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PERTURBATIONS OF NON-AUTONOMOUS SECOND-ORDER ABSTRACT CAUCHY PROBLEMS

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Abstract. In this paper we present time-dependent perturbations of second-order non-autonomous abstract Cauchy problems associated to a family of operators with constant domain. We make use of the equivalence to a first-order non-autonomous abstract Cauchy problem in a product space, which we elaborate in full detail. As an application we provide a perturbed non-autonomous wave equation.

1. Introduction

Autonomous second-order abstract Cauchy problems, which are of the form

$$(\mathrm{ACP}_2) \qquad \begin{cases} \ddot{u}(t) = Au(t), & t > 0, \\ u(0) = x, \\ \dot{u}(0) = y, \end{cases}$$

for some (unbounded) operator $(A, \mathrm{D}(A))$, which often occur in the context of wave equations, have been studied intensively by several authors in the

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past, e.g., Sova [44], Da Prato and Giusti [8], Fattorini [11], Neubrander [32], Xio and Jin [54] as well as Xiao and Liang [53]. One can also find more information in the monographs by Arendt et al. [4, Sects. 3.14, 3.15], Melnikova and Filinkov [29, Sect. 1.7] or Vasil'ev and Piskarev [49]. In contrast to the first-order problem, where (classical) solutions are given by C_0-semigroups, one needs another solution concept for (ACP_2), the so-called cosine and sine families. Similar to the Hille–Yosida generation theorem for strongly continuous semigroups, one can also characterize generators of cosine families, cf. [4, Theorem 3.15.3], [29, Theorem 1.7.2] or [54, Theorem A]. The classical operator theoretical approach to (ACP_2) is to reduce these to first-order ones, where one can apply the theory of C_0-semigroups. For a detailed overview on C_0-semigroups, we refer for example to the monographs by Engel and Nagel [9], Goldstein [12] or Pazy [35]. In [21], Kisyński gives an explicit correspondence between generators of cosine families and strongly continuous semigroups, see also [4, Theorem 3.14.11].

In contrast to the autonomous second-order problems, one has the non-autonomous second-order abstract Cauchy problems of the form

$$(\mathrm{nACP}_2) \qquad \begin{cases} \ddot{u}(t) = A(t)u(t), \quad t \in (0,T]\,, \\ u(0) = x, \\ \dot{u}(0) = y, \end{cases}$$

for some fixed $T > 0$, where $(A(t), \mathrm{D}(A(t)))_{t\in[0,T]}$ is a family of operators. These non-autonomous second-order abstract Cauchy problems have been studied first by Kozak [22–24] and later on by Bochenek [7], Winiarska [51, 52] and Lan [25], just to mention a few. The same idea as for (ACP_2) helps to reduce (nACP_2) again to a first-order problem. Solutions of non-autonomous first-order abstract Cauchy problems have been studied exhaustively by means of evolution families for example by Acquistapace and Terreni [2] and Kato and Tanabe [18,46,47]. A semigroup approach by so-called evolution semigroups was firstly introduced by Howland [16] and later on studied by several authors, e.g., Evans [10], Nagel [30], Nickel [33], Rhandi [31] and Schnaubelt [39].

The goal of this paper is to establish a perturbation result for (nACP_2). Perturbation theorems for cosine families associated to (ACP_2) have been developed for example by Piskarev and Shaw [36,37], Miyadera [43], Takenaka and Okazawa [45] as well as Travis and Webb [48]. Also time-dependent perturbations have been studied, cf. [26,28,42]. We want to perturb (nACP_2) in a time-dependent way as it has been done for first-order non-autonomous problems by Räbiger et al. [38,40]. Especially, we want to cover time-dependent perturbations of bounded type.

The paper is structured as follows. Section 2 consists of preliminary definitions regarding solutions of non-autonomous abstract Cauchy problems.

The following Section 3 provides a relation between existence of fundamental solutions to first- and second-order non-autonomous Cauchy problems. This can be viewed as a non-autonomous version of the generation theorem of Kisyński in [21], cf. Theorem 3.4. Section 4 provides our main result regarding perturbations of second-order non-autonomous Cauchy problems, cf. Theorem 4.2. The final Section 5 provides an example of a non-autonomous wave equation.

2. Preliminaries on non-autonomous abstract Cauchy problems

Let X be a Banach space, $T > 0$ and $(A(t), \mathrm{D}(A(t)))_{t\in[0,T]}$ be a family of closed operators on X.

We make the following crucial assumption throughout this article.

Assumption 2.1. The domains $\mathrm{D}(A(t))$ do not depend on time, i.e., there exists $D \subseteq X$ such that $\mathrm{D}(A(t)) = D$ for all $t \in [0, T]$.

Remark 2.2. It is noteworthy that Assumption 2.1 causes both restrictions and simplifications. In particular, we exclude wave equations with time-dependent coefficients and Neumann-type boundary conditions on $\mathrm{L}^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is open and bounded with smooth boundary. However, one has well-posedness results from Kato by hand [17], even without Assumption 2.1 [19,20] (however, the theorems in the constant domain case are much easier to apply). Note that the assumption of constant domains is regularly used in the literature, see for example [1,5,6], just to mention a few.

We write $\Delta := \Delta_T := \{(t, s) \in [0, T]^2 : \ t \geq s\}$.

2.1. Non-autonomous first-order Cauchy problems. Non-autonomous first-order abstract Cauchy problems are of the form

$$\text{(nACP)} \qquad \begin{cases} \dot{v}(t) = A(t)v(t), & t \in (0, T]\,, \\ v(0) = x, \end{cases}$$

where $x \in X$.

Definition 2.3. A function $v\colon [0, T] \to X$ is called a *(classical) solution* to (nACP) if v is continuously differentiable, $v(t) \in D$ for all $t \in (0, T]$ and v satisfies (nACP).

The following solution concept is important for non-autonomous first-order problems.

Definition 2.4. A *fundamental solution* to (nACP) associated with $(A(t), \mathrm{D}(A(t)))_{t\in[0,T]}$ satisfying Assumption 2.1 is a family of bounded linear operators $(U(t, s))_{(t,s)\in\Delta}$ on a Banach space X satisfying the following conditions:

(U1) $U(t,t)=I$ for all $t\in[0,T]$ and $U(t,s)U(s,r)=U(t,r)$ for all $(t,s),(s,r)\in\Delta$.

(U2) The mapping $\Delta\ni(t,s)\mapsto U(t,s)$ is strongly continuous on X.

(U3) $U(t,s)D\subseteq D$ for all $(t,s)\in\Delta$.

(U4) For all $(t,s)\in\Delta$ and $x\in D$ one has that $\frac{\partial}{\partial t}U(t,s)x$ and $\frac{\partial}{\partial s}U(t,s)x$ exist and

$$\frac{\partial}{\partial t}U(t,s)x=A(t)U(t,s)x \quad\text{and}\quad \frac{\partial}{\partial s}U(t,s)x=-U(t,s)A(s)x. \tag{2.1}$$

Note that (U4) requires (U3).

Remark 2.5. (a) Observe that a fundamental solution to (nACP) is bounded in $\mathcal{L}(X)$, which follows from (U2), compactness of Δ and the uniform boundedness principle.

(b) Fundamental solutions of (nACP) as defined in Definition 2.4 are also often called evolution families.

Proposition 2.6. *Let $(U(t,s))_{(t,s)\in\Delta}$ be a fundamental solution of (nACP). If $x\in D$ then $v(t):=U(t,0)x$ defines a classical solution of (nACP).*

Proof. The statement follows easily from the properties (U1)–(U4). ☐

2.2. Non-autonomous second-order Cauchy problems. Non-autonomous second-order abstract Cauchy problems are of the form

$$\begin{cases}\ddot{u}(t)=A(t)u(t), & t\in(0,T],\\ u(0)=x,\\ \dot{u}(0)=y,\end{cases} \tag{nACP$_2$}$$

where $x,y\in X$.

Definition 2.7. A function $u\colon[0,T]\to X$ is called a *(classical) solution* to (nACP$_2$) if u is twice continuously differentiable, $u(t)\in D$ for all $t\in(0,T]$ and u satisfies (nACP$_2$).

The solution concept of (nACP$_2$) is vastly more involved than this of (nACP). In fact, it goes back to Kozak [24, Defs. 2.1, 3.1]. Note that our definition is slightly different from the one of Kozak, see also Remark 2.9 below.

Definition 2.8. A *fundamental solution* to (nACP$_2$) associated with $(A(t),\mathrm{D}(A(t)))_{t\in[0,T]}$ satisfying Assumption 2.1 is a family of bounded linear operators $(S(t,s))_{(t,s)\in\Delta}$ on X satisfying the following conditions:

(S1) (a) $S(t,t)=0$ for all $t\in[0,T]$.

(b) The mapping $\Delta\ni(t,s)\mapsto S(t,s)$ in strongly continuous on X.

