Dynamic Stiffness Matrix of Sagging Cable

By Uwe Starossek[1]

Abstract: The dynamic behavior of an extensible sagging cable is investigated. A dynamic stiffness matrix is presented whose coefficients are functions of the frequency of motion, and that is suitable for dynamic direct-stiffness analysis of composed systems such as cable-stayed bridges and guyed masts. The study is restricted to small displacements (linear theory) and considers motion within the vertical cable plane only. Viscous damping due to external fluid is taken into account. Trigonometrical solution functions with complex arguments are utilized, which implies a substantial simplification in the analysis of damped vibrations. By means of example calculations, stiffness functions are discussed and compared to the results of other authors. For tightly stretched inclined cables, utilization of the more accurate theory presented in this paper is indispensable.

Introduction

The static analysis of a mechanical system usually requires knowledge of the load-deformation behavior of the system elements. This behavior can be described in compact form by stiffness matrices. Limited to the steady-state response, it is possible to transfer this concept to the investigation of dynamic processes, which implies the development of dynamic impedance or stiffness matrices (Clough and Penzien 1975).

In this paper the dynamic stiffness matrix of an extensible, flexible, sagging cable is presented. This matrix can be utilized for the dynamic analysis of composed systems such as cable-stayed bridges or guyed masts. The cable is considered as a continuum. Only small displacements are admitted (linear theory) and only motions and forces within the vertical cable plane are regarded. Viscous damping—for example due to external fluid forces—is taken into consideration.

As far as is known, previous solutions to the defined problem have been confined essentially to one element of the here computed 4 × 4 matrix: the horizontal stiffness at the upper end of an inclined cable that is fixed at the lower end. Treatises by Davenport and Steels (1965) and Irvine (1981) give formulas that according to the findings of this study are not sufficiently precise for tightly stretched inclined cables. The paper by Davenport and Steels (1965) also considers damping, but the numeric evaluation of infinite series is required. Veletsos and Darbre (1983) developed a more accurate closed-form expression for the damped cable, which can be transformed into the corresponding element of the matrix presented here, although a small correction must be made.

This study partly follows the course revealed by Irvine and Caughey (1974) and Irvine (1978, 1981), but the scope is different (stiffness matrix) and greater generality is obtained (damping). Initially, the derivation proceeds in a local coordinate system and is valid for a horizontal cable. The result is then generalized to apply to an inclined cable and transformed to global

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coordinates. A discussion of example calculations and a detailed comparison with the results of other authors conclude this paper.

The utilization of trigonometrical solution functions with complex arguments is novel and remarkable; with the help of these functions the analysis of damped cable vibrations can be managed with astonishing elegance. Beyond the particular problem discussed herein, this method may be applied to most kinds of damped oscillations that can be described by linear differential equations.

Basic Equations

The static form of a horizontal cable (see Fig. 1) is approximated by a quadratic parabola. For a steel cable with sufficiently small sag ($d/l \leq 1/20$), the condition of dynamic equilibrium of the vertical forces (see Fig. 2) leads to the equation of motion

$$H\frac{\partial^2 v}{\partial x^2} + h_\tau \frac{d^2 y}{dx^2} = m\frac{\partial^2 v}{\partial t^2} + c\frac{\partial v}{\partial t} \quad \text{(1)}$$

in which H = horizontal component of the static cable tension; m = cable mass per unit length; and c = damping force per unit length and velocity. The auxiliary quantity h_τ is defined as

