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A Note on the Connection between Ordered Semihyperrings

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Abstract: The notion of ordered semihyperrings is a generalization of ordered semirings and a generalization of semihyperrings. In this paper, the Galois connection between ordered semihyperrings are studied in detail and various interesting results are obtained. A construction of an ordered semihyperring via a regular relation is given. Furthermore, we present the Galois connection between homomorphisms and derivations on an ordered semihyperring.

Keywords: ordered semihyperring; Galois connection; homomorphism; derivation

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1. Introduction

In [1], Rasouli established a connection between stabilizers and Galois connection in residuated lattices. For more details about Galois connections, we refer the reader to Chapter 1 of [2]. Some examples for Galois connections can be found in [2]. Motivated by the recent work of S. Rasouli [1], on Galois connection of stabilizers in residuated lattices, we investigate Galois connections between ordered semihyperrings.

Hypergroups were introduced in 1934 by Marty [3] as group generalizations. The notion of hyperrings and hyperfields was introduced by Krasner [4] as a generalization of rings. Hyperrings and hyperfields were introduced by Krasner in connection with his work on valued fields. In [5], Jun studied algebraic and geometric aspects of Krasner hyperrings. Some principal applications of hyperring theory can be found in Chapter 8 of [6].

Derivations has been of great interest to different fields of science. The notion of derivation in a prime ring first appeared in Posner's classic paper [7]. The study of derivations is an interesting topic in hyperstructure theory. Asokkumar [8] and Kamali Ardekani and Davvaz [9] initiated the study of derivations on hyperrings and prime hyperrings. In 2017, Zhang and Li [10] studied derivations of partially ordered sets. In 2019, Omidi and Davvaz [11] studied derivation in (ordered) hyper(near)-rings. Derivations have been applied in coding theory [12].

The interested reader can find all relevant applications of monoids and categories in wreath products and graphs in [13]. Maximal ideals and congruences of the partial semiring $C^\infty(X)$ of all continuous functions are investigated in [14]. The tropical semiring has various applications, and forms the basis of tropical geometry. In [15], the authors found formulas for the subpolygroup commutativity degree of some polygroups and applied it to the polygroup associated to the dihedral group. The dihedral group D_n is the symmetry group of a regular polygon with n sides.

A mapping $\circ : S \times S \rightarrow \mathcal{P}^*(S)$ is called a *hyperoperation* on S. By *hypergroupoid* we mean a non-empty set S, endowed with a hyperoperation $\circ$. If $\emptyset \neq A, B \subseteq S$ and $x \in S$, then:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if for all $x, y, z \in S$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that:

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

Bijan Davvaz, in Chapter 3 of his book *"Semihypergroup Theory"* [16], studied ordered semihypergroups. An ordered semihypergroup $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order relation $\leq$ that is compatible with the hyperoperation $\circ$. The concept of an ordered semihypergroup was first defined by Heidari and Davvaz [17]. Many researchers worked on the ordered semihypergroups, (see [18–20]). For instance, in 2015, Davvaz et al. studied the pseudoorder of the product of ordered semihypergroups [18]. After that, in [19], Gu and Tang presented basic results of ordered regular equivalence relations on ordered semihypergroups. Later, Tang et al. [20] completely proved the following open problem on ordered semihypergroups.

Open Problem. [18] Is there a regular relation σ on an ordered semihypergroup $(S, \circ, \leq)$ for which S/σ is an ordered semihypergroup?

In [21], Cristea and Kankaraš studied the reducibility of hyperrings. In [22,23], the $(m$-$)k$-hyperideals of an ordered semihyperring are studied. In [23], the concept of derivation on ordered semihyperrings are studied and the relation between m-k-hyperideals and k-hyperideals on ordered semihyperrings are investigated. Omidi and Davvaz [24] tried to make connections between ordered semihyperrings by using 2-hyperideals. Let $(R, +, \cdot, \leq)$ be an ordered semihyerring and I a proper 2-hyperideal of R. Consider the Rees relation σ_I on R as follows:

$$\sigma_I := (I \times I) \cup \{(x, y) \in R \setminus I \times R \setminus I \mid x = y\}$$

By Theorem 3.8 of [24], σ_I is an ordered regular equivalence relation on R. In fact, the authors constructed an ordered regular equivalence relation σ_I on an ordered semihyperring R by a proper 2-hyperideal I of R, such that the corresponding quotient structure is also an ordered semihyperring. In [25], the authors studied ordered semihyperrings to find a strongly regular relation such that the constructed quotient structure is an ordered semiring. Motivated by the above works on ordered semihyperrings [22–25], in this study, we define the concept of an (antitone) Galois connection between ordered semihyperrings; several properties are provided. Furthermore, we discuss the relationship between ordered semihyperrings by using two fundamental notions, the homomorphisms of ordered semihyperrings and the homo-derivations. The study ends with conclusions and ideas for future work.

