Graphs of degree at least 3 with minimum algebraic connectivity

Maryam Abdi [a,*] Ebrahim Ghorbani [b,c]

[a] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran*

[b] *Department of Mathematics, K. N. Toosi University of Technology, P. O. Box 16765-3381, Tehran, Iran*

[c] *Hamburg University of Technology, Institute for Algorithms and Complexity, Germany*

Abstract

Guiduli and Mohar (1996) proposed a conjecture that predicts the structure of connected graphs with minimum degree δ and minimum algebraic connectivity. We settle this conjecture for the case $\delta = 3$. As a result, we conclude that the minimum algebraic connectivity of connected graphs with n vertices and $\delta \geq 3$ is $(1+o(1))\frac{2\pi^2}{n^2}$, where $o(1)$ is a function in n that tends to 0 as n goes to infinity. This enables us to provide a positive answer to the problem of whether graphs with $\delta = 3$ and asymptotic maximum diameter have asymptotic minimum algebraic connectivity.

Keywords: Algebraic connectivity, Minimum degree, Diameter

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1 Introduction

For a simple graph G, $L(G) = D - A$ is its *Laplacian matrix* in which D and A are the diagonal matrix of vertex degrees and the adjacency matrix of G, respectively. We denote by $\mu(G)$ the *algebraic connectivity* of G, which refers to the second smallest eigenvalue of

*Corresponding author

E-mail Addresses: m.abdi@ipm.ir (M. Abdi), ghorbani@tuhh.de (E. Ghorbani)

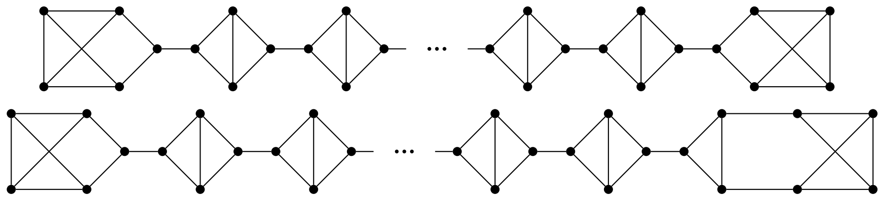

Figure 1: The cubic graphs with minimum algebraic connectivity of order $n \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$, resp.

$L(G)$. A graph G is said to be *μ-minimal* within a specified family $\mathcal{F}$ of connected graphs if G has the minimum algebraic connectivity among all graphs in $\mathcal{F}$ with the same order as G. The context should make the family $\mathcal{F}$ clear. Recall that a *block* of a graph is a maximal connected subgraph with no cut vertex. The blocks of a graph fit together in a tree-like structure, called the *block-tree* of G. When G has at least two blocks and its block-tree is a path, we say that G is *path-like*. In such a case, G has two pendant blocks, which are called *end blocks* of G.

Based on empirical observations by Bussemaker, Čobeljić, Cvetković, and Seidel ([7], see also [8]), L. Babai (see [15]) put forward a conjecture that described the structure of μ-minimal cubic graphs. Guiduli [15] (see also [14, p. 19]) proved that such graphs must be path-like, built from specific blocks. The result of Guiduli was improved later by Brand, Guiduli, and Imrich [6]. They confirmed Babai Conjecture by proving that for any even $n \geq 10$, the n-vertex graphs given in Figure 1 are the unique μ-minimal cubic graphs of order n. Continuing this line of work, Abdi and Ghorbani [2] established that μ-minimal quartic (i.e., 4-regular) graphs have a path-like structure in which the blocks, except for the first and last ones, have the structure shown in Figure 2. In general, in [1], we proposed the following conjecture:

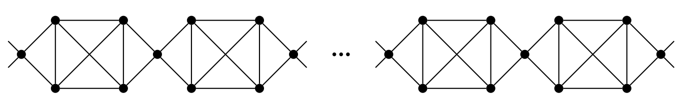

Figure 2: Structure of a μ-minimal quartic graph, excluding its end blocks.

Conjecture 1.1 (Abdi and Ghorbani [1]). For every integer $d \geq 3$, there exist constants C_1 and C_2 such that any μ-minimal d-regular graph G of order at least C_1 is path-like and except for a limited number of blocks at either end containing at most C_2 vertices in total, G has the same structure as Figure 3 for odd d and as Figure 4 for even d.

As mentioned above, Conjecture 1.1 has already been shown to be true for $d = 3$ in [6], for $d = 4$ in [2], and remains open for $d \geq 5$.

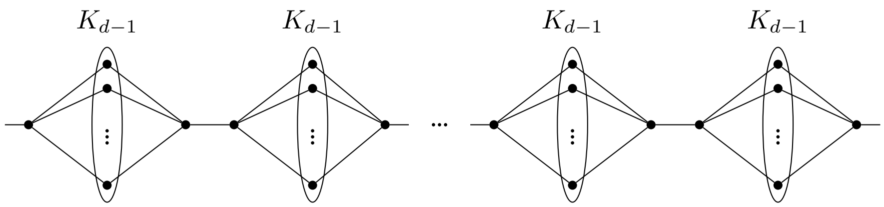

Figure 3: Structure of the conjectured μ-minimal graphs with $\delta = d$, excluding at most a constant number of vertices at either end. Here K_l is the complete graph of order l.

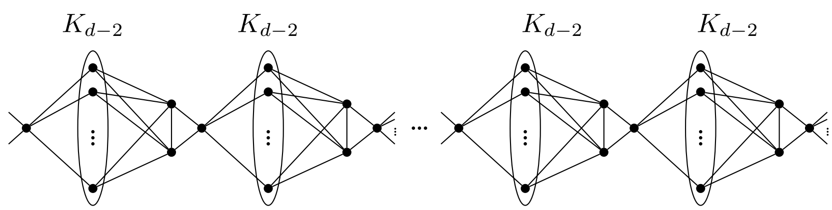

Figure 4: Structure of the conjectured μ-minimal d-regular graphs for even d, excluding at most a constant number of vertices at either end.

The problem addressed in this paper is somewhat dual to the famous problem of finding graphs with the largest algebraic connectivity, which is of interest in the study of graph connectivity and expanders. In that direction, there has been recent work on studying, for a given n and d, which d-regular graphs on n vertices have the largest $\mu(G)$ (see [18]) or, for a given d and $-1 \leq \theta \leq \sqrt{d-1}$, what is the largest order of a d-regular graph G with $\mu(G) \geq d - \theta$ (see [11]).

1.1 Guiduli–Mohar Conjecture

Guiduli and Mohar (see [14, p. 88]) proposed another generalization of Babai Conjecture by considering graphs with minimum degree $\delta = d$ rather than d-regular graphs. They put forward the following conjecture on the structure of μ-minimal graphs with $\delta = d$.

Conjecture 1.2 (Guiduli and Mohar [14, p. 88])**.** Let G be a μ-minimal graph with $\delta = d$. Then G is path-like, and except for some blocks near each end, the graph has the structure demonstrated in Figure 3.

A more precise phrasing of Conjecture 1.2 is that for every integer d, there exist constants C_1 and C_2 such that any μ-minimal graph with $\delta = d$ and order at least C_1 is path-like and except for a limited number of blocks positioned at either end of the path representing the block-tree of G and containing at most C_2 vertices in total, the remaining blocks exhibit the structure of Figure 3.

