Chapter 13
Continuous Dependence on the Coefficients I

The power of the functional analytic framework for evolutionary equations lies in its variety. In fact, as we have outlined in earlier chapters, it is possible to formulate many differential equations in the form

$$(\partial_t M(\partial_t) + A)\, U = F.$$

In this chapter we want to use this versatility and address continuity of the above expression (or more precisely of the solution operator) in $M(\partial_t)$. To see this more clearly, fix F and take a sequence of material laws $(M_n)_n$. We will address the following question: what are the conditions or notions of convergence of $(M_n)_n$ to some M in order that $(U_n)_n$ with U_n given as the solution of

$$(\partial_t M_n(\partial_t) + A)\, U_n = F$$

converges to U, which satisfies

$$(\partial_t M(\partial_t) + A)\, U = F?$$

In the first of two chapters on this subject, we shall specialise to $A = 0$; that is, we will discuss ordinary differential equations with infinite-dimensional state space. To begin with, we address the convergence of material laws pointwise in the Fourier–Laplace transformed domain and its relation to the convergence of material laws evaluated at the time derivative.

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13.1 Convergence of Material Laws

Throughout, let H be a Hilbert space. We briefly recall that a sequence $(T_n)_n$ in $L(H)$ converges in the *strong operator topology* to some $T \in L(H)$ if for all $x \in H$ we have

$$T_n x \to Tx \quad (n \to \infty).$$

$(T_n)_n$ is said to converge in the *weak operator topology* to $T \in L(H)$ if for all $x, y \in H$ we have

$$\langle y, T_n x\rangle \to \langle y, Tx\rangle \quad (n \to \infty).$$

We denote the set of material laws on H with abscissa of boundedness less than or equal to $\nu_0 \in \mathbb{R}$ by

$$\mathcal{M}(H, \nu_0) := \{M\colon \operatorname{dom}(M) \to L(H)\,;\; M \text{ material law}, \mathrm{s_b}(M) \leqslant \nu_0\}.$$

Remark 13.1.1 Let $\nu_0 \in \mathbb{R}$, $\nu > \nu_0$. Then $\mathcal{M}(H, \nu_0)$ is an algebra and $\mathcal{M}(H, \nu_0) \ni M \mapsto M(\partial_{t,\nu}) \in L\big(L_{2,\nu}(\mathbb{R}; H)\big)$ is an algebra homomorphism which is one-to-one by Theorem 8.2.1.

Definition Let $\nu_0 \in \mathbb{R}$. A sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(H, \nu_0)$ is called *bounded* if

$$\sup_{n\in\mathbb{N}} \|M_n\|_{\infty, \mathbb{C}_{\mathrm{Re}>\nu_0}} < \infty.$$

Theorem 13.1.2 *Let $\nu_0 \in \mathbb{R}$, $(M_n)_n$ in $\mathcal{M}(H, \nu_0)$ be bounded. Assume that for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ the sequence $(M_n(z))_n$ converges in the weak operator topology of $L(H)$ with limit $M(z)$ and let $\nu > \nu_0$. Then $M \in \mathcal{M}(H, \nu_0)$ and $M_n(\partial_{t,\nu}) \to M(\partial_{t,\nu})$ as $n \to \infty$ in the weak operator topology of $L\big(L_{2,\nu}(\mathbb{R}, H)\big)$.*

If, in addition, $(M_n(z))_n$ converges in the strong operator topology of $L(H)$ for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$, then, as $n \to \infty$, $M_n(\partial_{t,\nu}) \to M(\partial_{t,\nu})$ in the strong operator topology of $L\big(L_{2,\nu}(\mathbb{R}, H)\big)$.

Proof Let $z_0 \in \mathbb{C}_{\mathrm{Re}>\nu_0}$, $r \in (0, \operatorname{Re} z_0 - \nu_0)$. For $x, y \in H$, by Cauchy's integral formula, we deduce

$$\langle y, M_n(z_0)x\rangle = \frac{1}{2\pi \mathrm{i}} \int_{\partial B(z_0,r)} \frac{\langle y, M_n(z)x\rangle_H}{z - z_0}\,\mathrm{d}z \quad (n \in \mathbb{N}).$$

As $(M_n)_n$ is bounded, Lebesgue's dominated convergence theorem yields

$$\langle y, M(z_0)x\rangle = \frac{1}{2\pi \mathrm{i}} \int_{\partial B(z_0,r)} \frac{\langle y, M(z)x\rangle_H}{z - z_0}\,\mathrm{d}z.$$

Since

$$|\langle y, M(z)x\rangle|_H \leqslant \|x\|_H \|y\|_H \sup_{n\in\mathbb{N}} \|M_n\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu_0}} \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu_0}), \tag{13.1}$$

$\langle y, M(\cdot)x\rangle_H$ is holomorphic in a neighbourhood of z_0. By Exercise 5.3 we obtain that $M\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to L(H)$ is holomorphic. In fact, the estimate (13.1) even implies that $M \in \mathcal{M}(H, \nu_0)$.

If $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ and $(M_n(z))_n$ even converges in the strong operator topology, then the limit is clearly $M(z)$.

The convergence statements for $(M_n(\partial_{t,\nu}))_n$ (in the weak and strong operator topology) are then implied by Fourier–Laplace transformation. □

Remark 13.1.3 In Theorem 13.1.2, it suffices to assume that $(M_n(z))_n$ converges only for z belonging to a countable subset of $\mathbb{C}_{\mathrm{Re}>\nu_0}$ with an accumulation point in $\mathbb{C}_{\mathrm{Re}>\nu_0}$.