By a *semihyperring* [26], we mean a triple hypergroup $(R, +, \cdot)$ with two hyperoperations $+$ and $\cdot$ such that:

(1) $(R, +)$ is a (commutative) semihypergroup;
(2) $(R, \cdot)$ is a semi(hyper)group;
(3) $\cdot$ is distributive with respect to the hyperoperation $+$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$.

If there exists an element $0 \in R$ such that $x + 0 = 0 + x = \{x\}$ and $x \cdot 0 = 0 \cdot x = \{0\}$ for all $x \in R$; then 0 is called the *zero element* of R. The *ordered semihyperring* (po-semihyperring), $(R, +, \cdot)$ [25] introduced by Omidi and Davvaz, is defined as follows:

Definition 1. *[25] A semihyperring $(T, \oplus, \odot)$, together with a suitable (partial) order (reflexive, anti-symmetric, and transitive) relation $\leq$ that is compatible with the hyperoperations $\oplus$ and $\odot$, i.e.,*

(1) for any $a, b, x \in T$, $a \leq b$ implies $a \oplus x \preceq b \oplus x$;
(2) for any $a, b, x \in T$, $a \leq b$ implies $a \odot x \preceq b \odot x$ and $x \odot a \preceq x \odot b$,

is called an ordered semihyperring. Here, if U and V are non-empty subsets of R, then:

$$U \preceq V \Leftrightarrow \forall u \in U, \exists v \in V; u \leq v.$$

A non-empty subset T' of an ordered semihyperring $(T, \oplus, \odot, \leq)$ is called a *subsemihyperring* of T if for all $x, y \in T'$, $x \oplus y \subseteq T'$, $x \odot y \subseteq T'$ and $\leq_{T'}$ is the relation $\leq$ restricted to T'.

Definition 2. *[25] Let $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$ be two ordered semihyperrings. The map $\varphi : R \to T$ is called a homomorphism if for all $a, b \in R$, we have:*

(1) $\varphi(a + b) \subseteq \varphi(a) \oplus \varphi(b)$,
(2) $\varphi(a \cdot b) \subseteq \varphi(a) \odot \varphi(b)$,
(3) *φ is isotone, i.e., $a \leq_R b$ implies $\varphi(a) \leq_T \varphi(b)$.*

Furthermore, φ is an order-embedding homomorphism if for all $a, b \in R$, if $\varphi(a) \leq_T \varphi(b)$, then $a \leq_R b$.

Definition 3. *[1,27] Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be ordered sets. Suppose $f : P \to Q$ and $g : Q \to P$ are a pair of functions such that for all $x \in P$ and all $y \in Q$,*

$$x \leq_R g(y) \text{ if and only if } y \leq_T f(x).$$

Then the pair (f, g) is called a antitone Galois connection between $(P, \leq_P)$ and $(Q, \leq_Q)$.

2. Main Results

First of all, we consider an ordered semihyperring, where we define an ordered regular equivalence relation σ^*, such that the quotient is an ordered semihyperring. In fact, we can construct an ordered semihyperring on the quotient set.

Example 1. *Let $R = \{a, b, c, d, e\}$ be a set with two symmetrical hyperoperations $+$ and $\cdot$ and the (partial) order relation $\leq_R$ on R defined as follows:*

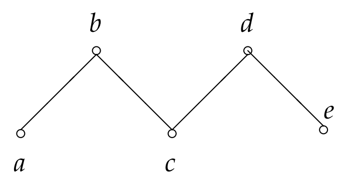
$+$	a	b	c	d	e
a	$\{b,c\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	e
b	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	e
c	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	e
d	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	e
e	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	e

$\cdot$	a	b	c	d	e
a	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$
b	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$
c	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$
d	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$
e	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$

$$\leq_R := \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(c,b),(c,d),(e,d)\}.$$

Then $(R, +, \cdot, \leq_R)$ is an ordered semihyperring. The Hasse diagram and the figure of R are given by:

$$\prec_R = \{(a,b),(c,b),(c,d),(e,d)\}.$$



We can now set:

$$\sigma = \{(a,a),(a,b),(b,a),(b,b),(c,b),(c,c),(c,d),(d,c),(d,d),(e,d),(e,e)\}.$$