The *relaxation time* of the random walk on a graph G is defined by $\tau = 1/(1-\eta_2)$, where η_2 is the second largest eigenvalue of the *transition matrix* of G, that is the matrix $D^{-1}A$. A central problem in the study of random walks is to determine the *mixing time*, a measure of how fast the random walk converges to the stationary distribution. As seen through the literature [5, 10], the relaxation time is the primary term controlling mixing time. Therefore, relaxation time is directly associated with the rate of convergence of the random walk. Guiduli–Mohar Conjecture is closely linked to the following conjecture by Aldous and Fill.

Conjecture 1.3 (Aldous and Fill [5, p. 217])**.** Over all regular graphs on n vertices, $\max \tau = (1+o(1))\frac{3n^2}{2\pi^2}$.

In [1], we proved that Guiduli–Mohar Conjecture implies Aldous–Fill Conjecture for odd d. More generally, we showed that Conjecture 1.1 implies Aldous–Fill Conjecture. Furthermore, Aldous–Fill Conjecture was proved to be true for $d = 3$ in [3] and for $d = 4$ in [2]. Noteworthy is also the result presented in [4], where it was proven that the maximum relaxation time for the random walk on a general graph on n vertices is $(1+o(1))\frac{n^3}{54}$, settling another conjecture by Aldous and Fill ([5, p. 216]).

The following theorem is our main result that settles Guiduli–Mohar Conjecture for $d = 3$. In fact, our result is even stronger as it pertains to graphs with $\delta \geq 3$, rather than just those with $\delta = 3$.

Theorem 1.4. *Among all connected graphs with given order and $\delta \geq 3$, the graph with minimum algebraic connectivity is path-like and except for at most four blocks at each end, it has the structure depicted in Figure 5.*

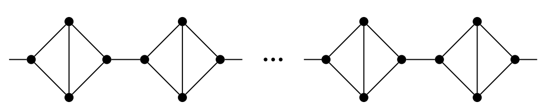

Figure 5: Structure of a μ-minimal graph with $\delta \geq 3$, excluding at most 27 vertices at both ends.

It is worth noting that the exceptional blocks, mentioned in Theorem 1.4, at either end contain collectively at most 27 vertices; see Remark 4.12 below.

In [1], we established that the algebraic connectivity of an n-vertex graph with the structure of Figure 3 is $(1+o(1))\frac{(d-1)\pi^2}{n^2}$. From this result and Theorem 1.4, we deduce the following quantitative version:

Theorem 1.5. *The minimum algebraic connectivity of connected graphs with n vertices and $\delta \geq 3$ is $(1+o(1))\frac{2\pi^2}{n^2}$.*

We point out that the μ-minimal graphs proposed in Conjecture 1.2 also show up in the context of maximizing the spectral radius (of the adjacency matrix) among non-regular graphs with fixed maximum degree. See [16, 17] for further details.

1.2 Diameter and algebraic connectivity

The diameter of a graph can be bounded in terms of its order and minimum degree. For $d \geq 3$ and $n \geq 2d+4$, the maximum diameter of a graph with order n and $\delta = d$ is $3\lfloor \frac{n}{d+1} \rfloor - \ell$ for some $\ell \in \{1,2,3\}$ (see [9]). The hypothetical μ-minimal graphs of Conjecture 1.2 have maximum diameter among the graphs with $\delta = d$ (see [1]). This observation suggests a potential interplay between graphs with maximum diameter and those with minimum algebraic connectivity, which has already been noted in the literature. Guiduli [14, p. 46] showed that the unique cubic graph with minimum μ has the maximum diameter among cubic graphs of order n. He then posed the converse as a problem (see [14, p. 87]): Is it true that the cubic graphs with maximal diameter have algebraic connectivity smaller than all others? Abdi and Ghorbani [1] showed that the answer to this problem in its general form, that is for d-regular graphs for every $d \geq 3$, is negative. They also considered the asymptotic formulation of the problem, which asks, for a fixed d, whether d-regular graphs or graphs with $\delta = d$ with asymptotic maximum diameter (that is with diameter $3n/(d+1) + O(1)$) have asymptotically minimum algebraic connectivity (i.e., they have $\mu = (1 + o(1))\mu_{\min}$).[1] They showed that:

- This holds true for 3- and 4-regular graphs.
- It is not true for d-regular graphs with $d \geq 5$ or graphs with $\delta = d$ for $d \geq 4$.

The case of graphs with $\delta = 3$ remained open:

Problem 1.6 (Abdi and Ghorbani [1])**.** Is it true that graphs with $\delta = 3$ and diameter $3n/4 + O(1)$ have algebraic connectivity $(1 + o(1))\frac{2\pi^2}{n^2}$?

We demonstrate a positive answer to Problem 1.6 as a corollary of Theorem 1.5.

It is natural to consider the converse of the question discussed above, which is conjectured to hold true.

Conjecture 1.7 (Abdi and Ghorbani [1])**.** *For any $d \geq 3$ if Γ_n is a sequence of graphs of $\delta = d$ (or a sequence of d-regular graphs) with asymptotic minimum algebraic connectivity, then it has asymptotically maximum diameter that is $3n/(d+1) + O(1)$.*

[1] Here, $\mu_{\min} = \mu_{\min}(n)$ is the minimum μ among all n-vertex d-regular graphs or graphs with $\delta = d$, respectively.

The rest of the paper is organized as follows. Section 2 provides basic results which will be used throughout the paper. In Section 3, we extract several structural properties of general μ-minimal graphs with $\delta = d$. In Section 4, we derive the structure of μ-minimal graphs with $\delta = 3$ and prove Theorem 1.4. Finally, in Section 5, we address Problem 1.6.

2 Preliminaries

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. It is well-known that $L(G)$ is a positive semidefinite matrix; it always possesses a 0 eigenvalue with eigenvector $\mathbf{1}$, the all-1 vector. Moreover, the multiplicity of the eigenvalue 0 is precisely the number of connected components of G, thus indicating that $\mu(G) > 0$ if and only if G is connected. An eigenvector corresponding to $\mu(G)$ is referred to as a *Fiedler vector* of G. For any vector, which is represented as a function $f : V(G) \longrightarrow \mathbb{R}$, the value $\frac{f^\top L(G) f}{\|f\|^2}$ is called a *Rayleigh quotient*. It is known that

$$\mu(G) = \min_{f \neq \mathbf{0},\, f \perp \mathbf{1}} \frac{f^\top L(G) f}{\|f\|^2}. \tag{1}$$

This implies that $f \neq \mathbf{0}$ is a Fiedler vector of G if and only if $f \perp \mathbf{1}$ and $\mu(G) = \frac{f^\top L(G) f}{\|f\|^2}$. The quantity $f^\top L(G) f$ can be expressed in the following useful manner:

$$f^\top L(G) f = \sum_{uv \in E(G)} (f(u) - f(v))^2. \tag{2}$$

If f is an eigenvector corresponding to μ, then from the equation $L(G)f = \mu f$, it can be inferred that for any vertex v,

$$\mu f(v) = \deg(v) f(v) - \sum_{u:\, vu \in E(G)} f(u). \tag{3}$$

We refer to (3) as the *eigen-equation*.

By (1), $\mu(G)$ is determined by the set of vectors that are orthogonal to $\mathbf{1}$. The following lemma demonstrates that vectors that are not necessarily orthogonal to $\mathbf{1}$ can also be utilized to obtain potentially good approximations for $\mu(G)$. It is called *eigenfunction pushing technique* in [13]. The notation $\langle \cdot, \cdot \rangle$ as usual denotes the standard inner product of real vectors.