(c) For all $x \in X$ and $s \in [0,T]$ the mapping $[s,T] \ni t \mapsto S(t,s)x$ is continuously differentiable, and $(t,s) \mapsto \frac{\partial}{\partial t} S(t,s)x$ is continuous with

$$\frac{\partial}{\partial t} S(t,s)x\Big|_{t=s} = x.$$

(d) For all $x \in D$ and $t \in [0,T]$ the mapping $[0,t] \ni s \mapsto S(t,s)x$ is continuously differentiable, and $(t,s) \mapsto \frac{\partial}{\partial s} S(t,s)x$ is continuous with

$$\frac{\partial}{\partial s} S(t,s)x\Big|_{t=s} = -x.$$

(S2) $S(t,s)D \subseteq D$ for all $(t,s) \in \Delta$, for $x \in D$ the mapping $\Delta \ni (t,s) \mapsto S(t,s)x$ is twice continuously differentiable and

(a) $\frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x$.

(b) $\frac{\partial^2}{\partial s^2} S(t,s)x = S(t,s)A(s)x$.

(c) $\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x\Big|_{t=s} = 0$

(S3) For all $(t,s) \in \Delta$ and $x \in D$ we have that $\frac{\partial}{\partial s} S(t,s)x \in D$ and, moreover, the partial derivatives $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s} S(t,s)x$ and $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t} S(t,s)x$ exist and the following properties hold:

(a) $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s} S(t,s)x = A(t)\frac{\partial}{\partial s} S(t,s)x$.

(b) $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t} S(t,s)x = \frac{\partial}{\partial t} S(t,s)A(s)x$.

(c) The mapping $\Delta \ni (t,s) \mapsto A(t)\frac{\partial}{\partial s} S(t,s)x$ is continuous.

Moreover, we call a fundamental solution $(S(t,s))_{(t,s)\in\Delta}$ *evolutionary* if additionally

(S4) For all $(t,s),(s,r) \in \Delta$ and $x \in D$ one has

$$\Big(-\frac{\partial}{\partial s} S(t,s)\Big)S(s,r)x + S(t,s)\frac{\partial}{\partial s} S(s,r)x = S(t,r)x.$$

REMARK 2.9. (a) Note that our Definition of fundamental solutions is slightly different from [24, Defs. 2.1, 3.1] in the sense that we do not assume $(t,s) \mapsto S(t,s)x$ to be continuously differentiable for all $x \in X$ in (S1), but allow for the partial derivative $\frac{\partial}{\partial s} S(t,s)x$ only for $x \in D$ in (S1)(d). Thus, our definition is also different from the ones used in [7,13–15]. The reason we adjusted the definition stems from [23, Theorem 4.1,5)] which seems to be only possible for $x \in \mathrm{D}(B)$ instead of $x \in X$ there. In fact, the reasoning in [23, Remark 4.2] does not apply for $\frac{\partial}{\partial s} S(t,s)$ in the topology of X, but only in the topology of $\mathrm{D}(B)$ there.

(b) Moreover, in contrast to [7,13–15,23,24], we only define the fundamental solutions on Δ instead of $[0,T]^2$, see also [22].

Lemma 2.10. *Let* $(S(t,s))_{(t,s)\in\Delta}$ *be a fundamental solution of* (nACP$_2$) *in* X. *Then* $(S(t,s))_{(t,s)\in\Delta}$ *and* $(\frac{\partial}{\partial t}S(t,s))_{(t,s)\in\Delta}$ *are bounded in* $\mathcal{L}(X)$.

Proof. Since (S1)(b)-(c) yield that the maps $(t,s)\mapsto S(t,s)x$ and $(t,s)\mapsto \frac{\partial}{\partial t}S(t,s)x$ are continuous on the compact set Δ for all $x\in X$, we observe that $(S(t,s))_{(t,s)\in\Delta}$ and $(\frac{\partial}{\partial t}S(t,s))_{(t,s)\in\Delta}$ are pointwise bounded. The uniform boundedness principle then yields boundedness in $\mathcal{L}(X)$. $\square$

Proposition 2.11. *Let* $(S(t,s))_{(t,s)\in\Delta}$ *be a fundamental solution of* (nACP$_2$). *If* $x,y\in D$, *then* $u(t) := -\frac{\partial}{\partial s}S(t,0)x + S(t,0)y$ *defines a classical solution of* (nACP$_2$).

Proof. The statement follows easily from the properties (S1)–(S3). $\square$

3. Existence of fundamental solutions: a generation type result

Let X be a Banach space, $T>0$, $(A(t),\mathrm{D}(A(t)))_{t\in[0,T]}$ a family of closed linear operators in X satisfying Assumption 2.1 with $D\subseteq X$ dense. We will further assume the following.

Assumption 3.1. There exists $C>0$ such that

$$\frac{1}{C}\|\cdot\|_{A(s)} \le \|\cdot\|_{A(t)} \le C\|\cdot\|_{A(s)} \quad (s,t\in[0,T]).$$

We equip D with the graph norm of $A(0)$ (equivalently, with the graph norm of any $A(t)$) such that D is a Banach space.

It is worth to mention that the equivalence of the graph norms is an assumption made regularly throughout the literature, see for example [41, Remark 4.5], [50, Remark 4.2] and [3, Section 7], just to mention a few.
Moreover, the following regularity assumption on $A(\cdot)$ will be made.

Assumption 3.2. Assume $A(\cdot)x\in \mathrm{C}^1([0,T];X)$ for all $x\in D$.

Note that the first-order non-autonomous abstract Cauchy problem (nACP) is well-posed according to [9, Chapter VI, Def. 9.1] if Assumptions 2.1 and 3.2 are satisfied and every operator generates a contractive C_0-semigroup, cf. [17,19].
The following lemma will be useful in what follows. For two Banach spaces Z,X we write $Z\subseteq X$ if Z continuously embeds into X.

Lemma 3.3. *Let* Z,X *be Banach spaces*, $Z\subseteq X$, $B\in\mathcal{L}(X)$.
(a) *Assume* $BX\subseteq Z$. *Then* $B\in\mathcal{L}(X,Z)$.
(b) *Assume* $BZ\subseteq Z$. *Then* $B\in\mathcal{L}(Z)$.

PROOF. (a) Let (x_n) in X, $x \in X$, $y \in Z$ such that $x_n \to x$ in X and $Bx_n \to y$ in Z. Since $Z \subseteq X$ we also have $Bx_n \to y$ in X. Hence, by continuity of B on X, we have $Bx = y$, so $B\colon X \to Z$ is closed and hence bounded by the closed graph theorem.

(b) Let (x_n) in Z, $x \in Z$, $y \in Z$ such that $x_n \to x$ in Z and $Bx_n \to y$ in Z. Since $Z \subseteq X$ we also have $x_n \to x$ in X and $Bx_n \to y$ in X. Hence, by continuity of B on X, we have $Bx = y$, so $B\colon Z \to Z$ is closed and hence bounded by the closed graph theorem. $\square$

Let Z be a Banach space such that $D \subseteq Z$ dense and $Z \subseteq X$ dense. Let $(\mathcal{A}(t), \mathrm{D}(\mathcal{A}(t)))_{t\in[0,T]}$ in $\mathcal{Z} := Z \times X$ be defined by

$$\mathcal{A}(t) := \begin{pmatrix} 0 & I \\ A(t) & 0 \end{pmatrix}, \quad \mathrm{D}(\mathcal{A}(t)) := \mathcal{D} := D \times Z.$$

THEOREM 3.4. *The following assertions are equivalent*:

(a) *There exists an evolutionary fundamental solution* $(S(t,s))_{(t,s)\in\Delta}$ *on* X *of* (nACP$_2$) *associated to* $(A(t), D)_{t\in[0,T]}$ *such that for all* $(t,s) \in \Delta$ *we have*

- $S(t,s)X \subseteq Z$, $S(t,s)Z \subseteq D$, $(t,s) \mapsto S(t,s)x \in Z$ *is continuous for all* $x \in X$,
- $\frac{\partial}{\partial t} S(t,s)Z \subseteq Z$, $\frac{\partial^2}{\partial t^2} S(t,s)x$ *exists for all* $x \in Z$ *and* $\frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x$,
- $\frac{\partial}{\partial s} S(t,s)x$ *exists for all* $x \in Z$, $\frac{\partial}{\partial s} S(t,s)Z \subseteq Z$ *and* $(t,s) \mapsto \frac{\partial}{\partial s} S(t,s)x \in Z$ *is continuous for all* $x \in Z$,
- $\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)D \subseteq Z$, $\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x$ *exists for all* $x \in Z$ *and there exists* $C \geq 0$ *such that* $\|\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x\|_X \leq C\|x\|_Z$ *for all* $x \in Z$ *and* $(t,s) \in \Delta$.

