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Louis Landweber

IV. Georg-Weinblum-Gedächtnis-Vorlesung

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IV. Georg-Weinblum-Gedächtnis-Vorlesung

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IRROTATIONAL FLOW WITHIN THE BOUNDARY LAYER AND WAKE

THE FOURTH GEORG WEINBLUM MEMORIAL LECTURE

by

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PREFACE

To be invited to present the Weinblum Memorial Lecture is one of the highest honors that can be bestowed in the field of ship hydrodynamics, and I would like to thank the binational Memorial Committee for selecting me to follow such eminent predecessors as John Wehausen, Otto Grim and Takao Inui. A common feature, of the first four lecturers is that we had been associated with Georg Weinblum, as friends and colleagues, for many years, Wehausen and I since he came to the David Taylor Model Basin in 1948, Grim since he returned to Germany as the first Director of the Institut für Schiffbau in 1952, and Inui since a visit to the Institut in 1960. This will not always be the case, so that it is important that those of us who enjoyed his warmth and inspiration should reminisce about the man and his accomplishments.

When he arrived at the David Taylor Model Basin in 1948 at the age of 51, he was already well known for his many publications on ship hydrodynamics. Soon he had many personal friends. John Wehausen, Phil Eisenberg and I would regularly stop at his office to take him to lunch. We invited him frequently to our homes in the evenings. When he decided he needed to walk several miles along the highway from the laboratory after work, he had to decline many offers of rides from passing employees. He was very popular.

Georg arrived at the David Taylor Model Basin at an opportune time when a relatively young group, with little previous experience in ship theory, had been assigned to study and perform research on almost all aspects of ship hydrodynamics. His influence was immeasurable. In a warm and persuasive manner, with humor, insight and reason, he communicated easily with all, from the Directors of the laboratory to the lowliest assistants. With his sound grasp of the fundamentals and his vast knowledge of the literature, he served as a catalyst for the research productivity of his colleagues.

His promotional activities assumed several forms. Most evident today are the TMB Reports that he wrote in which he indicated the state of the art and what he considered to be the problems requiring investigation. Another of his techniques was to invite the staff to his lecture on a topic on which he felt we should be more knowledgeable. I vividly recall two such lectures, one on hydrodynamic mass, the other on the Lagally theorem. Curiously, although, in the first lecture, he waved aside as impossibly large an inertia coefficient I had measured for an accelerating ship model, and, in the second lecture, I was skeptical about the validity of the Lagally theorem, the seed he planted took root and I have written many papers on these two subjects in the succeeding thirty years.

On the subject of ship wavemaking, for which he was renowned, he influenced many of the staff, not only by his published papers and his TMB Reports, but also by his personal inspiration. Then Hartley Pond wrote an important paper on the pitching moment on a submarine near a free surface, and William Cummins, in a milestone paper, generalized the Lagally theorem to unsteady flows and applied it to study the force and moment acting on a body of revolution moving under a train of surface waves. His greatest success, in my opinion, was in interesting John Wehausen in ship wavemaking, a field to which Wehausen and his students at Berkeley have been making major contributions for over twenty years.

These were also productive research years for Georg Weinblum. One work, in particular, with J. Kendrick and M.A. Todd, DTMB Report 840, November 1952, entitled "Investigation of Wave Effects Produced by a Thin Body - TMB Model 4215," is closely related to the theme of the present lecture. The aforementioned model was essentially a plank, thickened for structural reasons to a length-to-thickness ratio of 40. It was designed and tested in order to determine the effect of paint roughness on frictional resistance. Weinblum observed, however, that the measured resistance showed indications of wave resistance, and that the thinness of the form offered an unusual opportunity to test the Michell thin-ship theory. Hence, with his assistants, he undertook to calculate the wave resistance of the model from the Michell integral, a major task in pre-computer days. Although they obtained satisfactory agreement with the residuary resistance, perhaps of even greater significance is that their work has served as the basis for testing refinements of ship wave theory in a 1975 paper by K.W.H. Eggers and H.S. Choi at the First International Conference on Numerical Ship Hydrodynamics, and the basis for trying a procedure for including viscous effects in wave-resistance calculations in a 1980 Ph.D. thesis by S.-Y. Kang at The University of Iowa.

An area that he had promoted strongly was that of the behavior of a ship at sea. After convincing the Director of the importance of this field, he was dismayed to discover that it had been decided to invest millions of dollars in the design and construction of a huge seaworthiness facility. Although he was a firm believer in the need for experimental data to confirm theoretical results, or to guide the development of a mathematical model, his opinion was that a team of analysts could produce more useful results at a fraction of the cost. He emphasized this by presenting a paper, with Manley St. Denis, on the motions of a ship at sea, at the 1950 meeting of the Society of Naval Architects and Marine Engineers. St. Denis continued in this field, pioneering with W.J. Pierson in developing a theory for predicting ship motions in random seas. He also succeeded in interesting Victor Szebehely in the phenomenon of ship slamming, on which Szebehely continued to contribute for many years. These are two more examples of the far-reaching consequences of Georg's inspiration.

I know of only one case where Georg's persuasive power failed. St. Denis was studying for the Ph.D. degree at The Catholic University of America. His adviser, Max M. Munk, was developing a "lump" theory of turbulence at the time and wanted St. Denis to work in that field with him. St. Denis, however, was not interested in turbulence, preferred to select a problem on seaworthiness, and asked Weinblum to intercede for him. Georg, of course, knew about Max Munk, the famous aerodynamicist, but feared that the converse might not be true. Against his better judgment, he agreed, but as he told me later, the meeting was a disaster. Munk showed no respect for the stature and opinion of his eminent former countryman, and St. Denis had no choice but to write a thesis on the lump theory of turbulence.

Many of the staff at the David Taylor Model Basin who came into contact with Georg Weinblum during those years eventually departed to become Directors of laboratories, (John Breslin of the Davidson Laboratory, and Phil Eisenberg and Marshall Tulin of Hydronautics), or professors at universities, where research in ship hydrodynamics is vigorously pursued. In retrospect, it seems to me that the flowering of research and progress in this field in the United States in the last few decades is, in a good measure, attributable to his influence. The following quotation from a paper on both added masses and the Lagally theorem, published in 1956 in the

Journal of Fluid Mechanics, expresses my sentiments:

> "We are also pleased, here, to mention Georg Weinblum of the University of Hamburg, that most inspiring teacher, who pointed out the power of the Lagally theorem and new fields of research for many of us."

IRROTATIONAL FLOW WITHIN THE BOUNDARY LAYER AND WAKE

PART I

SOURCE DISTRIBUTION GENERATING THE IRROTATIONAL FLOW

INTRODUCTION

In a previous work [1], relationships between the flow exterior to a boundary layer and wake, and the concept of displacement thickness and source distributions which generate the outer irrotational flow were examined. Refinements of the source-distribution formulae of Preston [2] and Lighthill [3] were presented for two-dimensional and axisymmetric bodies in a uniform stream. An experimental result given by T.T. Huang et al [4], that the pressure distribution of the irrotational flow continued into the region of the boundary layer and wake is in good agreement with the measured pressures in that region, suggested that this equivalent irrotational flow might be useful in several current problems of ship hydrodynamics.

Many investigators have attempted to take the presence of the boundary layer and wake into account, in calculating the wavemaking resistance of a ship form, by thickening the body by its displacement thickness. In the present approach, one seeks a source distribution which generates the irrotational flow exterior, to the boundary layer and wake, and then determines the wave resistance associated with this source distribution. Application to the Weinblum-Kendrick-Todd form [5], which is essentially a thin plank of 40 to 1 length-to-thickness ratio, reported in the Ph.D. thesis of S.-Y. Kang [6], showed that agreement with the measured residuary was considerably improved by modifying the source strengths for the effects of viscosity.

Similar procedures are currently being applied to the Wigley parabolic ship form. For this application, an extension of the analysis in [1] to the case of a ship form is required. Although, in general, it will probably be necessary to use integral-equation methods to determine the equivalent irrotational flow, the special geometry of the Wigley form, with its sharp bow, stern and keel and rectangular centerplane, suggests that a centerplane source distribution could be assumed. Hence an extension of the "second-order" formulae of [1] to the case of a ship form with a centerplane source distribution was undertaken.

In the derivation in [1] for a two-dimensional section, advantage was taken of the existence of a stream function. Since a stream function is not available for the ship form, the result for the two-dimensional case was rederived without use of a stream function, to serve as a guide for the three-dimensional case. These analyses are presented in the following.

Axial Source Distribution for a Symmetrical Two-Dimensional Form

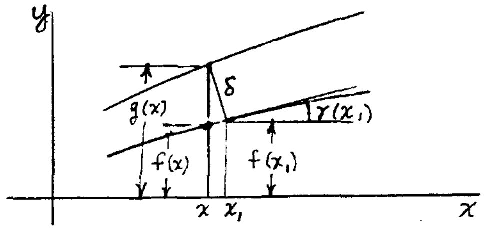

Figure 1

We shall first consider a flow about a symmetrical two-dimensional body in a uniform stream U_∞ in the direction of the body axis, including the boundary layer of thickness δ, and the wake. Coordinates parallel and normal to the surface will be denoted by (s,n) and the corresponding mean velocity components by (u,v). The curvature of the body profile K(s) will be assumed to be of the order of magnitude of L^{-1}, where L is the body length. Arc length in the direction of increasing s is given by h ds, where h = 1+Kn.