The next statement is essential for the convergence statement for "ordinary" evolutionary equations.

Proposition 13.1.4 *Let $(T_n)_n$ be a sequence in $L(H)$ converging in the strong operator topology to some $T \in L(H)$ with $0 \in \bigcap_{n\in\mathbb{N}} \rho(T_n)$, $\sup_{n\in\mathbb{N}} \left\|T_n^{-1}\right\| < \infty$ and* $\operatorname{ran}(T) \subseteq H$ *dense. Then T is continuously invertible and $(T_n^{-1})_n$ converges to T^{-1} in the strong operator topology.*

Proof We set $K := \sup_{n\in\mathbb{N}} \left\|T_n^{-1}\right\|$. We show that T is continuously invertible first. For this, let $x \in H$. Then

$$\|x\| = \left\|T_n^{-1}T_n x\right\| \leqslant K\,\|T_n x\| \to K\,\|Tx\| \quad (n \to \infty).$$

Hence, T is one-to-one and it follows that $\operatorname{ran}(T) \subseteq H$ is closed. Hence, $0 \in \rho(T)$. For $x \in H$ we conclude

$$\left\|T_n^{-1}x - T^{-1}x\right\| = \left\|T_n^{-1}(T - T_n)T^{-1}x\right\| \leqslant K\left\|(T - T_n)T^{-1}x\right\| \to 0$$

as $(n \to \infty)$. □

We are now in the position to obtain the first result on continuous dependence.

Theorem 13.1.5 *Let $\nu_0 \in \mathbb{R}$, $(M_n)_n$ a bounded sequence in $\mathcal{M}(H, \nu_0)$, $c > 0$ such that for all $n \in \mathbb{N}$ and $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ we have*

$$\operatorname{Re} zM_n(z) \geqslant c.$$

If $(M_n(z))_n$ converges in the strong operator topology for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ then for the limit $M(z)$ we have $M \in \mathcal{M}(H, \nu_0)$ with $\mathrm{Re}\, zM(z) \geqslant c$ *for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ and for $\nu > \nu_0$ we have*

$$\left(\partial_{t,\nu} M_n(\partial_{t,\nu})\right)^{-1} \to \left(\partial_{t,\nu} M(\partial_{t,\nu})\right)^{-1}$$

in the strong operator topology.

Proof By Theorem 13.1.2, we observe $M \in \mathcal{M}(H, \nu_0)$. Let $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. Then we have $\mathrm{Re}\, zM(z) = \lim_{n\to\infty} \mathrm{Re}\, zM_n(z) \geqslant c$ and hence $zM(z)$ is continuously invertible. Since $0 \in \bigcap_{n\in\mathbb{N}} \rho(zM_n(z))$ and $\left\|(zM_n(z))^{-1}\right\| \leqslant 1/c$ by Proposition 6.2.3(b), we deduce by Proposition 13.1.4 applied to $T_n = zM_n(z)$ that $(zM_n(z))^{-1} \to (zM(z))^{-1}$ in the strong operator topology. By Theorem 13.1.2, for $\nu > \nu_0$ we infer $\left(\partial_{t,\nu} M_n(\partial_{t,\nu})\right)^{-1} \to \left(\partial_{t,\nu} M(\partial_{t,\nu})\right)^{-1}$ in the strong operator topology. □

13.2 A Leading Example

We want to illustrate the findings of the previous section with the help of an ordinary differential equation. Also, we shall provide an argument on the limitations of the theory presented above. Let (Ω, Σ, μ) be a finite measure space.

Note that for $V \in L_\infty(\mu)$ with associated multiplication operator $V(\mathrm{m})$ as in Theorem 2.4.3 we have that

$$M : z \mapsto 1 + z^{-1} V(\mathrm{m}) \in L(L_2(\mu))$$

is a material law with $\mathrm{s_b}(M) = 0$ unless $V = 0$ (in case $V = 0$ we have $\mathrm{s_b}(M) = -\infty$). The corresponding evolutionary equation is given by

$$\partial_{t,\nu} u + V(\mathrm{m}) u = f.$$

We want to study sequences of material laws of this form; that is, material laws induced by sequences $(V_n)_n$ in $L_\infty(\mu)$. First, we provide the following characterisation of the convergence of multiplication operators. We recall that for a Banach space X the weak* topology $\sigma(X', X)$ on X' is the coarsest topology such that all the mappings $X' \ni x' \mapsto x'(x)$ $(x \in X)$ are continuous.

Proposition 13.2.1 *Let $(V_n)_n$ in $L_\infty(\mu)$ and $V \in L_\infty(\mu)$. Then the following statements hold.*

(a) $V_n(\mathrm{m}) \to V(\mathrm{m})$ *in $L(L_2(\mu))$ if and only if $V_n \to V$ in $L_\infty(\mu)$.*
(b) $V_n(\mathrm{m}) \to V(\mathrm{m})$ *in the strong operator topology of $L(L_2(\mu))$ if and only if (V_n) is bounded in $L_\infty(\mu)$ and $V_n \to V$ in $L_1(\mu)$.*

(c) *$V_n(\mathrm{m}) \to V(\mathrm{m})$ in the weak operator topology of $L(L_2(\mu))$ if and only if $V_n \to V$ in the weak* topology $\sigma\big(L_\infty(\mu), L_1(\mu)\big)$.*

Proof

(a) This is a direct consequence of Proposition 2.4.6.