(b) *There exists a fundamental solution* $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ *on* $\mathcal{L}(\mathcal{Z})$ *of* (nACP) *associated to* $(\mathcal{A}(t), \mathcal{D})_{t\in[0,T]}$.

PROOF. We first prove the implication (b) $\Rightarrow$ (a).

Define $(S(t,s))_{(t,s)\in\Delta}$ on X by

$$S(t,s)x := \pi_1 \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad x \in X, \tag{3.1}$$

where $\pi_1\colon Z \times X \to Z$ denotes the projection on the first component. Hence, $S(t,s) \in \mathcal{L}(X,Z)$ and since $Z \subseteq X$ we also have $S(t,s) \in \mathcal{L}(X)$ for all $(t,s) \in \Delta$. Since $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ is strongly continuous on $\mathcal{Z}$, also $(t,s) \mapsto S(t,s) \in \mathcal{L}(X,Z)$ strongly continuous.

Let us provide some useful formulas. First, we note that $\pi_1\mathcal{A}(t) = \pi_2|_{\mathcal{D}}$, where $\pi_2\colon Z \times X \to X$ is the projection on the second component. Second, for $x \in Z$ we have that $\mathcal{A}(s)\binom{0}{x} = \binom{x}{0}$ for all $s \in [0,T]$. In particular, for $x \in D$ we have $\mathcal{A}(s)\binom{0}{x} \in \mathcal{D}$ for all $s \in [0,T]$.

For $x \in Z$ we have $\binom{0}{x} \in \mathcal{D}$, so by (U3) we obtain $S(t,s)x \in D$ for all $(t,s) \in \Delta$. Moreover, $t \mapsto S(t,s)x$ is differentiable and

$$\frac{\partial}{\partial t} S(t,s)x = \pi_1 \frac{\partial}{\partial t} \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix} = \pi_1 \mathcal{A}(t)\mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix} = \pi_2 \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

By (U3) we observe $\frac{\partial}{\partial t} S(t,s)x \in Z$.

Let $x \in Z$. Then, by (U4), we have that $\mathcal{U}(t,s)\binom{0}{x}$ is differentiable with respect to s and we obtain

$$\frac{\partial}{\partial s} \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix} = -\mathcal{U}(t,s)\mathcal{A}(s) \begin{pmatrix} 0 \\ x \end{pmatrix} = -\mathcal{U}(t,s) \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Thus, $\frac{\partial}{\partial s} S(t,s)x$ exists, $\frac{\partial}{\partial s} S(t,s)x \in Z$ and

$$\frac{\partial}{\partial s} S(t,s)x = \pi_1 \frac{\partial}{\partial s} \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix} = -\pi_1 \mathcal{U}(t,s) \begin{pmatrix} x \\ 0 \end{pmatrix}$$

for all $(t,s) \in \Delta$. As $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ is strongly continuous, also $(t,s) \mapsto \frac{\partial}{\partial s} S(t,s)x \in Z$ is continuous for all $x \in Z$.

We now show existence of the mixed derivative of $(S(t,s))_{(t,s)\in\Delta}$. First, let $x \in D$. Then, by (U4) and the above, we have that $\frac{\partial}{\partial s} \mathcal{U}(t,s)\binom{0}{x}$ is differentiable with respect to t and

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix} = -\frac{\partial}{\partial t} \mathcal{U}(t,s) \begin{pmatrix} x \\ 0 \end{pmatrix} = -\mathcal{A}(t)\mathcal{U}(t,s) \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Thus, $\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x$ exists and

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x = -\pi_1 \mathcal{A}(t)\mathcal{U}(t,s) \begin{pmatrix} x \\ 0 \end{pmatrix} = \pi_2 \mathcal{U}(t,s) \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

By (U3) this shows that $\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x \in Z$.

Let $x \in Z$. Then $\binom{0}{x} \in \mathcal{D}$, so as above

$$\frac{\partial}{\partial t} S(t,s)x = \pi_2 \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

As the right-hand side is differentiable with respect to t, $\frac{\partial^2}{\partial t^2} S(t,s)x$ exists and

$$\frac{\partial^2}{\partial t^2} S(t,s)x = \pi_2 \mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix} = \pi_2\mathcal{A}(t)\begin{pmatrix}\pi_1\\\pi_2\end{pmatrix}\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix}$$
$$= A(t)\pi_1\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix} = A(t)S(t,s)x.$$

Now, since $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ is bounded on $\mathcal{Z}$, there exists $C \geq 0$ such that $\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{Z})} \leq C$ for all $(t,s) \in \Delta$. Thus,

$$\left\| \frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x \right\|_X \leq C\|x\|_Z \tag{3.2}$$

for all $(t,s) \in \Delta$ and $x \in Z$. Let $x \in Z$ and choose (x_n) in D such that $x_n \to x$ in Z. Then $t \mapsto \frac{\partial}{\partial s} S(t,s)x_n$ is continuously differentiable for all $n \in \mathbb{N}$ and these functions converge uniformly to $t \mapsto \frac{\partial}{\partial s} S(t,s)x$, by boundedness of $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$. Moreover, $t \mapsto \frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x_n$ is continuous for all $n \in \mathbb{N}$ and these functions converge uniformly by the estimate above. Thus, $t \mapsto \frac{\partial}{\partial s} S(t,s)x$ is continuously differentiable and (3.2) holds for all $x \in Z$.

We are now ready to prove (S1)–(S4) similarly as in [22, Section 3] and [23, Theorem 4.1].

(S1): We prove (a)–(d) step by step.

(a) For $t \in [0,T]$ we have $\mathcal{U}(t,t) = I$ and therefore $S(t,t) = 0$.

(b) Strong continuity of $(t,s) \mapsto S(t,s)$ follows from strong continuity of $(t,s) \mapsto \mathcal{U}(t,s)$ and continuity of π_1.

(c) Let $s \in [0,T]$ and $x \in Z$. Then $\begin{pmatrix}0\\x\end{pmatrix} \in \mathcal{D}$ and (U4) yields that $[s,T] \ni t \mapsto \mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix}$ is continuously differentiable. Thus, also $[s,T] \ni t \mapsto S(t,s)x$ is continuously differentiable, and by (U4) we compute

$$\frac{\partial}{\partial t} S(t,s)x = \frac{\partial}{\partial t}\pi_1\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix} = \pi_1\frac{\partial}{\partial t}\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix}$$
$$= \pi_1\mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix} = \pi_2\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix}.$$

Now, for $x \in X$ we choose (x_n) in Z such that $x_n \to x$ in X. Then $t \mapsto S(t,s)x_n$ is continuously differentiable for all $n \in \mathbb{N}$ and these functions converge uniformly to $t \mapsto S(t,s)x$, by boundedness of $(S(t,s))_{(t,s)\in\Delta}$. Moreover, $t \mapsto \frac{\partial}{\partial t} S(t,s)x_n$ is continuous for all $n \in \mathbb{N}$ and these functions converge uniformly by boundedness of $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$. Thus, $t \mapsto S(t,s)x$ is continuously differentiable with

$$\frac{\partial}{\partial t} S(t,s)x = \pi_2 \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

Since the right-hand side is continuous in (t,s), we have that $(t,s) \mapsto \frac{\partial}{\partial t}\, S(t,s)x$ is continuous. Moreover,

$$\frac{\partial}{\partial t}\, S(t,s)x\bigg|_{t=s} = \pi_2 \mathcal{U}(s,s) \begin{pmatrix} 0 \\ x \end{pmatrix} = x.$$

(d) We have already established that $\frac{\partial}{\partial s}\, S(t,s)x$ exists for all $x \in Z$ and $(t,s) \in \Delta$, and that

$$\frac{\partial}{\partial s}\, S(t,s)x = -\pi_1 \mathcal{U}(t,s) \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Since the right-hand side is continuous in s, $s \mapsto S(t,s)x$ is continuously differentiable with

$$\frac{\partial}{\partial s}\, S(t,s)x = -\pi_1 \mathcal{U}(t,s) \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Since the right-hand side is continuous in (t,s), we have that $(t,s) \mapsto \frac{\partial}{\partial s}\, S(t,s)x$ is continuous. Moreover,

$$\frac{\partial}{\partial s}\, S(t,s)x\bigg|_{t=s} = -\pi_1 \mathcal{U}(s,s) \begin{pmatrix} x \\ 0 \end{pmatrix} = -x.$$

(S2): We have already established $S(t,s)Z \subseteq D$, so since $D \subseteq Z$ we have $S(t,s)D \subseteq D$ for all $(t,s) \in \Delta$.

