Chapter 2
Unbounded Operators

We will gather some information on operators in Banach and Hilbert spaces. Throughout this chapter let X_0, X_1, and X_2 be Banach spaces and H_0, H_1, and H_2 be Hilbert spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

2.1 Operators in Banach Spaces

We define the set of continuous linear operators

$$L(X_0, X_1) := \left\{ B \colon X_0 \to X_1 \, ; \; B \text{ linear}, \; \|B\| := \sup_{x \in X_0 \setminus \{0\}} \frac{\|Bx\|}{\|x\|} < \infty \right\}$$

with the usual abbreviation $L(X_0) := L(X_0, X_0)$. In contrast to a bounded linear operator, a discontinuous or unbounded linear operator only needs to be defined on a proper albeit possibly dense subset of X_0. In order to define unbounded linear operators, we will first take a more general point of view and introduce (linear) relations. This perspective will turn out to be the natural setting later on.

Definition A subset $A \subseteq X_0 \times X_1$ is called a *relation in* X_0 *and* X_1. We define the *domain*, *range* and *kernel of* A as follows

$$\begin{aligned} \operatorname{dom}(A) &:= \{x \in X_0 \, ; \; \exists y \in X_1 \colon (x, y) \in A\}, \\ \operatorname{ran}(A) &:= \{y \in X_1 \, ; \; \exists x \in X_0 \colon (x, y) \in A\}, \\ \ker(A) &:= \{x \in X_0 \, ; \; (x, 0) \in A\}. \end{aligned}$$

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The *image, $A[M]$, of a set $M \subseteq X_0$ under A* is given by

$$A[M] := \{y \in X_1\,;\ \exists x \in M\colon (x, y) \in A\}\,.$$

A relation A is called *bounded* if for all bounded $M \subseteq X_0$ the set $A[M] \subseteq X_1$ is bounded. For a given relation A we define the *inverse relation*

$$A^{-1} := \{(y, x) \in X_1 \times X_0\,;\ (x, y) \in A\}\,.$$

A relation A is called *linear* if $A \subseteq X_0 \times X_1$ is a linear subspace. A linear relation A is called *linear operator* or just *operator from X_0 to X_1* if

$$A[\{0\}] = \{y \in X_1\,;\ (0, y) \in A\} = \{0\}.$$

In this case, we also write

$$A\colon\ \operatorname{dom}(A) \subseteq X_0 \to X_1$$

to denote a linear operator from X_0 to X_1. Moreover, we shall write $Ax = y$ instead of $(x, y) \in A$ in this case. A linear operator A, which is not bounded, is called *unbounded*.

For completeness, we also define the sum, scalar multiples, and composition of relations.

Definition Let $A \subseteq X_0 \times X_1$, $B \subseteq X_0 \times X_1$ and $C \subseteq X_1 \times X_2$ be relations, $\lambda \in \mathbb{K}$. Then we define

$$\begin{aligned} A + B &:= \{(x, y + w) \in X_0 \times X_1\,;\ (x, y) \in A, (x, w) \in B\}\,, \\ \lambda A &:= \{(x, \lambda y) \in X_0 \times X_1\,;\ (x, y) \in A\}\,, \\ CA &:= \{(x, z) \in X_0 \times X_2\,;\ \exists y \in X_1\colon (x, y) \in A, (y, z) \in C\}\,. \end{aligned}$$

For a relation $A \subseteq X_0 \times X_1$ we will use the abbreviation $-A := -1A$ (so that the minus sign only acts on the second component). We now proceed with topological notions for relations.

Definition Let $A \subseteq X_0 \times X_1$ be a relation. A is called *densely defined* if $\operatorname{dom}(A)$ is dense in X_0. We call *A closed* if A is a closed subset of the direct sum of the Banach spaces X_0 and X_1. If A is a linear operator then we will call A *closable*, whenever $\overline{A} \subseteq X_0 \times X_1$ is a linear operator.

Proposition 2.1.1 *Let $A \subseteq X_0 \times X_1$ be a relation, $C \in L(X_2, X_0)$ and $B \in L(X_0, X_1)$. Then the following statements hold.*

(a) *A is closed if and only if A^{-1} is closed. Moreover, we have $(\overline{A})^{-1} = \overline{A^{-1}}$.*
(b) *A is closed if and only if $A + B$ is closed.*
(c) *If A is closed, then AC is closed.*

Proof Statement (a) follows upon realising that $X_0 \times X_1 \ni (x, y) \mapsto (y, x) \in X_1 \times X_0$ is an isomorphism.

For statement (b), it suffices to show that the closedness of A implies the same for $A + B$. Let $((x_n, y_n))_n$ be a sequence in $A + B$ convergent in $X_0 \times X_1$ to some (x, y). Since $B \in L(X_0, X_1)$, it follows that $((x_n, y_n - Bx_n))_n$ in A is convergent to $(x, y - Bx)$ in $X_0 \times X_1$. Since A is closed, $(x, y - Bx) \in A$. Thus, $(x, y) \in A + B$.

For statement (c), let $((w_n, y_n))_n$ be a sequence in AC convergent in $X_2 \times X_1$ to some (w, y). Since C is continuous, $(Cw_n)_n$ converges to Cw. Hence, $(Cw_n, y_n) \to (Cw, y)$ in $X_0 \times X_1$ and since $(Cw_n, y_n) \in A$ and A is closed, it follows that $(Cw, y) \in A$. Equivalently, $(w, y) \in AC$, which yields closedness of AC. □

We shall gather some other elementary facts about closed operators in the following. We will make use of the following notion.

Definition Let $A \colon \operatorname{dom}(A) \subseteq X_0 \to X_1$ be a linear operator. Then the *graph norm* of A is defined by $\operatorname{dom}(A) \ni x \mapsto \|x\|_A := \sqrt{\|x\|^2 + \|Ax\|^2}$.