$$h_\tau := \tau\frac{dx}{ds} \quad \text{(2)}$$

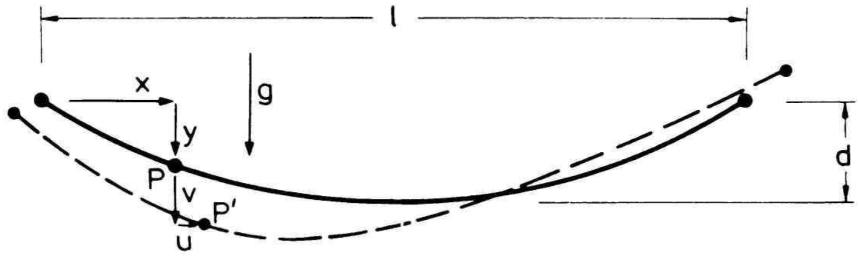

FIG. 1. Horizontal Cable

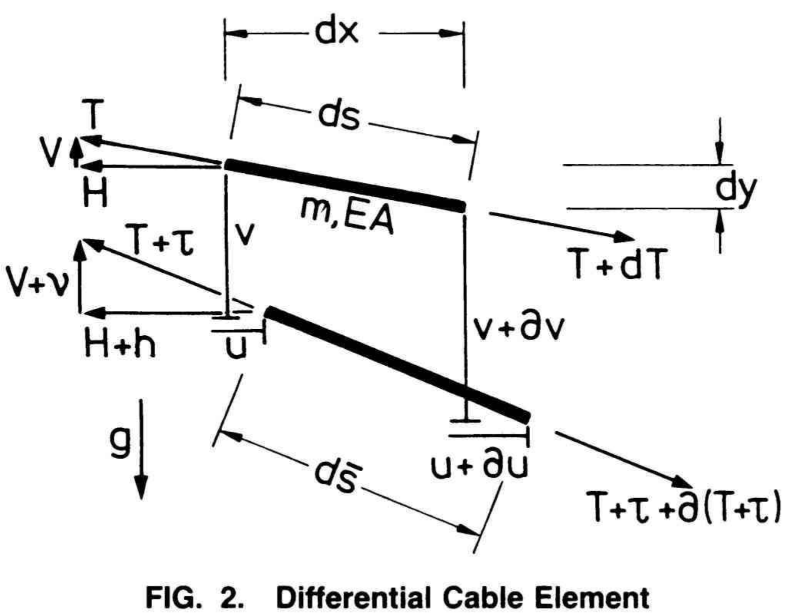

FIG. 2. Differential Cable Element

where τ = dynamic part of the total cable tension that is supposed to be invariable along the cable (quasi-static elastic deformation). The second basic equation

$$\frac{h_\tau}{EA}\left(\frac{ds}{dx}\right)^3 = \frac{dy}{dx}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \quad \text{(3)}$$

provides for the elastic and geometric compatibility of the cable element. Eqs. (1) and (3) were derived in Starossek (1990). When the damping term is omitted and the quantity h_τ is substituted by h (dynamic part of horizontal component of total cable tension), they conform to the equations presented by Irvine and Caughey (1974). As shown in Starossek (1990), these relations can be derived without limitation of the horizontal displacement u. They can therefore be taken as a basis for the analysis of a cable with displaceable boundaries presented herein.

Dynamic Behavior of Horizontal Cable

General Terms

The ends of the cable are supported at the same level. It is assumed that the vibration is described by the products

$$v(x, t) = \tilde{v}(x)e^{i\omega t} \quad \text{(4a)}$$

$$u(x, t) = \tilde{u}(x)e^{i\omega t} \quad \text{(4b)}$$

where $i^2 = -1$ and $\tilde{u}, \tilde{v}, \omega \in \mathbb{C}$Consequently, the expression

$$h_\tau(x, t) = \tilde{h}_\tau(x)e^{i\omega t}; \qquad \tilde{h}_\tau \in \mathbb{C} \quad \text{(4c)}$$

will hold and analogous product descriptions are valid for the boundary forces and displacements. That is, only harmonic vibrations and vibrations with an exponentially variable amplitude (modified-harmonic vibrations) are admitted. With this approach, the steady-state system response to harmonic excitation, as well as damped free vibrations, can be investigated. In every case, the dynamic stiffness functions are defined as time-independent relations between boundary forces and boundary displacements of the cable as a part of a vibrating system.

With the adoption of approach (4), the equation of motion (1) leads to the ordinary differential equation

$$H\frac{\partial^2\tilde{v}}{\partial x^2} + \omega_c^2 m\tilde{v} = \frac{8d}{l^2}\tilde{h}_\tau \quad \text{(5)}$$

The introduction of the auxiliary parameter

$$\omega_c := \omega\sqrt{1 - 2\xi i} \quad \text{(6a)}$$

where

$$\xi := \frac{c}{2m\omega} \quad \text{(6b)}$$

provides a substantial simplification of further derivations. They can now be formally carried out as if damping were not present. The definition of the dimensionless quantities

$$\Omega := \omega l \sqrt{\frac{m}{H}} \quad \text{(7a)}$$

$$\Omega_c := \omega_c l \sqrt{\frac{m}{H}} \quad \text{(7b)}$$

will later prove useful. From the compatibility condition (3)

$$\frac{\tilde{h}_\tau}{EA}\left(\frac{ds}{dx}\right)^3 = \frac{dy}{dx}\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial x} \quad \text{(8)}$$

is obtained.

The investigation proceeds separately for horizontal and vertical boundary displacements. A distinction between symmetric and antisymmetric contributions is made. Eqs. (5) and (8) are solved for the different sets of boundary conditions, and the corresponding stiffness functions are calculated. Trigonometrical solution functions with complex arguments are utilized. For example, in the case of symmetric horizontal boundary displacements, the utilized solution of (5) is

$$\tilde{v} = \frac{8d}{\Omega_c^2}\frac{\tilde{h}_\tau}{H}\left(1 - \tan\frac{\Omega_c}{2}\sin\frac{\Omega_c x}{l} - \cos\frac{\Omega_c x}{l}\right) \quad \text{(9)}$$

Displacement function $\tilde{u}(x)$ in (8) is eliminated by integration. Solutions for all necessary sets of boundary displacements and derivation of corresponding stiffness functions are presented in detail in Starossek (1990).