Lemma 2.1 ([2, 13])**.** *Let G be a graph of order n, f an arbitrary vector of length n which is not a multiple of $\mathbf{1}$ and $\theta = \langle f, \mathbf{1} \rangle$. Then*

$$\mu(G) \leq \frac{f^\top L(G) f}{\|f\|^2 - \frac{\theta^2}{n}}.$$

The following lemma is a useful tool that allows us to modify specific parts of a graph, thereby reducing the algebraic connectivity under certain conditions. A *unit vector* is a vector with magnitude 1. Also, when H is a subgraph of G, by $G-H$ we mean the graph obtained from G by deleting all the vertices of $V(H)$ together with all the edges with at least one end in $V(H)$.

Lemma 2.2. *Let G be a connected graph and f be a unit Fiedler vector of G. Let H be a subgraph of G such that $G-H$ has two connected components G_1 and G_2, and H' be a graph with the same number of vertices as H. In G we replace H by H' to obtain a new connected graph G'. Suppose that there is a vector f' on $V(G')$ such that*

$$\ell := \sum_{uv\in E(G)\setminus E(H)} (f(u)-f(v))^2 = \sum_{uv\in E(G')\setminus E(H')} (f'(u)-f'(v))^2.$$

Set

$$h := \sum_{uv\in E(H)} (f(u)-f(v))^2, \quad h' := \sum_{uv\in E(H')} (f'(u)-f'(v))^2, \quad \text{and} \quad \epsilon := \|f'\|^2 - \frac{\theta^2}{n} - 1,$$

where $\theta = \langle f', \mathbf{1}\rangle$. If $h'-h-\epsilon\mu(G) < 0$, then $\mu(G') < \mu(G)$.

Proof. We have $\mu(G) = f^\top L(G) f = \ell + h$, and by Lemma 2.1,

$$\mu(G') \le \frac{f'^\top L(G') f'}{\|f'\|^2 - \frac{\theta^2}{n}} = \frac{\ell + h'}{1+\epsilon}.$$

It follows that if $h'-h-\epsilon\mu(G) < 0$, then $\mu(G') < \mu(G)$. □

The symbols '$\sim$' and '$\nsim$' respectively represent adjacency and non-adjacency in graphs. When $ab \in E(G)$, we use the notation $G-ab$ to refer to the graph obtained by removing ab from G and when $ab \notin E(G)$, $G+ab$ represents the graph obtained by adding the edge ab to G. An *elementary move* or *switching* in G involves the swapping of parallel edges as follows: given $a \sim b$, $c \sim d$, $a \nsim c$, and $b \nsim d$, the move $\mathsf{sw}(a,b,c,d)$ removes the edges ab and cd and replaces them with the edges ac and bd. The subsequent lemma, originally presented in [15], demonstrates how switching can be utilized to decrease the algebraic connectivity. Our contribution to this result is the addition of the equality case. We present a complete proof below.

Lemma 2.3. *Let G be a connected graph and f be a Fiedler vector of G. If there are vertices a,b,c,d in G such that $a\sim b$, $c \sim d$, $a \nsim c$, $b \nsim d$, with $f(a) \ge f(d)$ and $f(c) \ge f(b)$, then $\mathsf{sw}(a,b,c,d)$ does not increase the algebraic connectivity. The algebraic connectivity does not change if and only if $f(a) = f(d)$ and $f(c) = f(b)$.*

Proof. We may suppose that $\|f\| = 1$. Let G' be the resulting graph obtained after switching. By considering the Rayleigh quotient (1), it suffices to show that $f^\top L(G)f \geq f^\top L(G')f$. This holds because $f^\top L(G)f - f^\top L(G')f = 2\,(f(c) - f(b))\,(f(a) - f(d)) \geq 0$. Thus $\mu(G) = \mu(G') := \mu$ if and only if $(f(c) - f(b))\,(f(a) - f(d)) = 0$ and f is an eigenvector of G'. Our aim is to show that under the above condition, if one the terms $(f(c) - f(b))$ or $(f(a) - f(d))$ is zero, then the other term must also be zero. Without loss of the generality, assume that $f(a) = f(d)$ while $f(c) \neq f(b)$. By applying the eigen-equation to the vertex a in G and G', respectively, we have

$$\mu f(a) = \deg_G(a) f(a) - \sum_{v\colon va \in E(G)} f(v),$$

and

$$\mu f(a) = \deg_G(a) f(a) - \sum_{v\colon va \in E(G)} f(v) + f(b) - f(c).$$

Comparing these equations, we get $f(c) = f(b)$, a contradiction. □

Let r be a real number. We define the induced subgraphs G_r^+ and G_r^- as follows:

$$G_r^+ := G\left[\{v \in V(G) : f(v) \geq r\}\right], \quad \text{and} \quad G_r^- := G\left[\{v \in V(G) : f(v) \leq r\}\right].$$

Lemma 2.4 ([12]). *Let G be a connected graph and f be a Fiedler vector of G. If $r \geq 0$, then G_{-r}^+ and G_r^- are connected.*

Lemma 2.5 ([12]). *Let G be a connected graph and f be a Fiedler vector of G. If $f(u) > 0$, then there exists a vertex v such that $u \sim v$ and $f(v) < f(u)$.*

3 Properties of general μ-minimal graphs

In this section, we extract some key structural properties of μ-minimal graphs that are essential for the proof of our main result. Our first aim is to establish that we can restrict ourselves to μ-minimal graphs in which the vertices with equal components under a Fiedler vector induce a clique.

We use the notation $\mathcal{G}_{n,d}$ to denote the family of connected graphs of order n and $\delta \geq d$.

Lemma 3.1. *Let $d \geq 2$, $H \in \mathcal{G}_{n,d+1}$, and h be a unit Fieldler vector of H. If every pair of distinct vertices u, v with $h(u) = h(v)$ are adjacent, then there is a graph $G \in \mathcal{G}_{n,d}$ with $\mu(G) < \mu(H)$.*

Proof. Let $P = v_1 v_2 \dots v_t$ be a longest path in H. First assume that $h(v_1) = h(v_t)$. Then $v_1 \sim v_t$, so H contains a cycle C of length t. If $t < n$, then there is a vertex not in C but adjacent to some vertex in C. This gives a longer path than P, a contradiction. Thus $t = n$, that is H is hamiltonian. As h is orthogonal to $\mathbf{1}$, there exists a vertex v with $h(v) > 0$. By Lemma 2.5, v has a neighbor w with $h(w) < h(v)$. Remove $\deg(v) - d$ edges incident to v including vw. The resulting graph G is connected (since in G, v is adjacent to d vertices of $C - v$ and all other vertices lie on $C - v$). Also $\delta(G) = \deg_G(v) = d$, and in view of (1) and (2), $\mu(G) \leq \mu(H) - (h(v) - h(w))^2 < \mu(H)$.