Lemma 2.1.2 *Let $A \colon \operatorname{dom}(A) \subseteq X_0 \to X_1$ be a linear operator. Then the following statements are equivalent:*

(i) *A is closed.*
(ii) $\operatorname{dom}(A)$ *equipped with the graph norm is a Banach space.*
(iii) *For all $(x_n)_n$ in* $\operatorname{dom}(A)$ *convergent in X_0 such that $(Ax_n)_n$ is convergent in X_1 we have* $\lim_{n\to\infty} x_n \in \operatorname{dom}(A)$ *and* $A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n$.

Proof For the equivalence (i)⇔(ii), it suffices to observe that $\operatorname{dom}(A) \ni x \mapsto (x, Ax) \in A$, where $\operatorname{dom}(A)$ is endowed with the graph norm, is an isomorphism. The equivalence (i)⇔(iii) is an easy reformulation of the definition of closedness of $A \subseteq X_0 \times X_1$. □

Unless explicitly stated otherwise (e.g. in the form $\operatorname{dom}(A) \subseteq X_0$, where we regard $\operatorname{dom}(A)$ as a subspace of X_0), for closed operators A we always consider $\operatorname{dom}(A)$ as a Banach space in its own right; that is, we shall regard it as being endowed with the graph norm.

Lemma 2.1.3 *Let $A \colon \operatorname{dom}(A) \subseteq X_0 \to X_1$ be a closed linear operator. Then A is bounded if and only if* $\operatorname{dom}(A) \subseteq X_0$ *is closed.*

Proof First of all note that boundedness of A is equivalent to the fact that the graph norm and the X_0-norm on $\mathrm{dom}(A)$ are equivalent. Hence, the closedness and boundedness of A implies that $\mathrm{dom}(A) \subseteq X_0$ is closed. On the other hand, the embedding

$$\iota\colon (\mathrm{dom}(A), \|\cdot\|_A) \hookrightarrow (\mathrm{dom}(A), \|\cdot\|_{X_0})$$

is continuous and bijective. Since the range is closed, the open mapping theorem implies that ι^{-1} is continuous. This yields the equivalence of the graph norm and the X_0-norm and, thus, the boundedness of A. □

For unbounded operators, obtaining a precise description of the domain may be difficult. However, there may be a subset of the domain which essentially (or approximately) describes the operator. This gives rise to the following notion of a core.

Definition Let $A \subseteq X_0 \times X_1$. A set $D \subseteq \mathrm{dom}(A)$ is called a *core for A* provided $\overline{A \cap (D \times X_1)} = \overline{A}$.

Proposition 2.1.4 *Let $A \in L(X_0, X_1)$, and $D \subseteq X_0$ a dense linear subspace. Then D is a core for A.*

Corollary 2.1.5 *Let $A\colon \mathrm{dom}(A) \subseteq X_0 \to X_1$ be a densely defined, bounded linear operator. Then there exists a unique $B \in L(X_0, X_1)$ with $B \supseteq A$. In particular, we have $B = \overline{A}$ and*

$$\|B\| = \sup_{x \in \mathrm{dom}(A), x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

The proofs of Proposition 2.1.4 and Corollary 2.1.5 are asked for in Exercise 2.2.

2.2 Operators in Hilbert Spaces

Let us now focus on operators on Hilbert spaces. In this setting, we can additionally make use of scalar products $\langle \cdot, \cdot \rangle$, which in this course are considered to be linear in the second argument (and anti-linear in the first, in the case when $\mathbb{K} = \mathbb{C}$).

For a linear operator $A\colon \mathrm{dom}(A) \subseteq H_0 \to H_1$ the graph norm of A is induced by the scalar product

$$(x, y) \mapsto \langle x, y \rangle + \langle Ax, Ay \rangle\,,$$

known as the *graph scalar product of A*. If A is closed then $\mathrm{dom}(A)$ (equipped with the graph norm) is a Hilbert space.

Of course, no presentation of operators in Hilbert spaces would be complete without the central notion of the adjoint operator. We wish to pose the adjoint within the relational framework just established. The definition is as follows.

Definition For a relation $A \subseteq H_0 \times H_1$ we define the *adjoint relation* A^* by

$$A^* := -\left(\left(A^{-1}\right)^{\perp}\right) \subseteq H_1 \times H_0,$$

where the orthogonal complement is computed in the direct sum of the Hilbert spaces H_1 and H_0; that is, the set $H_1 \times H_0$ endowed with the scalar product $\big((x, y), (u, v)\big) \mapsto \langle x, u\rangle_{H_1} + \langle y, v\rangle_{H_0}$.

Remark 2.2.1 Let $A \subseteq H_0 \times H_1$. Then we have

$$A^* = \left\{(u, v) \in H_1 \times H_0\,;\ \forall (x, y) \in A : \langle u, y\rangle_{H_1} = \langle v, x\rangle_{H_0}\right\}.$$

In particular, if A is a linear operator, we have

$$A^* = \left\{(u, v) \in H_1 \times H_0\,;\ \forall x \in \operatorname{dom}(A) : \langle u, Ax\rangle_{H_1} = \langle v, x\rangle_{H_0}\right\}.$$

Lemma 2.2.2 *Let $A \subseteq H_0 \times H_1$ be a relation. Then A^* is a linear relation. Moreover, we have*

$$A^* = -\left(\left(A^{\perp}\right)^{-1}\right) = \left((-A)^{-1}\right)^{\perp} = \left(-\left(A^{-1}\right)\right)^{\perp} = \left((-A)^{\perp}\right)^{-1} = \left(-\left(A^{\perp}\right)\right)^{-1}.$$

The proof of this lemma is left as Exercise 2.3.

Remark 2.2.3 Let $A \subseteq H_0 \times H_1$. Since A^* is the orthogonal complement of $-A^{-1}$, it follows immediately that A^* is closed. Moreover, $A^* = \left(\overline{A}\right)^*$ since $A^{\perp} = \left(\overline{A}\right)^{\perp}$.

