{\psi}$	Optimal value of control ψ
ω	Abbreviation, see Eq. (17)

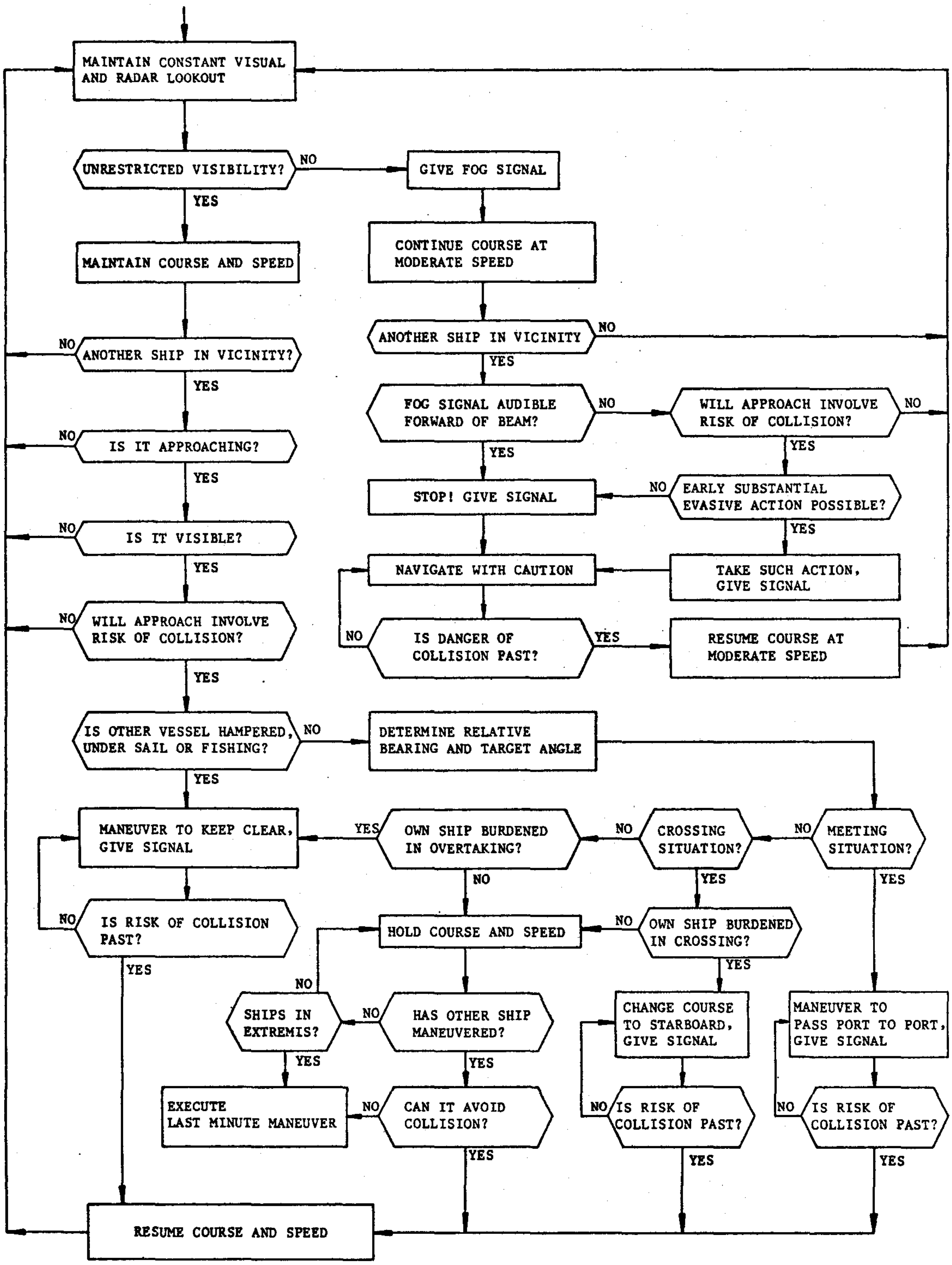

Fig. 1 - Logic Flow Diagram for Two-Ship Encounter in Open Sea following International Rules of the Nautical Road