Local Dynamic Stiffness Matrix

The elements k_{ij} $(i, j = 1, \ldots, 4)$ of the local dynamic stiffness matrix $\mathbf{k}$ are defined by the transformation

$$\mathbf{f} = \mathbf{k} \cdot \boldsymbol{\delta} \quad \text{(10)}$$

in which

$$\mathbf{f} := \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \quad \text{(11a)}$$

$$\boldsymbol{\delta} := \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} \quad \text{(11b)}$$

are the vectors of the local force and displacement quantities according to Fig. 3. Matrix $\mathbf{k}$ is found by superposing stiffness contributions, which are calculated for different sets of boundary conditions. When neglecting contributions of minor influence, three of the four antisymmetric stiffness contributions vanish, and the stiffness function $k^s_{h,u}$ (see the following) is slightly simplified (Starossek 1990). The local dynamic stiffness matrix then is given by

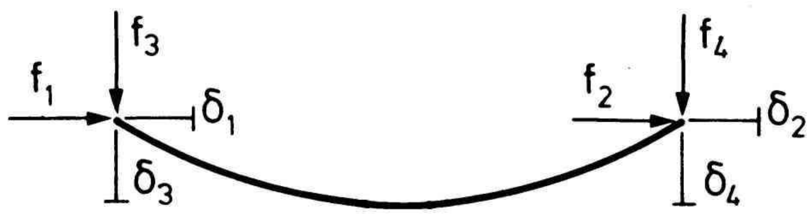

FIG. 3. Local Force and Displacement Quantities

$$\mathbf{k} = \begin{pmatrix} \mathbf{k}^s_{h,u} & \mathbf{k}^s_{h,v} \\ \mathbf{k}^s_{\nu,u} & (\mathbf{k}^s_{\nu,v} + \mathbf{k}^a_{\nu,v}) \end{pmatrix} \quad \text{(12)}$$

where

$$\mathbf{k}^s_{h,u} := k^s_{h,u} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{(13a)}$$

$$\mathbf{k}^s_{h,v} := k^s_{h,v} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{(13b)}$$

$$\mathbf{k}^s_{\nu,u} := k^s_{\nu,u} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{(13c)}$$

$$\mathbf{k}^s_{\nu,v} := k^s_{\nu,v} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{(13d)}$$

$$\mathbf{k}^a_{\nu,v} := k^a_{\nu,v} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{(13e)}$$

The stiffness functions are

$$k^s_{h,u} = \frac{EA}{L_e} \frac{1}{1 + \frac{\lambda^2}{\Omega_c^2}(\kappa - 1)} \quad \text{(14)}$$

$$k^s_{h,v} = k^s_{\nu,u} = \frac{EA}{L_e} \frac{\frac{1}{2}\varepsilon(\kappa - 1)}{1 + \frac{\lambda^2}{\Omega_c^2}(\kappa - 1)} \quad \text{(15)}$$

$$k^s_{\nu,v} = -\frac{EA}{L_e} \frac{\frac{1}{4}\frac{\varepsilon^2}{\lambda^2}\Omega_c^2\left[\kappa + \frac{\lambda^2}{\Omega_c^2}(\kappa - 1)\right]}{1 + \frac{\lambda^2}{\Omega_c^2}(\kappa - 1)} \quad \text{(16)}$$

$$k^a_{\nu,v} = \frac{EA}{L_e} \frac{\varepsilon^2}{\lambda^2} \frac{1}{\kappa} \quad \text{(17)}$$

where

$$\lambda^2 := \left(\frac{mgl}{H}\right)^2 \frac{EAl}{HL_e} \qquad (18)$$

$$\varepsilon := \frac{mgl}{H} = \frac{8d}{l} \qquad (19)$$

$$L_e := \int_0^l \left(\frac{ds}{dx}\right)^3 dx \simeq l\left[1 + 8\left(\frac{d}{l}\right)^2\right] \qquad (20)$$

are cable parameters and

$$\kappa = \kappa(\Omega_c) := \frac{\tan\left(\frac{\Omega_c}{2}\right)}{\left(\frac{\Omega_c}{2}\right)} \qquad (21)$$

is an auxiliary function dependent solely on Ω_c. Because $k^s_{h,v} = k^s_{v,u}$ and hence $\mathbf{k}^s_{h,v} = (\mathbf{k}^s_{v,u})^T$, the stiffness matrix is symmetric (but non-Hermitian if damping is taken into account).

It should be noted that the stiffness terms appearing in (12) become infinite for certain values of Ω_c. In the case of real Ω_c, these values must coincide with the natural frequencies of an undamped cable suspended from rigid end supports. This condition leads to frequency equations that conform to the findings of Irvine and Caughey (1974).