Consider the regular equivalence relation on R as follows:

$$\sigma^* = \{(u,v) \in R \times R \mid (u,v) \in \sigma \text{ and } (v,u) \in \sigma\}$$

By definition of σ^**, we obtain:*

$$\sigma^* = \{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e)\}.$$

Then $R/\sigma^* = \{u_1, u_2, u_3\}$*, where* $u_1 = \{a,b\}$*,* $u_2 = \{c,d\}$ *and* $u_3 = \{e\}$*.*
Clearly, $(R/\sigma^*, \oplus, \odot, \preceq_R)$ *is an ordered semihyperring, where* $\oplus$*,* $\odot$ *and* $\preceq_R$ *are defined by:*

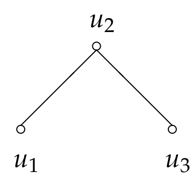
$\oplus$	u_1	u_2	u_3
u_1	$\{u_1,u_2\}$	$\{u_1,u_2\}$	u_3
u_2	$\{u_1,u_2\}$	$\{u_1,u_2\}$	u_3
u_3	$\{u_1,u_2\}$	$\{u_1,u_2\}$	u_3

$\odot$	u_1	u_2	u_3
u_1	$\{u_1,u_2\}$	$\{u_1,u_2\}$	$\{u_1,u_2\}$
u_2	$\{u_1,u_2\}$	$\{u_1,u_2\}$	$\{u_1,u_2\}$
u_3	$\{u_1,u_2\}$	$\{u_1,u_2\}$	$\{u_1,u_2\}$

$$\preceq_R = \{(u_1,u_1),(u_1,u_2),(u_2,u_2),(u_3,u_2),(u_3,u_3)\}.$$

The Hasse diagram and the figure of R/σ^* *are given by:*

$$\prec = \{(u_1,u_2),(u_3,u_2)\}.$$



The aim of this study is to introduce (antitone) the Galois connection between ordered semihyperrings. Moreover, some properties of these connections are investigated.

Theorem 1. *Suppose that* φ *is a homomorphism of an ordered semihyperring* $(R, +, \cdot, \leq_R)$ *onto an ordered semihyperring* $(T, \oplus, \odot, \leq_T)$*. If* φ *is an order-embedding homomorphism, then* φ *is injective.*

Proof. Let $\varphi(x) = \varphi(y)$. Then $\varphi(x) \leq_T \varphi(y)$ and $\varphi(y) \leq_T \varphi(x)$. Since φ is an order-embedding homomorphism, we get $x \leq_R y$ and $y \leq_R x$. This implies that $x = y$. Thus, φ is an injective homomorphism. □

An *isomorphism* from $(R, +, \cdot, \leq_R)$ into $(T, \oplus, \odot, \leq_T)$ is a bijective good homomorphism. Then φ is an isomorphism if and only if φ is a surjection and is an order-embedding homomorphism. We continue this section with the following definition.

Definition 4. *Let* $(R, +, \cdot, \leq_R)$ *and* $(T, \oplus, \odot, \leq_T)$ *be ordered semihyperrings. Suppose* $\varphi : R \to T$ *and* $\psi : T \to R$ *are a pair of functions such that for all* $x \in R$ *and all* $y \in T$*,*

$$x \leq_R \psi(y) \text{ if and only if } \varphi(x) \leq_T y.$$

Then the pair (φ, ψ) *is called a Galois connection between* $(R, +, \cdot, \leq_R)$ *and* $(T, \oplus, \odot, \leq_T)$*.*

Example 2. *Let* φ *be an isomorphism between* $(R, +, \cdot, \leq_R)$ *and* $(T, \oplus, \odot, \leq_T)$*. Then* φ^{-1} *is also one. Thus, we have:*

$$\varphi(x) \leq_T y \text{ if and only if } \varphi^{-1}(\varphi(x)) \leq_R \varphi^{-1}(y).$$

So,

$$\varphi(x) \leq_T y \text{ if and only if } x \leq_R \varphi^{-1}(y).$$

Then (φ, φ^{-1}) *forms a Galois connection between* $(R, +, \cdot, \leq_R)$ *and* $(T, \oplus, \odot, \leq_T)$.