(b) Assume $V_n \to V$ in $L_1(\mu)$ and that $(V_n)_n$ is bounded in $L_\infty(\mu)$. Then $(V_n - V)_n$ is also bounded in $L_\infty(\mu)$. For $f \in L_\infty(\mu) \subseteq L_2(\mu)$ we obtain

$$\|V_n(\mathrm{m})f - V(\mathrm{m})f\|^2_{L_2(\mu)} = \int_\Omega |V_n - V|^2 \, |f|^2 \, \mathrm{d}\mu$$
$$\leqslant \sup_{n\in\mathbb{N}} \|V_n - V\|_{L_\infty(\mu)} \, \|f\|^2_{L_\infty(\mu)} \int_\Omega |V_n - V| \, d\mu \to 0.$$

Since $L_\infty(\mu)$ is dense in $L_2(\mu)$ and $(V_n(\mathrm{m}) - V(\mathrm{m}))_n$ is bounded by Proposition 2.4.6, we obtain $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mu))$.

Now, let $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mu))$. Then $(V_n(\mathrm{m}))_n$ is bounded in $L(L_2(\mu))$ by the uniform boundedness principle. Now Proposition 2.4.6 yields boundedness of $(V_n)_n$ in $L_\infty(\mu)$. Moreover, since $\mathbb{1}_\Omega \in L_2(\mu)$, we deduce $V_n = V_n(\mathrm{m})\mathbb{1}_\Omega \to V(\mathrm{m})\mathbb{1}_\Omega = V$ in $L_2(\mu)$. Since $L_2(\mu)$ embeds continuously into $L_1(\mu)$ we obtain $V_n \to V$ in $L_1(\mu)$.

(c) The assertion follows easily upon realising that $\phi \in L_1(\mu)$ if and only if there exists $\psi_1, \psi_2 \in L_2(\mu)$ such that $\phi = \psi_1\psi_2$. □

With the latter result at hand together with the results in the previous section, we easily deduce the next theorem on continuous dependence on the coefficients.

Theorem 13.2.2 *Let $(V_n)_n$ in $L_\infty(\mu)$ be bounded, $V \in L_\infty(\mu)$, and $V_n \to V$ in $L_1(\mu)$. Then there exists $\nu > 0$ such that*

$$\big(\partial_{t,\nu} + V_n(\mathrm{m})\big)^{-1} \to \big(\partial_{t,\nu} + V(\mathrm{m})\big)^{-1}$$

in the strong operator topology of $L\big(L_{2,\nu}(\mathbb{R}; L_2(\mu))\big)$.

Note that the convergence statement can be improved, see Exercise 13.3.

Proof By Proposition 13.2.1(b) we obtain $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mu))$. Note that for $\nu \geqslant 1 + \sup_{n\in\mathbb{N}} \|V_n\|_{L_\infty(\mu)}$ we have

$$\mathrm{Re}(z + V_n(\mathrm{m})) \geqslant 1 \quad (z \in \mathbb{C}_{\mathrm{Re}>\nu}, n \in \mathbb{N}).$$

Now Theorem 13.1.5 applied to $M_n(z) = 1 + z^{-1}V_n(\mathrm{m})$ yields the assertion. □

Remark 13.2.3 Theorem 13.2.2 can be generalized in the following way. Let $(B_n)_n$ in $L(H)$, $B \in L(H)$, $B_n \to B$ in the strong operator topology. Then there exists $\nu > 0$ such that

$$\left(\partial_{t,\nu} + B_n\right)^{-1} \to \left(\partial_{t,\nu} + B\right)^{-1}$$

in the strong operator topology of $L\big(L_{2,\nu}(\mathbb{R}; L_2(\mu))\big)$.

In Theorem 13.2.2 we assumed strong convergence of the sequence of multiplication operators $(V_n(\mathrm{m}))_n$. A natural question to ask is whether the stated result can be improved to $(V_n)_n$ converging in the weak* topology $\sigma\big(L_\infty(\mu), L_1(\mu)\big)$ only. The answer is neither 'yes' nor 'no', but rather 'not quite', as we will show in the following. We start with a result on weak* limits of scaled periodic functions, which will serve as the prototypical example for a sequence converging in the weak* topology of L_∞.

Theorem 13.2.4 *Let $f \in L_\infty(\mathbb{R}^d)$ be $[0,1)^d$-periodic; that is,*

$$f(\cdot + k) = f \quad (k \in \mathbb{Z}^d).$$

Then

$$f(n\cdot) \to \int_{[0,1)^d} f(x)\,\mathrm{d}x\,\mathbb{1}_{\mathbb{R}^d}$$

in the weak topology $\sigma\big(L_\infty(\mathbb{R}^d), L_1(\mathbb{R}^d)\big)$ as $n \to \infty$.*

Proof Without loss of generality, we may assume $\int_{[0,1)^d} f(x)\,\mathrm{d}x = 0$. By the density of simple functions in $L_1(\mathbb{R}^d)$ and the boundedness of $(f(n\cdot))_n$ in $L_\infty(\mathbb{R}^d)$, it suffices to show

$$\int_Q f(nx)\,\mathrm{d}x \to 0 \quad (n \to \infty)$$

for $Q = [a,b] := [a_1, b_1] \times \ldots \times [a_d, b_d]$ where $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathbb{R}^d$. By translation and the periodicity of f we may assume $a = 0$. Thus, it suffices to show

$$\int_{[0,b]} f(nx)\,\mathrm{d}x \to 0 \quad (n \to \infty)$$

for all $b \in (0, \infty)^d$. So, let $b = (b_1, \ldots, b_d) \in (0, \infty)^d$. Let $n \in \mathbb{N}$. Then we find $z \in \mathbb{N}_0^d$ and $\zeta \in [0, 1)^d$ such that $nb = z + \zeta$. We compute

$$\begin{aligned}
&\int_{[0,b]} f(nx)\,\mathrm{d}x \\
&= \frac{1}{n^d} \int_{[0,nb]} f(x)\,\mathrm{d}x \\
&= \frac{1}{n^d} \int_{[0,z_1]\times[0,nb_2]\times\ldots\times[0,nb_d]} f(x)\,\mathrm{d}x + \frac{1}{n^d} \int_{(z_1,z_1+\zeta_1]\times[0,nb_2]\times\ldots\times[0,nb_d]} f(x)\,\mathrm{d}x.
\end{aligned}$$