Generalization to Inclined Cable

Extending the theory of the horizontal cable developed in Irvine and Caughey (1974), Irvine (1978) gave solutions for the free vibration of an inclined rigidly supported cable. It can be shown that only one additional assumption is necessary for this extension: that the weight component parallel to the cable chord can be neglected. The results of Irvine (1978) correspond well with those of the more precise theory of Triantafyllou (1984) and Triantafyllou and Grinfogel (1986), as long as the cable parameters λ^2 and ε as well as the angle of inclination Θ do not exceed certain limits. In particular, λ^2 should maintain a certain distance (about 20%) from the so-called crossover points $4n^2\pi^2$ ($n = 1, 2, \ldots$); and ε and Θ should not be too large. The theory presented here corresponds to Irvine and Caughey (1974) in its essential assumptions. Neglecting once again the weight component parallel to the cable chord, the transition to an inclined cable is made by the following substitutions: (1) g is replaced by the gravitational component $g \cos \Theta$, which acts perpendicular to the cable chord (where Θ is the chord inclination); and (2) the quantity $T_\Theta = H/\cos \Theta$ must be substituted for the horizontal component H of the static cable tension—it represents the static cable tension at the section where the cable is parallel to the chord, and corresponds approximately to the average cable tension.

The other free parameters remain unchanged but now l denotes the chord length and d is the sag perpendicular to the chord. All coordinates and displacements relate to the local cordinate system, where the x-axis is parallel to the chord. The dependent parameters become

$$\Omega = \omega l \sqrt{\frac{m}{T_\Theta}} \quad (22a)$$

$$\Omega_c = \omega_c l \sqrt{\frac{m}{T_\Theta}} \quad (22b)$$

$$\lambda^2 = \left(\frac{mgl}{T_\Theta}\right)^2 \frac{EAl}{T_\Theta L_e} \cos^2\Theta \quad (23)$$

$$\varepsilon = \frac{mgl}{T_\Theta} \cos \Theta = \frac{8d}{l} \quad (24)$$

where

$$T_\Theta = \frac{H}{\cos \Theta} \quad (25)$$

$$L_e \simeq l \left[1 + 8\left(\frac{d}{l}\right)^2\right] \quad (26)$$

With these new terms, the theory presented here can be used for an inclined cable, if the limiting conditions

$$\lambda^2 \leq 24 \quad (27a)$$

and

$$\Theta \leq 60° \text{ and } \varepsilon \leq 0.10 \left(\frac{d}{l} \leq \frac{1}{80}\right)$$

or

$$\Theta \leq 30° \text{ and } \varepsilon \leq 0.24 \left(\frac{d}{l} \leq \frac{1}{33}\right) \quad (27b)$$

are satisfied. The specified range of validity is inferred from the numeric results of Irvine (1978) and Triantafyllou and Grinfogel (1986); a conservative criterion relative to λ^2 is employed, hereby taking into account that the problems investigated are similar but not identical [movable supports in this paper, fixed supports in Irvine (1978) and Triantafyllou and Grinfogel (1986)].

Transformation to Global Coordinates

The global force and displacement vectors (according to Fig. 4)

$$\mathbf{F} := \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \quad (28a)$$

$$\mathbf{\Delta} := \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{pmatrix} \quad (28b)$$

are converted into the local quantities given by Fig. 3 through the orthogonal transformations

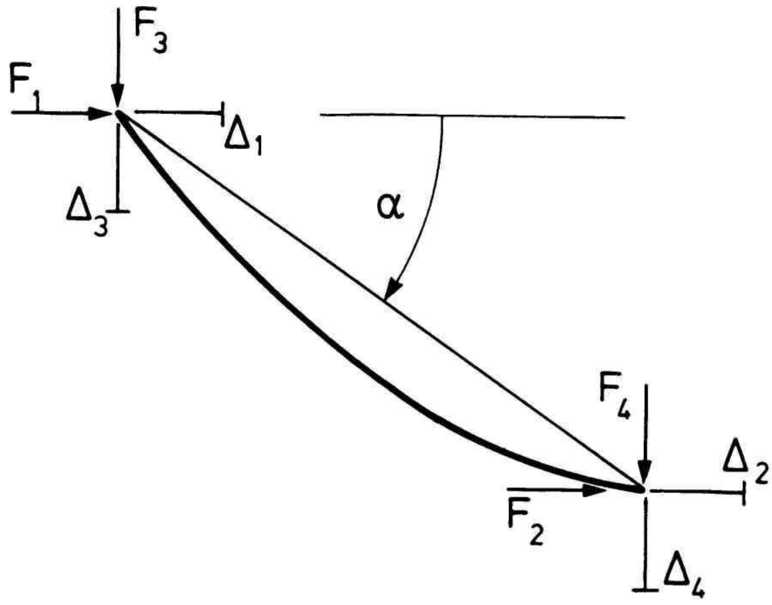

FIG. 4. Global Force and Displacement Quantities

$$\mathbf{f} = \mathbf{T} \cdot \mathbf{F} \quad (29a)$$

$$\boldsymbol{\delta} = \mathbf{T} \cdot \boldsymbol{\Delta} \quad (29b)$$

$$\mathbf{T} := \begin{pmatrix} \mathbf{E} \cos \alpha & \mathbf{E} \sin \alpha \\ -\mathbf{E} \sin \alpha & \mathbf{E} \cos \alpha \end{pmatrix} \quad (29c)$$