Lemma 2.2.4 *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then*

$$A^{**} := \left(A^*\right)^* = \overline{A}.$$

Proof We compute using Lemma 2.2.2

$$A^{**} = \left(\left(-\left(A^*\right)\right)^{-1}\right)^{\perp} = \left(\left(-\left(-\left(\left(A^{\perp}\right)^{-1}\right)\right)\right)^{-1}\right)^{\perp} = \left(A^{\perp}\right)^{\perp} = \overline{A}.$$ □

Theorem 2.2.5 *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then*

$$\operatorname{ran}(A)^{\perp} = \ker(A^*) \quad \textit{and} \quad \overline{\operatorname{ran}}(A^*) = \ker(\overline{A})^{\perp}.$$

Proof Let $u \in \ker(A^*)$ and let $y \in \operatorname{ran}(A)$. Then we find $x \in \operatorname{dom}(A)$ such that $(x, y) \in A$. Moreover, note that $(u, 0) \in A^*$. Then, we compute

$$\langle u, y \rangle_{H_1} = \langle 0, x \rangle_{H_0} = 0.$$

This equality shows that $\operatorname{ran}(A)^\perp \supseteq \ker(A^*)$. If on the other hand, $u \in \operatorname{ran}(A)^\perp$ then for all $(x, y) \in A$ we have that

$$0 = \langle u, y \rangle_{H_1} ,$$

which implies $(u, 0) \in A^*$ and hence $u \in \ker(A^*)$. The remaining equation follows from Lemma 2.2.4 together with the first equation applied to A^*. □

The following decomposition result is immediate from the latter theorem and will be used frequently throughout the text.

Corollary 2.2.6 *Let $A \subseteq H_0 \times H_1$ be a closed linear relation. Then*

$$H_1 = \overline{\operatorname{ran}}(A) \oplus \ker(A^*) \quad \text{and} \quad H_0 = \ker(A) \oplus \overline{\operatorname{ran}}(A^*).$$

We will now turn to the case where the adjoint relation is actually a linear operator.

Lemma 2.2.7 *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then A^* is a linear operator if and only if A is densely defined. If, in addition, A is a linear operator, then A is closable if and only if A^* is densely defined.*

Proof For the first equivalence, it suffices to observe that

$$A^*[\{0\}] = \operatorname{dom}(A)^\perp. \tag{2.1}$$

Indeed, A being densely defined is equivalent to having $\operatorname{dom}(A)^\perp = \{0\}$. Moreover, A^* is an operator if and only if $A^*[\{0\}] = \{0\}$. Next, we show (2.1). For this, apply Theorem 2.2.5 to the linear relation A^{-1}. One obtains $(\operatorname{ran} A^{-1})^\perp = \ker(A^{-1})^*$. Hence, $(\operatorname{dom}(A))^\perp = \ker(A^*)^{-1} = A^*[\{0\}]$, which is (2.1). For the remaining equivalence, we need to characterise $\overline{A}$ being an operator. Using Lemma 2.2.4 and the first equivalence, we deduce that $\overline{A} = (A^*)^*$ is a linear operator if and only if A^* is densely defined. □

Remark 2.2.8 Note that the statement "A^* is an operator if A is densely defined" asserted in Lemma 2.2.7 is also true for *any* relation. For this, it suffices to observe that (2.1) is true for any relation $A \subseteq H_0 \times H_1$. Indeed, let $A \subseteq H_0 \times H_1$ be a relation; define $B := \operatorname{lin} A$. Then $\operatorname{dom}(B) = \operatorname{lin} \operatorname{dom}(A)$. Also, we have

$$A^* = -(A^\perp)^{-1} = -(B^\perp)^{-1} = B^*.$$

With these preparations, we can write

$$\operatorname{dom}(A)^{\perp} = (\operatorname{lin}\operatorname{dom}(A))^{\perp} = \operatorname{dom}(B)^{\perp} = B^{*}[\{0\}] = A^{*}[\{0\}],$$

where we used that (2.1) holds for linear relations.

Lemma 2.2.9 *Let $A \subseteq H_0 \times H_1$ be a linear relation. Then $\overline{A} \in L(H_0, H_1)$ if and only if $A^* \in L(H_1, H_0)$. In either case, $\|A^*\| = \|\overline{A}\|$.*

Proof Note that $\overline{A} \in L(H_0, H_1)$ implies that A is closable and densely defined. Thus, by Lemma 2.2.7, A^* is a densely defined, closed linear operator. For $u \in \operatorname{dom}(A^*)$ we compute using Lemma 2.2.4

$$\left\|A^{*}u\right\| = \sup_{x \in H_0 \setminus \{0\}} \frac{|\langle A^{*}u, x\rangle|}{\|x\|} = \sup_{x \in H_0 \setminus \{0\}} \frac{\left|\langle u, \overline{A}x\rangle\right|}{\|x\|} \leqslant \left\|\overline{A}\right\| \|u\|,$$

yielding $\|A^*\| \leqslant \|\overline{A}\|$. On the one hand, this implies that A^* is bounded, and on the other, since A^* is densely defined we deduce $A^* \in L(H_1, H_0)$ by Lemma 2.1.3. The other implication (and the other inequality) follows from the first one applied to A^* instead of A using $A^{**} = \overline{A}$. □

We end this section by defining some special classes of relations and operators.

Definition Let H be a Hilbert space and $A \subseteq H \times H$ a linear relation. We call A *(skew-)Hermitian* if $A \subseteq A^*$ ($A \subseteq -A^*$). We say that A is *(skew-)symmetric* if A is (skew-)Hermitian and densely defined (so that A^* is a linear operator), and A is called *(skew-)selfadjoint* if $A = A^*$ ($A = -A^*$). Additionally, if A is densely defined, then we say that A is *normal* if $AA^* = A^*A$.