We now estimate

$$\begin{aligned}
\left| \frac{1}{n^d} \int_{(z_1,z_1+\zeta_1]\times[0,nb_2]\times\ldots\times[0,nb_d]} f(x)\,\mathrm{d}x \right| &\leqslant \frac{1}{n^d} \int_{(z_1,z_1+\zeta_1]\times[0,nb_2]\times\ldots\times[0,nb_d]} |f(x)|\,\mathrm{d}x \\
&\leqslant \frac{1}{n^d} \int_{(0,1]\times[0,nb_2]\times\ldots\times[0,nb_d]} \mathrm{d}x\, \|f\|_{L_\infty(\mu)} \\
&= \frac{1}{n} b_2 \cdot \ldots \cdot b_d \, \|f\|_{L_\infty(\mu)} \,.
\end{aligned}$$

Continuing in this manner and using $z_j \leqslant nb_j$ for all $j \in \{1, \ldots, d\}$, we obtain

$$\left| \int_{[0,b]} f(nx)\,\mathrm{d}x \right| \leqslant \frac{1}{n^d} \left| \int_{[0,z]} f(x)\,\mathrm{d}x \right| + \frac{1}{n} \sum_{j=1}^{d} \frac{b_1 \cdot \ldots \cdot b_d}{b_j} \|f\|_{L_\infty(\mu)} \,.$$

Since f is $[0, 1)^d$-periodic and $z \in \mathbb{N}_0^d$ we observe

$$\int_{[0,z]} f(x)\,\mathrm{d}x = \prod_{j=1}^{d} z_j \int_{[0,1)^d} f(x)\,\mathrm{d}x = 0.$$

Thus,

$$\left| \int_{[0,b]} f(nx)\,\mathrm{d}x \right| \leqslant \frac{1}{n} \sum_{j=1}^{d} \frac{b_1 \cdot \ldots \cdot b_d}{b_j} \|f\|_{L_\infty(\mu)} \,,$$

which tends to 0 as $n \to \infty$. □

Remark 13.2.5 Note that Theorem 13.2.4 also yields

$$f(n\cdot) \to \int_{[0,1)^d} f(x)\,\mathrm{d}x \mathbb{1}_\Omega$$

in the weak* topology $\sigma(L_\infty(\Omega), L_1(\Omega))$ for all measurable subsets $\Omega \subseteq \mathbb{R}^d$ with non-zero Lebesgue measure.

We now present an example which shows that weak* convergence of $(V_n)_n$ does not yield the result of Theorem 13.2.2.

Example 13.2.6 Let $(\Omega, \Sigma, \mu) = ((0,1), \mathcal{B}((0,1)), \lambda|_{(0,1)})$. For $n \in \mathbb{N}$ let V_n be given by $V_n(x) := \sin(2\pi n x)$ for $x \in (0,1)$. Then, by Theorem 13.2.4, we obtain $V_n \to 0$ in $\sigma\big(L_\infty((0,1)), L_1((0,1))\big)$ as $n \to \infty$. Let $\nu > 1$. Then $\big(\partial_{t,\nu} + V_n(\mathrm{m})\big)$ is continuously invertible as an operator in $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$. Let $\widetilde{f} \in C([0,1])$ and denote $f : t \mapsto \mathbb{1}_{[0,\infty)}(t)\widetilde{f}$. Then $f \in L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$. The solution $u_n \in L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$ of

$$\big(\partial_{t,\nu} + V_n(\mathrm{m})\big)u_n = f$$

is given by the variations of constants formula; that is,

$$u_n(t,x) = \mathbb{1}_{[0,\infty)}(t)\int_0^t \exp\big(-(t-s)\sin(2\pi n x)\big)\,\mathrm{d}s\,\widetilde{f}(x) \quad (t \in \mathbb{R}, x \in (0,1)).$$

Thus, if a variant of Theorem 13.2.2 were true also in this case, $(u_n)_n$ needs to converge (in some sense) to the solution u of

$$\partial_{t,\nu} u = f,$$

which is given by

$$u(t,x) = \mathbb{1}_{[0,\infty)}(t)t\widetilde{f}(x) \quad (t \in \mathbb{R}, x \in (0,1)).$$