Next, suppose that $h(v_1) \neq h(v_t)$, so at least one of them is non-zero. We can assume that $h(v_1) > 0$ (if $h(v_1) < 0$, then consider $-h$ instead). So by Lemma 2.5, v_1 has a neighbor w with $h(w) < h(v_1)$. We remove $\deg(v_1) - d$ edges incident to v_1 including wv_1. As all the neighbors of v_1 are already in P, the resulting graph G is connected. Moreover, $\delta(G) = \deg_G(v_1) = d$ and $\mu(G) \leq \mu(H) - (h(v_1) - h(w))^2 < \mu(H)$. □

Lemma 3.2. *Let $d \geq 2$, G be a μ-minimal graph with $\delta(G) = d$, and g be a unit Fieldler vector of G. Then, the graph H obtained from G by adding the edges $\{uv : g(u) = g(v)\}$ is also a μ-minimal graph with $\delta(H) = d$.*

Proof. We have

$$\mu(G) \leq \mu(H) \leq \sum_{uv \in E(H)} (g(u) - g(v))^2 = \sum_{uv \in E(G)} (g(u) - v(v))^2 = \mu(G).$$

Therefore, $\mu(H) = \mu(G)$ and g is a Fiedler vector for H too. It remains to show that $\delta(H) = d$. If $\delta(H) \geq d+1$, then $H \in \mathcal{G}_{n,d+1}$, and so it satisfies the conditions of Lemma 3.1. From this and minimality of G, it follows that $\mu(H) > \mu(G)$, a contradiction. □

Throughout the remainder of this section, we assume that Γ is a graph with the minimum algebraic connectivity among all connected graphs with n vertices and $\delta = d$ in which the vertices with equal components induce a clique. By the term *component* of a vertex v, we refer to $f(v)$, where f is assumed to be a fixed unit Fiedler vector of Γ. Lemma 3.2 guarantees the existence of such a Γ, and the subsequent lemma shows that, to prove Conjecture 1.2, it is sufficient to establish it for such a Γ.

Lemma 3.3. *If Conjecture 1.2 is true for μ-minimal graphs in which vertices with equal components are adjacent, then it is true for all μ-minimal graphs.*

Proof. Let G be a μ-minimal graph with $\delta(G) = d$, and $V = V(G)$. According to Lemma 3.2, G is a subgraph of a graph Γ that is also a μ-minimal graph with the same vertex set V and $\delta(\Gamma) = d$. Let us assume that Conjecture 1.2 is true for Γ. Hence, for some constant C_2, Γ is path-like and for a subset $U \subset V$ with $|U| \leq C_2$ containing some blocks from both ends of Γ, the vertices in $V \setminus U$ induce a graph as depicted in Figure 3. Consequently, for all $v \in V \setminus U$, $\deg_\Gamma(v) = d$. Since $\delta(G) = d$, we must have $\deg_G(v) = d$ for all $v \in V \setminus U$ too. Thus, we can conclude that G and Γ coincide on $V \setminus U$, completing the proof. □

Next theorem demonstrates that Γ is not only μ-minimal among the graphs with $\delta = d$ but also among the graphs with $\delta \geq d$.

Theorem 3.4. *The graph Γ has minimum algebraic connectivity in $\mathcal{G}_{n,d}$. More precisely, if $\Gamma' \in \mathcal{G}_{n,d+1}$, then $\mu(\Gamma') > \mu(\Gamma)$.*

Proof. Let Γ' be a μ-minimal graph in $\mathcal{G}_{n,d+1}$. By Lemma 3.2, we may assume that in Γ' the vertices with equal components are adjacent. Now from Lemma 3.1, it follows that $\mu(\Gamma') > \mu(\Gamma)$. □

Our next goal is to demonstrate that f is strictly monotone on the parts of an appropriate distance-partition of Γ.

We use the notation $N(v)$ for the neighborhood of a vertex v, $N[v] := N(v) \cup \{v\}$ and $\triangle$ to denote symmetric difference.

Lemma 3.5. *Let $u, v \in V(\Gamma)$ with $f(u) = f(v)$ such that $N[u] \triangle N[v] \neq \emptyset$. Then* $\deg(u) = \deg(v) = d$*, and all the vertices in $N[u] \triangle N[v]$ have equal components.*

Proof. As $f(u) = f(v)$, $u \sim v$. By the assumption, there is a vertex w such that $v \sim w \nsim u$. If $\deg_\Gamma(v) > d$, then set $\Gamma' := \Gamma - wv + wu$. As $u \sim v$, the path wuv is present in Γ', so Γ' is connected. We have $\mu(\Gamma') \leq \mu(\Gamma)$, and so by Theorem 3.4, we must have $\mu(\Gamma') = \mu(\Gamma) =: \mu$ and f is a Fiedler vector of Γ'. By applying the eigen-equation to the vertex v in Γ and Γ', respectively, we obtain

$$\mu f(v) = \deg_\Gamma(v) f(v) - \sum_{x\colon vx \in E(\Gamma')} f(x) - f(w),$$

and

$$\mu f(v) = (\deg_\Gamma(v) - 1)\, f(v) - \sum_{x\colon vx \in E(\Gamma')} f(x).$$

Comparing these equations, we get $f(v) = f(w)$, so $f(u) = f(w)$. Hence we must have $u \sim w$. But this contradicts the assumption that $u \nsim w$ implying that $\deg_\Gamma(v) = d$. Next, if $\deg_\Gamma(u) > d$, then u has a neighbor not in $N(v) \cup \{v\}$ and we can obtain a contradiction similarly. So $\deg_\Gamma(u) = d$.

Let $N[u]\triangle N[v] = \{y_1, \dots, y_{2t}\}$ and

$$f(y_1) \geq f(y_2) \geq \cdots \geq f(y_{2t}). \tag{4}$$

By performing a switching, we can obtain a graph Γ' in which $\{y_1, \dots, y_t\} \subset N_{\Gamma'}(v)$, $\{y_{t+1}, \dots, y_{2t}\} \subset N_{\Gamma'}(u)$, and $\mu(\Gamma') \leq \mu(\Gamma)$. Since $u \sim v$, the graph Γ' is connected. The minimality of $\mu(\Gamma)$ now implies that $\mu(\Gamma') = \mu(\Gamma)$, with f being a Fiedler vector of Γ'. Applying the eigen-equation to vertices u and v in Γ' yields $f(y_1) + \cdots + f(y_t) = f(y_{t+1}) + \cdots + f(y_{2t})$; in view of (4) this is possible only if $f(y_1) = f(y_2) = \cdots = f(y_{2t})$. □

Lemma 3.6. *Let $u, v \in V(\Gamma)$ with $f(u) = f(v)$. Then* $\deg(u) = \deg(v)$. *Moreover, if $w \sim v$, then u has a neighbor w' with $f(w') = f(w)$.*

Proof. If $N[u] = N[v]$, there is nothing to prove. Otherwise, the result follows from Lemma 3.5. □

According to Lemma 2.4, for $r \geq 0$, the subgraphs G^+_{-r} and G^-_r are connected. Both these graphs include all vertices with zero components. We show that this property can be refined for subgraphs of our minimal graph Γ by eliminating the necessity to include vertices with zero components.