We shall also employ a rectangular coordinate system (x,y), with the x-axis along the axis of symmetry of the body. The velocity components in the x- and y- directions of the irrotational flow exterior to the boundary layer and wake will be denoted by U(x,y), V(x,y). The equation of the given profile will be represented by y = f(x), that of the edge of the boundary layer and wake (EBLW) by y = g(x), and the slope angle γ by $\tan\gamma = df/dx$. Then, as is seen from Fig. 1, we have the relations

$$g(x) = f(x_1) + \delta(x_1)\cos\gamma_1, \quad \gamma_1 = \gamma(x_1) \tag{1}$$

$$x_1 - x = \delta(x_1)\sin\gamma_1, \qquad s_1 = s(x_1) \tag{2}$$

The curvature K is given by

$$K = - d\gamma/ds \tag{3}$$

It will be useful to eliminate x_1 from (1) and (2).

The Taylor expansion of $f(x_1)$ and eqs. (2) and (3) give

$$f(x_1) = f(x) + \delta(x_1) \sin\gamma_1 \tan\gamma - \frac{K}{2} \delta(x)^2 \tan^2\gamma \sec\gamma + \ldots$$

which, substituted into (1), yields

$$g(x) = f(x) + \delta(x_1)(\cos\gamma_1 + \sin\gamma_1 \tan\gamma) - \frac{K}{2} \delta^2 \tan^2\gamma \sec\gamma + \ldots \tag{4}$$

But

$$\delta(x_1)(\cos\gamma_1 + \sin\gamma_1 \tan\gamma) = \delta(x)(\cos\gamma + \sin\gamma \tan\gamma)$$

$$+ (x_1 - x)[\frac{d}{dx}(\delta\cos\gamma) + \tan\gamma \frac{d}{dx} (\delta\sin\gamma)]$$

$$= \delta\sec\gamma + \delta\delta'\tan\gamma + \ldots$$

where $\delta' = d\delta/dx$. Then (4) becomes

$$g(x) = f(x) + \delta\sec\gamma + \delta\delta'\tan\gamma - \frac{1}{2}K\delta^2\tan^2\gamma \sec\gamma + O(\delta^3) \tag{5}$$

The equation of continuity in the (s,n) coordinate system is

$$\frac{\partial u}{\partial s} + \frac{\partial(hv)}{\partial n} = 0$$

This gives for the normal velocity component at EBLW, v_e,

$$(1+K\delta)v_e = - \int_0^\delta \frac{\partial u}{\partial s_1} dn = - \delta(x_1)\frac{du_e}{ds_1} + \frac{d}{ds_1} [u_e\delta_1(x_1)] \tag{6}$$

where

$$u_e = u(s,\delta), \; v_e = v(s,\delta), \qquad \delta_1 = \int_0^{\delta} (1 - \frac{u}{u_e})\, dn$$

But

$$\frac{du_e}{ds_1} = \frac{du_e}{dx_1}\cos\gamma_1 = \frac{du_e}{dx}\frac{dx}{dx_1}(\cos\gamma + K\delta\sin\gamma\tan\gamma + \ldots)$$

and, from (2),

$$\frac{dx}{dx_1} = 1 + K\delta - \delta'\sin\gamma + O(\delta^2)$$

Then

$$\frac{du_e}{ds_1} = \frac{du_e}{dx}(\cos\gamma + K\delta\sec\gamma - \delta'\sin\gamma\cos\gamma + \ldots) \tag{7}$$

Similarly

$$\frac{d}{ds_1}[u_e\delta_1(x_1)] = \frac{d}{ds_1}[u_e(\delta_1 + \delta\delta_1'\sin\gamma)]$$

$$\doteq (\cos\gamma + K\delta\sec\gamma - \delta'\sin\gamma\cos\gamma)\frac{d}{dx}[u_e(\delta_1 + \delta\delta_1'\sin\gamma)] \tag{8}$$

Then (6) becomes

$$v_e\sec\gamma \doteq [-\delta\frac{du_e}{dx} + \frac{d}{dx}(u_e\delta_1)](1 + K\delta f'^2 - \delta'\sin\gamma) + \frac{d}{dx}(u_e\delta\delta_1'\sin\gamma) \tag{9}$$

Let us assume that the irrotational flow is generated by an axial source distribution M(x). Then we have

$$V(x,o^+) = \pi M(x) \tag{10}$$

where o^+ indicates that $y \to o$ through positive values. Since $V_y = -U_x$, and

$$V_{yy}(x,o^+) = -V_{xx}(x,o^+) = -\pi M''(x)$$

the Taylor expansion of V(x,g) yields

$$M(x) \doteq \frac{1}{\pi} V(x,g) + \frac{1}{\pi} gU_o' + \frac{1}{2} g^2 M'' \tag{11}$$

in which $U_o = U(x,o)$, and the primes denote differentiation with respect to x.

At EBLW, the velocity vectors (u_e, v_e) and (U,V) coincide and we have the relations

$$u_e \cos\gamma_1 - v_e \sin\gamma_1 = U(x,g) \tag{12}$$

$$V(x,g) \cos\gamma_1 - U(x,g) \sin\gamma_1 = v_e \tag{13}$$

The Taylor expansion of U(x,g) gives

$$U(x,g) = U_o + g(U_y)_o + \frac{1}{2} g^2 (U_{yy})_o + \ldots = U_o + g(V_x)_o - \frac{1}{2} g^2 U_o'' + \ldots$$

$$= U_o + \pi g M' - \frac{1}{2} g^2 U_o'' + O(g^3) \tag{14}$$

which, substituted into (13), yields

$$V(x,g) = U_o f_1' + v_e \sec\gamma_1 + (\pi g M' - \frac{1}{2} g^2 U_o'') \tan\gamma_1 + \ldots \tag{15}$$

The Taylor expansion of $f_1' = \tan\gamma_1$ gives

$$f_1' = f' + \delta f'' \sin\gamma + \delta \sin\gamma(\delta' f'' \sin\gamma + \delta f''^2 \cos^3\gamma + \frac{\delta}{2} f''' \sin\gamma) + \ldots$$

Substituting this into (15) and applying (5) and (11) gives

$$\begin{aligned}\pi M(x) = {} & \frac{d}{dx}(U_o f) + U_o \delta \sin\gamma (f'' + \delta' f'' \sin\gamma + \delta f''^2 \cos^3\gamma + \frac{\delta}{2} f''' \sin\gamma) \\ & + U_o' \delta (\sec\gamma + \delta' f' + \frac{\delta}{2} f'' \sin^2\gamma) - \frac{g^2}{2} (U_o'' f' - \pi M'') + v_e \sec\gamma_1 \\ & + \pi M' [ff' + \delta f' \sec\gamma + \delta\delta' f'^2 + \delta^2 f' f'' (1 + \frac{1}{2} \sin^2\gamma) + \delta f f'' \sin\gamma]\end{aligned} \tag{16}$$

We still need to determine $v_e \sec\gamma_1$. Since

$$\sec\gamma_1 = \sec\gamma\cdot(1 + \delta f'' \sin^2\gamma \cos\gamma) + O(\delta^2)$$

we have, by (9),

$$v_e \sec\gamma_1 = [-\delta u_e' + \frac{d}{dx}(u_e\delta_1)](1 - \delta'\sin\gamma) + O(\delta^3) \qquad (17)$$

Here u_e and u_e' are required to first order only. From (12) and (14), we get

$$u_e = u_{eo} + u_{eo}\,\delta f'' \sin^2\gamma \cos\gamma - \frac{1}{2} g^2 U_o'' \sec\gamma + v_e \tan\gamma_1 + O(\delta^2) \qquad (18)$$

where

$$u_{eo} = (U_o + \pi g M')\sec\gamma \qquad (19)$$

Here the factor g^2 is retained since its derivative may be of first order. Substitution of u_{eo} into (17) then gives, correct to first order,

$$v_e \tan\gamma_1 = -\delta u_{eo}' \sin\gamma + \sin\gamma \frac{d}{dx}(u_{eo}\delta_1) + O(\delta^2)$$

and hence (18) becomes, to terms of first order,

$$u_{e1} = u_{eo} + u_{eo}\delta f'' \sin^2\gamma \cos\gamma - \frac{1}{2} g^2 U_o'' \sec\gamma - \delta u_{eo}' \sin\gamma + \sin\gamma \frac{d}{dx}(u_{eo}\delta_1) \qquad (20)$$

Then, from (17), we obtain

$$v_e \sec\gamma_1 = -\delta u_{e1}' + \frac{d}{dx}(u_{e1}\delta_1) + \delta'\sin\gamma[\delta u_{eo}' - \frac{d}{dx}(u_{eo}\delta_1)] + O(\delta^3) \qquad (21)$$

and $v_e \sec\gamma_1$ is given by (19), (20), and (21).