However, by Theorem 13.2.4, for $x \in (0,1)$ we deduce

$$\int_0^t \exp\big(-(t-s)\sin(2\pi n x)\big)\,\mathrm{d}s \to \int_0^t J(-(t-s))\,\mathrm{d}s \quad (n \to \infty)$$

in $\sigma\big(L_\infty((0,1)), L_1((0,1))\big)$ for each $t \geqslant 0$, where

$$J(s) := \int_0^1 \exp\big(s\sin(2\pi x)\big)\,\mathrm{d}x \quad (s \in \mathbb{R})$$

denotes the 0-th order modified Bessel function of the first kind, cf. [1, p. 9.6.19]. Moreover, for $\varphi \in C_c^\infty(\mathbb{R})$, $A \in \mathcal{B}((0,1))$ and using dominated convergence we obtain

$$
\begin{aligned}
&\langle u_n, \varphi \mathbb{1}_A \rangle_{L_{2,\nu}(\mathbb{R}; L_2((0,1)))} \\
&= \int_0^\infty \int_0^1 \int_0^t \exp\big(-(t-s)\sin(2\pi n x)\big)\, \mathrm{d}s\, \widetilde{f}(x)^* \mathbb{1}_A(x)\, \mathrm{d}x\, \varphi(t) \mathrm{e}^{-2\nu t}\, \mathrm{d}t \\
&\to \int_0^\infty \int_0^1 \int_0^t J(-(t-s))\, \mathrm{d}s\, \widetilde{f}(x)^* \mathbb{1}_A(x)\, \mathrm{d}x\, \varphi(t) \mathrm{e}^{-2\nu t}\, \mathrm{d}t \\
&= \langle \widetilde{u}, \varphi \mathbb{1}_A \rangle_{L_{2,\nu}(\mathbb{R}; L_2((0,1)))}
\end{aligned}
$$

with

$$
\widetilde{u}(t,x) \coloneqq \mathbb{1}_{[0,\infty)}(t) \int_0^t J(-(t-s))\, \mathrm{d}s\, \widetilde{f}(x) \quad (t \in \mathbb{R}, x \in (0,1)).
$$

Since $(u_n)_n$ is bounded in $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$ and, by Lemma 3.1.9, the set $\big\{\varphi \mathbb{1}_A\,;\, A \in \mathcal{B}((0,1)),\ \varphi \in C_c^\infty(\mathbb{R})\big\}$ is total in $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$, we infer $u_n \to \widetilde{u}$ weakly in $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$ as $n \to \infty$. In particular, $\widetilde{u} \neq u$. Furthermore, $\widetilde{u}$ is *not* of the form

$$
\int_0^t \exp\big(-(t-s)\widetilde{V}(x)\big)\, \mathrm{d}s\, \widetilde{f}(x)
$$

for some $\widetilde{V} \in L_\infty((0,1))$ and hence, we *cannot* hope for $\widetilde{u}$ to satisfy an equation of the type

$$
\big(\partial_{t,\nu} + \widetilde{V}(\mathrm{m})\big)\widetilde{u} = f.
$$

As we shall see next, in the framework of evolutionary equations it is possible to derive an equation involving suitable limits of $(V_n)_n$ and f as a right-hand side.

13.3 Convergence in the Weak Operator Topology

In this section, we consider a particular class of material laws and characterise convergence of the solution operators of the corresponding evolutionary equations in the weak operator topology. The main theorem that will serve to compute the limit equation satisfied by $\widetilde{u}$ in Example 13.2.6 reads as follows.

Theorem 13.3.1 *Let H be a Hilbert space, $(B_n)_n$ a bounded sequence in $L(H)$ and $\nu > \sup_{n\in\mathbb{N}} \|B_n\|$. Then $\big((\partial_{t,\nu} + B_n)^{-1}\big)_n$ converges in the weak operator topology of $L(L_{2,\nu}(\mathbb{R}; H))$ if and only if for all $k \in \mathbb{N}$ the sequence $(B_n^k)_n$ converges in the weak operator topology of $L(H)$. In either case, we have*

$$(\partial_{t,\nu} + B_n)^{-1} \to \sum_{k=0}^{\infty} \big(- \partial_{t,\nu}^{-1}\big)^k C_k \partial_{t,\nu}^{-1}$$

in the weak operator topology of $L(L_{2,\nu}(\mathbb{R}; H))$, where $C_k \in L(H)$ denotes the weak limit of $(B_n^k)_n$ for $k \in \mathbb{N}$ and $C_0 \coloneqq 1_H$.

Remark 13.3.2 In the situation of Theorem 13.3.1, let $B_n^k \to C_k$ in the weak operator topology for all $k \in \mathbb{N}$. Let $L \coloneqq \sup_{n\in\mathbb{N}} \|B_n\|$, $\nu > 2L$, and $f \in L_{2,\nu}(\mathbb{R}; H)$. By Theorem 13.3.1, if $(\partial_{t,\nu} + B_n)u_n = f$ for all $n \in \mathbb{N}$, then $(u_n)_n$ converges weakly in $L_{2,\nu}(\mathbb{R}; H)$ to some element $\widetilde{u} \in L_{2,\nu}(\mathbb{R}; H)$. In order to determine the differential equation satisfied by $\widetilde{u}$, we make the following observations: by weak convergence,

$$\|C_k\| \leqslant \liminf_{n\to\infty} \left\| B_n^k \right\| \leqslant L^k.$$

Hence, since $\left\| \partial_{t,\nu}^{-1} \right\|_{L_{2,\nu}} \leqslant \frac{1}{\nu}$ (see Sect. 3.2) we infer that

$$\sum_{k=1}^{\infty} \big(- \partial_{t,\nu}^{-1}\big)^k C_k$$

converges in $L(L_{2,\nu}(\mathbb{R}; H))$ and

$$\left\| \sum_{k=1}^{\infty} \big(- \partial_{t,\nu}^{-1}\big)^k C_k \right\| \leqslant \sum_{k=1}^{\infty} \left\| \partial_{t,\nu}^{-1} \right\|^k \|C_k\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Hence, since $C_0 = 1_H$ we deduce that $\sum_{k=0}^{\infty} \big(- \partial_{t,\nu}^{-1}\big)^k C_k$ is boundedly invertible by the Neumann series. Thus, we obtain