Let $x \in D$. By (U4) and the formulas at the beginning of the proof we have that $\Delta \ni (t,s) \mapsto \mathcal{U}(t,s)\begin{pmatrix} 0 \\ x \end{pmatrix}$ is twice continuously differentiable. Thus, also $\Delta \ni (t,s) \mapsto S(t,s)x = \pi_1 \mathcal{U}(t,s)\begin{pmatrix} 0 \\ x \end{pmatrix}$ is twice continuously differentiable. For the second derivatives, with (U4) we compute:

(a)

$$\begin{aligned} \frac{\partial^2}{\partial t^2} S(t,s)x &= \frac{\partial^2}{\partial t^2} \pi_1 \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix} = \frac{\partial}{\partial t} \pi_1 \frac{\partial}{\partial t} \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix} \\ &= \frac{\partial}{\partial t} \pi_1 \mathcal{A}(t) \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix} = \frac{\partial}{\partial t} \pi_2 \mathcal{U}(t,s) \begin{pmatrix} 0 \\ x \end{pmatrix} \end{aligned}$$

$$= \pi_2 \frac{\partial}{\partial t}\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix} = \pi_2\mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix} = A(t)S(t,s)x.$$

(b)

$$\frac{\partial^2}{\partial s^2}S(t,s)x = \pi_1\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix} = -\pi_1\frac{\partial}{\partial s}\mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix}0\\x\end{pmatrix}$$
$$= -\pi_1\frac{\partial}{\partial s}\mathcal{U}(t,s)\begin{pmatrix}x\\0\end{pmatrix} = \pi_1\mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix}x\\0\end{pmatrix}$$
$$= \pi_1\mathcal{U}(t,s)\begin{pmatrix}0\\A(s)x\end{pmatrix} = S(t,s)A(s)x.$$

(c)

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}S(t,s)x = \frac{\partial}{\partial s}\pi_1\mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix}$$
$$= -\pi_1\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix}0\\x\end{pmatrix} = -\pi_2\mathcal{U}(t,s)\begin{pmatrix}x\\0\end{pmatrix}$$

This also yields the last assertion by (U1).

(S3): Let $(t,s)\in\Delta$ and $x\in D$. Then by (U4) we have $\frac{\partial}{\partial s}S(t,s)x = -\pi_1\mathcal{U}(t,s)\binom{x}{0}\in D$ by (U3).

Moreover, by (U4) we observe that $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}S(t,s)x$ and $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}S(t,s)x$ exist and we can compute these derivatives:

(a)

$$\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}S(t,s)x = -\frac{\partial^2}{\partial t^2}\pi_1\mathcal{U}(t,s)\begin{pmatrix}x\\0\end{pmatrix} = -\frac{\partial}{\partial t}\pi_1\mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix}x\\0\end{pmatrix}$$
$$= -\frac{\partial}{\partial t}\pi_2\mathcal{U}(t,s)\begin{pmatrix}x\\0\end{pmatrix} = -\pi_2\mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix}x\\0\end{pmatrix}$$
$$= A(t)\left(-\pi_1\mathcal{U}(t,s)\begin{pmatrix}x\\0\end{pmatrix}\right) = A(t)\frac{\partial}{\partial s}S(t,s)x.$$

(b)

$$\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}S(t,s)x = \frac{\partial^2}{\partial s^2}\pi_2\mathcal{U}(t,s)\begin{pmatrix}0\\x\end{pmatrix} = -\frac{\partial}{\partial s}\pi_2\mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix}0\\x\end{pmatrix}$$
$$= -\frac{\partial}{\partial s}\pi_2\mathcal{U}(t,s)\begin{pmatrix}x\\0\end{pmatrix} = \pi_2\mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix}x\\0\end{pmatrix}$$

$$= \pi_2 \mathcal{U}(t,s) \begin{pmatrix} 0 \\ A(s)x \end{pmatrix} = \frac{\partial}{\partial t} S(t,s)A(s)x,$$

where the last equality follows from (S1)(c).

(c) The continuity of $\Delta \ni (t,s) \mapsto A(t)\frac{\partial}{\partial s} S(t,s)x$ follows from strong continuity of $t \mapsto A(t)y$ for all $y \in D$, continuity of $(t,s) \mapsto \frac{\partial}{\partial s} S(t,s)x$ for all $x \in D$ and the fact that $\frac{\partial}{\partial s} S(t,s)D \subseteq D$.

(S4): Let $(t,s), (s,r) \in \Delta$ and $x \in D$. Then, by (U1), we compute

$$\Big(-\frac{\partial}{\partial s} S(t,s)\Big) S(s,r)x + S(t,s)\frac{\partial}{\partial s} S(s,r)x = \pi_1 \mathcal{U}(t,s) \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \mathcal{U}(s,r) \begin{pmatrix} 0 \\ x \end{pmatrix}$$
$$= \pi_1 \mathcal{U}(t,s)\mathcal{U}(s,r) \begin{pmatrix} 0 \\ x \end{pmatrix} = \pi_1 \mathcal{U}(t,r) \begin{pmatrix} 0 \\ x \end{pmatrix} = S(t,r)x.$$

Let us prove the converse implication (a) $\Rightarrow$ (b).

For $(t,s) \in \Delta$ and $\begin{pmatrix} x \\ y \end{pmatrix} \in D \times X$ we define

$$\mathcal{U}(t,s) \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} -\frac{\partial}{\partial s} S(t,s)x + S(t,s)y \\ -\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x + \frac{\partial}{\partial t} S(t,s)y \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{\partial}{\partial s} S(t,s) & S(t,s) \\ -\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s) & \frac{\partial}{\partial t} S(t,s) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in Z \times X.$$

We first show that $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ can be extended to a family in $\mathcal{L}(\mathcal{Z})$. Let $x \in Z$. Then there exists (x_n) in D such that $x_n \to x$ in Z. Since $S(t,s) \in \mathcal{L}(X,Z)$ by Lemma 3.3(a), by (S2) and $D \subseteq Z$ we have that the functions $s \mapsto S(t,s)x_n \in Z$ converge pointwise to $s \mapsto S(t,s)x \in Z$. Moreover, we have

$$\frac{\partial}{\partial s} S(t,s)x_n = \frac{\partial}{\partial s} S(s,s)x_n + \int_s^t \frac{\partial}{\partial t}\frac{\partial}{\partial s} S(\tau,s)x_n \,\mathrm{d}\tau$$
$$= -x_n + \int_s^t \frac{\partial}{\partial t}\frac{\partial}{\partial s} S(\tau,s)x_n \,\mathrm{d}\tau$$

for all $n \in \mathbb{N}$, by (S1)(c). By assumption, we have that the functions $t \mapsto \frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x_n$ converge uniformly to $t \mapsto \frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x$. Thus, also the functions $t \mapsto \frac{\partial}{\partial s} S(t,s)x_n \in Z$ converge uniformly. Thus, $t \mapsto S(t,s)x$ is differentiable. Lemma 3.3(b) now yields $\frac{\partial}{\partial s} S(t,s) \in \mathcal{L}(Z)$ for all $(t,s) \in \Delta$. As $\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s) \in \mathcal{L}(Z,X)$ for all $(t,s) \in \Delta$ by assumption, we can continuously extend $\mathcal{U}(t,s)$ to $\mathcal{Z}$ for all $(t,s) \in \Delta$.

We now check (U1)–(U4).

(U1): By (S1)(c)–(d) and (S2)(c), for $\binom{x}{y} \in D \times X$ we obtain $\mathcal{U}(t,t)\binom{x}{y} = \binom{x}{y}$ for all $t \in [0,T]$. Thus, by continuity and denseness of D in Z, $\mathcal{U}(t,t) = I$ for all $t \in [0,T]$.

Let $\binom{x}{y} \in D \times D$ and $(t,s),(s,r) \in \Delta$. By (S3) we have $\frac{\partial}{\partial r} S(s,r)x \in D \subseteq Z$, and $S(s,r)y \in Z$ by (S2). Thus, by (S4), we observe

$$
\begin{aligned}
&\mathcal{U}(t,s)\mathcal{U}(s,r)\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\partial}{\partial s} S(t,s) & S(t,s) \\ -\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s) & \frac{\partial}{\partial t} S(t,s) \end{pmatrix} \begin{pmatrix} -\frac{\partial}{\partial r} S(s,r)x + S(s,r)y \\ -\frac{\partial}{\partial s}\frac{\partial}{\partial r} S(s,r)x + \frac{\partial}{\partial s} S(s,r)y \end{pmatrix} \\
&= \begin{pmatrix} \big(\frac{\partial}{\partial s} S(t,s)\big) \frac{\partial}{\partial r} S(s,r)x - S(t,s) \frac{\partial}{\partial s} \frac{\partial}{\partial r} S(s,r)x \\ \qquad + \big(-\frac{\partial}{\partial s} S(t,s)\big) S(s,r)y + S(t,s) \frac{\partial}{\partial s} S(s,r)y \\ \big(\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)\big) \frac{\partial}{\partial r} S(s,r)x - \frac{\partial}{\partial t} S(t,s) \frac{\partial}{\partial s}\frac{\partial}{\partial r} S(s,r)x \\ \qquad - \big(\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)\big) S(s,r)y + \frac{\partial}{\partial t} S(t,s) \frac{\partial}{\partial s} S(s,r)y \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\partial}{\partial r} \big(\big(-\frac{\partial}{\partial s} S(t,s)\big) S(s,r)x + S(t,s) \frac{\partial}{\partial s} S(s,r)x\big) \\ \qquad + \big(-\frac{\partial}{\partial s} S(t,s)\big) S(s,r)y + S(t,s) \frac{\partial}{\partial s} S(s,r)y \\ -\frac{\partial}{\partial t}\frac{\partial}{\partial r} \big(\big(-\frac{\partial}{\partial s} S(t,s)\big) S(s,r)x + S(t,s) \frac{\partial}{\partial s} S(s,r)x\big) \\ \qquad + \frac{\partial}{\partial t} \big(\big(-\frac{\partial}{\partial s} S(t,s)\big) S(s,r) + S(t,s) \frac{\partial}{\partial s} S(s,r)\big) y \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\partial}{\partial r} S(t,r)x + S(t,r)y \\ -\frac{\partial}{\partial t}\frac{\partial}{\partial r} S(t,r)x + \frac{\partial}{\partial t} S(t,r)y \end{pmatrix} = \mathcal{U}(t,r)\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}
$$