Equation (16) for $M(x)$ is not yet in explicit form since M' and M'' appear in its right member, as well as in the expression (21) for $v_e \sec\gamma_1$ through the quantity u_{eo}. These derivatives can be determined in the following manner. Put $M = M_o + O(\delta) = M_1 + O(\delta^2)$ and $M' = M_o' + O(\delta) = M_1' + O(\delta^2)$. Then, from (16), we obtain

$$\pi M_o = U_o f'$$

from the derivative of (16),

$$\pi M_o' = U_o f'' + 2U_o' f' + \pi M_o' f'^2$$

or

$$\pi M_o' = \frac{U_o f'' + 2U_o' f'}{1-f'^2} \tag{22}$$

and from the second derivative of (16),

$$\pi M_o'' = U_o f''' + 3U_o' f'' + 3U_o'' f' - f'^2(U_o'' f' - \pi M_o'') + 2\pi M_o'' f'^2 + 3\pi M_o' f' f''$$

or

$$\pi M_o'' = \frac{U_o f''' + 3U_o' f'' + 3U_o'' f' - U_o'' f'^2 + 3\pi M_o' f' f''}{1-3f'^2} \tag{23}$$

Next, we obtain from (16),

$$\pi M_1' = \pi M_o' + \{ U_o(\delta_1 f''' \sin\gamma + \delta_1 f''^2 \cos^3\gamma + 2\delta_1' f'' \sin\gamma + \delta_1'' \sec\gamma)$$
$$+ 2U_o'(\delta_1 f'' \sin\gamma + \delta_1' \sec\gamma) + U_o''(h_1 - g f'^2) + \pi M_o' (h_1 f'' + 2\delta_1' f' \sec\gamma + 2\delta_1 f' f'' \sin\gamma)$$
$$+ 2\pi M_o'' h_1 f'\}/(1-f'^2) \tag{24}$$

where $h_1 = f + \delta_1 \sec\gamma$, and we may take $g = f + \delta\sec\gamma$. The source distribution (16) then becomes

$$\pi M(x) = \frac{d(U_o h_1)}{dx} + U_o \delta \sin\gamma(\delta' f'' \sin\gamma + \delta f''^2 \cos^3\gamma + \frac{\delta}{2} f''' \sin\gamma) + U_o' \delta(\delta' f'$$
$$+ \frac{\delta}{2} f'' \sin^2\gamma) - \frac{1}{2} U_o'' g^2 f' + \pi M_o' [-\delta\delta' - \delta^2 f' f''(1 - \frac{1}{2}\sin^2\gamma) + \delta_1(\delta' \sec^2\gamma$$
$$+ 2\delta f' f'' + f f'' \sin\gamma) + \delta_1' g \sec\gamma] + \pi M_1' h_1 f' + \pi M_o'' g[\frac{g}{2} - (\delta - \delta_1) \sec\gamma] \tag{25}$$

In deriving (25), it was not assumed that f', f'' and f''' are small. If some of these quantities are small, then many of the terms of (25) may be omitted. Except near the body extremities, these quantities would usually be small, and, in fact, of order δ^3, and (25) would reduce to

$$M(x) = \frac{1}{\pi}\frac{d}{dx}\,[U_0(f + \delta_1 \sec\gamma)] + O(\delta^3) \qquad (26)$$

in agreement with [1].

Derivation for a Thin Ship Form

Procedures similar to those for the two-dimensional case will now be applied to derive a second-order approximation for a centerplane source distribution for a ship form. Although no explicit assumptions are made concerning the smallness of various dimensions and angles, until a result free of such assumptions has been obtained, the procedures used imply certain restrictions. Truncation of Taylor expansions in powers of g(x,y), the distance from the centerplane to the edge of the boundary layer and wake, can be expected to give a good approximation only if g is small relative to the body length. This requires that not only the body but also the region bounded by the EBLW must be thin.

Nor could one expect that a solution in terms of a centerplane distribution could give a good approximation for a ship form with a blunt bow or stern, large curvature at the turn of the bilge, and a flat or nearly flat bottom. For such forms, however, a distribution on the hull surface should be sought. The primary application of the present work will be to mathematical ship forms which are serving as research vehicles.

Geometric consideration

A double ship form is defined by the equation

$$z = \pm f(x,y) \qquad (27)$$

Here (x,y,z) are coordinates of a right-handed rectangular, cartesian coordinate system with the x- and y-axes in the centerplane, the x-axis at the undisturbed level of the free surface, and the y-axis vertical, positive upwards. The z-axis is then horizontal.

We shall also require a right-handed, curvilinear, surface-related, orthogonal coordinate system (s,t,n), where n denotes distance in the direction of the outward normal to the hull surface S, (23), and n = 0 on S. Such a coordinate system is generated only by the lines of principal curvature on S which define two families of orthogonally intersecting curves. The developable surfaces generated by the normals to S along these curves,

$$s(x,y,z) = \text{const.}, \qquad t(x,y,z) = \text{const.}$$

together with the surfaces n = const., form the desired orthogonal coordinate system; see references [7] and [8]. Let $h_1\,ds$ and h_2dt denote elements of arc in the directions of increasing s and t, and put $H_1 = h_1(s,t,o)$, $H_2 = h_2(s,t,o)$.

Let $k_1(s,t,n)$ and $k_2(s,t,n)$ denote the principal curvatures of the developable surfaces s = const. and t = const., and put $K_1 = k_1(s,t,o)$, $K_2 = k_2(s,t,o)$. Let K_3 and K_4 be the principal curvatures of S corresponding to the directions of increasing s and t. Then, as is shown in [7], we have the relations

$$h_1 = H_1(1 + K_3n), \qquad h_2 = H_2(1 + K_4n) \tag{28}$$

$$K_1 = k_1(1 + K_4n), \qquad K_2 = k_2(1 + K_3n) \tag{29}$$

$$\frac{1}{h_1}\frac{\partial h_2}{\partial s} = H_2K_1, \qquad \frac{1}{h_2}\frac{\partial h_1}{\partial t} = H_1K_2 \tag{30}$$

Let ℓ_{ij} denote the matrix of direction cosines relating the two coordinate systems, where i = 1, 2, 3 refer to the x, y, z axes and j = 1, 2, 3 refer to the s, t, n axes, respectively. We shall require the well-known relations

$$\ell_{ij}\ell_{ik} = \delta_{jk}, \quad \ell_{ij}\ell_{kj} = \delta_{ik} \tag{31}$$

$$\varepsilon_{ijk}\ell_{j\ell}\ell_{km} = \varepsilon_{\ell mn}\ell_{in} \tag{32}$$

Here the convention that a repeated index implies summation is employed, δ_{ij} is the kronecker delta, and ε_{ijk} is the permutation tensor, $\varepsilon_{ijk} = 0$ if i,j,k are not all different, and if i, j, k are different, $\varepsilon_{ijk} = \pm 1$ according as i, j, k are in cyclic or countercyclic order.

In the present system of parallel coordinates, the direction cosines ℓ_{ij} are independent of n, and are given by their values on S. The derivatives of ℓ_{ij} with respect to s or t along S can be obtained from a formula due to Rodrigues [9]. If $\bar{n}$ and $\bar{\sigma}$ denote unit vectors along the normal and along the tangent to a line of principal curvature of curvature K, respectively, and σ denotes arclength along this line, then the formula of Ridriques is

$$\frac{d\bar{n}}{d\sigma} = K\,\bar{\sigma} \tag{33}$$

For the surface S and the line of principal curvature of parameter s, this gives, for example,

$$\frac{\partial \ell_{13}}{H_1 \partial s} = D_s \ell_{13} = K_3 \ell_{11}, \qquad D_s = \ell_{11}\frac{\partial}{\partial x} + \ell_{21}\frac{\partial}{\partial y}$$

and

$$\frac{\partial \ell_{13}}{H_2 \partial t} = D_t \ell_{13} = K_4 \ell_{12}, \qquad D_t = \ell_{12}\frac{\partial}{\partial x} + \ell_{22}\frac{\partial}{\partial y}$$

For the developable surface s = constant, of curvature K_1, the formula similarly yields

$$D_t \ell_{11} = K_1 \ell_{12}$$

In this way, one can obtain the results for $D_s \ell_{ij}$ for j = 2 and 3, and for $D_t \ell_{ij}$ for j = 1 and 3, given in Tables 1a and 1b. The procedure

Table 1a $D_s \ell_{ij}$

i \ j	1	2	3
1	$-K_2\ell_{12}-K_3\ell_{13}$	$K_2\ell_{11}$	$K_3\ell_{11}$
2	$-K_2\ell_{22}-K_3\ell_{23}$	$K_2\ell_{21}$	$K_3\ell_{21}$
3	$-K_2\ell_{32}-K_3\ell_{33}$	$K_2\ell_{31}$	$K_3\ell_{31}$

Table 1b $D_t \ell_{ij}$

i \ j	1	2	3
1	$K_1\ell_{12}$	$-K_1\ell_{11}-K_4\ell_{13}$	$K_4\ell_{12}$
2	$K_1\ell_{22}$	$-K_1\ell_{21}-K_4\ell_{23}$	$K_4\ell_{22}$
3	$K_1\ell_{32}$	$-K_1\ell_{31}-K_4\ell_{33}$	$K_4\ell_{32}$

The procedure for deriving the other values in the tables will be illustrated by the case $D_s\ell_{21}$. We have, by (32) and (33),

$$D_s\ell_{21} = D_s(\ell_{13}\ell_{32}-\ell_{12}\ell_{33}) = K_2(\ell_{13}\ell_{31}-\ell_{11}\ell_{33})+K_3(\ell_{11}\ell_{32}-\ell_{31}\ell_{12})$$

$$= -K_2\ell_{22} - K_3\ell_{23}$$

We see from (33) that the curvature is given by the scalar product

$$K = \bar{\sigma}\cdot\frac{d\bar{n}}{d\sigma}$$

We then obtain from the tables

$$K_1 = \ell_{i2}D_t\ell_{i1},\quad K_2 = \ell_{i1}D_s\ell_{i2},\quad K_3 = \ell_{i1}D_s\ell_{i3},\quad K_4 = \ell_{i2}D_t\ell_{i3} \tag{34}$$

in which the repeated index implies summation over i = 1, 2, 3.