$$\begin{aligned} f &= \partial_{t,\nu} \left(\sum_{k=0}^{\infty} \big(- \partial_{t,\nu}^{-1}\big)^k C_k \right)^{-1} \widetilde{u} = \partial_{t,\nu} \left(1_H + \sum_{k=1}^{\infty} \big(- \partial_{t,\nu}^{-1}\big)^k C_k \right)^{-1} \widetilde{u} \\ &= \partial_{t,\nu} \sum_{\ell=0}^{\infty} \left(- \sum_{k=1}^{\infty} \big(- \partial_{t,\nu}^{-1}\big)^k C_k \right)^{\ell} \widetilde{u} = \partial_{t,\nu}\widetilde{u} + \partial_{t,\nu} \sum_{\ell=1}^{\infty} \left(- \sum_{k=1}^{\infty} \big(- \partial_{t,\nu}^{-1}\big)^k C_k \right)^{\ell} \widetilde{u}. \end{aligned}$$

Before we prove Theorem 13.3.1 we revisit Example 13.2.6.

Example 13.3.3 (Example 13.2.6 Continued) By Theorem 13.3.1, we need to compute the limit of $(\sin^k(2\pi n\cdot))_n$ in the weak* topology of $L_\infty((0,1))$ for all $k \in \mathbb{N}$. By Theorem 13.2.4, we obtain for all $k \in \mathbb{N}$

$$\begin{aligned}\lim_{n\to\infty} \sin^k(2\pi n\cdot) &= \int_0^1 \sin^k(2\pi\xi)\,\mathrm{d}\xi\,\mathbb{1}_{(0,1)} \\ &= \begin{cases} \frac{(2m)!}{(m!2^m)^2}\mathbb{1}_{(0,1)}, & k = 2m \text{ for some } m \in \mathbb{N}, \\ 0, & k \text{ odd}, \end{cases}\end{aligned}$$

in $\sigma\big(L_\infty((0,1)), L_1((0,1))\big)$. Hence, $u_n \to \widetilde{u}$ weakly, where $\widetilde{u}$ satisfies

$$\partial_{t,\nu}\widetilde{u} + \partial_{t,\nu}\sum_{\ell=1}^{\infty}\left(-\sum_{m=1}^{\infty}\partial_{t,\nu}^{-2m}\frac{(2m)!}{(m!2^m)^2}\right)^{\ell}\widetilde{u} = f$$

for $\nu > 2$ by Remark 13.3.2.

Proof of Theorem 13.3.1 Before we prove the equivalence, we make some observations. Since $\nu > \sup_{n\in\mathbb{N}}\|B_n\| =: L$, by a Neumann series argument we deduce that

$$\big(\partial_{t,\nu} + B_n\big)^{-1} = \sum_{k=0}^{\infty}\big(-\partial_{t,\nu}^{-1}B_n\big)^k\partial_{t,\nu}^{-1} = \sum_{k=0}^{\infty}\big(-\partial_{t,\nu}^{-1}\big)^k B_n^k\partial_{t,\nu}^{-1}.$$

The series $\sum_{k=0}^{\infty}\big(-\partial_{t,\nu}^{-1}\big)^k B_n^k\partial_{t,\nu}^{-1}$ is absolutely convergent in $L(L_{2,\nu}(\mathbb{R};H))$. Also note that for $M_n\colon \mathbb{C}_{\mathrm{Re}>L} \ni z \mapsto \sum_{k=0}^{\infty}(-\frac{1}{z})^k B_n^k\frac{1}{z}$ we have $M_n \in \mathcal{M}(H,\nu)$.

Assume now that $(B_n^k)_n$ converges in the weak operator topology to some C_k for all $k \in \mathbb{N}$. A little computation reveals that as $n \to \infty$,

$$M_n(z) \to \sum_{k=0}^{\infty}\left(-\frac{1}{z}\right)^k C_k\frac{1}{z} =: M(z) \quad (z \in \mathbb{C}_{\mathrm{Re}>L})$$

in the weak operator topology, where the series on the right-hand side converges in $L(H)$ since

$$\|C_k\| \leqslant \liminf_{n\to\infty}\left\|B_n^k\right\| \leqslant L^k \quad (k \in \mathbb{N}).$$

Moreover, since $\nu > L$, the sequence $(M_n)_n$ is bounded in $\mathcal{M}(H,\nu)$ and thus, $M \in \mathcal{M}(H,\nu)$ and

$$M_n(\partial_{t,\nu}) \to M(\partial_{t,\nu})$$

in the weak operator topology by Theorem 13.1.2.