Since $D \times D$ is dense in $\mathcal{Z}$ and $\mathcal{U}(t,r)$, $\mathcal{U}(t,s)$ and $\mathcal{U}(s,r)$ are continuous, we obtain the assertion.

(U2): We show componentwise the strong continuity of $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$. To do so for the first component, let $\binom{x}{y} \in Z \times X$, then

$$\pi_1 \mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\partial}{\partial s} S(t,s)x + S(t,s)y \in Z.$$

By assumption on $(S(t,s))_{(t,s)\in\Delta}$, we obtain strong continuity in the first component.

Likewise, for $\binom{x}{y} \in D \times X$ we observe that

$$\pi_2 \mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x + \frac{\partial}{\partial t} S(t,s)y.$$

By (S2) we know that $(t,s) \mapsto S(t,s)x$ is twice continuously differentiable. As we also assumed that $\| \frac{\partial}{\partial t} \frac{\partial}{\partial s} S(t,s)x\|_X \le C\|x\|_Z$ for all $x \in Z$ and $(t,s) \in \Delta$ we also conclude the strong continuity of the first term. For the second term, (S1)(c) directly yields the strong continuity. Thus, we obtain strong continuity in the second component.

(U3): Let $\binom{x}{y} \in \mathcal{D}$ and $(t,s) \in \Delta$. Then $x \in D$ and therefore $\frac{\partial}{\partial s} S(t,s)x \in D$ by (S3). Moreover, $S(t,s)y \in D$ by assumption, and $\frac{\partial}{\partial t} \frac{\partial}{\partial s} S(t,s)x \in Z$ and $\frac{\partial}{\partial t} S(t,s)y \in Z$ by assumption. Hence,

$$\mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial s} S(t,s)x + S(t,s)y \\ -\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x + \frac{\partial}{\partial t} S(t,s)y \end{pmatrix} \in \mathcal{D}.$$

(U4): Let $\binom{x}{y} \in \mathcal{D}$. Then $\Delta \ni (t,s) \mapsto S(t,s)x$ is two times continuously differentiable by (S2), and $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s} S(t,s)x$ exists by (S3). Moreover, by assumption $\frac{\partial^2}{\partial t^2} S(t,s)y$ exists and $\frac{\partial^2}{\partial t^2} S(t,s)y = A(t)S(t,s)y$. Thus, we observe that

$$\frac{\partial}{\partial t}\mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x + \frac{\partial}{\partial t} S(t,s)y \\ -\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s} S(t,s)x + \frac{\partial^2}{\partial t^2} S(t,s)y \end{pmatrix}$$

exists, and by assumption and (S3)(a) we conclude

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x + \frac{\partial}{\partial t} S(t,s)y \\ -\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s} S(t,s)x + \frac{\partial^2}{\partial t^2} S(t,s)y \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x + \frac{\partial}{\partial t} S(t,s)y \\ -A(t)\frac{\partial}{\partial s} S(t,s)x + A(t)S(t,s)y \end{pmatrix} \\ &= \begin{pmatrix} 0 & I \\ A(t) & 0 \end{pmatrix}\mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix}.\end{aligned}$$

To show the other statement, we first let $\binom{x}{y} \in D \times D$. Then $\Delta \ni (t,s) \mapsto S(t,s)x$ and $\Delta \ni (t,s) \mapsto S(t,s)y$ are two times continuously differentiable by (S2). Moreover, $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t} S(t,s)x$ exists by (S3). Thus,

$$\frac{\partial}{\partial s}\mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2}{\partial s^2} S(t,s)x + \frac{\partial}{\partial s} S(t,s)y \\ -\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t} S(t,s)x + \frac{\partial}{\partial s}\frac{\partial}{\partial t} S(t,s)y \end{pmatrix}$$

exists, and by (S2)(b) and (S3)(b) we conclude

$$\begin{aligned}\frac{\partial}{\partial s}\mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -\frac{\partial^2}{\partial s^2} S(t,s)x + \frac{\partial}{\partial s} S(t,s)y \\ -\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t} S(t,s)x + \frac{\partial}{\partial s}\frac{\partial}{\partial t} S(t,s)y \end{pmatrix} \\ &= \begin{pmatrix} -S(t,s)A(s)x + \frac{\partial}{\partial s} S(t,s)y \\ -\frac{\partial}{\partial t} S(t,s)A(s)x + \frac{\partial}{\partial s}\frac{\partial}{\partial t} S(t,s)y \end{pmatrix}\end{aligned}$$

$$= -\mathcal{U}(t,s)\begin{pmatrix} 0 & I \\ A(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Now, let $\binom{x}{y} \in \mathcal{D}$. There exists (y_n) in D such that $y_n \to y$ in Z. Then the sequence of continuous functions $s \mapsto \mathcal{U}(t,s)\binom{x}{y_n}$ converge uniformly to $s \mapsto \mathcal{U}(t,s)\binom{x}{y}$ by boundedness of $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$. Moreover, the sequence of functions $t \mapsto \frac{\partial}{\partial s}\mathcal{U}(t,s)\binom{x}{y_n} = \mathcal{U}(t,s)\mathcal{A}(s)\binom{x}{y_n} = \mathcal{U}(t,s)\binom{y_n}{A(s)x}$ converges uniformly to $s \mapsto \mathcal{U}(t,s)\mathcal{A}(s)\binom{x}{y}$, since $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ is bounded and $s \mapsto A(s)x$ is continuous. Thus, $\frac{\partial}{\partial s}\mathcal{U}(t,s)\binom{x}{y}$ exists and

$$\frac{\partial}{\partial s}\mathcal{U}(t,s)\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix} x \\ y \end{pmatrix}. \quad \square$$

REMARK 3.5. If Theorem 3.4((a)) is true the statement in Proposition 2.11 can be extended to $y \in Z$.

REMARK 3.6 (the Choice of the Space Z). (a) Let us consider the autonomous case, i.e., let A be a densely defined closed operator and $A(t) = A$ for all $t \in [0,T]$. Then Z is the Kisyński-space given by

$$Z = \{x \in X : \ S(\cdot,\cdot)x \in C(\Delta; D)\},$$

cf. [21] or [4, Theorem 3.14.11]. Note that in this case the space is uniquely defined. In this case, A generates a so-called cosine family.

(b) For all $t \in [0,T]$ let $A(t)$ generate a cosine family. Then by [48], without loss of generality we may assume that there exists $(B(t))_{t\in[0,T]}$ such that $B(t)^2 = A(t)$ for all $t \in [0,T]$. Let us assume that $(B(t))_{t\in[0,T]}$ satisfies Assumptions 2.1 and 3.1. Then, under some further assumptions, the space Z can be chosen to be $Z := \mathrm{D}(B(t))$ for some/all $t \in [0,T]$ (equipped with the graph norm); cf. [22–24].

As we have seen in the above remark, in the autonomous case we have an explicit description of the space Z.

CONJECTURE 3.7. *Let $(S(t,s))_{(t,s)\in\Delta}$ be an evolutionary fundamental solution on X of* (nACP$_2$) *associated to $(A(t),\mathrm{D}(A(t)))_{t\in[0,T]}$. We conjecture that Z has the form*

$$Z = \{x \in X : \ S(\cdot,\cdot)x \in \mathrm{C}(\Delta; D)\}$$

equipped with the norm $\|\cdot\|_Z$ given by

$$\|x\|_Z := \|x\|_X + \sup_{(t,s)\in\Delta} \|A(t)S(t,s)x\|_X.$$

4. A bounded perturbation type result

Let $(B(t))_{t\in[0,T]}$ be a family of bounded operators on X. We first prove a first-order non-autonomous bounded perturbation type result similar to [9, Chapter VI, Corollary 9.20] which fits into our framework of fundamental solutions.