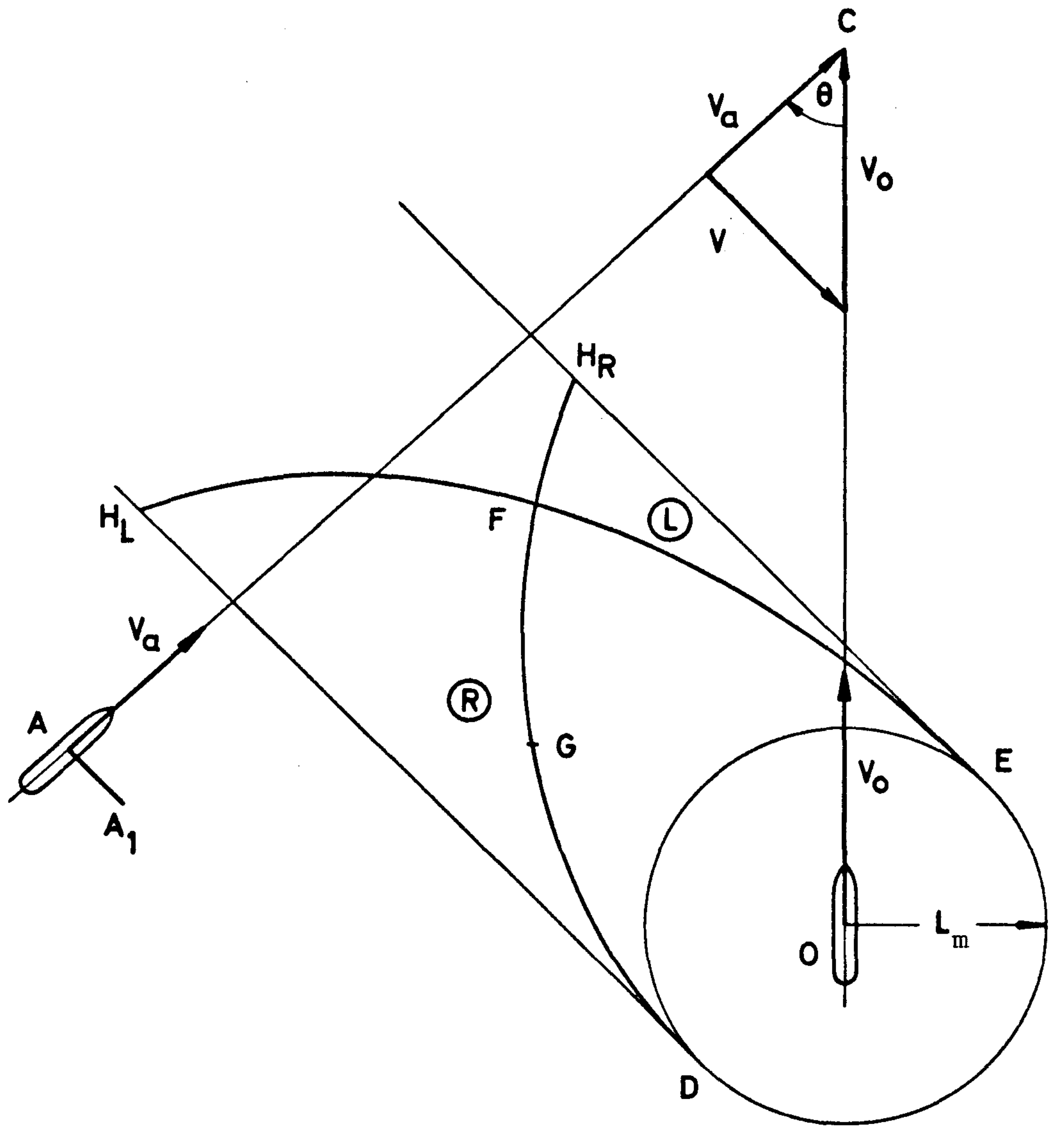

Fig. 2 - Sample Illustration of Barriers and Critical Maneuvers for Collision Avoidance in a Two-Ship Encounter