The derivatives of ℓ_{ij} with respect to x or y can also be obtained from the foregoing tables. These are given as the solutions of the pair of linear equations

$$D_s\ell_{ij} = \ell_{11}\ell_{ijx} + \ell_{21}\ell_{ijy}$$

$$D_t\ell_{ij} = \ell_{12}\ell_{ijx} + \ell_{22}\ell_{ijy}$$

Since $\ell_{11}\ell_{22} - \ell_{12}\ell_{21} = \ell_{33}$, these yield

$$\ell_{ijx} = \frac{\ell_{22}D_s\ell_{ij} - \ell_{21}D_t\ell_{ij}}{\ell_{33}},\quad \ell_{ijy} = \frac{\ell_{11}D_t\ell_{ij} - \ell_{12}D_s\ell_{ij}}{\ell_{33}} \tag{35}$$

Their values are given in Tables 2a and 3a.

TABLE 2a $\ell_{33}\ell_{ijx}$

i\j	1	2	3
1	$-(K_1\ell_{12}\ell_{21}+K_2\ell_{12}\ell_{22}+K_3\ell_{13}\ell_{22})$	$K_1\ell_{11}\ell_{21}+K_2\ell_{11}\ell_{22}+K_4\ell_{13}\ell_{21}$	$K_3\ell_{11}\ell_{22}-K_4\ell_{12}\ell_{21}$
2	$-(K_1\ell_{22}\ell_{21}+K_2\ell_{22}^2+K_3\ell_{23}\ell_{22})$	$K_1\ell_{21}^2+K_2\ell_{21}\ell_{22}+K_4\ell_{23}\ell_{21}$	$K_3\ell_{21}\ell_{22}-K_4\ell_{22}\ell_{21}$
3	$-(K_1\ell_{32}\ell_{21}+K_2\ell_{32}\ell_{22}+K_3\ell_{33}\ell_{22})$	$K_1\ell_{31}\ell_{21}+K_2\ell_{31}\ell_{22}+K_4\ell_{33}\ell_{21}$	$K_3\ell_{31}\ell_{22}-K_4\ell_{32}\ell_{21}$

TABLE 2b $\ell_{33}\ell_{ijy}$

i \ j	1	2	3
1	$K_1\ell_{12}\ell_{11}+K_2\ell_{12}^2+K_3\ell_{13}\ell_{12}$	$-(K_1\ell_{11}^2+K_2\ell_{11}\ell_{12}+K_4\ell_{13}\ell_{11})$	$K_4\ell_{12}\ell_{11}-K_3\ell_{11}\ell_{12}$
2	$K_1\ell_{22}\ell_{11}+K_2\ell_{22}\ell_{12}+K_3\ell_{23}\ell_{12}$	$-(K_1\ell_{21}\ell_{11}+K_2\ell_{21}\ell_{12}+K_4\ell_{23}\ell_{11})$	$K_4\ell_{22}\ell_{11}-K_3\ell_{21}\ell_{12}$
3	$K_1\ell_{32}\ell_{11}+K_2\ell_{32}\ell_{12}+K_3\ell_{33}\ell_{12}$	$-(K_1\ell_{31}\ell_{11}+K_2\ell_{31}\ell_{12}+K_4\ell_{33}\ell_{11})$	$K_4\ell_{32}\ell_{11}-K_3\ell_{31}\ell_{12}$

Put $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$. Since direction numbers of the normal to S are $(-f_x, -f_y, 1)$, we have

$$f_x = -\ell_{13}/\ell_{33}, \quad f_y = -\ell_{23}/\ell_{33}, \quad \ell_{33} = [1 + f_x^2 + f_y^2]^{-1/2} \tag{36}$$

We shall assume that f/L is small of the first order, but that f_x and f_y, and ℓ_{23}, ℓ_{32}, ℓ_{31}, ℓ_{13}, ℓ_{12}, ℓ_{21} may not be small. Here L denotes the length of the body.

The equation of the surface S' bounding the boundary layer and wake about S will be denoted by

$$z = g(x,y) \tag{37}$$

If the normal at (x_1, y_1, f_1) of S passes through (x, y, g) of S', its length is δ, and we have the relations

$$\frac{x-x_1}{(\ell_{13})_1} = \frac{y-y_1}{(\ell_{23})_1} = \frac{g-f_1}{(\ell_{33})_1} = \delta(x_1,y_1) \tag{38}$$

We shall assume that δ/L and g/L are also quantities of the first order.

Velocity fields

Let u(s,t,n), v(s,t,n), w(s,t,n) denote the velocity components in the directions of increasing s, t, n within the boundary layer and wake. In this coordinate system, the equation of continuity is

$$(h_1h_2)_{n=\delta}\, w_e = -\int_0^\delta \left[\frac{\partial(uh_2)}{\partial s} + \frac{\partial(vh_1)}{\partial t}\right] dn$$

$$= \int_0^\delta [H_2(1+K_4n)\frac{\partial}{\partial s}(u_e-u)+H_1(1+K_3n)\frac{\partial}{\partial t}(v_e-v)-H_2(1+K_4n)u_{es}$$
$$-H_1(1+K_3n)v_{et}+H_1H_2K_1(1+K_3n)(u_e-u)+H_1H_2K_2(1+K_4n)(v_e-v)$$
$$-H_1H_2K_1(1+K_3n)u_e-H_1H_2K_2(1+K_4n)v_e]dn$$

in which (30) has been applied. Hence we obtain, to terms of second order,

$$w_e[1+(K_3+K_4)\delta] = U_\infty(D_s\delta_1+D_t\delta_2+K_1\delta_1+K_2\delta_2)-\delta(D_su_e+D_tv_e+K_1u_e+K_2v_e)$$
$$-\frac{1}{2}\delta^2(K_4D_su_e+K_3D_tv_e+K_1K_3u_e+K_2K_4v_e)$$
$$+U_\infty(K_4D_s\bar{\delta}_1^2+K_3D_t\bar{\delta}_2^2+K_1K_3\bar{\delta}_1^2+K_2K_4\bar{\delta}_2^2) \qquad (39)$$

where u_e, v_e, w_e denote values of u, v, w at n = δ,

$$\delta_1 = \int_0^\delta \frac{u_e-u}{U_\infty}\,dn,\quad \bar{\delta}_1^2 = \int_0^\delta \frac{u_e-u}{U_\infty}\,ndn,\quad \delta_2 = \int_0^\delta \frac{v_e-v}{U_\infty}\,dn\ ,\quad \bar{\delta}_2^2 = \int_0^\delta \frac{v_e-v}{U_\infty}\,ndn \qquad (40)$$

subscripts s, t denote derivatives with respect to s or t, and all quantities are evaluated at (x_1,y_1).

Let U, V, W denote the x, y, z - components of the irrotational velocity field exterior to S'. At S', we have

$$w_e = \ell_{13}U(x,\ y,\ g) + \ell_{23}V(x,\ y,\ g) + \ell_{33}W(x,\ y,\ g) \qquad (41)$$

$$u_e = (\ell_{22}U - \ell_{12}V + \ell_{31}\,w_e)/\ell_{33} \qquad (42)$$

$$v_e = (\ell_{11}V - \ell_{21}U + \ell_{32}w_e)/\ell_{33} \qquad (43)$$

in which the ℓ_{ij} are evaluated at (x_1,y_1) and U, V, W at (x,y,g). Equations (42) and (43) can be verified by applying (32).