$$\mathbf{E} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (29d)$$

(In practical applications the angle of rotation α is often equal to the inclination angle Θ, defined in the previous section.) When substituting for the local vectors into (10) and premultiplying by $\mathbf{T}^{-1}$, this leads to

$$\mathbf{F} = \mathbf{K} \cdot \boldsymbol{\Delta} \quad (30)$$

where

$$\mathbf{K} = \mathbf{T}^{-1}\mathbf{k}\mathbf{T} = \mathbf{T}^{T}\mathbf{k}\mathbf{T} \quad (31)$$

is the global dynamic stiffness matrix. In systems such as guyed masts and cable-stayed bridges, the axial deformations of the beams are relatively small. If the axes of the global coordinate system lie parallel to the beams, the elements K_{11}, K_{44}, and also K_{14}, K_{41} of the dynamic stiffness matrix $\mathbf{K}$ will be of special interest. By following (31) and utilizing the properties of symmetry previously stated, it is found that

$$K_{11} = k^{s}_{h,u} \cos^2\alpha + 2k^{s}_{v,u} \cos \alpha \sin \alpha + (k^{s}_{v,v} + k^{a}_{v,v})\sin^2\alpha \quad (32a)$$

$$K_{14} = K_{41} = -[k^{s}_{h,u} + (k^{s}_{v,v} - k^{a}_{v,v})] \cos \alpha \sin \alpha - k^{s}_{v,u} \quad (32b)$$

$$K_{44} = k^{s}_{h,u} \sin^2\alpha + 2k^{s}_{v,u} \cos \alpha \sin \alpha + (k^{s}_{v,v} + k^{a}_{v,v})\cos^2\alpha \quad (32c)$$

By neglecting several components of frequently minor influence, these equations are reduced to the simplified expressions

$$K_{11} = k^{s}_{h,u} \cos^2\alpha + 2k^{s}_{v,u} \cos \alpha \sin \alpha \quad (33a)$$

$$K_{14} = K_{41} = -k^s_{h,u}\cos\alpha\sin\alpha - k^s_{v,u} \quad \text{(33b)}$$

$$K_{44} = k^s_{h,u}\sin^2\alpha + 2k^s_{v,u}\cos\alpha\sin\alpha \quad \text{(33c)}$$

which will often give sufficiently accurate results.

Extension of the given theory to the spatial problem is possible without major difficulty. In linearized theory the in-plane motion is uncoupled from the transverse horizontal motion [see Irvine and Caughey (1974)]. The spatial dynamic cable stiffness can therefore be described by a block diagonal matrix if an appropriate coordinate system is chosen. Its first diagonal block is already given by the 4 × 4 matrix of (31); the second one, a 2 × 2 matrix, must still be derived. The action of fluid forces must be considered if the investigated cable is immersed in weighty mediums like water. This can be accomplished by adequately establishing the parameters m, g, and ξ [see Davenport and Steels (1965)]. For a cable in air, this only concerns the damping parameter ξ.

Example Analysis and Comment

Example 1

The dynamic stiffness K_{11} is calculated for the parameters

$$\frac{mgl}{T_\Theta} = 0.217 \quad \text{(34a)}$$

$$\frac{T_\Theta}{EA} = 0.000633 \quad \text{(34b)}$$

$$\alpha = \Theta = 55.86° \quad \text{(34c)}$$

$$\xi = \frac{0.14}{\left(\dfrac{\Omega}{\pi}\right)} \quad \text{(34d)}$$

as function of real Ω. From (26), (23) and (24) follows

$$\frac{L_e}{l} = 1.002 \simeq 1 \quad \text{(35a)}$$

$$\lambda^2 = 23.48 \quad \text{(35b)}$$

$$\varepsilon = 0.1218 \quad \text{(35c)}$$

The limits of validity established by equations (27) are nearly met. The stiffness function K_{11} was computed according to (32a) and—for purposes of comparison—according to the simplified equation (33a). In the range of extreme values the K_{11} determined from various equations deviate at most 3% in their real and imaginary parts.