Lemma 3.7. *For all $r \in \mathbb{R}$, Γ^+_r and Γ^-_r are connected (if not empty).*

Proof. We show that Γ^+_r is connected, for Γ^-_r we can consider $-f$ as a Fiedler vector. If $r \leq 0$, then Γ^+_r is connected by Lemma 2.4. Now, suppose that there exists $r > 0$ such that Γ^+_r is disconnected with (at least) two connected components Γ_1 and Γ_2. Let s be the maximum component of f such that $s < r$. By Lemma 2.4, Γ^-_s is connected. Let $f(u)$ be the minimum component of f in Γ^+_r. With no loss of generality assume that $u \in V(\Gamma_2)$. Some vertex w of Γ_1 has to have a neighbor v in Γ^-_s. By minimality of $f(u)$ and since $u \nsim w$, $f(u) < f(w)$. By Lemma 2.5, u has a neighbor in Γ^-_s. First, suppose that $u \nsim v$. We have $f(v) < f(u) < f(w)$ implying that $(f(w) - f(v))^2 > (f(w) - f(u))^2 + (f(u) - f(v))^2$. So $\Gamma - wv + wu + uv \in \mathcal{G}_{n,d}$ with $\mu(\Gamma') < \mu(\Gamma)$, which contradicts Theorem 3.4. Hence, we must have $u \sim v$. If $\deg(v) > d$, then $\mu(\Gamma - wv + wu) < \mu(\Gamma)$. So $\deg(v) = d$. This in turn implies that u has a neighbor y with $y \nsim v$. If $f(w) < f(y)$, then consider the graph $\Gamma - uy + uw + wy$. Otherwise perform $\mathsf{sw}(w, v, u, y)$. In either cases the resulting graph has $\mu < \mu(\Gamma)$, a contradiction. □

Definition 3.8. Let S_0 be the set of vertices of Γ with maximum component and $\mathcal{S}_f = \{S_0, \dots, S_k\}$ be the *distance-partition* of $V(\Gamma)$ with respect to S_0, that is, S_i is the set of vertices in $V(\Gamma) \setminus \bigcup_{j=0}^{i-1} S_j$ which are adjacent to some vertex in S_{i-1}.

The following theorem shows that the components of f on the parts of $\mathcal{S}_f$ form a strictly decreasing sequence that changes sign exactly once.

Theorem 3.9. *For any two vertices $v_i \in S_i$ and $v_j \in S_j$ with $i > j$, we have $f(v_i) > f(v_j)$.*

Proof. By the definition of $\mathcal{S}_f$, the statement holds for $i = 0$. Let $i \geq 1$ and by induction assume that the statement holds for all $j < i$. Let v be a vertex with smallest component in S_i and let $u \in S_{i-1}$ with $u \sim v$. Toward a contradiction, assume that there exists a vertex $w \in \bigcup_{j=i+1}^{k} S_j$ such that $f(w) \geq f(v)$. By our induction hypothesis, $f(v) \leq f(w) < f(u)$. If $f(v) = f(w)$, by Lemma 3.6, w has a neighbor y with $f(y) = f(u)$. Therefore $u \sim y$ and we must have $y \in S_{i-1}$ and consequently $w \in S_i$ which is impossible. Therefore, $f(v) < f(w) < f(u)$. Also $w \nsim u$ as $w \notin S_i$.

First, suppose that $\deg(v) > d$. We have $\mu(\Gamma - uv + uw) < \mu(\Gamma)$. By Lemma 3.7, we know that $\Gamma^-_{f(w)}$ is connected, so there exists a path from w to v in $\Gamma^-_{f(w)}$. This path is preserved in $\Gamma - uv + uw$, which shows that $\Gamma - uv + uw$ is connected. Thus, we obtain a contradiction by Theorem 3.4.

Next, assume that $\deg(v) = d$. If $w \nsim v$, then $\Gamma - uv + uw + wv \in \mathcal{G}_{n,d}$ is a graph with $\mu < \mu(\Gamma)$. Therefore, we must have $w \sim v$. As $\deg(w) \geq \deg(v) = d$ and $v \sim u \nsim w$, w has a neighbor y such that $y \nsim v$. If $f(u) > f(y)$, then applying $\mathsf{sw}(u, v, w, y)$ results in a graph with $\mu < \mu(\Gamma)$. Thus, we must have $f(u) \leq f(y)$. If $u \nsim y$, then $f(u) < f(y)$. In this case, $\Gamma - wy + wu + uy \in \mathcal{G}_{n,d}$ is a graph with $\mu < \mu(\Gamma)$. Therefore, $u \sim y$. Since $f(y) \geq f(u)$, the induction hypothesis implies that $y \in S_{i-1}$, and consequently, $w \in S_i$, which leads to a contradiction. □

4 Structure of μ-minimal graphs with $\delta = 3$

In this section, we present the proof of Theorem 1.4. We assume that Γ is a graph with minimum algebraic connectivity among all connected graphs with order n and minimum degree $\delta \geq 3$. Based on Theorem 3.4, the minimum algebraic connectivity among connected graphs with a minimum degree of $\delta \geq 3$ is achieved only by graphs with a minimum degree of $\delta = 3$. Therefore, we can focus solely on graphs where the minimum degree is equal to 3.

4.1 Path-like structure

In this subsection, we will demonstrate that Γ has a path-like structure. Throughout this section, we will be utilizing the distance-partition $\{S_0, \ldots, S_k\}$ of $V(\Gamma)$ as defined in

Definition 3.8.

Lemma 4.1. *For* $i \in \{2, \dots, k-1\}$, $|S_i| \leq 2$, *for* $i \in \{1, k\}$, $|S_i| \leq 4$, *and* $|S_0| \leq 3$.

Proof. Let $i \in \{2, \dots, k-1\}$ and u be a vertex with minimum component in S_{i-1}.

First, assume that all vertices of S_i are adjacent to u. Let $|S_i| > 2$. By construction, u has a neighbor in S_{i-2}, so $\deg(u) \geq 4$. Let v be a vertex with minimum component in S_i. By Lemma 3.7, $\Gamma^-_{f(v)}$ is connected, and so we can assume that v has a neighbor in S_{i+1}. If v has non-neighbor $w \in S_i$, then $\Gamma - uv + vw$ is a graph in $\mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$, which contradicts Theorem 3.4. Thus, we have v adjacent to all vertices in S_i, which implies that $\deg(v) \geq 4$. Then $\Gamma - uv$ is another graph in $\mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$. Therefore, we must have $|S_i| \leq 2$, and we are done.

In the remainder of the proof, we assume that u has a non-neighbor $v \in S_i$. Let w be a vertex with minimum component among the neighbors of v in S_{i-1}.

If $f(w) = f(u)$, then $u \sim w$ and as $v \in N[w] \triangle N[u]$, by Lemma 3.5, $\deg(w) = \deg(u) = 3$. If $(N[w] \triangle N[u]) \cap S_{i-2} \neq \emptyset$, then again by Lemma 3.5, some vertex in S_{i-2} has an equal component with v which is impossible by Theorem 3.9. So $(N[w] \triangle N[u]) \cap S_{i-2} = \emptyset$ which means that w and u have a common neighbor in S_{i-2} and further u has a neighbor $z \in S_i \setminus \{v\}$ with $f(z) = f(v)$. Let $p \in S_i \setminus \{v, z\}$ which has a neighbor $y \in S_{i-1}$. We have $y \neq w, u$ since otherwise the degree of either w or u would be greater than 3. Now $\Gamma' := \Gamma - yp + yu + up$ is a graph in $\mathcal{G}_{n,3}$. As $f(y) > f(u) > f(p)$, we have $\mu(\Gamma') < \mu(\Gamma)$. This contradiction implies that $S_i = \{v, z\}$ and thus, we are done.

From this point onward, we will assume that $f(w) > f(u)$. Let us set

$$S'_{i-1} := \{p \in S_{i-1} : f(p) < f(w)\}, \quad S'_i := \{p \in S_i : f(p) > f(v)\}.$$

In the following Steps 1–5, we will establish that $|S_i| = 2$ for $i \in \{2, \dots, k-1\}$.