We shall assume that the disturbance potential is generated by a centerplane source distribution M(x,y), and that M is of order $O(U_\infty\delta/L)$. Then we have

$$2\pi M(x_1,y_1) = W(x_1,y_1,0^+) \qquad (44)$$

All quantities, except U, V, W in (41), (42), and (43), have been expressed in terms of x_1, y_1. The latter can also be so expressed by means of Taylor expansions, such as

$$U(x,y,g) = U_o + (x-x_1)U_{ox} + (y-y_1)U_{oy} + gU_{zo} + \frac{1}{2}[(x-x_1)^2U_{oxx}$$
$$+ (y-y_1)^2U_{oyy} + g^2U_{zzo}] + g(y-y_1)U_{yzo} + g(x-x_1)U_{zxo}+(x-x_1)(y-y_1)U_{oxy}+\dots$$

where $U_o = U(x_1, y_1, o)$, subscripts x, y, z indicate partial differentiations with respect to the variable, and the derivatives are evaluated at (x_1, y_1, o). Since the flow due to the source distribution is irrotational and harmonic, we have

$$U_{zo} = W_{ox} = 2\pi M_x, \quad U_{zzo} = - U_{oxx} - U_{oyy} \equiv - \nabla^2 U_o, \text{ etc.}$$

Applying these and equations (36) and (38) in the Taylor expansions, we obtain, to $O(\delta^2)$,

$$U(x,y,g) = U_o + U_1 + U_2, \qquad V(x,y,g) = V_o + V_1 + V_2$$

where $V_o = V(x_1, y_1, o)$,

$$U_1 = \delta(\ell_{13}U_{ox} + \ell_{23}U_{oy}), \quad V_1 = (\ell_{13}V_{ox} + \ell_{23}V_{oy}) \tag{45}$$

$$U_2 = 2\pi M_x g + \frac{1}{2}\delta^2(\ell_{13}^2 U_{oxx} + \ell_{23}^2 U_{oyy} + 2\ell_{13}\ell_{23}U_{oxy}) - \frac{1}{2}g^2\nabla^2 U_o$$

$$V_2 = 2\pi M_{yg} + \frac{1}{2}\delta^2(\ell_{13}V_{oxx} + \ell_{23}V_{oyy} + 2\ell_{13}\ell_{23}V_{oxy}) - \frac{1}{2}g^2\nabla^2 V_o \tag{46}$$

and

$$W(x,y,g) = 2\pi M - 2\pi\delta\ell_{33}(f_x M_x + f_y M_y) - g(U_{ox} + V_{oy})$$

$$- g\delta(\ell_{13}\nabla^2 U_o + \ell_{23}\nabla^2 V_o) \tag{47}$$

Since, hereafter, all quantities are functions of x_1, y_1, the subscripts of x and y will be omitted.

Equation (39) will now be simplified. We have, by (42) and Table 1,

$$D_s u_e = U(K_2 \frac{\ell_{21}}{\ell_{23}} - K_3 \frac{\ell_{31}\ell_{22}}{\ell_{33}^2}) - V(K_2 \frac{\ell_{11}}{\ell_{33}} - K_3 \frac{\ell_{12}\ell_{31}}{\ell_{33}^2}) - w_e(K_2 \frac{\ell_{32}}{\ell_{33}} +$$

$$K_3 + K_3 \frac{\ell_{31}^2}{\ell_{33}^2}) + \frac{\ell_{22}}{\ell_{33}} D_s U - \frac{\ell_{12}}{\ell_{33}} D_s V + \frac{\ell_{31}}{\ell_{33}} D_s w_e \tag{48}$$

and similarly

$$D_t v_e = V(K_1 \frac{\ell_{12}}{\ell_{33}^2} - K_4 \frac{\ell_{32}\ell_{11}}{\ell_{33}^2}) - U(K_1 \frac{\ell_{22}}{\ell_{33}} - K_4 \frac{\ell_{21}\ell_{32}}{\ell_{33}^2}) - w_e(K_1 \frac{\ell_{31}}{\ell_{33}} +$$

$$K_4 + K_4 \frac{\ell_{32}^2}{\ell_{33}^2}) + \frac{\ell_{11}}{\ell_{33}} D_t V - \frac{\ell_{21}}{\ell_{33}} D_t U + \frac{\ell_{32}}{\ell_{33}} D_t w_e \qquad (49)$$

Hence, expressing D_s and D_t in terms of derivatives with respect to x and y, applying properties of the ℓ_{ij}, and again using (42) and (43), we obtain, from (48) and (49),

$$D_s u_e + D_t v_e = -(K_1 + K_3 \frac{\ell_{31}}{\ell_{33}})u_e - (K_2 + K_4 \frac{\ell_{32}}{\ell_{33}})v_e - (K_3 + K_4)w_e$$

$$+ U_x + V_y - \ell_{13}w_{ex} - \ell_{23}w_{ey} + O(\delta^2) \qquad (50)$$

Similarly, we obtain from (48) and (49), by also using (41),

$$K_4 D_s u_e + K_3 D_t v_e = -K_1 K_3 u_e - K_2 K_4 v_e$$

$$+ K_4(\frac{\ell_{22}}{\ell_{33}} D_s U - \frac{\ell_{12}}{\ell_{33}} D_s V) + K_3(\frac{\ell_{11}}{\ell_{33}} D_t V - \frac{\ell_{21}}{\ell_{33}} D_t U) + O(\delta) \qquad (51)$$

Substitution of (50) and (51) into (39) then yields

$$w_e = U_\infty(D_s\delta_1 + D_t\delta_2 + K_1\delta_1 + K_2\delta_2) + \delta[K_3 \frac{\ell_{31}}{\ell_{33}} u_e + K_4 \frac{\ell_{32}}{\ell_{33}} v_e + \ell_{13}w_{ex}$$

$$\ell_{23}w_{ey} - U_x - V_y] + \frac{\delta^2}{2\ell_{33}} [K_4(\ell_{12}D_s V - \ell_{22}D_s U) + K_3(\ell_{21}D_t U - \ell_{11}D_t V)]$$

$$+ U_\infty(K_4 D_s \bar{\delta}_1^2 + K_3 D_t \bar{\delta}_2^2 + K_1 K_3 \bar{\delta}_1^2 + K_2 K_4 \bar{\delta}_2^2) + O(\delta^3) \qquad (52)$$

Put, for the terms of w_e of $O(\delta)$,

$$w_{e1} = U_\infty[(K_1 + D_s)\delta_1 + (K_2 + D_t)\delta_2] + \delta(K_3 \frac{\ell_{31}}{\ell_{33}} u_{eo} + K_4 \frac{\ell_{32}}{\ell_{33}} v_{eo} - U_{ox} - V_{oy}) \qquad (53)$$

where

$$u_{eo} = (\ell_{22}U_o - \ell_{12}V_o)/\ell_{33}, \quad v_{eo} = (\ell_{11}V_o - \ell_{21}U_o)/\ell_{33} \qquad (54)$$

Also, for the terms of u_e and v_e of $O(\delta)$, by (42), (43) and (45), we have

$$u_{e1} = -\delta[\ell_{22}f_xU_{ox} + \ell_{22}f_yU_{oy} - \ell_{12}f_xV_{ox} - \ell_{12}f_yV_{oy}] + \frac{\ell_{31}}{\ell_{33}}w_{e1} \quad (55)$$

$$v_{e1} = -\delta[\ell_{11}f_xV_{ox} + \ell_{11}f_yV_{oy} - \ell_{21}f_xU_{ox} - \ell_{21}f_yU_{oy}] + \frac{\ell_{32}}{\ell_{33}}w_{e1} \quad (56)$$

In (52), terms of $O(\delta^2)$ may be neglected in the factor of δ, and of $O(\delta)$ in the factor of δ^2. Thus, by (45) and (46), U_x in the factor of δ may be replaced by

$$U_x \doteqdot U_{ox} + U_{1x} + 2\pi f_xM_x - gf_x\,\nabla^2U_o$$

with a similar expression for V_y. The terms w_{ex} and w_{ey} also cannot be simply replaced by w_{e1x} and w_{e1y} since the derivatives of g yield the derivatives of f, which are of lower order. These additional terms can be obtained by considering the derivative of equation (52). For example, we have, by (45) and (46),

$$w_{ex} = -\delta(U_{xx} + V_{xy}) + \ldots = -\delta(U_{xx} + U_{yy}) + \ldots.$$

$$= \delta\nabla^2(\tfrac{1}{2}g^2\nabla^2U_o) + \ldots = \delta(f_x^2 + f_y^2)\nabla^2U_o + \ldots$$

in which only the additional terms of $O(\delta)$ are displayed. Similarly,

$$w_{ey} = \delta(f_x{}^2 + f_y{}^2)\nabla^2V_o + \ldots$$

Hence, after combining the additional terms, (52) becomes

$$\begin{aligned} w_e = w_{e1} &+ \delta(K_3\frac{\ell_{31}}{\ell_{33}}u_{e1} + K_4\frac{\ell_{32}}{\ell_{33}}v_{e1} + \ell_{13}w_{e1x} + \ell_{23}w_{e1y} - U_{1x} - V_{1y}) \\ &+ \frac{\delta^2}{2\ell_{33}}[K_3(\ell_{21}D_tU_o - \ell_{11}D_tV_o) - K_4(\ell_{22}D_sU_o - \ell_{12}D_sV_o)] \\ &+ U_\infty(K_4D_s\bar{\delta}_1^2 + K_3\bar{D}_t\bar{\delta}_2^2 + K_1K_3\bar{\delta}_1^2 + K_2K_4\bar{\delta}_2^2) \\ &+ \delta[f + \delta(2\ell_{33} - \frac{1}{\ell_{33}})](f_x\nabla^2U_o + f_y\nabla^2V_o) - 2\pi\delta(f_xM_x + f_yM_y) + O(\delta^3) \end{aligned} \quad (57)$$