2.3 Computing the Adjoint

In general it is a very difficult task to compute the adjoint of a given (unbounded) operator. There are, however, cases, where the adjoint of a sum or the product can be computed more readily. We start with the most basic case of bounded linear operators.

Proposition 2.3.1 *Let $A, B \in L(H_0, H_1), C \in L(H_2, H_0)$. Then $(A + B)^* = A^* + B^*$ and $(AC)^* = C^*A^*$.*

The latter results are special cases of more general statements to follow.

Theorem 2.3.2 *Let $A, B \subseteq H_0 \times H_1$ be relations. Then $A^* + B^* \subseteq (A + B)^*$. If, in addition, $B \in L(H_0, H_1)$, then $(A + B)^* = A^* + B^*$.*

Proof In order to show the claimed inclusion, let $(u, r) \in A^* + B^*$. By definition of the sum of relations, we find $v, w \in H_0$, $r = v + w$, with $(u, v) \in A^*$ and

$(u, w) \in B^*$. We compute for all $(x, s) \in A + B$, that is, $(x, y) \in A$ and $(x, z) \in B$ for some $y, z \in H_1$ with $s = y + z$

$$\begin{aligned}\langle x, r\rangle_{H_0} &= \langle x, v + w\rangle_{H_0} = \langle x, v\rangle_{H_0} + \langle x, w\rangle_{H_0} \\ &= \langle y, u\rangle_{H_1} + \langle z, u\rangle_{H_1} = \langle y + z, u\rangle_{H_1} = \langle s, u\rangle_{H_1} .\end{aligned}$$

This shows the desired inclusion. Next, we assume in addition that $B \in L(H_0, H_1)$. For the equality, it remains to show that $(A+B)^* \subseteq A^* + B^*$, which in conjunction with the above follows if $\mathrm{dom}((A+B)^*) \subseteq \mathrm{dom}(A^*+B^*) = \mathrm{dom}(A^*)\cap\mathrm{dom}(B^*)$. By Lemma 2.2.9, we have $\mathrm{dom}(B^*) = H_1$. Hence, it suffices to show that $\mathrm{dom}((A+B)^*) \subseteq \mathrm{dom}(A^*)$. For this, let $(u, v) \in (A + B)^*$. Then we compute for all $(x, y) \in A$ using Lemma 2.2.9 again

$$\langle x, v\rangle_{H_0} = \langle y + Bx, u\rangle_{H_1} = \langle y, u\rangle_{H_1} + \left\langle x, B^*u\right\rangle_{H_0} .$$

Thus, $\langle x, v - B^*u\rangle_{H_0} = \langle y, u\rangle_{H_1}$, which yields $(u, v - B^*u) \in A^*$; whence, $u \in \mathrm{dom}(A^*)$ as desired. □

Corollary 2.3.3 *Let $A \subseteq H_0 \times H_1$, $B \in L(H_0, H_1)$. If A is densely defined, then $A^* + B^*$ is an operator and $(A + B)^* = A^* + B^*$.*

Theorem 2.3.4 *Let $A \subseteq H_0 \times H_1$ and $C \subseteq H_2 \times H_0$. Then $\overline{C^*A^*} \subseteq (AC)^*$. If, in addition, $A \subseteq H_0 \times H_1$ is closed and linear as well as $C \in L(H_2, H_0)$, then $(AC)^* = \overline{C^*A^*}$.*

Proof For the first inclusion, let $(u, w) \in C^*A^*$. Thus, we find $v \in H_0$ such that $(u, v) \in A^*$ and $(v, w) \in C^*$. Next, let $(r, y) \in AC$. Then we find $x \in H_0$ such that $(r, x) \in C$ and $(x, y) \in A$. We compute

$$\langle y, u\rangle_{H_1} = \langle x, v\rangle_{H_0} = \langle r, w\rangle_{H_2} .$$

Since $(r, y) \in AC$ were chosen arbitrarily, we infer $C^*A^* \subseteq (AC)^*$. As every adjoint is closed, we obtain $\overline{C^*A^*} \subseteq (AC)^*$.

Next, we assume that A is closed and linear as well as that C is bounded and linear. Then, by what we have just shown, we obtain $AC \subseteq (C^*A^*)^*$. Next, let $(w, y) \in (C^*A^*)^*$. Then for all $(u, v) \in A^*$ and $z = C^*v$ we obtain

$$\langle u, y\rangle_{H_1} = \langle z, w\rangle_{H_2} = \left\langle C^*v, w\right\rangle_{H_2} = \langle v, Cw\rangle_{H_0} .$$

Thus, we obtain $(Cw, y) \in A^{**} = \overline{A} = A$. Thus, $(w, y) \in AC$. Hence,

$$AC = \left(C^*A^*\right)^* ,$$

which yields the assertion by adjoining this equation. □

Corollary 2.3.5 *Let $A \subseteq H_0 \times H_1$ be a linear relation and $C \in L(H_2, H_0)$. Then $\left(\overline{A}C\right)^* = \overline{C^*A^*}$.*

Proof The result follows upon realising that $A^* = A^{***} = \left(\overline{A}\right)^*$. □

Corollary 2.3.6 *Let $A \subseteq H_0 \times H_1$ be a linear relation and $C \in L(H_2, H_0)$. If $\overline{A}C$ is densely defined, then C^*A^* is a closable linear operator with $\overline{C^*A^*} = \left(\overline{A}C\right)^*$.*

Remark 2.3.7 Let us comment on the equalities in the prevoius statements.