Proposition 1. *Let* $(R, +, \cdot, \leq_R)$, $(T, \oplus, \odot, \leq_T)$ *and* $(S, \uplus, \otimes, \leq_S)$ *be ordered semihyperrings. Suppose* $\varphi : R \to T$ *and* $\psi : T \to R$ *are a pair of functions, such that* (φ, ψ) *is a Galois connection between* $(R, +, \cdot, \leq_R)$ *and* $(T, \oplus, \odot, \leq_T)$. *Let* $\mu : T \to S$ *and* $\theta : S \to T$ *be a pair of functions, such that* (μ, θ) *is a Galois connection between* $(T, \oplus, \odot, \leq_T)$ *and* $(S, \uplus, \otimes, \leq_S)$. *Then,* $(\mu \circ \varphi, \psi \circ \theta)$ *forms a Galois connection between* $(R, +, \cdot, \leq_R)$ *and* $(S, \uplus, \otimes, \leq_S)$.

Proof. Let $x \in R$, $y \in T$ and $z \in S$. Since (φ, ψ) is a Galois connection, then we have:

$$\varphi(x) \leq_T y \text{ if and only if } x \leq_R \psi(y).$$

So,

$$\varphi(x) \leq_T \theta(z) \text{ if and only if } x \leq_R \psi(\theta(z)).$$

On the other hand, by Definition 4, we have:

$$\mu(y) \leq_S z \text{ if and only if } y \leq_T \theta(z).$$

Which implies that:

$$\mu(\varphi(x)) \leq_S z \text{ if and only if } \varphi(x) \leq_T \theta(z).$$

Now, it is easy to see that:

$$\mu(\varphi(x)) \leq_S z \text{ if and only if } x \leq_R \psi(\theta(z)).$$

Therefore, by Definition 4, $(\mu \circ \varphi, \psi \circ \theta)$ forms a Galois connection between $(R, +, \cdot, \leq_R)$ and $(S, \uplus, \otimes, \leq_S)$. □

Theorem 2. *Let* $(R, +, \cdot, \leq_R)$ *and* $(T, \oplus, \odot, \leq_T)$ *be ordered semihyperrings. Suppose* $\varphi : R \to T$ *and* $\psi : T \to R$ *are a pair of functions. Then* (φ, ψ) *is a Galois connection between* $(R, +, \cdot, \leq_R)$ *and* $(T, \oplus, \odot, \leq_T)$ *if and only if:*

(1) *for all* $x \in R, y \in T$, $x \leq_R \psi\varphi(x)$ *and* $y \leq_T \varphi\psi(y)$;
(2) φ *and* ψ *are both isotone.*

Proof. Let (φ, ψ) form a Galois connection. Consider $x \in R$. Since $\leq_T$ is reflexive, we get $\varphi(x) \leq_T \varphi(x)$. As a particular case of Definition 4, we have $x \leq_R \psi\varphi(x)$. Similarly, we have $y \leq_T \varphi\psi(y)$. Now, let $x_1, x_2 \in R$. If $x_1 \leq_R x_2$, then by condition (1), we get $x_2 \leq_R \psi\varphi(x_2)$. Since $\leq_R$ is transitive, it follows that $x_1 \leq_R \psi\varphi(x_2)$. But, by Definition 4, we have $\varphi(x_1) \leq_T \varphi(x_2)$. So, φ is isotone. Similarly, we can show that ψ is isotone.

Conversely, let (1) and (2) be holds. Let $\varphi(x) \leq_T y$ for some $x \in R$ and $y \in T$. Since ψ is isotone, it follows that $\psi(\varphi(x)) \leq_R \psi(y)$. By condition (1), we have $x \leq_R \psi\varphi(x)$. Since $\leq_R$ is transitive, we get $x \leq_R \psi(y)$. Therefore, (φ, ψ) forms a Galois connection between $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$. □

For a semihyperring $(T, \oplus, \odot)$ with (partial) order relation $\leq_T$, a *derivation* on T [23] is a function $d : T \to T$ having the property that (i) $d(x \oplus y) \subseteq d(x) \oplus d(y)$ for all $x, y \in T$; (ii) $d(x \odot y) \subseteq d(x) \odot y \oplus x \odot d(y)$ for all $x, y \in T$ and (iii) d is isotone, i.e., for any $x, y \in T$, $x \leq_T y$ implies $d(x) \leq_T d(y)$. A derivation d on an ordered semihyperring $(R, +, \cdot, \leq)$ is called *positive* if $d(x) \geq 0$ for $x \geq 0$. We say that a positive derivation d on R is a *homo-derivation* on R if $d(a \cdot b) = d(a) \cdot d(b)$. In the following example, we present the Galois connection between homomorphisms and derivations on an ordered semihyperring.