Proposition 4.1. *Let $(A(t), \mathrm{D}(A(t)))_{t\in[0,T]}$ be a family of densely defined closed operators on a Banach space X satisfying Assumptions 2.1, 3.1 and 3.2. Let $(S(t,s))_{(t,s)\in\Delta}$ be an evolutionary fundamental solution of the corresponding (nACP) on X and let Z be the space occuring in Theorem 3.4 such that for all $(t,s)\in\Delta$ we have*

- *$S(t,s)X \subseteq Z$, $S(t,s)Z \subseteq D$, $(t,s) \mapsto S(t,s)x \in Z$ is continuous for all $x \in X$,*
- *$\frac{\partial}{\partial t} S(t,s)Z \subseteq Z$, $\frac{\partial^2}{\partial t^2} S(t,s)x$ exists for all $x \in Z$ and $\frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x$,*
- *$\frac{\partial}{\partial s} S(t,s)x$ exists for all $x \in Z$, $\frac{\partial}{\partial s} S(t,s)Z \subseteq Z$ and $(t,s) \mapsto \frac{\partial}{\partial s} S(t,s)x \in Z$ is continuous for all $x \in Z$,*
- *$\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)D \subseteq Z$, $\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x$ exists for all $x \in Z$ and there exists $C \geq 0$ such that $\|\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x\|_X \leq C\|x\|_Z$ for all $x \in Z$ and $(t,s)\in\Delta$.*

Let $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ be a fundamental solution of (nACP) on $\mathcal{Z} := Z \times X$ corresponding to $(\mathcal{A}(t), \mathrm{D}(\mathcal{A}(t)))_{t\in[0,T]}$. Let $B(\cdot) \in \mathrm{C}([0,T]; \mathscr{L}_{\mathrm{s}}(Z))$. Then there exists a fundamental solution $(\mathcal{V}(t,s))_{(t,s)\in\Delta}$ of (nACP) on $\mathcal{Z}$ corresponding to the family of operators $(\mathcal{A}(t) + \big(t\big), \mathrm{D}(\mathcal{A}(t)))_{t\in[0,T]}$, where

$$\big(t\big) := \begin{pmatrix} 0 & 0 \\ B(t) & 0 \end{pmatrix}, \quad t \in [0,T].$$

Proof. Firstly, by Theorem 3.4 we know that $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ is indeed a fundamental solution of (nACP) on $\mathcal{Z} = Z \times X$. By [9, Chapter VI, Cor. 9.20] there exists a family of operators $(\mathcal{V}(t,s))_{(t,s)\in\Delta}$ satisfying (U1) and (U2). Moreover, we know that the variation of constants formula holds, i.e., one has

$$\mathcal{V}(t,s)\begin{pmatrix} z \\ x \end{pmatrix} = \mathcal{U}(t,s)\begin{pmatrix} z \\ x \end{pmatrix} + \int_s^t \mathcal{U}(t,r)\big(r\big)\mathcal{V}(r,s)\begin{pmatrix} z \\ x \end{pmatrix} \,\mathrm{d}r$$
$$= \mathcal{U}(t,s)\begin{pmatrix} z \\ x \end{pmatrix} + \int_s^t \mathcal{V}(t,r)\big(r\big)\mathcal{U}(r,s)\begin{pmatrix} z \\ x \end{pmatrix} \,\mathrm{d}r, \tag{4.1}$$

for all $z \in Z$ and $x \in X$. In order to show that $(\mathcal{V}(t,s))_{(t,s)\in\Delta}$ is a fundamental solution according to Definition 2.4 we have to show that also (U3) and (U4) hold. To do so, let $\binom{x}{y} \in \mathcal{D} := D \times Z$ be arbitrary. Then $\mathcal{U}(t,s)\binom{x}{y} \in \mathcal{D}$

by the assumption that $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ is a fundamental solution, cf. Definition 2.4. As $(\mathcal{V}(t,s))_{(t,s)\in\Delta}$ is a family of operators in $Z\times X$ we obviously have $\mathcal{V}(t,s)(D\times X)\subseteq Z\times X$. Now, we observe that by the assumption that $B(\cdot)\in \mathrm{C}\left([0,T];\mathscr{L}_{\mathrm{s}}(Z)\right)$ and the explicit representation of the operators we have $\big(t\big)(Z\times X)\subseteq\{0\}\times Z\subseteq D\times Z$ so that the integral term appearing in the variation of constant formula (4.1) is in $D\times Z$ as well. This shows that (U3) holds. For (U4), let $\binom{x}{y}\in\mathcal{D}$. Then we have

$$\begin{gathered}
\frac{\partial}{\partial t}\mathcal{V}(t,s)\begin{pmatrix}x\\y\end{pmatrix}=\frac{\partial}{\partial t}\mathcal{U}(t,s)\begin{pmatrix}x\\y\end{pmatrix}+\frac{\partial}{\partial t}\int_s^t\mathcal{U}(t,r)\big(r\big)\mathcal{V}(r,s)\begin{pmatrix}x\\y\end{pmatrix}\,\mathrm{d}r\\
=\mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix}x\\y\end{pmatrix}+\mathcal{U}(t,t)\big(t\big)\mathcal{V}(t,s)\begin{pmatrix}x\\y\end{pmatrix}+\int_s^t\frac{\partial}{\partial t}\mathcal{U}(t,r)\big(r\big)\mathcal{V}(r,s)\begin{pmatrix}x\\y\end{pmatrix}\,\mathrm{d}r\\
=\mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix}x\\y\end{pmatrix}+\big(t\big)\mathcal{V}(t,s)\begin{pmatrix}x\\y\end{pmatrix}+\int_s^t\mathcal{A}(t)\mathcal{U}(t,r)\big(r\big)\mathcal{V}(r,s)\begin{pmatrix}x\\y\end{pmatrix}\,\mathrm{d}r\\
=\mathcal{A}(t)\mathcal{U}(t,s)\begin{pmatrix}x\\y\end{pmatrix}+\big(t\big)\mathcal{V}(t,s)\begin{pmatrix}x\\y\end{pmatrix}+\mathcal{A}(t)\int_s^t\mathcal{U}(t,r)\big(r\big)\mathcal{V}(r,s)\begin{pmatrix}x\\y\end{pmatrix}\,\mathrm{d}r\\
=(\mathcal{A}(t)+\big(t\big))\mathcal{V}(t,s)\begin{pmatrix}x\\y\end{pmatrix},
\end{gathered}$$

where we have used Hille's theorem. Moreover,

$$\begin{gathered}
\frac{\partial}{\partial s}\mathcal{V}(t,s)\begin{pmatrix}x\\y\end{pmatrix}=\frac{\partial}{\partial s}\mathcal{U}(t,s)\begin{pmatrix}x\\y\end{pmatrix}+\frac{\partial}{\partial s}\int_s^t\mathcal{V}(t,r)\big(r\big)\mathcal{U}(r,s)\begin{pmatrix}x\\y\end{pmatrix}\,\mathrm{d}r\\
=-\mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix}x\\y\end{pmatrix}-\mathcal{V}(t,s)\big(s\big)\mathcal{U}(s,s)\begin{pmatrix}x\\y\end{pmatrix}+\int_s^t\frac{\partial}{\partial s}\mathcal{V}(t,r)\big(r\big)\mathcal{U}(r,s)\begin{pmatrix}x\\y\end{pmatrix}\,\mathrm{d}r\\
=-\mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix}x\\y\end{pmatrix}-\mathcal{V}(t,s)\big(s\big)\begin{pmatrix}x\\y\end{pmatrix}+\int_s^t\mathcal{V}(t,r)\big(r\big)\frac{\partial}{\partial s}\mathcal{U}(r,s)\begin{pmatrix}x\\y\end{pmatrix}\,\mathrm{d}r\\
=-\mathcal{U}(t,s)\mathcal{A}(s)\begin{pmatrix}x\\y\end{pmatrix}-\mathcal{V}(t,s)\big(s\big)\begin{pmatrix}x\\y\end{pmatrix}-\int_s^t\mathcal{V}(t,r)\big(r\big)\mathcal{U}(r,s)\mathcal{A}(s)\begin{pmatrix}x\\y\end{pmatrix}\,\mathrm{d}r\\
=-\mathcal{V}(t,s)(\mathcal{A}(s)+\big(s\big))\begin{pmatrix}x\\y\end{pmatrix}.
\end{gathered}$$

Hence, we conclude that there exists a fundamental solution of (nACP) corresponding to the family of operators $(\mathcal{A}(t)+\big(t\big),\mathrm{D}(\mathcal{A}(t)))_{t\in[0,T]}$ according to Definition 2.4. □

Now, by using Theorem 3.4 in combination with Proposition 4.1 we obtain the following result.