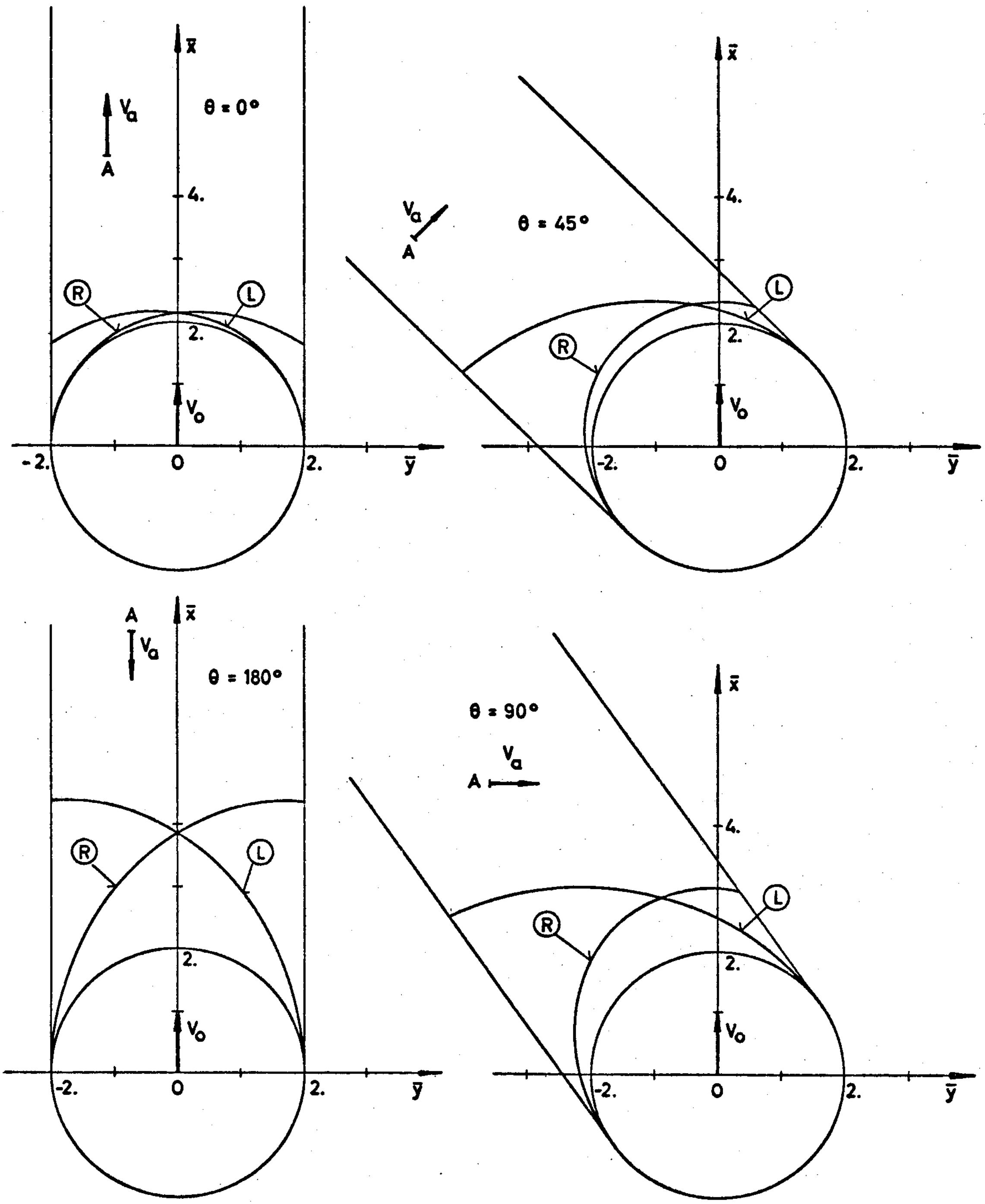

Fig. 3 - Calculated Barriers and Critical Maneuvers for $\eta = 1/\sqrt{2}$, $l = 2$, and selected values of initial course angle θ. Game version (a): Ship 0 evades, ship A holds course and speed.