Step 1. $\deg(w) = 3$, $|S'_{i-1}| = 1$ and $w \sim u$: If $\deg(w) > 3$, let $\Gamma' := \Gamma - vw + vu$. Then $\mu(\Gamma') < \mu(\Gamma)$ and by Lemma 3.7, $\Gamma^+_{f(u)}$ is connected, so $\Gamma' \in \mathcal{G}_{n,3}$. Therefore, $\deg(w) = 3$. Let $p \in S'_{i-1}$. If $w \nsim p$, then $\Gamma - wv + wp + pv$ is a graph in $\mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$. So w is adjacent with any vertex in S'_{i-1}. By definition, w also has a neighbor in S_{i-2}. As $\deg(w) = 3$, this is only possible when $|S'_{i-1}| = 1$, that is $S'_{i-1} = \{u\}$.

Step 2. Since $\Gamma^-_{f(u)}$ is connected, u has a neighbor $z \in S_i$. If $z \in S'_i$, then applying $\mathsf{sw}(w, v, z, u)$ leads to a graph in $\mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$. Therefore, u has no neighbor in S'_i. We can then assume that $z \in S_i \setminus (S'_i \cup \{v\})$ and that it has the maximum component among the neighbors of u in this set. Lemma 3.5 implies that $f(v) \neq f(z)$.

Hence, $f(v) > f(z)$. If $v \nsim z$, then $\Gamma - uz + uv + vz$ is a graph in $\mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$. Therefore, we have $v \sim z$.

Step 3. $S'_i = \emptyset$: Assume, by contradiction, that there exists a vertex $p \in S'_i$. By Steps 1 and 2, we know that $p \nsim w, u$. Thus, p has a neighbor $y \in S_{i-1} \setminus \{w, u\}$. As $S'_{i-1} = \{u\}$ by Step 1, we have $f(y) \geq f(w) > f(u)$. Now, consider the graph $\Gamma' := \Gamma - yp + yu + up$. Since $\mu(\Gamma') < \mu(\Gamma)$, we must have $y \sim u$, and $\deg(u) \geq 4$. Similarly, by considering $\Gamma - uz + up + pz$, we see that $z \sim p$. Since $\Gamma^-_{f(z)}$ is connected, $\deg(z) \geq 4$. Then, $\Gamma - uz$ is a graph in $\mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$, which leads to a contradiction.

Step 4. $\deg(z) = 3$ and v is the unique neighbor of z in S_i: Recall that from Step 2, we have $f(v) > f(z)$ and $v \sim z$. If $\deg(z) > 3$, then $\Gamma - uz + uv$ has $\mu < \mu(\Gamma)$. So, $\deg(z) = 3$. As $\Gamma^-_{f(z)}$ is connected, z has a neighbor q with $f(q) \leq f(z)$. Assume that $q \in S_i$. If $q \sim u$, then $q \sim v$; otherwise, $\mu(\Gamma - qu + uv + vq) < \mu(\Gamma)$, which leads to a contradiction. Therefore, $\deg(u) \geq 4$, and since q has a neighbor with component less than or equal to $f(q)$, we have $\deg(q) \geq 4$. Now, $\Gamma - qu$ is a graph in $\mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$. Hence, $q \nsim u$, which means that q must have a neighbor $y \in S_{i-1} \setminus \{w, u\}$. Therefore, $y \sim u$; otherwise, we obtain $\Gamma - yq + yu + uq \in \mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$, a contradiction. So we can apply $\mathsf{sw}(y, u, w, v)$ to produce a graph with $\mu < \mu(\Gamma)$. Thus, $q \in S_{i+1}$ and v is the only neighbor of z in S_i.

Step 5. $S_i = \{v, z\}$: Let $p \in S_i \setminus \{v, z\}$. If $f(v) \geq f(p) > f(z)$, then $\Gamma - uz + up + pz$ is a graph with $\mu < \mu(\Gamma)$. So, we must have $p \sim u$. However, this is a contradiction by the maximality of $f(z)$ given in Step 2. Thus $f(p) < f(z)$. If $p \sim u$, then $\deg(u) \geq 4$. Then $\Gamma - pu + pz$ is a graph with $\mu < \mu(\Gamma)$. Therefore $p \nsim u$ and p has a neighbor $y \in S_{i-1} \setminus \{w, u\}$. Now $\Gamma - yp + yz + zp$ is a graph with $\mu < \mu(\Gamma)$, a contradiction.

Next, we will prove that $|S_0| \leq 3$. Assume, by contradiction, that $|S_0| \geq 4$. We know that S_0 induces a clique. We will now show that S_1 also induces a clique. Suppose, by contradiction, that there exist $u, v \in S_1$ with $u \nsim v$. Let $f(u) > f(v)$, and let $w \in S_0$ be a neighbor of v in S_0. Then, $\Gamma - vw + vu \in \mathcal{G}_{n,3}$ has $\mu < \mu(\Gamma)$, which leads to a contradiction. On the other hand, Lemma 3.6 implies that each vertex of S_0 has a neighbor in S_1, which means that there are at least 4 edges between S_0 and S_1. Therefore, S_1 has a vertex p with $\deg(p) \geq 4$. Let y be a neighbor of p in S_0 which has also degree ≥ 4. Then, $\Gamma - py \in \mathcal{G}_{n,3}$ has $\mu < \mu(\Gamma)$, a contradiction. Hence, we conclude that $|S_0| \leq 3$.

Finally, we show that $|S_i| \leq 4$ for $i \in \{1, k\}$. Assume, by contradiction, that $|S_j| \geq 5$ for $j = 1$ or k. Let v be a vertex with minimum component in S_j and $u \in S_{j-1}$ be a neighbor of v. If $\deg(u) \geq 4$, then v is adjacent with all other vertices of S_j (otherwise if

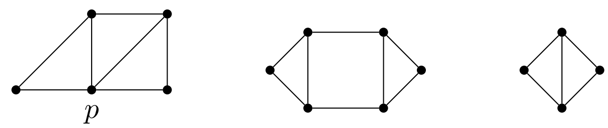

Figure 6: Three possible non-trivial middle blocks.

$w \in S_j$ and $v \nsim w$, then $\mu(\Gamma - vu + vw) < \mu(\Gamma))$, that is, $\deg(v) \geq 4$ and $\Gamma - uv$ leads to a contradiction. Therefore, $\deg(u) = 3$. So u is not adjacent with at least three vertices p, y, z of S_j. If $v \nsim p$, then $\Gamma - uv + up + pv$ has $\mu < \mu(\Gamma)$. Thus, $v \sim p$. Similarly $v \sim y$ and $v \sim z$. So $\deg(v) \geq 4$. Now, Lemma 3.5 implies that $f(v) \neq f(p)$. Hence, $f(p) > f(v)$ and $\mu(\Gamma - uv + up) < \mu(\Gamma)$, a contradiction. □

By a non-trivial block, we mean a block other than K_2.

Lemma 4.2. *If $p \in \bigcup_{i=3}^{k-2} S_i$, then $\deg(p) \leq 4$. Moreover, if $\deg(p) = 4$, then either p is a cut vertex or it is in the same situation as depicted by M_1 in Figure 6.*

Proof. Consider $i \in \{3, \ldots, k-2\}$ and $p \in S_i$. Assume, for the sake of contradiction, that $\deg(p) \geq 5$. By Lemma 4.1, $\deg(p) = 5$, and this can only happen if $|S_{i-1}| = |S_i| = |S_{i+1}| = 2$. Let $S_i = \{p, w\}$, $S_{i-1} = \{u, v\}$, and $f(u) \geq f(v)$. We have $p \sim w, u, v$. If $u \nsim v$, then $\Gamma - up + uv \in \mathcal{G}_{n,3}$ has $\mu < \mu(\Gamma)$. Hence, we have $u \sim v$. By definition, u, v, and w have neighbors in their previous part. Therefore, $w \sim u$ or $w \sim v$. If $w \sim u$, then $\deg(u) \geq 4$, and $\Gamma - up \in \mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$. The case where $w \sim v$ leads to a similar contradiction. Hence, we deduce that $\deg(p) \leq 4$.