Solution for M

We now obtain, from (47), (41), and (36)

$$2\pi M = \frac{w_e}{\ell_{33}} + f_x U + f_y V + g(U_{ox} + V_{oy}) - g\delta\ell_{33}(f_x\nabla^2 U_o + f_y\nabla^2 V_o)$$

$$+ 2\pi\delta\ell_{33}(f_x M_x + f_y M_y) + O(\delta^3) \qquad (58)$$

Here M_x and M_y are required to $O(\delta)$ only. These quantities can be eliminated by deriving the two additional equations given by the derivatives of (58) with respect to x and y,

$$2\pi M_x = \frac{\partial}{\partial x}\left[\frac{w_{e1}}{\ell_{33}} + f_x(U_o+U_1) + f_y(V_o+V_1) + g(U_{ox}+V_{oy})\right]$$

$$+ 2\pi f_x(f_x M_x + f_y M_y) - f_x(f_x\nabla^2 U_o + f_y\nabla^2 V_o)[f + (2\ell_{33} - \frac{1}{\ell_{33}})] \qquad (59)$$

and a similar equation for M_y. Here the additional term of $O(\delta)$ comes from (57) in evaluating w_{ex}, and the terms $M_x f_x^2$ and $M_y f_x f_y$ from U_{2x} and V_{2x} in (46). This gives the pair of equations of the form

$$M_x(1 - f_x^2) - M_y f_x f_y = A$$

$$-M_x f_x f_y + M_y(1-f_y^2) = B$$

which have the solutions

$$M_x = \frac{A(1-f_y^2)+Bf_x f_y}{1-f_x^2-f_y^2}, \qquad M_y = \frac{B(1-f_x^2)+Af_x f_y}{1-f_x^2-f_y^2} \qquad (60)$$

Here

$$A = \frac{1}{2\pi}\left\{\frac{\partial}{\partial x}\left[\frac{w_{e1}}{\ell_{33}} + f_x(U_o+U_1) + f_y(V_o+V_1)+g(U_{ox}+V_{oy})\right]\right.$$

$$\left. - f_x(f_x\nabla^2 U_o+f_y\nabla^2 V_o)[f+\delta(2\ell_{33}-\frac{1}{\ell_{33}})]\right\} \qquad (61)$$

$$B = \frac{1}{2\pi}\{\frac{\partial}{\partial y}[\frac{w_{e1}}{\ell_{33}} + f_x(U_o+U_1)+f_y(V_o+V_1)+g(U_{ox}+V_{oy})]$$

$$-f_y(f_x\nabla^2 U_o+f_y\nabla^2 V_o)[f+\delta(2\ell_{33}-\frac{1}{\ell_{33}})]\} \qquad (62)$$

The last term of (58) then becomes

$$2\pi\delta\ell_{33}\frac{Af_x+Bf_y}{1-f_x^{\,2}-f_y^{\,2}} \qquad (63)$$

This completes the derivation of an expression for the source distribution M. It is given by (58), (61), (62) and (63) with U, V, u_{eo}, v_{eo}, u_{e1}, v_{e1}, w_{e1}, and w_e given by (45), (46), (54), (55), (56), (53), and (57), respectively.

The appearance of terms of $O(\delta^0)$ in (58) apparently contradicts the assumption that M is $O(\delta)$. We see, however, that all the terms of zero order in (58) are included in the terms

$$f_xU_o + f_yV_o + f(U_{ox} + V_{oy}) = \frac{\partial}{\partial x}(U_of) + \frac{\partial}{\partial y}(V_of) \qquad (64)$$

which is the well-known second-order formula for the centerplane distribution for a thin ship without a boundary layer. Thus the assumption that $M = O(\delta)$ is consistent with the result derived for the contribution of the effect of the boundary layer on the centerplane source distribution.

For a thin ship, with the usual assumption that f and its derivatives, and, consequently, the curvatures are small of first order, the direction cosines ℓ_{ij}, $i \neq j$, would be of first order, and $1 - \ell_{11}$, $1 - \ell_{22}$, $1 - \ell_{33}$ would be of second order. Expression (58) for M would then reduce to

$$2\pi M = \frac{\partial}{\partial x}(U_of) + \frac{\partial}{\partial y}(V_of)$$

$$+ \frac{U_\infty}{\ell_{33}}(\ell_{11}\delta_{1x} + \ell_{21}\delta_{1y} + \ell_{12}\delta_{2x} + \ell_{22}\delta_{2y} + K_1\delta_1 + K_2\delta_2) + O(\delta^3) \qquad (65)$$

as is seen from the sum of the remaining second-order terms

$$-\delta(U_{ox} + V_{oy})(\frac{1}{\ell_{33}} - \ell_{33})$$

This may serve as a useful approximation for a thin ship form. When some, but not all of the ℓ_{ij}, $i \neq j$, are small, an explicit form for M, somewhat longer than (65), can be readily obtained for particular cases.

PART II

PRESSURE DISTRIBUTIONS

An important property of thin boundary-layer theory is that the pressure within it is equal to the pressure in the irrotational flow at the edge of the boundary layer. One could hardly expect that there would be an equally simple relation for a thick boundary layer; yet it was found by Huang et al [4] that the measured pressure in the boundary layer of a body of revolution agreed within one percent with that computed for the irrotational flow continued into the boundary-layer region.

In an attempt to explain this unexpected correspondence, derivations of expressions for the pressure distributions in the boundary layer and in the continued irrotational flow have been undertaken. Results for only the two-dimensional case have thus far been obtained and these will be presented in the following.

Formulation of Two-Dimensional Mathematical Model

Consider a boundary layer of thickness $\delta(s)$ along a streamlined cylindrical form immersed in a uniform stream. Here s denotes arc length along the profile. We shall assume that the vorticity within the boundary layer vanishes at $n = \delta$, and that the flow is irrotational for $n \geq \delta$, where n is distance along the normal to the profile, measured from the body surface.

The equations governing the flow will be taken to be

$$\frac{1}{h} u v_s + v v_n - \frac{K}{h} u^2 + \frac{1}{\rho} p_n = 0 \qquad (1)$$

$$u_s + \frac{\partial}{\partial n}(hv) = 0, \quad h = 1 + nK(s) \qquad (2)$$

$$u(s,o) = v(s,o) = o \qquad (3)$$

Here u(x,n) and v(s,n) denote velocity components in the direction of increasing s and n, respectively, h is the linearizing factor in the direction of increasing s, K(s) is the longitudinal curvature of the body, p is the pressure, and ρ is the mass density of the fluid. Subscripts s and n indicate partial differentiation with respect to the indicated variables. The momentum equation in the s-direction will not be required, since we shall assume particular vorticity distributions in its stead. In equation (1), we have assumed that the normal components of the viscous and turbulent stress, other than the mean pressure, may be neglected in comparison with the inertia terms, in accordance with the equations for a thick boundary layer [10]. Equation (3) is the nonslip condition.

In the (s,n) coordinate system, the vorticity ζ is given by

$$\zeta = \frac{1}{h}\left[v_s - \frac{\partial}{\partial n}(hu)\right] = \frac{1}{h}(v_s - hu_n - Ku) \qquad (4)$$

At $n = \delta(s)$, we then have

$$[v_s - (1 + K\delta)u_n - Ku]_{n=\delta} = 0 \qquad (5)$$

The derivatives of u and v with respect to s and n are continuous at $n = \delta$, but the second derivatives are not, unless both ζ_s and ζ_n are zero at that boundary. This can be shown by eliminating u or v between (2) and (4).

An expression for the pressure in terms of vorticity, derived by eliminating v_s between (1) and (4), and then integrating with respect to n and applying the Bernoulli equation at $n = \delta$, is

$$\frac{p}{\rho} = \frac{1}{2}(U_\infty^2 - u^2 - v^2) + \int_n^\delta u\zeta dn \qquad (6)$$

where U_∞ is the free-stream velocity. This gives for the pressure at the wall, p_w,

$$\frac{p_w}{\rho} = \frac{1}{2}U_\infty^2 + \int_0^\delta u\zeta dn$$

with pressure coefficient

$$c_{pw} = \frac{2p_w}{\rho U_\infty^2} = 1 + \frac{2}{U_\infty^2} \int_0^\delta u\zeta dn \qquad (7)$$

The velocity distribution U(s,n), V(s,n) in the outer irrotational flow satisfies the continuity relations

$$U(s,\delta) = u(s,\delta), \quad V(s,\delta) = v(s,\delta) \qquad (8)$$

$$U_n(s,\delta) = u_n(s,\delta), \; V_n(s,\delta) = v_n(s,\delta) \qquad (9)$$

This irrotational flow can be extrapolated into the boundary layer by means of the Taylor expansions

$$U(s,n) = u(s,\delta) + (n-\delta)\, u_n(s,\delta) + \frac{1}{2}(n-\delta)^2\, U_{nn}(s,\delta) + \ldots \qquad (10)$$

$$V(s,n) = v(s,\delta) + (n-\delta)\, v_n(s,\delta) + \frac{1}{2}(n-\delta)^2\, V_{nn}(s,\delta) + \ldots \qquad (11)$$

which yield the extrapolated pressure distribution

$$P = \frac{\rho}{2}\,[U_\infty^2 - U^2(s,n) - V^2(s,n)]$$

and the pressure coefficient at the wall,

$$C_{Pw} = 1 - \frac{U^2(s,o) + V^2(s,o)}{U_\infty^2}. \qquad (12)$$

We shall compare the pressure coefficients given by (7) and (12) for several cases.