THEOREM 4.2. *Let $(A(t), \mathrm{D}(A(t)))_{t\in[0,T]}$ be a family of densely defined closed operators on a Banach space X satisfying Assumptions 2.1, 3.1 and 3.2. Let $(S(t,s))_{(t,s)\in\Delta}$ be an evolutionary fundamental solution of (nACP$_2$) such that for all $(t,s)\in\Delta$ we have*

- $S(t,s)X \subseteq Z$, $S(t,s)Z \subseteq D$, $(t,s)\mapsto S(t,s)x \in Z$ *is continuous for all* $x\in X$,
- $\frac{\partial}{\partial t} S(t,s)Z \subseteq Z$, $\frac{\partial^2}{\partial t^2} S(t,s)x$ *exists for all* $x\in Z$ *and* $\frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x$,
- $\frac{\partial}{\partial s} S(t,s)x$ *exists for all* $x\in Z$, $\frac{\partial}{\partial s} S(t,s)Z\subseteq Z$ *and* $(t,s)\mapsto \frac{\partial}{\partial s} S(t,s)x \in Z$ *is continuous for all* $x\in Z$,
- $\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)D \subseteq Z$, $\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x$ *exists for all* $x\in Z$ *and there exists* $C\geq 0$ *such that* $\|\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)x\|_X \leq C\|x\|_Z$ *for all* $x\in Z$ *and* $(t,s)\in\Delta$.

Let $B(\cdot)\in \mathrm{C}([0,T];\mathscr{L}_{\mathrm{s}}(X))\cap \mathrm{C}([0,T];\mathscr{L}_{\mathrm{s}}(Z))$. Then there exists an evolutionary fundamental solution of (nACP$_2$) on X associated to a family of operator $(A(t)+B(t), \mathrm{D}(A(t)))_{t\in[0,T]}$.

PROOF. By the assumptions on the evolutionary fundamental solution $(S(t,s))_{(t,s)\in\Delta}$ and Theorem 3.4, there exists a fundamental solution $(\mathcal{U}(t,s))_{(t,s)\in\Delta}$ of (nACP) on $\mathcal{Z} := Z\times X$. For $t\in[0,T]$ we define a family of operators on $\mathcal{Z} = Z\times X$ by

$$\mathcal{B}(t) := \begin{pmatrix} 0 & 0 \\ B(t) & 0 \end{pmatrix}.$$

By Proposition 4.1 we obtain a fundamental solution $(\mathcal{V}(t,s))_{(t,s)\in\Delta}$ of (nACP) on $\mathcal{Z}$ corresponding to the family of operators

$$(\mathcal{A}(t) + (t), \mathrm{D}(\mathcal{A}(t)))_{t\in[0,T]}.$$

By using Theorem 3.4 again we conclude the result. □

REMARK 4.3. We observe that we do not need any additional assumption on $(S(t,s))_{(t,s)\in\Delta}$ for Theorem 4.2 besides the ones already appearing in Theorem 3.4. Hence, perturbation just relies on the continuity assumption on the perturbing operators.

5. Example: Non-autonomous wave equation

Motivated by [15, Sec. 5], we consider the following perturbed non-autonomous wave equation on $\mathrm{L}^2(0,\pi)$ given by

$$\begin{cases} \frac{\partial^2}{\partial t^2} w(t,\xi) = \alpha(t)\frac{\partial^2}{\partial \xi^2} w(t,\xi) + \beta(t,\xi)w(t,\xi), & t\in(0,T],\ \xi\in(0,\pi), \\ w(t,0) = w(t,\pi) = 0, & t\in(0,T], \\ w(0,\xi) = \varphi(\xi), & \xi\in[0,\pi], \\ \frac{\partial}{\partial t} w(0,\xi) = \psi(\xi), & \xi\in[0,\pi], \end{cases} \tag{5.1}$$

where $T > 0$, $\varphi, \psi \in \mathrm{L}^2(0,\pi)$ and $\alpha\colon [0,T] \to \mathbb{R}$ is continuously differentiable such that $\alpha(t) \geq 1$ for all $t \in [0,T]$. Moreover, we assume that $\beta \in \mathrm{C}^1([0,T] \times [0,\pi])$.

Remark 5.1. Littman [27] has shown that the standard wave equation on $\mathrm{L}^p(\mathbb{R}^n)$ is well-posed if and only if $p = 2$ or $n = 1$.

For $t \in [0,T]$, we define a family of operators $(A(t), \mathrm{D}(A(t)))_{t\in[0,T]}$ on $X := \mathrm{L}^2(0,\pi)$ by $A(t) = \alpha(t)A_0$ with dense domain $\mathrm{D}(A(t)) = \mathrm{D}(A_0) =: D$, $t \in [0,T]$, where

$$A_0 f := f'', \quad \mathrm{D}(A_0) := \big\{ f \in \mathrm{H}^2(0,\pi) : f(0) = f(\pi) = 0 \big\}$$

is the Dirichlet Laplacian. In particular, Assumption 2.1 is satisfied. Since α is continuous, there exists $C > 0$ such that $\alpha(t) \leq C$ for all $t \in [0,T]$, and therefore by the assumption $\alpha \geq 1$, we observe

$$\|\cdot\|_{A_0} \leq \|\cdot\|_{A(t)} \leq C\|\cdot\|_{A_0}$$

for all $t \in [0,T]$, which implies that Assumption 3.1 holds. Moreover, as α is continuously differentiable, Assumption 3.2 is satisfied as well.

Furthermore, we introduce a family of bounded operators $(B(t))_{t\in[0,T]}$ by

$$B(t)f := \beta(t,\cdot)f, \quad t \in [0,T]$$

Then (5.1) has an abstract form of a non-autonomous second-order abstract Cauchy problem

$$\begin{cases} \ddot{u}(t) = (A(t) + B(t))u(t), & t \in (0,T], \\ u(0) = \varphi, \\ \dot{u}(0) = \psi. \end{cases} \tag{5.2}$$

As elaborated by Henríquez and Pozo in [15], for $n \in \mathbb{N}$ let $z_n\colon [0,\pi] \to \mathbb{R}$, $z_n(\xi) := \sqrt{\frac{2}{\pi}}\sin(n\xi)$. Then $(z_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathrm{L}^2(0,\pi)$ of eigenfunctions of A_0 corresponding to the sequence of eigenvalues $(-n^2)_{n\in\mathbb{N}}$ and the family of bounded linear operators $(S(t,s))_{(t,s)\in\Delta}$ on $\mathrm{L}^2(0,\pi)$ defined by

$$S(t,s)x := \sum_{n=1}^{\infty} r_n(t,s)\,\langle x, z_n\rangle\, z_n,$$

provides a fundamental solution to the second-order non-autonomous abstract Cauchy problem associated to $(A(t), \mathrm{D}(A(t)))_{t\in[0,T]}$, where the functions r_n denote the solution of the initial value problem

$$\begin{cases} r''(t) + n^2\alpha(t)r(t) = 0, \quad 0 \le s \le t \le T, \\ r(s) = 0, \\ r'(s) = 1. \end{cases} \tag{5.3}$$

Note that we have $|r_n(t,s)| \le \frac{1}{\sqrt{\alpha(s)}n} \le \frac{1}{n}$, $\left|\frac{\partial}{\partial t} r_n(t,s)\right| \le 1$, $\left|\frac{\partial}{\partial s} r_n(t,s)\right| \le 1$ and $\left|\frac{\partial}{\partial t}\frac{\partial}{\partial s} r_n(t,s)\right| \le n$ for all $(t,s) \in \Delta$ and $n \in \mathbb{N}$; cf. [15, (5.12)] as well as [34]. By spectral theory,

$$D = \Big\{x \in \mathrm{L}^2(0,\pi) : \sum_{n=1}^{\infty} n^4 \langle x, z_n\rangle^2 < \infty\Big\}.$$

Let

$$\begin{aligned} Z := \mathrm{H}_0^1(0,\pi) &= \big\{f \in \mathrm{H}^1(0,\pi) : f(0) = f(\pi) = 0\big\} \\ &= \Big\{x \in \mathrm{L}^2(0,\pi) : \sum_{n=1}^{\infty} n^2 \langle x, z_n\rangle^2 < \infty\Big\}. \end{aligned}$$

The estimates on the r_n imply that the hypotheses on $(S(t,s))_{(t,s)\in\Delta}$ in Theorem 4.2 are satisfied, due to uniform convergence of the series representations. Note that these hypotheses are the same as those in Theorem 3.4((a)).

Since $\beta \in \mathrm{C}^1([0,T] \times [0,\pi])$ we easily obtain $B(\cdot) \in \mathrm{C}([0,T];\mathscr{L}_{\mathrm{s}}(X)) \cap \mathrm{C}([0,T];\mathscr{L}_{\mathrm{s}}(Z))$. Thus, Theorem 4.2 yields a fundamental solution to (5.2).