To prove the second part of the statement, let $\deg(p) = 4$.

If $|S_i| = 1$, then p is a cut vertex. In what follows, we assume that $S_i = \{p, w\}$. If p is adjacent to two vertices in S_{i-1}, then a similar argument as above leads to a contradiction. Hence, we assume that p is adjacent to only one vertex in S_{i-1}. Let $S_{i+1} = \{y, z\}$. Then, we have $p \sim w, y, z$. We may assume that $f(y) \geq f(z)$. If $y \nsim z$, then $\Gamma - pz + yz \in \mathcal{G}_{n,3}$ has $\mu < \mu(\Gamma)$. Hence, we have $y \sim z$. If $N[w] = N[p]$, then at least one of y or z has degree more than 3, and subsequently either $\Gamma - yp$ or $\Gamma - zp$ is a graph with $\mu < \mu(\Gamma)$. So, $N[w] \neq N[p]$. If $f(p) = f(w)$, then by Lemma 3.5, we have $\deg(p) = 3$, which is not the case. Therefore, we conclude that $f(p) \neq f(w)$. If $f(p) > f(w)$, then we have $w \sim y, z$ (otherwise, either $\Gamma - py + wy$ or $\Gamma - pz + wz$ leads to a contradiction). Thus, at least one of y or z has degree at least 4, and subsequently either $\Gamma - py$ or $\Gamma - pz$ is a graph in $\mathcal{G}_{n,3}$ with $\mu < \mu(\Gamma)$. Therefore, $f(p) < f(w)$.

We claim that $|S_{i-1}| = 1$. Assume, by contradiction, that $|S_{i-1}| = 2$ with $S_{i-1} = \{u, v\}$ and $f(u) \geq f(v)$. First assume that u is the only neighbor of p in S_{i-1}. Then, we have

$u \sim w$ and $u \sim v$ (otherwise either $\Gamma - up + uw$ or $\Gamma - up + uv$ leads to a contradiction). Then $\deg(u) > 3$ and $\mu(\Gamma - up) < \mu(\Gamma)$. Now assume that v is the only neighbor of p in S_{i-1}. Then $v \sim w$ (otherwise $\mu(\Gamma - vp + vw) < \mu(\Gamma)$). If $\deg(v) > 3$, then $\mu(\Gamma - vp) < \mu(\Gamma)$. So $\deg(v) = 3$ and $u \nsim v$. As $\deg(p) = 4$ and $\Gamma^{-}_{f(u)}$ is connected, we have $u \sim w$. So $|N(u) \cap S_{i-2}| \geq 2$ and v is adjacent to only one vertex t in S_{i-2}. If $\deg(t) > 3$, then $v \sim u$ (otherwise $\mu(\Gamma - vt + vu) < \mu(\Gamma)$), a contradiction. Therefore $\deg(t) = 3$. So there exists a vertex $q \in N(u) \cap S_{i-2}$ with $q \nsim t$. We can apply $\mathsf{sw}(t, v, q, u)$ to produce a graph with $\mu < \mu(\Gamma)$. These contradictions establish the claim.

Since $N[w] \neq N[p]$, w is adjacent to only one vertex in S_{i+1}. Lemma 3.5 implies that $f(y) \neq f(z)$. Thus $f(y) > f(z)$. If $w \sim z$, then $\deg(z) > 3$ and $\mu(\Gamma - wz + wy) < \mu(\Gamma)$. Thus $w \sim y$. Also if $\deg(y) \geq 4$, then $\mu(\Gamma - py) < \mu(\Gamma)$. So $\deg(y) = 3$ and y has no neighbor in S_{i+2}. Therefore, p can only be in the same situation as M_1 in Figure 6. □

Lemma 4.3. *If $i \in \{3, \ldots, k-2\}$, then S_i induces a clique.*

Proof. By Lemma 4.1, $|S_i| \leq 2$. If $|S_i|=1$, there is nothing to prove. Otherwise, let $S_i = \{u, v\}$ with $u \nsim v$. If $f(u) = f(v)$, then $u \sim v$ and we are done. Thus, we may assume that $f(u) > f(v)$. From Lemma 4.2, it can be inferred that $\deg(u) = \deg(v) = 3$. If $|N(u) \cap N(v) \cap S_{i+1}| = 2$, then let $S_{i+1} = \{p, y\}$. If $p \sim y$, then at least one of p or y has degree at least 4, a contradiction with Lemma 4.2. Thus $p \nsim y$ and we can apply $\mathsf{sw}(u, p, v, y)$ to obtain a graph with $\mu < \mu(\Gamma)$, a contradiction. If $|N(u) \cap N(v) \cap S_{i+1}| < 2$, then necessarily $|S_{i-1}| = 2$. Let $S_{i-1} = \{w, z\}$ with $f(w) \geq f(z)$. If $w \sim z$, then at least one of w or z has degree at least 4, a contradiction with Lemma 4.2. Thus $w \nsim z$. If $w \sim v$ and $z \sim u$, then apply $\mathsf{sw}(w, v, z, u)$. Otherwise perform $\mathsf{sw}(w, u, z, v)$. In either cases the resulting graph has $\mu < \mu(\Gamma)$, a contradiction. □

For the sake of simplicity, we denote the induced subgraph of Γ by $S_i \cup \cdots \cup S_j$, $j > i$, by $\Gamma[i:j]$.

Theorem 4.4. *The graph Γ has a path-like structure in which each non-trivial middle block is one of the graphs M_1, M_2, or M_3 of Figure 6.*

Proof. Consider three consecutive parts S_{i-1}, S_i, S_{i+1} for $4 \leq i \leq k-3$. By Lemma 4.1, these sets have a size of at most 2. We claim that at least one of them has size 1. Assume, by contradiction, that $|S_{i-1}| = |S_i| = |S_{i+1}| = 2$. By Lemma 4.3, each of these parts induces a K_2, and by Lemma 4.2, each vertex in $S_{i-1} \cup S_i \cup S_{i+1}$ has degree 3. Thus for $j \in \{i, i+1\}$, the vertices of S_j have no common neighbor in S_{j-1}. It turns out that Γ_i is isomorphic to the graph of Figure 7a. Then we can apply $\mathsf{sw}(a, c, b, d)$ to obtain the graph of Figure 7b with $\mu < \mu(\Gamma)$, a contradiction. This contradiction establishes the claim.

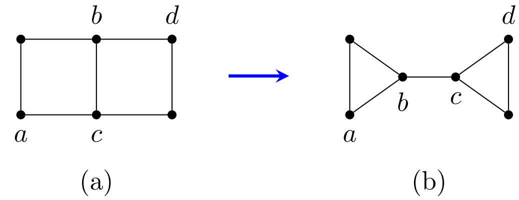

Figure 7: Induced subgraph on S_{i-1}, S_i, S_{i+1} when each of them has size 2, and the outcome of applying $\mathsf{sw}(a, c, b, d)$.