Expressions for $U_{nn}(s,\delta)$ and $V_{nn}(s,\delta)$

In order to compute U(s,o) and V(s,o) in (12), we need to obtain expressions for $U_{nn}(s,\delta)$ and $V_{nn}(s,\delta)$ in (10) and (11). Put

$$U_E = U(s,\delta), \quad V_E = V(s,\delta)$$

Then

$$U_E' = U_s + \delta' U_n(s,\delta), \quad V_E' = V_s + \delta' V_n(s,\delta) \qquad (13)$$

in which the prime indicates differentiation with respect to s. We also have, from (2) and (4), with $\zeta = o$,

$$U_s + hV_n = - KV_E \tag{14}$$

$$V_s - hU_n = KU_E \tag{15}$$

Equations (13), (14) and (15) yield the solutions at $n = \delta$,

$$U_s = \frac{1}{D}[h^2U_E' + h\delta'(KU_E - V_E') - K\delta'^2 V_E] \tag{16}$$

$$U_n = \frac{1}{D}[h(V_E' - KU_E) + \delta'(KV_E + U_E')] \tag{17}$$

$$V_s = \frac{1}{D}[h^2V_E' + h\delta'(KV_E + U_E') + K\delta'^2 U_E] \tag{18}$$

$$V_n = - \frac{1}{D}[h(KV_E + U_E'] + \delta'(KU_E - V_E')] \tag{19}$$

where

$$D = (1 + K\delta)^2 + \delta'^2$$

Next, from (13), (14), and (15) we obtain at $n = \delta$,

$$V_{ss} - hU_{sn} = K'\delta U_n + K'U_E + KU_s \tag{20}$$

$$V_{sn} - hU_{nn} = 2KU_n \tag{21}$$

$$U_{ss} + hV_{sn} = - KV_s - K'\delta V_n - K'V \tag{22}$$

$$U_{sn} + hV_{nn} = - 2KV_n \tag{23}$$

$$U_{ss} + 2\delta' U_{sn} + \delta'^2 U_{nn} = U_E'' - \delta'' U_n \tag{24}$$

$$V_{ss} + 2\delta' U_{sn} + \delta'^2 V_{nn} = V_E'' - \delta'' V_n \tag{25}$$

Eliminating U_{sn} and V_{sn} gives

$$h^2V_{nn} + V_{ss} = KU_s + K'\delta U_n - 2hKV_n + K'U_E \quad (26)$$

$$h^2U_{nn} + U_{ss} = - KV_s - K'\delta V_n - 2KhU_n - K'V_E \quad (27)$$

$$U_{ss} + \delta'^2U_{nn} - 2h\delta'V_{nn} = U_E'' + 4K\delta'V_n - \delta''U_n \quad (28)$$

$$V_{ss} + \delta'^2V_{nn} + 2h\delta'U_{nn} = V_E'' - 4K\delta'U_n - \delta''V_n \quad (29)$$

Next, eliminating U_{ss} and V_{ss} gives

$$(h^2-\delta'^2)U_{nn}+2h\delta'V_{nn} = - U_E'' - KV_s + (\delta'' - 2Kh)U_n - (K'\delta + 4K\delta')V_n-K'V_E \quad (30)$$

$$(h^2-\delta'^2)V_{nn} - 2h\delta'U_{nn} = - V_E'' + KU_s + (\delta'' - 2Kh)V_n + (K'\delta + 4K\delta')U_n+K'U_E \quad (31)$$

which yield the solutions

$$U_{nn} = \frac{1}{D^2} \{2h\delta'(V_E'' - K'U_E-KU_s)-(h^2-\delta'^2)(U_E'' +K'V_E+KV_s)$$
$$+[\delta''(h^2-\delta'^2)-2Kh(h^2+3\delta'^2)-2K'h\delta\delta']U_n$$
$$-[K'h^2\delta - K'\delta\delta'^2 - 4K\delta'^3+2h\delta'\delta'']V_n\} \quad (32)$$

$$V_{nn} = \frac{1}{D^2} \{- 2h\delta'(U_E'' + K'V_E+KV_s)+(h^2-\delta'^2)(-V_E''+K'U_E+KU_s)$$
$$+[K'h^2\delta-K'\delta\delta'^2-4K\delta'^3+2h\delta'\delta'']U_n$$
$$+[\delta''(h^2-\delta'^2)-2Kh(h^2+3\delta'^2) - 2K'h\delta\delta']V_n\} \quad (33)$$

or, if (14) and (15) are applied to eliminate U_s and V_s,

$$U_{nn} = \frac{1}{D^2} \{2h\delta'(V_E'' - K'U_E + K^2V_E)-(h^2-\delta'^2)(U_E'' +K'V_E +K^2U_E)$$
$$+[K'\delta(h^2-\delta'^2)-2K\delta'(h^2+2\delta'^2) + 2h\delta'\delta'']V_n$$
$$+[\delta''(h^2-\delta'^2)-Kh(3h^2+5\delta'^2)-2K'h\delta\delta']U_n\} \quad (34)$$

$$V_{nn} = \frac{1}{D^2}\{2h\delta'(U_E'' + K'V_E + K^2U_E)+(h^2-\delta'^2)(-V_E''+K'U_E-K^2V_E)$$

$$+[K'\delta(h^2-\delta'^2)-2K\delta'(h^2+2\delta'^2)+2h\delta'\delta'']U_n$$

$$+[\delta''(h^2-\delta'^2)-Kh(3h^2+5\delta'^2)-2K'h\delta\delta']V_n\} \qquad (35)$$

With U_n and V_n given by (17) and (19), (34) and (35) express U_{nn} and V_{nn} in terms of quantities that are assumed to be known on the boundary $n = \delta(s)$.

Application to Similar Velocity Profiles

Consider velocity profiles of the form

$$u = U_E\ f(\eta),\ \eta = n/\delta(s),\quad 0 \leq \eta \leq 1 \qquad (36)$$

where $f(o) = o$, $f(1) = 1$. Then we have

$$u_s = -\frac{\delta'}{\delta}\eta f'(\eta)\ U_E + U_E' f(\eta) \qquad (37)$$

and hence, by (2),

$$v = \frac{1}{h}\int_0^\eta [\delta' U_E \eta f'(\eta) - \delta U_E' f(\eta)]d\eta$$

or since

$$\int_0^\eta \eta f'(\eta)\ d\eta = \eta f - \int_0^\eta f(\eta)\ d\eta$$

$$v = \frac{1}{h}[\delta' U_E \eta f - (\delta' U_E + \delta U_E')F(\eta)],\ F = \int_0^\eta f d\eta \qquad (38)$$

The partial derivative of (38) with respect to s then gives

$$v_s = \frac{1}{h}\{U_E[\delta''\ \eta f - (\frac{\delta'^2}{\delta} f' + \frac{K'\delta\delta'}{h} f)\eta^2] + 2\delta' U_E' \eta f$$

$$+ [U_E(\frac{K'\delta\delta'}{h}\eta - \delta'') + U_E'(\frac{K'\delta^2}{h}\eta - 2\delta') - \delta U_E'']F(\eta)\} \qquad (39)$$

We also have

$$u_n = \frac{U_E}{\delta} f'(\eta) \qquad (40)$$

Hence, substituting (39) and (40) into (4), we obtain the expression for the vorticity,

$$\zeta(s,\eta) = \frac{1}{h^2} \{U_E[\delta''\eta f - (\frac{\delta'^2}{\delta} f' + \frac{K'\delta\delta'}{h} f)\eta^2 - \frac{h^2}{\delta} f' - Khf] + 2\delta' U_E' \eta f$$
$$+ [U_E(\frac{K'\delta\delta'}{h}\eta - \delta'') + U_E'(\frac{K'\delta^2}{h}\eta - 2\delta') - \delta U_E''] F(\eta)\} \qquad (41)$$

where now

$$h = 1 + \lambda\eta \ , \quad \lambda = K\delta \qquad (42)$$

The condition $\zeta(s,1) = 0$ is then

$$U_E[\alpha(1+\lambda)^2+\lambda(1+\lambda)-\delta_1\delta'' + \delta'^2\alpha+ \frac{K'\delta\delta_1\delta'}{1+\lambda}]$$
$$- U_E'[2\delta_1\delta' + \frac{K'\delta^2}{1+\lambda}(\delta-\delta_1)]+U_E''(\delta-\delta_1) \quad \delta = 0 \qquad (43)$$

where $\alpha = f'(1)$ and $\delta_1 = \delta \int_0^1 (1-f)d\eta$ is the displacement thickness.