To obtain dimensionless graphs, the dynamic stiffness K_{11} is related to the elastic part $K^{t,e}_{11}$ of the static stiffness K^t_{11} of a straight rod

$$K^t_{11} = K^{t,e}_{11}\left(1 + \frac{T_\Theta}{EA}\tan^2\alpha\right) \quad \text{(36a)}$$

$$K_{11}^{l,e} := \frac{EA}{l}\cos^2\alpha \qquad (36b)$$

Utilizing the simplified equation (33*a*), and neglecting the factor l/L_e, the expression

$$K_{11}^{*} := \frac{K_{11}}{K_{11}^{l,e}} = \frac{1 + \varepsilon \tan\alpha\,(\kappa - 1)}{1 + \dfrac{\lambda^2}{\Omega_c^2}(\kappa - 1)} \qquad (37)$$

is obtained, which provides a reasonably accurate approximation for the cable considered here. The real and imaginary parts of this function are depicted in Fig. 5. A comparison of these curves with the curves that Davenport and Steels (1965) calculated for the same parameters shows similarity, but not total agreement. Their solution requires numerical evaluation of infinite series and so cannot be directly compared to the closed-form solution given here. It appears, however, that the contribution of $k_{v,u}^{s}$ was omitted. This term is equal to $k_{h,v}^{s}$ (which they considered) and, according to (31), has the same influence on the total stiffness. When expression (37) is truncated by this contribution (by dividing the second term in the numerator by 2), the curves plotted in Davenport and Steels (1965) can be precisely reproduced. Moreover, it then coincides with a closed-form expression given there for the special case of an undamped cable. In this example, $k_{v,u}^{s}$ contributes approximately 15% to the total stiffness and therefore should not be neglected.

For the special case of an undamped cable, Irvine (1981) gave a closed-form expression for the dynamic stiffness K_{11}. When the simplified expression (37) is adapted to apply to an undamped cable and the second term in the numerator (contributions of $k_{h,v}^{s}$ and $k_{v,u}^{s}$) is omitted this expression coincides with Irvine's equation; depending on the respective values of the omitted contributions, the equation given in Irvine (1981) is less accurate.

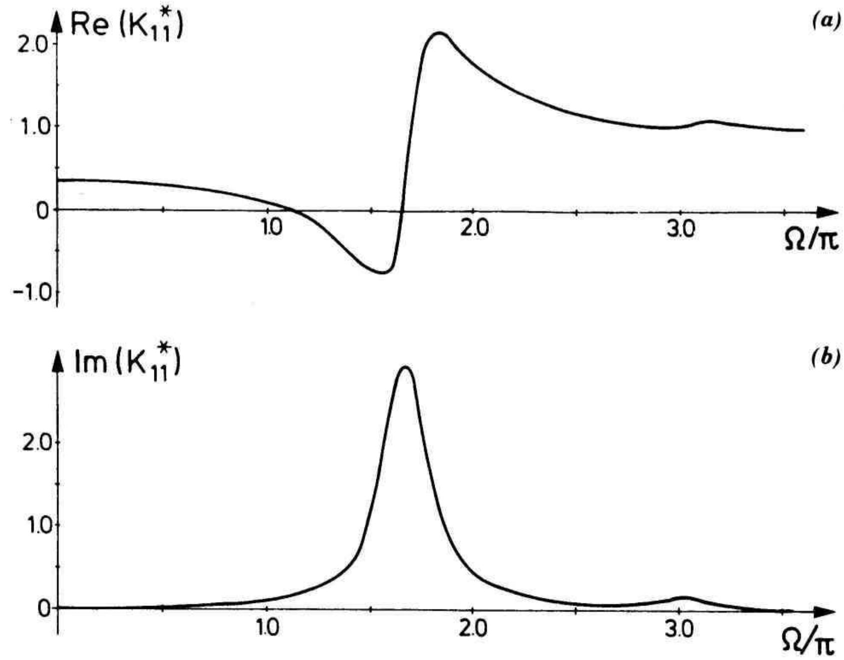

FIG. 5. Dynamic Stiffness Function (Example 1)

Example 2

The dynamic stiffness K_{44} is calculated for the parameters

$$l = 200 \text{ m}; \quad \frac{mg}{A} = 0.08 \text{ MN/m}^3; \quad \frac{T_\Theta}{A} = 500 \text{ MPa};$$

$$E = 200{,}000 \text{ MPa}; \quad \alpha = \Theta = 30°; \quad \xi = 0.01 \qquad (38)$$

as function of real Ω. From (26), (23), and (24)

$$\frac{L_e}{l} = 1.0001 \simeq 1; \quad \lambda^2 = 0.3072; \quad \varepsilon = 0.02771 \qquad (39)$$

is obtained. The limits of validity defined by (27) are safely met. The influence of simplifications in the analysis of the dynamic stiffness is also investigated here, as in example 1. In this case, however, the use of the simplified equation (33c)—that is, the neglect of $k^s_{v,v}$ and $k^a_{v,v}$—leads to completely wrong results. It is of particular interest that the imaginary part K''_{44} of the total stiffness becomes negative due to the omission of $k^s_{v,v}$. The work performed by the boundary force F_4 within one period is given by

$$W = \int_0^{2\pi/\omega} (K_{44}\Delta_4)' \left(\frac{\partial \Delta_4}{\partial t}\right)' dt = \pi K''_{44}|\Delta_4|^2 \qquad (40)$$

Because this work is positive (as long as it is ξ), negative K''_{44} may not occur; otherwise, this system would be a *perpetuum mobile*. The contribution of $k^a_{v,v}$ must be considered, too, because considerable variations occur within the range of the natural frequencies associated with antisymmetric modes (i.e. of a cable fixed at both ends). These results seem to be typical for tightly stretched inclined cables as they are employed in cable-stayed bridges.