Now, we assume that $\big((\partial_{t,\nu} + B_n)^{-1}\big)_n$ converges in the weak operator topology. Then $(M_n(\partial_{t,\nu}))_n$ converges in the weak operator topology. Let $k \in \mathbb{N}$. We need to show that for all $\phi, \psi \in H$ the sequence $(\langle \phi, B_n^k \psi\rangle_H)_n$ is convergent to some number $c_{k,\phi,\psi}$ as $n \to \infty$. The Riesz representation theorem then yields the existence of $C_k \in L(H)$ with $\langle \phi, C_k\psi\rangle = c_{k,\phi,\psi}$. So, let $\phi, \psi \in H$. Moreover, we consider the functions m_n and h_n given by

$$m_n(z) := \sum_{k=0}^{\infty} (-z)^k z \left\langle \phi, B_n^k \psi \right\rangle_H \quad (z \in B(0, 1/L), n \in \mathbb{N})$$

and

$$h_n(z) := \langle \phi, M_n(z)\psi\rangle_H = \sum_{k=0}^{\infty} \frac{1}{z}\left(-\frac{1}{z}\right)^k \left\langle \phi, B_n^k \psi \right\rangle_H \quad (z \in \mathbb{C}_{\mathrm{Re}>L}, n \in \mathbb{N}).$$

Clearly, m_n and h_n are holomorphic on their respective domains for each $n \in \mathbb{N}$ and the sequences $(m_n)_n$ and $(h_n)_n$ are uniformly bounded on compact subsets (in other words they form normal families). Moreover,

$$m_n(z) = h_n\left(\frac{1}{z}\right) \quad \big(z \in B\big(1/(2L), 1/(2L)\big), n \in \mathbb{N}\big).$$

We aim to show that the coefficients of the power series of m_n converge as n tends to infinity. The proof will be done in two steps. In step 1, we will prove that the sequence $(h_n)_n$ converges to a holomorphic function $h\colon \mathbb{C}_{\mathrm{Re}>L} \to \mathbb{C}$ uniformly on compact sets. Then, in the second step, we will use this to deduce that $(m_n)_n$ also converges uniformly on compact sets and prove the assertion with the help of Cauchy's integral formula.

Step 1: By Proposition 5.3.2, $(M_n(\mathrm{im} + \nu))_n$ converges in the weak operator topology of $L(L_2(\mathbb{R}; H))$. For $f, g \in L_2(\mathbb{R})$ we thus obtain that

$$\big(\langle f, h_n(\mathrm{im} + \nu)g\rangle_{L_2(\mathbb{R})}\big)_n = \big(\langle f\phi, M_n(\mathrm{im} + \nu)g\psi\rangle_{L_2(\mathbb{R};H)}\big)_n$$

is convergent. Thus, using $L_2(\mathbb{R}) \cdot L_2(\mathbb{R}) = L_1(\mathbb{R})$, we obtain that

$$\Psi\colon L_1(\mathbb{R}) \ni u \mapsto \lim_{n\to\infty} \left(\int_{\mathbb{R}} h_n(\mathrm{i}t + \nu)u(t)\,\mathrm{d}t\right) \in \mathbb{C}$$

defines a linear functional, which is continuous, since

$$\sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} \|M_n(\mathrm{i}t + \nu)\|_{L(H)} = \sup_{n\in\mathbb{N}} \|M_n(\mathrm{im} + \nu)\|_{L(L_2(\mathbb{R};H))} < \infty$$

by boundedness of $(B_n)_n$. Hence, since $L_1(\mathbb{R})' = L_\infty(\mathbb{R})$, we find a unique $\widetilde{h} \in L_\infty(\mathbb{R})$ with

$$\lim_{n\to\infty} \int_{\mathbb{R}} h_n(\mathrm{i}t + \nu)u(t)\,\mathrm{d}t = \int_{\mathbb{R}} \widetilde{h}(t)u(t)\,\mathrm{d}t \quad (u \in L_1(\mathbb{R})).$$

We now show that every subsequence $(h_{n_k})_k$ of $(h_n)_n$ has a subsequence $(h_{n_{k_l}})_l$ which converges locally uniformly to a holomorphic function $h\colon \mathbb{C}_{\mathrm{Re}>L} \to \mathbb{C}$ such that $h(\mathrm{i}\cdot +\nu) = \widetilde{h}$ a.e., and that this implies that the limit h does not depend on the subsequences. Then we conclude that $(h_n)_n$ itself converges locally uniformly to h.

So, let $(h_{n_k})_k$ be a subsequence of (h_n). By Montel's theorem (see [104, Theorem 6.2.2]), we find a subsequence $(h_{n_{k_l}})_l$ of $(h_{n_k})_k$ such that $h_{n_{k_l}} \to h$ as $l \to \infty$ uniformly on compact subsets of $\mathbb{C}_{\mathrm{Re}>L}$ for some holomorphic function $h\colon \mathbb{C}_{\mathrm{Re}>L} \to \mathbb{C}$. In particular, we obtain

$$\lim_{l\to\infty} \int_{\mathbb{R}} h_{n_{k_l}}(\mathrm{i}t + \nu)\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}} h(\mathrm{i}t + \nu)\varphi(t)\,\mathrm{d}t \quad (\varphi \in C_{\mathrm{c}}(\mathbb{R}))$$

by dominated convergence and hence, $h(\mathrm{i}t + \nu) = \widetilde{h}(t)$ for almost every $t \in \mathbb{R}$. This shows that the limit h is independent of choice of the subsequences $(h_{n_k})_k$ and $(h_{n_{k_l}})_l$. Indeed, if $\widehat{h}\colon \mathbb{C}_{\mathrm{Re}>L} \to \mathbb{C}$ is the limit of another subsubsequence of $(h_n)_n$ as above, then $\widehat{h}(\mathrm{i}\cdot+\nu) = \widetilde{h} = h(\mathrm{i}\cdot+\nu)$ a.e. Since $\widehat{h}$ and h are holomorphic, the identity theorem yields $\widehat{h} = h$.