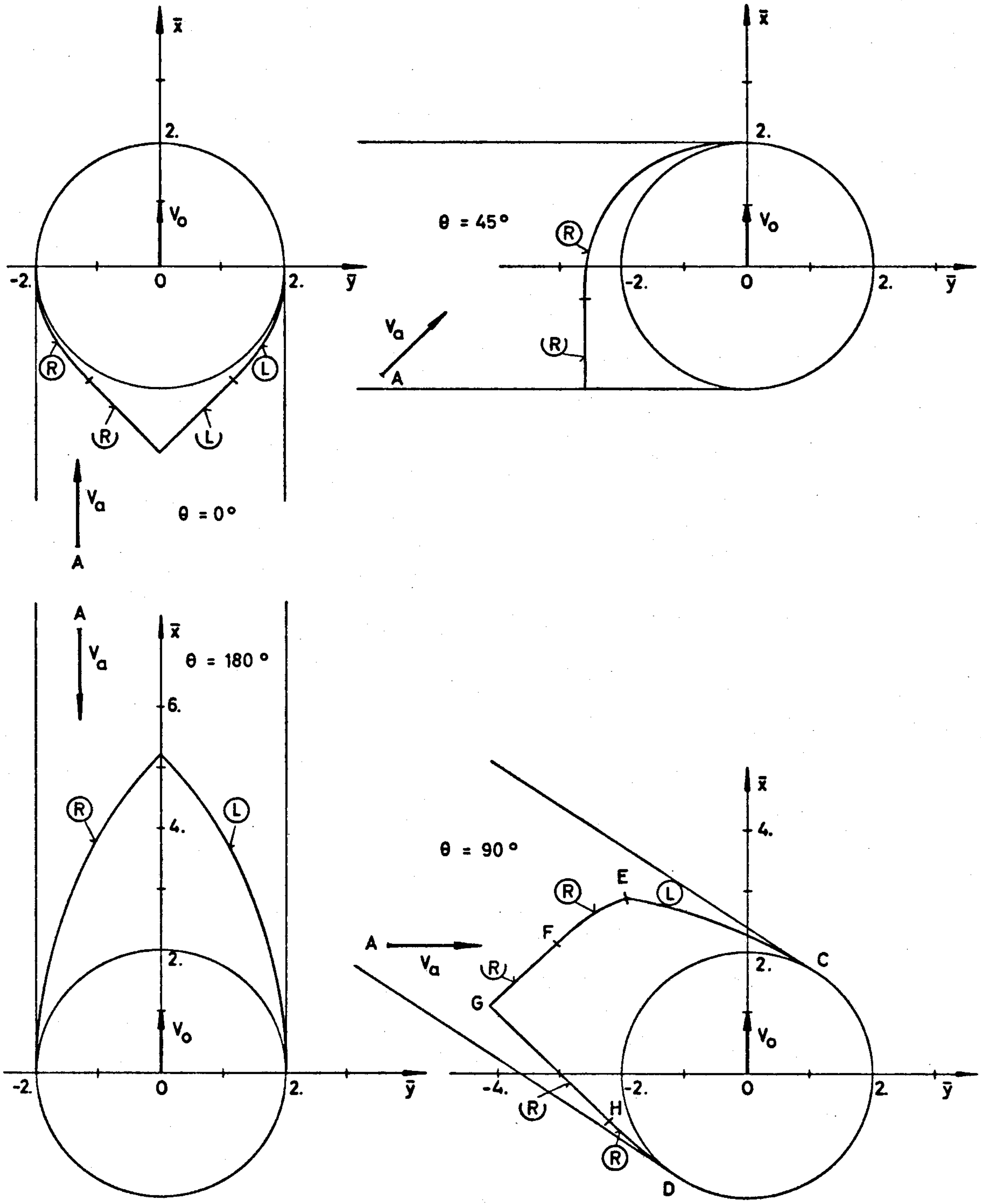

Fig. 4 - Calculated Barriers and Critical Maneuvers for $\eta = \sqrt{2}$, $l = 2$, and selected values of initial course angle θ. Game version (a): Ship O evades, ship A holds course and speed.