The dynamic stiffness is again related to the elastic part $K^{t,e}_{44}$ of the static stiffness K^t_{44} of a straight rod

$$K^t_{44} = K^{t,e}_{44}\left(1 + \frac{T_\Theta}{EA}\cot^2\alpha\right) \qquad (41a)$$

$$K^{t,e}_{44} := \frac{EA}{l}\sin^2\alpha \qquad (41b)$$

From the complete equation (32c) it is found that

$$K^*_{44} := \frac{K_{44}}{K^{t,e}_{44}} = \frac{1 + \varepsilon\cot\alpha(\kappa - 1) - \dfrac{1}{4}\rho\Omega_c^2\left[\kappa + \dfrac{\lambda^2}{\Omega_c^2}(\kappa - 1)\right]}{1 + \dfrac{\lambda^2}{\Omega_c^2}(\kappa - 1)} + \frac{\rho}{\kappa} \qquad (42)$$

where the abbreviation

$$\rho := \frac{\varepsilon^2}{\lambda^2}\cot^2\alpha \simeq \frac{T_\Theta}{EA}\cot^2\alpha \qquad (43)$$

is used. The factor l/L_e is insignificant and is again neglected. The curves resulting from (42) are shown in Fig. 6. The variations in the range of the natural frequency associated with the first symmetric mode are small. The

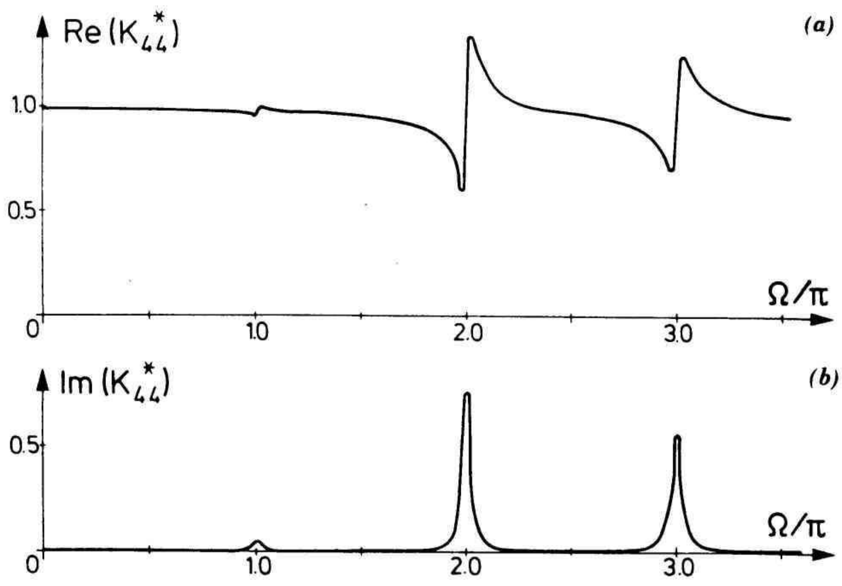

FIG. 6. Dynamic Stiffness Function (Example 2)

numeric analysis showed that here the various stiffness contributions nearly cancel each other. The variations in the vicinity of the first antisymmetric and second symmetric natural frequencies predominate.

Compared to example 1, the "resonance tubes" of the stiffness functions are much narrower here; the variations are smaller, and the real part, K'_{44}, remains positive. As a review of the applied equations shows, these differences are due to ε and λ^2 becoming much smaller. Consequently, the sag as well as the influence of the geometrical stiffness (which is mainly induced by transverse cable displacements) decrease. The elastic stiffness, now being relatively small and therefore of dominant influence, is quasi-static. Hence the resulting stiffness function approximates in a wide range (and especially for small Ω) the constant value that would have been determined for a massless straight rod. Appreciable transverse displacements and the implied dynamic stiffness effects occur only in the direct proximity to resonance frequencies. Therefore, in the case of tightly stretched cables, less damping seems to be necessary in order to limit the variations of the dynamic stiffness functions.

The greater prominence of higher natural frequencies has its numeric origin in a larger ρ. In addition to ε and λ^2, this parameter is likewise of importance. It corresponds to the ratio of rotational to elastic resistance of a straight rod [see (41*a*) and (43)].