Example 3. *Consider the semihyperring $R = \{0, a, b\}$ with the symmetrical hyperaddition $+$ and the hypermultiplication $\cdot$ defined as follows:*

$+$	0	a	b
0	0	a	b
a	a	a	$\{a, b\}$
b	b	$\{a, b\}$	b

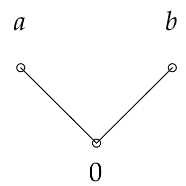
$\cdot$	0	a	b
0	0	0	0
a	0	$\{0, a\}$	$\{0, a\}$
b	0	$\{0, b\}$	$\{0, b\}$

We can now set:

$$\leq_R := \{(0,0), (a,a), (b,b), (0,a), (0,b)\}.$$

Then $(R, +, \cdot, \leq_R)$ is an ordered semihyperring The covering relation and the Hasse diagram of R are given by:

$$\prec = \{(0,a), (0,b)\}$$



Define a function $d : R \to R$ by $d(0) = 0, d(a) = b$ and $d(b) = a$. Now, we have:

$$d(a + a) = d(a) = b = b + b = d(a) + d(a),$$

and:

$$d(a \cdot a) = d(\{0, a\}) = \{0, b\} \subseteq \{0, b\} + \{0, a\} = b \cdot a + a \cdot b = d(a) \cdot a + a \cdot d(a).$$

Furthermore,

$$d(a + b) = d(\{a, b\}) = \{a, b\} = b + a = d(a) + d(b),$$

$$d(a \cdot b) = d(\{0, a\}) = \{0, b\} \subseteq \{0, b\} + \{0, a\} = b \cdot b + a \cdot a = d(a) \cdot b + a \cdot d(b),$$

$$d(b + b) = d(b) = a = a + a = d(b) + d(b),$$

$$d(b \cdot b) = d(\{0, b\}) = \{0, a\} \subseteq \{0, a\} + \{0, b\} = a \cdot b + b \cdot a = d(b) \cdot b + b \cdot d(b),$$

and,

$$d(b \cdot a) = d(\{0, b\}) = \{0, a\} \subseteq \{0, a\} + \{0, b\} = a \cdot a + b \cdot b = d(b) \cdot a + b \cdot d(a).$$

We can easily verify that $x \leq_R y$ implies $d(x) \leq_R d(y)$, for all $x, y \in R$. Hence, d is a derivation on R. On the other hand, we have:

$$d(a \cdot a) = d(\{0, a\}) = \{0, b\} = b \cdot b = d(a) \cdot d(a),$$

$$d(a \cdot b) = d(\{0, a\}) = \{0, b\} = b \cdot a = d(a) \cdot d(b),$$

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$$d(b \cdot b) = d(\{0, b\}) = \{0, a\} = a \cdot a = d(b) \cdot d(b).$$

Hence, d is a homo-derivation on R. By Theorem 2, (d, d^{-1}) is a Galois connection for $(R, +, \cdot, \leq_R)$.

Theorem 3. *Let $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$ be ordered semihyperrings. If (φ, ψ) is a Galois connection between $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$, then:*

(1) $\psi(y) = Max\{x \in R \mid \varphi(x) \leq_T y\}$;
(2) $\varphi(x) = Min\{y \in T \mid x \leq_R \psi(y)\}$.

Proof. (1): By Theorem 2, we have $\varphi(\psi(y)) \leq_T y$. So, $\psi(y) \in \{x \in R \mid \varphi(x) \leq_T y\}$. Now, let $x \in R$ and $\varphi(x) \leq_T y\}$. By Theorem 2, ψ is isotone. It implies that $\psi(\varphi(x)) \leq_R \psi(y)$. On the other hand, $x \leq_R \psi(\varphi(x))$. Therefore, $x \leq_R \psi(y)$ and hence $\psi(y)$ is the maximum of $\{x \in R \mid \varphi(x) \leq_T y\}$.

(2): It is an interesting exercise. □

Definition 5. *Let $(R, +, \cdot, \leq_R)$ be an ordered semihyperring. A closure function for $(R, +, \cdot, \leq_R)$ is a function φ such that, for all $a, b \in R$, we have:*

(1) φ is extensive, i.e., $a \leq_R \varphi(a)$;
(2) If $a \leq_R b$, then $\varphi(a) \leq_R \varphi(b)$;
(3) φ is idempotent, i.e., $\varphi(\varphi(a)) = \varphi(a)$.