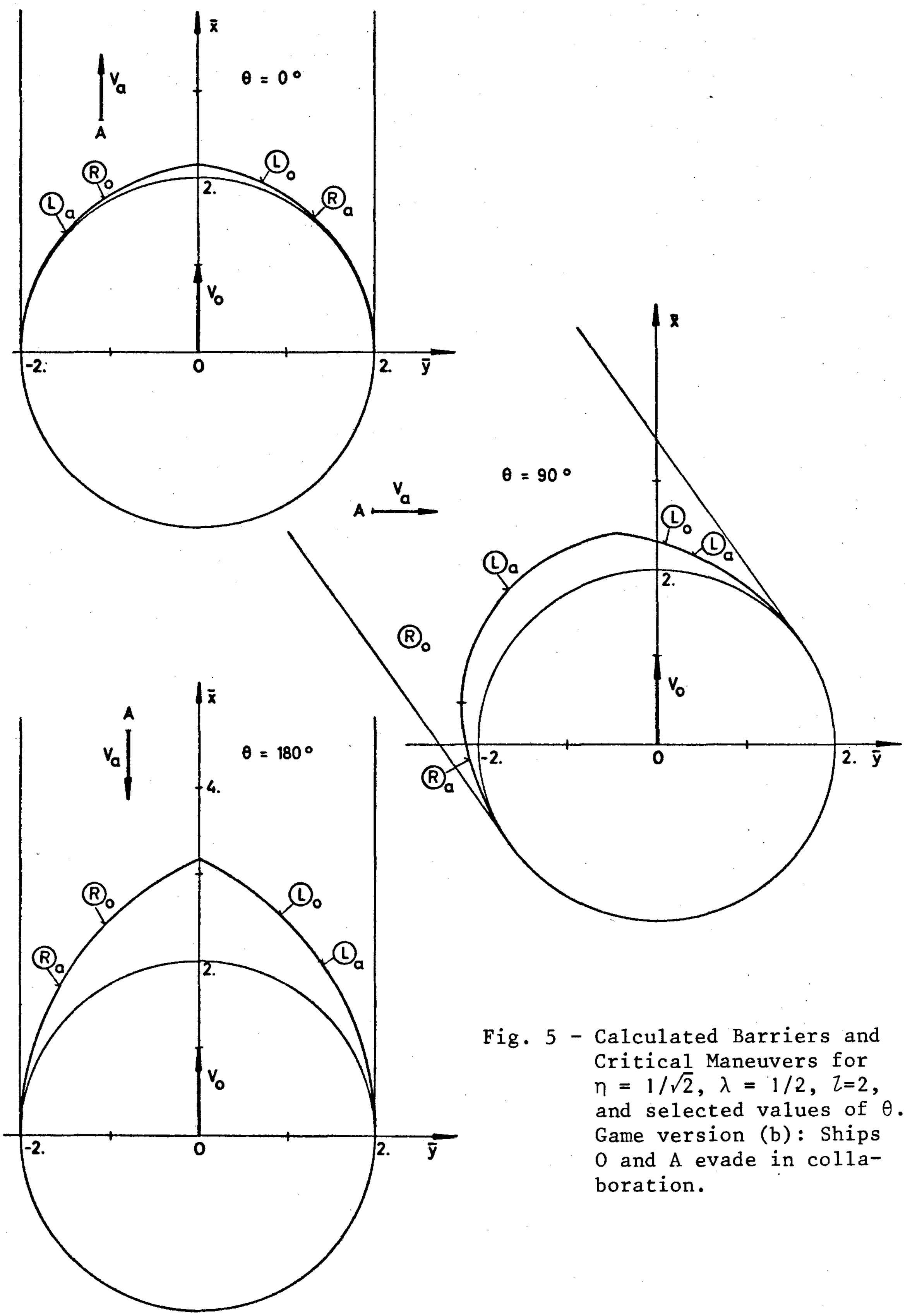

Fig. 5 - Calculated Barriers and Critical Maneuvers for $\eta = 1/\sqrt{2}$, $\lambda = 1/2$, $l=2$, and selected values of θ. Game version (b): Ships O and A evade in collaboration.