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References

[1] P. Acquistapace and B. Terreni, On the abstract nonautonomous parabolic Cauchy problem in the case of constant domains, *Ann. Mat. Pura Appl. (4)*, **140** (1985), 1–55.

[2] P. Acquistapace and B. Terreni, A unified approach to abstract linear nonautonomous parabolic equations, *Rend. Sem. Mat. Univ. Padova*, **78** (1987), 47–107.

[3] H. Amann, Parabolic evolution equations in interpolation and extrapolation spaces, *J. Funct. Anal.*, **78** (1988), 233–270.

[4] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, vol. 96, Birkhäuser/Springer Basel AG (Basel, 2011).

[5] S. Bertoni, Multiplicative perturbations of constant-domain evolution equations, *J. Evol. Equ.*, **5** (2005), 291–316.

[6] S. Bertoni, Stability of CD-systems under perturbations in the Favard class, *Mediterr. J. Math.*, **11** (2014), 1195–1204.

[7] J. Bochenek, Existence of the fundamental solution of a second order evolution equation, *Ann. Polon. Math.*, **66** (1997), 15–35 (volume dedicated to the memory of Włodzimierz Mlak).

[8] G. Da Prato and E. Giusti, Una caratterizzazione dei generatori di funzioni coseno astratte, *Boll. Un. Mat. Ital. (3)*, **22** (1967), 357–362.

[9] K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag (New York, 2000).

[10] D. E. Evans, Time dependent perturbations and scattering of strongly continuous groups on Banach spaces, *Math. Ann.*, **221** (1976), 275–290.

[11] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Mathematics Studies, vol. 108, North-Holland Publishing Co. (Amsterdam, 1985).

[12] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Dover Publications, Inc. (Mineola, NY, 2017).

[13] H. R. Henríquez, Existence of solutions of non-autonomous second order functional differential equations with infinite delay, *Nonlinear Anal.*, **74** (2011), 3333–3352.

[14] H. R. Henríquez, Existence of solutions of the nonautonomous abstract Cauchy problem of second order, *Semigroup Forum*, **87** (2013), 277–297.

[15] H. R. Henríquez and J. C. Pozo, Existence of solutions of abstract non-autonomous second order integro-differential equations, *Bound. Value Probl.* (2016), Paper No. 168, 24 pp.

[16] J. S. Howland, Stationary scattering theory for time-dependent Hamiltonians, *Math. Ann.*, **207** (1974), 315–335.

[17] T. Kato, Integration of the equation of evolution in a Banach space, *J. Math. Soc. Japan*, **5** (1953), 208–234.

[18] T. Kato, Abstract evolution equations of parabolic type in Banach and Hilbert spaces, *Nagoya Math. J.*, **19** (1961), 93–125.

[19] T. Kato, Linear evolution equations of “hyperbolic” type, *J. Fac. Sci. Univ. Tokyo Sect. I*, **17** (1970), 241–258.

[20] T. Kato, Linear evolution equations of “hyperbolic” type. II, *J. Math. Soc. Japan*, **25** (1973), 648–666.

[21] J. Kisyński, On cosine operator functions and one-parameter groups of operators, *Studia Math.*, **44** (1972), 93–105.

[22] M. Kozak, An abstract linear second-order temporally inhomogeneous differential equation. I, *Univ. Iagel. Acta Math.* (1994), no. 31, 21–30.

[23] M. Kozak, An abstract second-order temporally inhomogeneous linear differential equation. II, *Univ. Iagel. Acta Math.* (1995), no. 32, 263–274.

[24] M. Kozak, A fundamental solution of a second-order differential equation in a Banach space, *Univ. Iagel. Acta Math.* (1995), no. 32, 275–289.

[25] N. T. Lan, On nonautonomous second-order differential equations on Banach space, *Int. J. Math. Math. Sci.*, **26** (2001), 45–54.

[26] Y. Lin, Time-dependent perturbation theory for abstract evolution equations of second order, *Studia Math.*, **130** (1998), 263–274.

[27] W. Littman, The wave operator and L_p norms, *J. Math. Mech.*, **12** (1963), 55–68.

[28] D. Lutz, On bounded time-dependent perturbations of operator cosine functions, *Aequationes Math.*, **23** (1981), 197–203, 1981.

[29] I. V. Melnikova and A. Filinkov, *Abstract Cauchy Problems: Three Approaches*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 120, Chapman & Hall/CRC (Boca Raton, FL, 2001).

[30] R. Nagel, Semigroup methods for nonautonomous Cauchy problems, in: *Evolution Equations* (Baton Rouge, LA, 1992), Lecture Notes in Pure and Appl. Math., vol. 168, Dekker (New York, 1995), pp. 301–316.

[31] R. Nagel and A. Rhandi, A characterization of Lipschitz continuous evolution families on Banach spaces, in: *Operator Theory in Function spaces and Banach Lattices*, Oper. Theory Adv. Appl., vol. 75, Birkhäuser (Basel, 1995), pp. 275–288.

[32] F. Neubrander, Well-posedness of higher order abstract Cauchy problems, *Trans. Amer. Math. Soc.*, **295** (1986), 257–290.

[33] G. Nickel, Evolution semigroups for nonautonomous Cauchy problems, *Abstr. Appl. Anal.*, **2** (1997), 73–95.

[34] S. Noyo, R. Obaya, and J. Rojo, *Ecuaciones y Sistemas Diferenciales*, McGraw-Hill (Madrid, 1995).

[35] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, *Applied Math. Sci.*,vol. 44, Springer-Verlag (New York, 1983).

[36] S. Piskarev and S.-Y. Shaw, Perturbation and comparison of cosine operator functions, *Semigroup Forum*, **51** (1995), 225–246.

[37] S. Piskarev and S.-Y. Shaw, On certain operator families related to cosine operator functions, *Taiwanese J. Math.*, **1** (1997), 527–546.

[38] F. Räbiger, A. Rhandi, and R. Schnaubelt, Perturbation and an abstract characterization of evolution semigroups, *J. Math. Anal. Appl.*, **198(2)** (1996), 516–533.

[39] F. Räbiger and R. Schnaubelt, The spectral mapping theorem for evolution semigroups on spaces of vector-valued functions, *Semigroup Forum*, **52(2)** (1996), 225–239.

[40] F. Räbiger, R. Schnaubelt, A. Rhandi, and J. Voigt, Non-autonomous Miyadera perturbations, *Differential Integral Equations*, **13** (2000), 341–368.

[41] R. Schnaubelt, Asymptotic behaviour of parabolic nonautonomous evolution equations, in: *Functional Analytic Methods for Evolution Equations*, Lecture Notes in Math., vol. 1855, Springer (Berlin, 2004), pp. 401–472.

[42] H. Serizawa and M. Watanabe, Time-dependent perturbation for cosine families in Banach spaces, *Houston J. Math.*, **12** (1986), 579–586.

[43] M. Shimizu and I. Miyadera, Perturbation theory for cosine families on Banach spaces, *Tokyo J. Math.*, **1** (1978), 333–343.

[44] M. Sova, Cosine operator functions, *Rozprawy Mat.*, **49** (1966), 47 pp.

[45] T. Takenaka and N. Okazawa, A Phillips–Miyadera type perturbation theorem for cosine functions of operators, *Tohoku Math. J. (2)*, **30** (1978), 107–115.

[46] H. Tanabe, On the equations of evolution in a Banach space, *Osaka Math. J.*, **12** (1960), 363–376.

[47] H. Tanabe, *Equations of Evolution*, Monographs and Studies in Mathematics, vol. 6, Pitman (Advanced Publishing Program) (Boston, Mass., London, 1979).

[48] C. C. Travis and G. F. Webb, Perturbation of strongly continuous cosine family generators, *Colloq. Math.*, **45** (1981), 277–285.

[49] V. V. Vasilev and S. I. Piskarev, Differential equations in Banach spaces. II. Theory of cosine operator functions, *J. Math. Sci. (N.Y.)*, **122** (2004), 3055–3174.

[50] M. Veraar and J. Zimmerschied, Non-autonomous stochastic Cauchy problems in Banach spaces, *Studia Math.*, **185** (2008), 1–34.

[51] T. Winiarska, Second order Cauchy problem with a damping operator, *Univ. Iagel. Acta Math.* (2005), no. 43, 11–20.

[52] T. Winiarska, Second order evolution problem with dependent on t and not densely defined operators, *Demonstratio Math.*, **42** (2009), 513–520.

[53] T.-J. Xiao and J. Liang, Higher order abstract Cauchy problems: their existence and uniqueness families, *J. London Math. Soc. (2)*, **67** (2003), 149–164.

[54] T. Xio and L. Jin, On complete second order linear differential equations in Banach spaces, *Pacific J. Math.*, **142** (1990), 175–195.