(a) Note that if $B \in L(H_1, H_2)$ and $A \subseteq H_0 \times H_1$ is linear, then $\left(B\overline{A}\right)^* = A^*B^*$. Indeed, this follows from Theorem 2.3.4 applied to A^* and B instead of A and C^*, respectively, since then we obtain $(A^*B^*)^* = \overline{B^{**}A^{**}} = \overline{B\overline{A}}$. Computing adjoints on both sides again and using that A^*B^* is closed by Proposition 2.1.1, we get the assertion.
(b) We note here that in Corollary 2.3.5 and Corollary 2.3.6 $\overline{A}C$ cannot be replaced by $\overline{AC}$ and encourage the reader to find a counterexample for A being a closable linear operator. We also refer to [94] for a counterexample due to J. Epperlein.

We have already seen that $A^* = \overline{A}^*$. We can even restrict A to a core and still obtain the same adjoint.

Proposition 2.3.8 *Let $A \subseteq H_0 \times H_1$ be a linear relation, $D \subseteq \operatorname{dom}(A)$ a linear subspace. Then D is a core for A if and only if $(A \cap (D \times H_1))^* = A^*$.*

Proof We set $A|_D := A \cap (D \times H_1)$. Then

$$D \text{ core} \iff \overline{A|_D} = \overline{A} \iff \overline{A|_D}^{\perp} = \overline{A}^{\perp} \iff A|_D{}^{\perp} = A^{\perp} \iff A|_D^* = A^*. \quad \square$$

2.4 The Spectrum and Resolvent Set

In this section, we focus on operators acting on a single Banach space. As such, throughout this section let X be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $A\colon \operatorname{dom}(A) \subseteq X \to X$ be a closed linear operator.

Definition The set

$$\rho(A) := \left\{\lambda \in \mathbb{K}\,;\ (\lambda - A)^{-1} \in L(X)\right\}$$

is called the *resolvent set* of A. We define

$$\sigma(A) := \mathbb{K} \setminus \rho(A)$$

to be the *spectrum* of A.

We state and prove some elementary properties of the spectrum and the resolvent set. We shall see natural examples for A which satisfy that $\sigma(A) = \mathbb{K}$ or $\sigma(A) = \varnothing$ later on.

For a metric space (X, d), we will write $B(x, r) = \{y \in X ; d(x, y) < r\}$ for the open ball around x of radius r and $B[x, r] = \{y \in X ; d(x, y) \leqslant r\}$ for the closed ball.

Proposition 2.4.1 *If $\lambda, \mu \in \rho(A)$, then the* resolvent identity *holds. That is*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}.$$

Moreover, the set $\rho(A)$ is open. More precisely, if $\lambda \in \rho(A)$ then $B\left(\lambda, 1/\left\|(\lambda - A)^{-1}\right\|\right) \subseteq \rho(A)$ and for $\mu \in B\left(\lambda, 1/\left\|(\lambda - A)^{-1}\right\|\right)$ we have

$$(\mu - A)^{-1} = \sum_{k=0}^{\infty} (\lambda - \mu)^k \left((\lambda - A)^{-1}\right)^{k+1}$$

as well as

$$\left\|(\mu - A)^{-1}\right\| \leqslant \frac{\left\|(\lambda - A)^{-1}\right\|}{1 - |\lambda - \mu| \left\|(\lambda - A)^{-1}\right\|}.$$

The mapping $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in L(X)$ is analytic.

Proof For the first assertion, we let $\lambda, \mu \in \rho(A)$ and compute

$$\begin{aligned}
(\lambda - A)^{-1} - (\mu - A)^{-1} &= (\lambda - A)^{-1}\big((\mu - A) - (\lambda - A)\big)(\mu - A)^{-1} \\
&= (\lambda - A)^{-1}(\mu - \lambda)(\mu - A)^{-1} \\
&= (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}.
\end{aligned}$$

Next, let $\lambda \in \rho(A)$ and $\mu \in B\left(\lambda, 1/\left\|(\lambda - A)^{-1}\right\|\right)$. Then

$$\left\|(\lambda - \mu)(\lambda - A)^{-1}\right\| < 1.$$

Hence, $1 - (\lambda - \mu)(\lambda - A)^{-1}$ admits an inverse in $L(X)$ satisfying

$$\left(1 - (\lambda - \mu)(\lambda - A)^{-1}\right)^{-1} = \sum_{k=0}^{\infty} \left((\lambda - \mu)(\lambda - A)^{-1}\right)^k. \tag{2.2}$$

We claim that $\mu \in \rho(A)$. For this, we compute

$$\mu - A = \lambda - A - (\lambda - \mu) = (\lambda - A)\left(1 - (\lambda - \mu)(\lambda - A)^{-1}\right).$$

Since $\left(1-(\lambda-\mu)(\lambda-A)^{-1}\right)$ is an isomorphism in $L(X)$, we deduce that the right-hand side admits a continuous inverse if and only if the left-hand side does. As $\lambda \in \rho(A)$, we thus infer $\mu \in \rho(A)$. The estimate follows from (2.2). Indeed, we have

$$\begin{aligned}\left\|(\mu-A)^{-1}\right\| &\leqslant \left\|(\lambda-A)^{-1}\right\| \left\|\sum_{k=0}^{\infty}\left((\lambda-\mu)(\lambda-A)^{-1}\right)^k\right\| \\ &\leqslant \left\|(\lambda-A)^{-1}\right\| \sum_{k=0}^{\infty}\left\|(\lambda-\mu)(\lambda-A)^{-1}\right\|^k = \frac{\left\|(\lambda-A)^{-1}\right\|}{1-\left\|(\lambda-\mu)(\lambda-A)^{-1}\right\|}.\end{aligned}$$

For the final claim of the present proposition, we observe that

$$\begin{aligned}(\mu-A)^{-1} &= \left(1-(\lambda-\mu)(\lambda-A)^{-1}\right)^{-1}(\lambda-A)^{-1} \\ &= \sum_{k=0}^{\infty}(\lambda-\mu)^k\left((\lambda-A)^{-1}\right)^{k+1},\end{aligned}$$

which is an operator norm convergent power series expression for the resolvent at μ about λ. Thus, analyticity follows. □

For a given measure space (Ω, Σ, μ) we shall consider multiplication operators in $L_2(\mu)$ next. For a measurable function $V\colon \Omega \to \mathbb{R}$ we will use the notation $[V \leqslant c] := V^{-1}\left[(-\infty, c]\right]$ for some constant $c \in \mathbb{R}$ (and similarly for other relational symbols).