Now, assume for a contradiction that $(h_n)_n$ does not converge locally uniformly to h. Then we find a subsequence $(h_{n_k})_k$ of $(h_n)_n$, a compact set $K \subseteq \mathbb{C}_{\mathrm{Re}>L}$ and $\varepsilon > 0$ such that

$$\left\| h_{n_k} - h \right\|_{\infty,K} \geqslant \varepsilon \quad (k \in \mathbb{N}). \tag{13.2}$$

However, the subsequence $(h_{n_k})_k$ has a subsequence $(h_{n_{k_l}})_l$ which converges locally uniformly to h, contradicting (13.2). Thus, $(h_n)_n$ itself converges locally uniformly to h, and, in particular, $h_n \to h$ pointwise on $\mathbb{C}_{\mathrm{Re}>L}$.

Step 2: By what we have shown in Step 1, the sequence $(m_n)_{n\in\mathbb{N}}$ converges pointwise on $B\big(1/(2L), 1/(2L)\big)$. Since $(m_n)_n$ is also uniformly bounded on compact subsets of $B(0, 1/L)$, we derive that $(m_n)_n$ converges uniformly on compact subsets of $B(0, 1/L)$ by Vitali's theorem (see [104, Theorem 6.2.8]). Choosing $0 < r < 1/L$, we thus obtain by Cauchy's integral formula

$$\left\langle \phi, B_n^k \psi \right\rangle_H = (-1)^k \frac{1}{2\pi\mathrm{i}} \int_{\partial B(0,r)} \frac{m_n(z)}{z^{k+2}}\,\mathrm{d}z.$$

Thus $(B_n^k)_n$ converges in the weak operator topology as $n \to \infty$. □

13.4 Comments

The problems discussed here are contained in [133, 138] for both the weak and the strong operator topology. The case of differential-algebraic equations has been invoked as well.

The appearance of memory effects; that is, the occurrence of higher order integral operators due to a weak convergence of the coefficients has been first observed by Tartar and can, for instance, be found in [113]. The limit equation, however, is described by a convolution term rather than a power series of integral operators. It is, however, possible to reformulate these resulting equations into one another, see [135].

The last characterisation of weak convergence in Theorem 13.3.1 was formulated for the first time in [89].

Exercises

Exercise 13.1 Let $(V_n)_n$ in $L_\infty(\mathbb{R}^d)$ and $V \in L_\infty(\mathbb{R}^d)$. Characterise convergence of $V_n(\mathrm{m}) \to V(\mathrm{m})$ in the strong operator topology of $L(L_2(\mathbb{R}^d))$ in terms of convergence of $(V_n)_n$ similar to as was done in Proposition 13.2.1.

Exercise 13.2 Show that there exists an unbounded sequence $(V_n)_n$ in $L_\infty((0,1))$ and $V \in L_\infty((0,1))$ with $V_n \to V$ in $L_1((0,1))$.

Exercise 13.3 Let (Ω, Σ, μ) be a finite measure space, $(V_n)_n$ a bounded sequence in $L_\infty(\mu)$ and assume that $V_n \to V$ in $L_1(\mu)$ for some $V \in L_\infty(\mu)$. Show that there exists $\nu > 0$ such that

$$\big(\partial_{t,\nu} + V_n(\mathrm{m})\big)^{-1} \to \big(\partial_{t,\nu} + V(\mathrm{m})\big)^{-1}$$

in the strong operator topology of $L\big(L_{2,\nu}(\mathbb{R}; L_2(\mu)), H^1_\nu(\mathbb{R}; L_2(\mu))\big)$.

Exercise 13.4 Let $D = \bigcup_{n\in\mathbb{Z}} [n + 1/2, n + 1]$, $V_n := \mathbb{1}_D(n\cdot)$. For suitable $\nu > 0$ compute the limit of

$$\big((\partial_{t,\nu} + V_n(\mathrm{m}))^{-1}\big)_n$$

in the weak operator topology of $L_{2,\nu}\big(\mathbb{R}; L_2((0,1))\big)$.

Exercise 13.5 Let H be a Hilbert space, $c > 0$ and $c \leqslant B_n = B_n^* \in L(H)$ for all $n \in \mathbb{N}$. Characterise, in terms of convergence of $(B_n)_n$ in a suitable sense, that

$$\big((\partial_{t,\nu} B_n)^{-1}\big)_n$$

converges in the weak operator topology. In the case of convergence, find its limit and a sufficient condition for which there exists a $B \in L(H)$ such that

$$(\partial_{t,\nu} B_n)^{-1} \to (\partial_{t,\nu} B)^{-1}$$

in the weak operator topology.

Exercise 13.6 Let H be a Hilbert space. Show that $B_{L(H)} := \{B \in L(H)\,;\, \|B\| \leqslant 1\}$ is a compact subset under the weak operator topology. If, in addition, H is separable, show that $B_{L(H)}$ is also metrisable under the weak operator topology.

Exercise 13.7 Let H be a separable Hilbert space, $(B_n)_n$ in $L(H)$ bounded. Show that there exists a subsequence $(B_{n_k})_k$ of $(B_n)_n$, a material law $M\colon \operatorname{dom}(M) \to L(H)$ and $\nu > 0$ such that given $f \in L_{2,\nu}(\mathbb{R}; H)$ and $(u_k)_k$ in $L_{2,\nu}(\mathbb{R}; H)$ with

$$\partial_{t,\nu} u_k + B_{n_k} u_k = f \quad (k \in \mathbb{N}),$$

we deduce that $(u_k)_k$ converges weakly to some $u \in L_{2,\nu}(\mathbb{R}; H)$ with the property that

$$\partial_{t,\nu} M(\partial_{t,\nu}) u = f.$$

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