We shall also need $U_n(s,\delta)$ and $V_n(s,\delta)$. From (40), (14), (37), and (38), we obtain

$$U_n(s,\delta) = \frac{\alpha}{\delta} U_E \qquad (44)$$

and

$$V_n(s,\delta) = [\frac{\alpha\delta'}{(1+\lambda)} - \frac{K\delta_1\delta'}{(1+\lambda)^2}] \frac{U_E}{\delta} - [\frac{1}{1+\lambda} - \frac{K(\delta-\delta_1)}{(1+\lambda)^2}]U_E' \qquad (45)$$

These results should coincide with those in (17) and (19) when condition (43) is taken into account.

Thin Boundary-Layer Approximation

Assume that $\delta/L << 1$, where L is the length of the body, that KL is of the order O(1), and δ' and δ'' are of the orders $O(\delta/L)$ and $O(\delta/L^2)$ respectively. Also assume that U_E' is of the order $O(U_E/L)$. Then, retaining only terms that are $O(\delta)$ in (43), we obtain the approximate relation

$$\alpha = - \lambda = - K\delta \qquad (46)$$

which, substituted into (44), gives the well-known result [7]

$$U_n(s,\delta) = - KU_E \qquad (47)$$

To the same degree of approximation, the vorticity (41) is

$$\zeta = - \frac{U_E}{\delta} (f' + \lambda f) \qquad (48)$$

The pressure coefficient (7) then becomes

$$\begin{aligned} c_{pw} &= 1 - \frac{2U_E^2}{U_\infty^2} \int_0^1 (ff' + \lambda f^2) d\eta \\ &= 1 - \frac{U_E^2}{U_\infty^2} [1 + 2\lambda \int_0^1 f^2 d\eta] \end{aligned} \qquad (49)$$

For determining the comparable pressure coefficient at the wall for the extrapolated irrotational flow from (12), we have, by (10) and (47),

$$U(s,o) = u(s,\delta) - \delta u_n(s,\delta) = (1+\lambda)U_E \qquad (50)$$

The term $V^2(s,o)$ in (12) is of order $O(\delta^2)$ and may be neglected in the present approximation. Hence, we obtain

$$C_{P_W} = 1 - \frac{U_E^2}{U_\infty^2} (1+\lambda)^2 \doteqdot 1 - \frac{U_E^2}{U_\infty^2} (1+2\lambda) \qquad (51)$$

The difference between the pressure coefficients in (49) and (51) can be expressed in terms of the displacement thickness δ_1 and the momentum thickness δ_2 of the boundary layer,

$$\delta_2 = \delta \int_0^1 f(1-f)d\eta$$

since

$$\int_0^1 f^2 d\eta \equiv 1 - \Delta, \quad \Delta = (\delta_1+\delta_2)/\delta \qquad (52)$$

Thus we have

$$c_{pw} - C_{Pw} = 2\lambda \frac{U_E^2}{U_\infty^2} \Delta = 2K(\delta_1+\delta_2) \frac{U_E^2}{U_\infty^2} \qquad (53)$$

This result was previously given by Sasajima and Tanaka [11]. The pressure coefficient at $n = \delta$ is given by

$$(c_p)_\delta = 1 - \frac{U_E^2 - V_E^2}{U_\infty^2} \doteqdot 1 - \frac{U_E^2}{U_\infty^2}$$

Hence we also have

$$(c_p)_\delta - c_{pw} = 2\lambda \frac{U_E^2}{U_\infty^2} (1-\Delta) = 2K(\delta-\delta_1-\delta_2) \frac{U_E^2}{U_\infty^2} \qquad (54)$$

If the curvature K is small of order δ, then the pressure-coefficient differences given in (53) and (54) would be of second order and could be set equal to zero in the thin boundary-layer approximation. This result agrees with the usual assumption.

Second Approximation

Let us assume that $U_E' = O(\frac{U_E\delta}{L^2})$ and $U_E'' = O(\frac{U_E\delta}{L^3})$. Then, to terms of order $O(\delta/L)$, (41) reduces to

$$\zeta = \frac{U_E}{h^2} [\delta''(\eta f - F) - \frac{f'}{\delta}(\delta'^2\eta^2 + h^2) - Khf] \qquad (55)$$

and (43) becomes

$$\alpha(1+\lambda)^2 + \lambda + \lambda^2 - \delta_1\delta'' = 0$$

or

$$\alpha \doteqdot -\lambda + \lambda^2 + \delta_1\delta'' \qquad (56)$$

Then, from (7) to terms of order $O(\delta^2/L^2)$, we obtain

$$c_{pw} \doteq 1 - \frac{U_E^2}{U_\infty^2}\{1+2\int_0^1[\lambda f^2-\lambda^2\eta f^2+\delta\delta''(Ff-\eta f^2)+\delta'^2\eta^2 ff']d\eta\}$$

or, by applying (52) and

$$\int_0^1 Ffd\eta = \frac{1}{2}F^2\Big|_0^1 = \frac{1}{2}(1-\frac{\delta_1}{\delta})^2, \quad \int_0^1\eta^2 ff'd\eta = \frac{1}{2} - \int_0^1\eta f^2 d\eta$$

$$c_{pw} \doteq 1 - \frac{U_E^2}{U_\infty^2}\{1 + \delta\delta''(1-\frac{\delta_1}{\delta})^2 + \delta'^2+2\int_0^1[\lambda f^2-(\lambda^2+\delta\delta''+\delta'^2)\eta f^2 d\eta\} \quad (57)$$

To determine C_{Pw} we first obtain from (44) and (56),

$$U_n(s,\delta) \doteq -\frac{U_E}{\delta}(\lambda - \lambda^2 - \delta_1\delta'') \quad (58)$$

and from (38),

$$V(s,\delta) \doteq \frac{\delta_1\delta' U_E}{\delta} \quad (59)$$

The contributions to the pressure coefficient from the omitted terms of $V(s,\delta)$ and from $V_n(s,\delta)$ and from $V_{nn}(s,\delta)$ would be of order greater than $O(\delta^2)$. Thus $V_n(s,\delta)$ and $V_{nn}(s,\delta)$ will not be needed. We do, however, require $U_{nn}(s,\delta)$. We obtain, from (32) and (47),

$$U_{nn} \doteq -2KU_n \doteq 2K^2 U_E \quad (60)$$

Then, from (10) and (11), we obtain

$$U(s,o) \doteq U_E(1+\lambda-\delta_1\delta'') \quad (61)$$

$$V(s,o) = U_E\frac{\delta_1\delta'}{\delta} \quad (62)$$

and from (12),

$$C_{Pw} = 1 - (\frac{U_E}{U_\infty})^2 [1+2\lambda+\lambda^2-2\delta_1\delta'' + (\frac{\delta_1\delta'}{\delta})^2] \quad (63)$$

Comparison of (57) and (63) then gives

$$c_{pw} - C_{P_W} = \frac{U_E^2}{U_\infty^2} \Big[2\lambda\Delta + \lambda^2 - \delta\delta''\Big(1+ \frac{\delta_1^2}{\delta^2}\Big) - \delta'^2\Big(1- \frac{\delta_1^2}{\delta^2}\Big) + 2(\lambda^2+\delta\delta''+\delta'^2) \int_0^1 \eta\, f^2\, d\eta\Big] \tag{64}$$

The integral in (64) can be approximately evaluated by assuming the power law

$$f(\eta) = \eta^m \tag{65}$$

which yields

$$\int_0^1 \eta f^2 d\eta \doteq \frac{1}{2m+2}, \quad \frac{\delta_1}{\delta} = \int_0^1 (1-f)d\eta \doteq \frac{m}{m+1} \tag{66}$$

The pressure difference (64) then becomes

$$c_{pw} - C_{P_W} \doteq \frac{U_E^2}{U_\infty^2} \Big[2\lambda\Delta + \lambda^2\Big(2 - \frac{\delta_1}{\delta}\Big) - \frac{\delta_1}{\delta}(\delta'^2+\delta\delta'') + \frac{\delta_1^2}{\delta^2}(\delta'^2-\delta\delta'')\Big] \tag{67}$$

or, if the last two terms are negligible

$$c_{pw} - C_{P_W} \doteq \frac{U_E^2}{U_\infty^2} \Big[2K(\delta_1+\delta_2) + K^2\delta^2\Big(2 - \frac{\delta_1}{\delta}\Big)\Big] \tag{68}$$

If the curvature K is also small, the result (68) would be the same as for the thin boundary layer.

Summary

Expressions have been derived for centerplane source distributions which generate an irrotational flow field about a laterally symmetrical body that matches that exterior to the boundary layer and wake. Results are given for two-dimensional bodies and thin ship forms.

Pressure distributions at the wall of a two-dimensional body have been determined for the flow in the boundary-layer region with and without vorticity. The difference between the pressure coefficients was shown to be principally proportional to the product of the surface curvature of the body by the sum of the displacement and momentum thicknesses.

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