Remark 2.4.2 Before we turn to more general multiplication operators, we like to reason our notation for them by illustrating the example case of multiplication operators in $L_2(\mathbb{R})$. A multiplication operator that immediately comes to mind is the so-called multiplication-by-the-argument operator on $L_2(\mathbb{R})$, which we shall denote by m. Expressed differently, let

$$\mathrm{m}\colon \operatorname{dom}(\mathrm{m}) \subseteq L_2(\mathbb{R}) \to L_2(\mathbb{R}),\, f \mapsto (x \mapsto xf(x)),$$

where dom(m) consists of all those $L_2(\mathbb{R})$-functions f such that $(x \mapsto xf(x)) \in L_2(\mathbb{R})$. Being a multiplication operator, m admits what is called a 'functional calculus': It is possible to define functions of m, which will turn out to be operators themselves. Thus, if $V\colon \mathbb{R} \to \mathbb{C}$ is measurable, we can define $V(\mathrm{m})$ to denote an operator in $L_2(\mathbb{R})$ acting as follows

$$(V(\mathrm{m})f)(x) := V(x)f(x)$$

for suitable f. To apply V to m turns out to be the same as the operator of multiplication by V. This correspondence serves to justify the notation of multiplication operators acting on $L_2(\mu)$ for some measure space (Ω, Σ, μ). We will re-use the notation $V(\mathrm{m})$ to denote the operator of multiplication-by-V, even in cases where there is no well-defined multiplication-by-argument-operator m in $L_2(\mu)$.

Theorem 2.4.3 *Let (Ω, Σ, μ) be a measure space and $V\colon \Omega \to \mathbb{K}$ a measurable function. Then the operator*

$$\begin{aligned} V(\mathrm{m})\colon\ \mathrm{dom}(V(\mathrm{m})) \subseteq L_2(\mu) &\to L_2(\mu) \\ f &\mapsto \big(\omega \mapsto V(\omega)f(\omega)\big), \end{aligned}$$

with $\mathrm{dom}(V(\mathrm{m})) := \big\{f \in L_2(\mu)\,;\ \big(\omega \mapsto V(\omega)f(\omega)\big) \in L_2(\mu)\big\}$ *satisfies the following properties:*

(a) $V(\mathrm{m})$ *is densely defined and closed.*
(b) $(V(\mathrm{m}))^* = V^*(\mathrm{m})$ *where $V^*(\omega) = V(\omega)^*$ for all $\omega \in \Omega$ (here $V(\omega)^*$ denotes the complex conjugate of $V(\omega)$).*
(c) *If V is μ-almost everywhere bounded, then $V(\mathrm{m})$ is continuous. Moreover, we have $\|V(\mathrm{m})\|_{L(L_2(\mu))} \leqslant \|V\|_{L_\infty(\mu)}$.*
(d) *If $V \neq 0$ μ-a.e. then $V(\mathrm{m})$ is injective and $V(\mathrm{m})^{-1} = \frac{1}{V}(\mathrm{m})$, where*

$$\frac{1}{V}(\omega) := \begin{cases} \frac{1}{V(\omega)}, & V(\omega) \neq 0, \\ 0, & V(\omega) = 0, \end{cases}$$

for all $\omega \in \Omega$.

Proof For the whole proof we let $\Omega_n := [|V| \leqslant n]$ and put $\mathbb{1}_n := \mathbb{1}_{\Omega_n}$.

(a) We first show that $V(\mathrm{m})$ is densely defined. Let $f \in L_2(\mu)$. Then, we have for all $n \in \mathbb{N}$ that $\mathbb{1}_n f \in \mathrm{dom}(V(\mathrm{m}))$. From $\Omega = \bigcup_n \Omega_n$ and $\Omega_n \subseteq \Omega_{n+1}$ it follows that $\mathbb{1}_n f \to f$ in $L_2(\mu)$ as $n \to \infty$.
Next, we confirm that $V(\mathrm{m})$ is closed. Let $(f_k)_k$ in $\mathrm{dom}(V(\mathrm{m}))$ convergent in $L_2(\mu)$ with $(V(\mathrm{m})f_k)_k$ be convergent in $L_2(\mu)$. Denote the respective limits by f and g. It is clear that for all $n \in \mathbb{N}$ we have $\mathbb{1}_n f_k \to \mathbb{1}_n f$ as $k \to \infty$. Also, we have

$$\mathbb{1}_n g = \lim_{k\to\infty} \mathbb{1}_n V(\mathrm{m}) f_k = \lim_{k\to\infty} V(\mathrm{m})(\mathbb{1}_n f_k) = V(\mathrm{m})(\mathbb{1}_n f) = \mathbb{1}_n V f.$$

Hence, $g = Vf$ μ-almost everywhere and since $g \in L_2(\mu)$, we have that $f \in \mathrm{dom}(V(\mathrm{m}))$.
(b) It is easy to see that $V^*(\mathrm{m}) \subseteq V(\mathrm{m})^*$. For the other inclusion, we let $u \in \mathrm{dom}(V(\mathrm{m})^*)$. Then, for all $f \in L_2(\mu)$ and $n \in \mathbb{N}$ we have $\mathbb{1}_n f \in \mathrm{dom}(V(\mathrm{m}))$ and, hence,

$$\begin{aligned} \langle f, \mathbb{1}_n V^* u\rangle = \textstyle\int_{\Omega_n} f^* V^* u \,\mathrm{d}\mu &= \langle V(\mathrm{m})(\mathbb{1}_n f), u\rangle = \langle \mathbb{1}_n f, V(\mathrm{m})^* u\rangle \\ &= \langle f, \mathbb{1}_n V(\mathrm{m})^* u\rangle\,. \end{aligned}$$

It follows that $\mathbb{1}_n V^* u = \mathbb{1}_n V(\mathrm{m})^* u$ for all $n \in \mathbb{N}$. Thus, $\Omega = \bigcup_n \Omega_n$ implies $V^* u = V(\mathrm{m})^* u$ and therefore $u \in \operatorname{dom}(V^*(\mathrm{m}))$ and $V^*(\mathrm{m})u = V(\mathrm{m})^* u$.