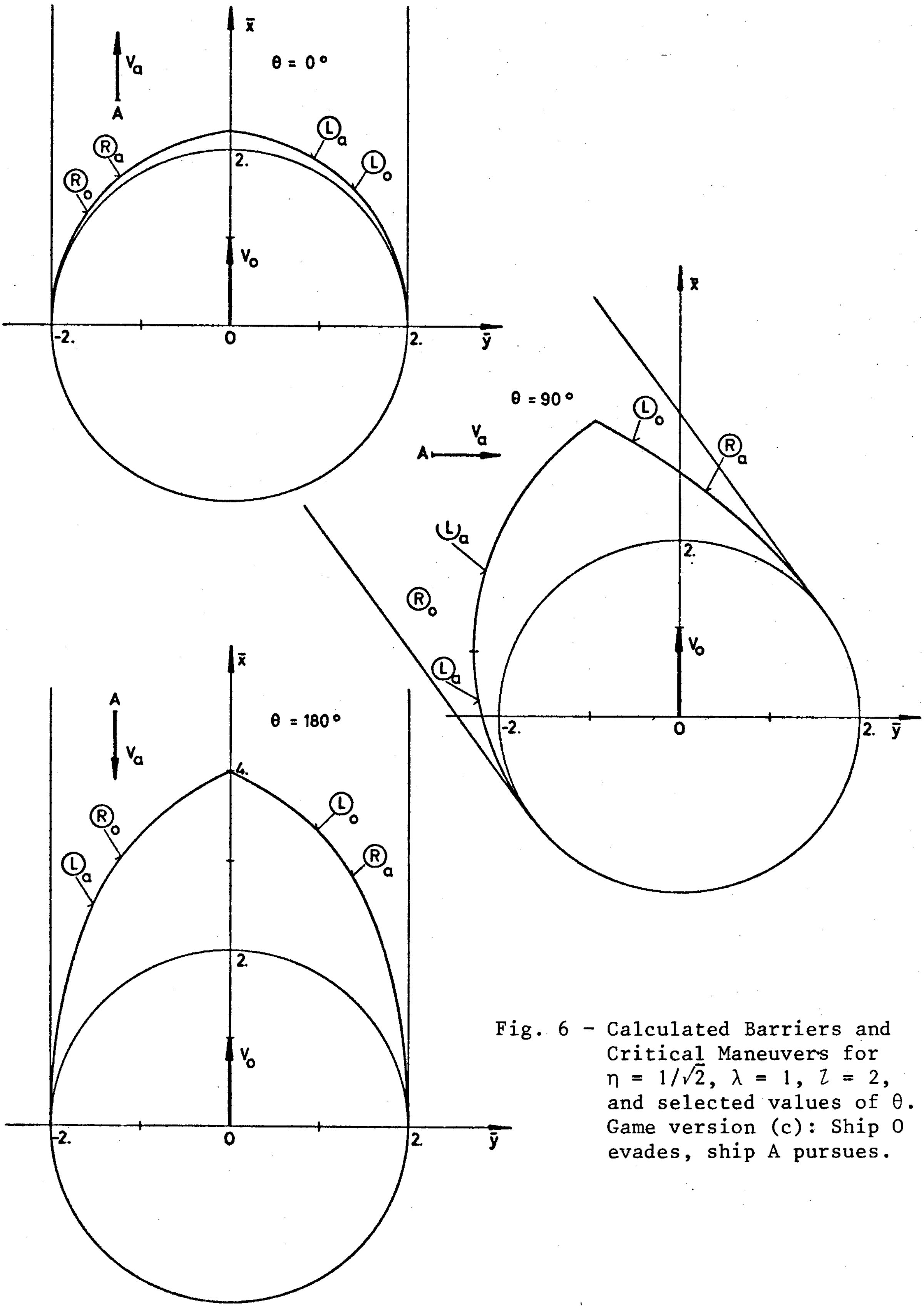

Fig. 6 – Calculated Barriers and Critical Maneuvers for $\eta = 1/\sqrt{2}$, $\lambda = 1$, $l = 2$, and selected values of θ. Game version (c): Ship O evades, ship A pursues.