Under the same conditions, Veletsos and Darbre (1983) derived a formula for the dynamic stiffness K_{11}. It can be compared with expression (42), provided cot α is first replaced by tan α, in order to accomplish a transition from K^*_{44} to K^*_{11}. Initially, exact agreement could not be established. A detailed review of Veletsos and Darbre (1983), however, showed that an error apparently occurred at the derivation of its central equation [(38)]. When carrying out this derivation as specified, an expression that differs from that given in Veletsos and Darbre is obtained. This result is also somewhat simpler: the denominator $[1 - i(2\pi\zeta/\Phi)]$ within the first part of [(38) in Veletsos and Darbre (1983)] must be replaced by unity. The equation modified in this way can be exactly transformed into expression (42) (except for the factor l/L_e which is here neglected). Both expressions remain slightly different in their external form, for they were obtained in different ways:

the separation into symmetric and antisymmetric contributions is less distinct in Veletsos and Darbre (1983) than in the formula given in this paper. Without separation, equation (42) can be converted into the somewhat shorter expression

$$K_{44}^{*} = \frac{\left[1 + \frac{1}{2}\varepsilon \cot \alpha\,(\kappa - 1)\right]^2}{1 + \frac{\lambda^2}{\Omega_c^2}(\kappa - 1)} + \rho\Omega_c \cot \Omega_c \quad (44)$$

which corresponds in its formal arrangement with [(38) and (41) of Veletsos and Darbre (1983)]. A comparison of the derivations and results presented in this study with those given in Veletsos and Darbre shows the advantage of the limitation to trigonometrical solution functions with complex arguments. Algebraic effort is reduced to a minimum. The resulting equations are more concise and take the same external shape for damped and undamped cables (for an undamped cable, Ω_c is simply replaced by Ω).

An experimental verification of the theory presented here would be valuable, especially for a more accurate delimitation of the range of validity. In view of the partial similarity of the theoretical results, the tests of Davenport and Steels (1965) are cited. Their comparison of analytical results with dynamic tests of a model cable immersed in a liquid bath produced good agreement. The previously established and analytically considered parameters (especially the damping coefficient), however, had to be greatly altered in order to obtain a closer fit to the experimental data. This difficulty seems to diminish when using the equations given in this study. The experimental results in Davenport and Steels (1965) also indicate a resonance effect in the range of the antisymmetric natural frequency $\Omega = 2\pi$. This is predicted by the complete solutions (32) and the resulting (42).

Conclusions

A dynamic stiffness matrix for a damped cable that is suitable for dynamic direct-stiffness analysis of composed systems was presented. By utilization of trigonometrical solution functions with complex arguments, it was possible to strongly simplify the derivation. By means of example calculations, stiffness functions were discussed and compared to the results of other authors. For tightly stretched inclined cables, the use of the more accurate theory presented here is indispensable. For very taut cables, the stiffness functions approximate in a wide range the static stiffness terms of a straight rod.

Appendix I. References

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APPENDIX II. NOTATION

The following symbols are used in this paper:

A = effective cross-sectional area of cable;
c = damping force per unit length and velocity;
d = cable sag perpendicular to cable chord;
E = Young's modulus of elasticity;
g = gravitational acceleration (decreased by buoyancy effect);
H = horizontal component of static cable tension;
h = dynamic part of horizontal component of total cable tension;
h_τ = force quantity according to (2);
K_{ij} = element of global dynamic stiffness matrix;
K^*_{ij} = dimensionless dynamic stiffness function defined by (37) and (42);
$\mathbf{K}$ = global dynamic stiffness matrix defined by (30);
$k^r_{p,q}$ = dynamic stiffness function related to specified set of boundary displacements;
$\mathbf{k}^r_{p,q}$ = submatrices according to (13);
$\mathbf{k}$ = local dynamic stiffness matrix defined by (10);
L_e = cable parameter according to (20) or (26);
l = length of cable chord;
m = cable mass per unit length (increased by virtual mass effect due to fluid);
T = static cable tension;
T_Θ = static cable tension at section where cable is parallel to chord, (25);
t = time;
u = displacement parallel to cable chord;
V = vertical component of static cable tension;
v = displacement perpendicular to cable chord;
x = coordinate of static cable line parallel to cable chord;
y = coordinate of static cable line perpendicular to cable chord;
α = angle of rotation of transformation to global coordinates, Fig. 4;
ε = cable parameter according to (19) or (24);
Θ = angle of inclination of cable chord (measured from horizontal line);
κ = Ω_c-dependent auxiliary term, (21);
λ^2 = fundamental cable parameter according to (18) or (23);
ν = dynamic part of vertical component of total cable tension;
ξ = dimensionless damping parameter defined by (6*b*);
ρ = cable parameter according to (43);

τ = dynamic part of total cable tension;
Ω = dimensionless frequency, (7*a*) or (22*a*);
Ω_c = dimensionless frequency-damping parameter, (7*b*) or (22*b*);
ω = circular frequency of motion according to (4); and
ω_c = frequency-damping parameter defined by (6*a*).