(c) If $|V| \leqslant \kappa$ μ-almost everywhere for some $\kappa \geqslant 0$, then for all $f \in L_2(\mu)$ we have $|V(\omega)f(\omega)| \leqslant \kappa\,|f(\omega)|$ for μ-almost every $\omega \in \Omega$. Squaring and integrating this inequality yields boundedness of $V(\mathrm{m})$ and the asserted inequality.

(d) Assume that $V \neq 0$ μ-a.e. and $V(\mathrm{m})f = 0$. Then, $f(\omega) = 0$ for μ-a.e. $\omega \in \Omega$, which implies $f = 0$ in $L_2(\mu)$. Moreover, if $V(\mathrm{m})f = g$ for $f, g \in L_2(\mu)$, then for μ-a.e. $\omega \in \Omega$ we deduce that $f(\omega) = \frac{1}{V}(\omega)g(\omega)$, which shows $\frac{1}{V}(\mathrm{m}) \supseteq V(\mathrm{m})^{-1}$. If on the other hand $g \in \operatorname{dom}\left(\frac{1}{V}(\mathrm{m})\right)$, then a similar computation reveals that $\frac{1}{V}(\mathrm{m})g \in \operatorname{dom}(V(\mathrm{m}))$ and $V(\mathrm{m})\frac{1}{V}(\mathrm{m})g = g$. □

The spectrum of $V(\mathrm{m})$ from the latter example can be computed once we consider a less general class of measure spaces. We provide a characterisation of these measure spaces first.

Proposition 2.4.4 *Let (Ω, Σ, μ) be a measure space. Then the following statements are equivalent:*

(i) (Ω, Σ, μ) *is* semi-finite*, that is, for every $A \in \Sigma$ with $\mu(A) = \infty$, there exists $B \in \Sigma$ with $0 < \mu(B) < \infty$ such that $B \subseteq A$.*

(ii) *For all measurable $V \colon \Omega \to \mathbb{K}$ with $V(\mathrm{m}) \in L(L_2(\mu))$, we have $V \in L_\infty(\mu)$ and $\|V\|_{L_\infty(\mu)} \leqslant \|V(\mathrm{m})\|_{L(L_2(\mu))}$.*

Proof (i)⇒(ii): Let $\varepsilon > 0$ and $A_\varepsilon := [|V| \geqslant \|V(\mathrm{m})\|_{L(L_2(\mu))} + \varepsilon]$. Assume that $\mu(A_\varepsilon) > 0$. Since (Ω, Σ, μ) is semi-finite we find $B_\varepsilon \subseteq A_\varepsilon$ such that $0 < \mu(B_\varepsilon) < \infty$. Define $f := \mu(B_\varepsilon)^{-1/2}\mathbb{1}_{B_\varepsilon} \in L_2(\mu)$ with $\|f\|_{L_2(\mu)} = 1$. Consequently, we obtain

$$\|V(\mathrm{m})\|_{L(L_2(\mu))} \geqslant \|V(\mathrm{m})f\|_{L_2(\mu)} \geqslant \|V(\mathrm{m})\|_{L(L_2(\mu))} + \varepsilon,$$

which yields a contradiction, and hence (ii).

(ii)⇒(i): Assume that (Ω, Σ, μ) is not semi-finite. Then we find $A \in \Sigma$ with $\mu(A) = \infty$ such that for each $B \subseteq A$ measurable, we have $\mu(B) \in \{0, \infty\}$. Then $V := \mathbb{1}_A$ is bounded and measurable with $\|V\|_{L_\infty(\mu)} = 1$. However, $V(\mathrm{m}) = 0$. Indeed, if $f \in L_2(\mu)$ then $[f \neq 0] = \bigcup_{n\in\mathbb{N}}[|f|^2 \geqslant n^{-1}]$. Thus,

$$[V(\mathrm{m})f \neq 0] = [f \neq 0] \cap A = \bigcup_{n\in\mathbb{N}}[|f|^2 \geqslant n^{-1}] \cap A.$$

Since $\mu([|f|^2 \geqslant n^{-1}]) < \infty$ as $f \in L_2(\mu)$, we infer $\mu([|f|^2 \geqslant n^{-1}] \cap A) = 0$ by the property assumed for A. Thus, $\mu([V(\mathrm{m})f \neq 0]) = 0$ implying $V(\mathrm{m}) = 0$. Hence, $\|V(\mathrm{m})\|_{L(L_2(\mu))} = 0 < 1 = \|V\|_{L_\infty(\mu)}$. □

Remark 2.4.5 Any σ-finite measure space is semi-finite. Indeed, let (Ω, Σ, μ) be σ-finite and $A \in \Sigma$ with $\mu(A) = \infty$. We find a sequence $(G_n)_n$ of pairwise disjoint, measurable sets with finite measure satisfying $\bigcup_n G_n = \Omega$. Hence, $\mu(G_n \cap A) \leqslant \mu(G_n) < \infty$. If $\mu(G_n \cap A) = 0$ for all n, then $\mu(A) = 0$ by the σ-additivity of μ. Thus, as $\mu(A) \neq 0$, we find n such that $0 < \mu(G_n \cap A) < \infty$ and (Ω, Σ, μ) is semi-finite.