Theorem 4. *Let $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$ be ordered semihyperrings. If (φ, ψ) is a Galois connection between $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$, then $\psi\varphi$ is a closure function for $(R, +, \cdot, \leq_R)$.*

Proof. The following are the steps of our definition:

Step 1. By Theorem 2, we have $x \leq_R \psi(\varphi(x))$ for all $x \in R$.

Step 2. By Theorem 2, φ and ψ are both isotone. If $x \leq_R y$, then $\varphi(x) \leq_T \varphi(y)$. It implies that $\psi(\varphi(x) \leq_R \psi(\varphi(y)$.

Step 3. Let $x \in R$. By Theorem 2, we have $x \leq_R \psi(\varphi(x))$. Since φ is isotone, it follows that $\varphi(x) \leq_T \varphi(\psi(\varphi(x)))$. As $\leq_R$ is reflexive, $\psi(\varphi(x)) \leq_R \psi(\varphi(x))$ But by Definition 4 we have:

$$\varphi(\psi(\varphi(x))) \leq_T \varphi(x) \text{ if and only if } \psi(\varphi(x)) \leq_R \psi(\varphi(x)).$$

Since $\leq_T$ is antisymmetric, we get $\varphi(\psi(\varphi(x))) = \varphi(x)$. Thus,

$$\psi(\varphi(\psi(\varphi(x)))) = \psi(\varphi(x)).$$

Therefore, $\psi\varphi$ is a closure function for $(R, +, \cdot, \leq_R)$. □

Let $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$ be ordered semihyperrings. Suppose $\varphi : R \to T$ and $\psi : T \to R$ are a pair of functions. φ is *antitone* if $a \leq_R b$ implies $\varphi(b) \leq_T \varphi(a)$, for all $a, b \in R$. There is an *antitone Galois connection* between $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$, meaning a pair of functions (φ, ψ) such that:

$$x \leq_R \psi(y) \text{ if and only if } y \leq_T \varphi(x).$$

Theorem 5. *Let $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$ be ordered semihyperrings. Suppose $\varphi : R \to T$ and $\psi : T \to R$ are a pair of functions. Then (φ, ψ) is an antitone Galois connection between $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$ if and only if:*

(1) for all $x \in R$, $y \in T$, $x \leq_R \psi\varphi(x)$ and $y \leq_T \varphi\psi(y)$;
(2) φ and ψ are both antitone.

Proof. Let (φ, ψ) form an antitone Galois connection. Consider $x \in R$. Since $\leq_T$ is reflexive, $\varphi(x) \leq_T \varphi(x)$. As a particular case of Definition 4, we have $x \leq_R \psi(\varphi(x))$. Then, $id_R \leq_R \psi\varphi$. Similarly, we have $id_T \leq_T \varphi\psi$. Now, let $x_1, x_2 \in R$. If $x_1 \leq_R x_2$, then by condition (1), we get $x_2 \leq_R \psi(\varphi(x_2))$. Since $\leq_R$ is transitive, it follows that $x_1 \leq_R \psi(\varphi(x_2))$. As (φ, ψ) is an antitone Galois connection, $\varphi(x_2) \leq_T \varphi(x_1)$. Then, φ is antitone. Similarly, we can prove that ψ is antitone.

Conversely, let (1) and (2) hold. Let $x \leq_R \psi(y)$ for some $x \in R$ and $y \in T$. Since φ is antitone, it follows that $\varphi(\psi(y)) \leq_T \varphi(x)$. By condition (1), $id_T \leq_T \varphi\psi$, then we get:

$$y \leq_T \varphi(\psi(y)) \leq_T \varphi(x).$$

Thus, $x \leq_R \psi(y)$ implies $y \leq_T \varphi(x)$. Similarly, $y \leq_T \varphi(x)$ implies $x \leq_R \psi(y)$. Therefore, (φ, ψ) forms an antitone Galois connection between $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$. □

3. Conclusions

In this study, we introduced the notion of (antitone) Galois connections between ordered semihyperrings and obtained some of their useful properties. The results related to homomorphisms were investigated. Moreover, we tried to generalize these results to homo-derivations of ordered semihyperrings. We hope that this work offers a foundation for further study of the (antitone) Galois connection between ordered hyperstructures.

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