A straightforward consequence of Theorem 2.4.3 (c) and Proposition 2.4.4 is the following.

Proposition 2.4.6 *Let (Ω, Σ, μ) be a semi-finite measure space, $V : \Omega \to \mathbb{K}$ measurable and bounded. Then $\|V(\mathrm{m})\|_{L(L_2(\mu))} = \|V\|_{L_\infty(\mu)}$.*

Theorem 2.4.7 *Let (Ω, Σ, μ) be a semi-finite measure space and let $V : \Omega \to \mathbb{K}$ be measurable. Then*

$$\sigma\left(V(\mathrm{m})\right) = \text{ess-ran}\, V := \{\lambda \in \mathbb{K};\ \forall \varepsilon > 0 \colon \mu\left([|\lambda - V| < \varepsilon]\right) > 0\}.$$

Proof Let $\lambda \in \text{ess-ran}\, V$. For all $n \in \mathbb{N}$ we find $B_n \in \Sigma$ with non-zero, but finite measure such that $B_n \subseteq \left[|\lambda - V| < \frac{1}{n}\right]$. We define $f_n := \sqrt{\frac{1}{\mu(B_n)}}\mathbb{1}_{B_n} \in L_2(\mu)$. Then $\|f_n\|_{L_2(\mu)} = 1$ and

$$|V(\omega) f_n(\omega)| \leqslant |V(\omega) - \lambda|\, |f_n(\omega)| + |\lambda|\, |f_n(\omega)| \leqslant \left(\frac{1}{n} + |\lambda|\right) |f_n(\omega)|$$

for $\omega \in \Omega$, which shows that $(f_n)_n$ is in $\operatorname{dom}(V(\mathrm{m}))$. A similar estimate, on the other hand, shows that

$$\|(V(\mathrm{m}) - \lambda)\, f_n\|_{L_2(\mu)} \to 0 \quad (n \to \infty).$$

Thus, $(V(\mathrm{m}) - \lambda)^{-1}$ cannot be continuous as $\|f_n\|_{L_2(\mu)} = 1$ for all $n \in \mathbb{N}$.

Let now $\lambda \in \mathbb{K} \setminus \text{ess-ran}\, V$. Then there exists $\varepsilon > 0$ such that $N := [|\lambda - V| < \varepsilon]$ is a μ-nullset. In particular, $\lambda - V \neq 0$ μ-a.e. Hence, $(\lambda - V(\mathrm{m}))^{-1} = \frac{1}{\lambda - V}(\mathrm{m})$ is a linear operator. Since, $\left|\frac{1}{\lambda - V}\right| \leqslant 1/\varepsilon$ μ-almost everywhere, we deduce that $(\lambda - V(\mathrm{m}))^{-1} \in L(L_2(\mu))$ and hence, $\lambda \in \rho(V(\mathrm{m}))$. □

We conclude this chapter by sketching that multiplication operators as discussed in Theorem 2.4.3, Propositions 2.4.4, 2.4.6, and Theorem 2.4.7 are *the* prototypical example for normal operators. In fact it can be shown that normal operators are unitarily equivalent to multiplication operators on some $L_2(\mu)$. This fact is also known as the 'spectral theorem'. It is also important to note that, as we have seen in Theorem 2.4.3, a multiplication operator in $L_2(\mu)$ is self-adjoint if and only if V assumes values in the real numbers, only.

2.5 Comments

The material presented in this chapter is basic textbook knowledge. We shall thus refer to the monographs [54, 139]. Note that spectral theory for self-adjoint operators is a classical topic in functional analysis. For a glimpse on further theory of linear relations we exemplarily refer to [7, 14, 25]. The restriction in Proposition 2.4.6 and Theorem 2.4.7 to semi-finite measure spaces is not very severe. In fact, if (Ω, Σ, μ) was not semi-finite, it is possible to construct a semi-finite measure space $(\Omega_{\mathrm{loc}}, \Sigma_{\mathrm{loc}}, \mu_{\mathrm{loc}})$ such that $L_p(\mu)$ is isometrically isomorphic to $L_p(\mu_{\mathrm{loc}})$, see [129, Section 2].

Exercises

Exercise 2.1 Let $A \subseteq X_0 \times X_1$ be an unbounded linear operator. Show that for every linear operator $B \subseteq X_0 \times X_1$ with $B \supseteq A$ and $\operatorname{dom}(B) = X_0$, we have that B is not closed.

Exercise 2.2 Prove Proposition 2.1.4 and Corollary 2.1.5. Hint: One might use that bounded linear relations are always operators.

Exercise 2.3 Prove Lemma 2.2.2.

Exercise 2.4 Let $A \colon \operatorname{dom}(A) \subseteq H_0 \to H_0$ be a closed and densely defined linear operator. Show that for all $\lambda \in \mathbb{K}$ we have

$$\lambda \in \rho(A) \iff \lambda^* \in \rho(A^*).$$

Exercise 2.5 Let $U \subseteq H_0 \times H_1$ satisfy $U^{-1} = U^*$. Show that $U \in L(H_0, H_1)$ and that U is *unitary*, that is, U is onto and for all $x \in H_0$ we have $\|Ux\|_{H_1} = \|x\|_{H_0}$.

Exercise 2.6 Let $\delta \colon C[0, 1] \subseteq L_2(0, 1) \to \mathbb{K}$, $f \mapsto f(0)$, where $C[0, 1]$ denotes the set of $\mathbb{K}$-valued continuous functions on $[0, 1]$. Show that δ is not closable. Compute $\overline{\delta}$.

Exercise 2.7 Let $C \subseteq \mathbb{C}$ be closed. Provide a Hilbert space H and a densely defined closed linear operator A on H such that $\sigma(A) = C$.

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