Firstly, assume that $|S_i| = 1$. We have either $|S_{i+1}| = 2$ or $|S_{i-1}| = 2$. Let the former be the case. If S_{i+1} has a vertex of degree 4, this vertex cannot be a cut vertex and so by Lemma 4.2, $\Gamma[i:i+2] = M_1$. Otherwise, either $|S_{i+2}| = 1$ and $\Gamma[i:i+2] = M_3$; or $|S_{i+2}| = 2$ and since at least one of $S_{i+1}, S_{i+2}, S_{i+3}$ has size 1, we infer that $\Gamma[i:i+3] = M_2$. Next, assume that $|S_{i-1}| = 2$. We have either $|S_{i-2}| = 1$, so $\Gamma[i-2:i] = M_3$; or $|S_{i-2}| = 2$ and thus either $\Gamma[i-3:i] = M_2$ or $\Gamma[i-3:i-1] = M_1$.

Next, assume that $|S_i| = 2$. If $|S_{i-1}| = |S_{i+1}| = 1$, then $\Gamma[i-1:i+1] = M_3$. If $|S_{i-1}| = 2$, then $|S_{i-2}| = |S_{i+1}| = 1$ and either $\Gamma[i-2:i+1] = M_2$ or $\Gamma[i-2:i] = M_1$. If $|S_{i+1}| = 2$, then $|S_{i-1}| = |S_{i+2}| = 1$ and either $\Gamma[i-1:i+1] = M_1$ or $\Gamma[i-1:i+2] = M_2$.

We observe that, the first block of Γ is $\Gamma[0:p]$ where p is the smallest index in $\{2, 3, 4\}$ such that $|S_p| = 1$. Similarly the last block of Γ is $\Gamma[q:k]$ where q is the largest index in $\{k-3, k-2, k-1\}$ such that $|S_q| = 1$. We conclude that in the subgraph $\Gamma[p:q]$ any block is one of M_1, M_2, or M_3. $\square$

The following result immediately follows from Lemma 4.2 and Theorem 4.4.

Lemma 4.5. *If $p \in \bigcup_{i=4}^{k-3} S_i$ is a cut vertex of degree 4 which does not belong to M_1 and two end blocks, then p belongs to two blocks of types M_2 or M_3.*

4.2 Middle blocks

Our goal is to show that except possibly the first and last three non-trivial middle blocks, any non-trivial middle block of Γ is M_3. In what follows, it is more convenient to represent vectors by their components, such as $\mathbf{x} = (x_1, \ldots, x_n)^\top$, rather than as functions, which was used thus far.

Let G be a quartic graph of order n. For some edge e of G, $G - e$ is connected and obviously $\delta(G - e) = 3$. So $\mu(\Gamma) \leq \mu(G - e) \leq \mu(G)$. In [2] the following upper bound is given on minimum algebraic connectivity of quartic graphs. So we can use it as an upper bound on $\mu(\Gamma)$.

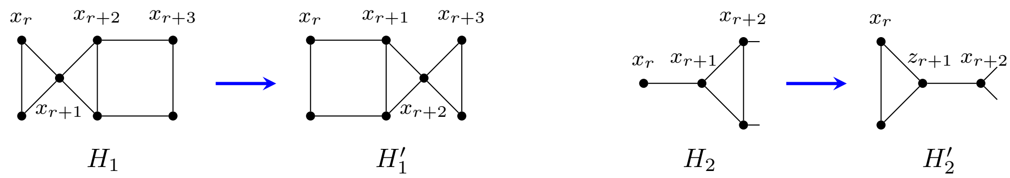

Figure 8: A middle block M_2 gives rise to subgraphs H_1 or H_2 in Γ, which are then replaced by H'_1 and H'_2 to obtain a graph with a smaller μ.

Lemma 4.6 ([2])**.** *If $n \geq 26$, then $\mu(\Gamma) < 0.059$.*

In what follows we assume that $n \geq 26$ and so the above upper bound holds. In the next Lemma, we show that no middle block in Γ can be M_2.

Lemma 4.7. *The graph Γ does not have a middle block M_2, with the exception of the first or last middle block.*

Proof. Assume, for a contradiction that Γ contains a middle block M_2 which is not the first or last middle block. Then in view of Theorem 4.4, Γ contains either H_1 of H_2 from Figure 8 as a subgraph. By the eigen-equation and Theorem 3.9, Γ has a *decreasing* unit Fiedler vector $\mathbf{x}$ which is *constant* on the parts of M_2. We assume that the components of $\mathbf{x}$ are as the ones depicted in Figure 8, so we have $x_r > x_{r+1} > x_{r+2} > x_{r+3} > \cdots$.

We first assume that $x_{r+2} \geq 0$.

Suppose that H_1 is a subgraph of Γ. We replace H_1 by the subgraph H'_1 given in Figure 8 to obtain $\Gamma' \in \mathcal{G}_{n,3}$. Define a vector $\mathbf{x}'$ on $V(\Gamma')$ such that its components on H'_1 are as given in Figure 8 and on the rest of vertices of Γ' agree with $\mathbf{x}$. We observe that $\langle \mathbf{x}', \mathbf{1}\rangle - \langle \mathbf{x}, \mathbf{1}\rangle = x_{r+1} - x_{r+2}$ and since $\mathbf{x} \perp \mathbf{1}$, $\theta = \langle \mathbf{x}', \mathbf{1}\rangle = x_{r+1} - x_{r+2}$. Also $\mathbf{x}'^\top L(\Gamma')\mathbf{x}' = \mathbf{x}^\top L(\Gamma)\mathbf{x}$, and

$$\begin{aligned}
\|\mathbf{x}'\|^2 - \frac{\theta^2}{n} &= \|\mathbf{x}\|^2 + x_{r+1}^2 - x_{r+2}^2 - \frac{\theta^2}{n} \\
&= 1 + (x_{r+1} - x_{r+2})\left(x_{r+1} + x_{r+2} - \frac{x_{r+1} - x_{r+2}}{n}\right) \\
&\geq 1 + (x_{r+1} - x_{r+2})^2\left(1 - \frac{1}{n}\right),
\end{aligned}$$

where the last inequality follows from the assumption that $x_{r+2} \geq 0$. Therefore $\|\mathbf{x}'\|^2 - \frac{\theta^2}{n} > 1$. So by Lemma 2.1, $\mu(\Gamma') \leq \frac{\mathbf{x}'^\top L(\Gamma')\mathbf{x}'}{\|\mathbf{x}'\|^2 - \frac{\theta^2}{n}} < \mu(\Gamma)$, a contradiction.

Assuming that H_2 is a subgraph of Γ, let us define Γ' and $\mathbf{x}'$ in a similar manner as described earlier, but by replacing H_2 with H'_2 as illustrated in Figure 8. Set $z_{r+1} :=$

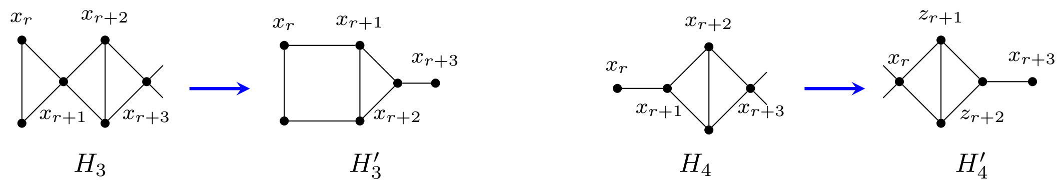

Figure 9: Two M_3-blocks with a common vertex give rise to subgraphs H_3 or H_4 in Γ, which are then replaced by H_3' and H_4' to obtain a graph with a smaller μ.

$x_r + x_{r+2} - x_{r+1}$. We see that $\theta = \langle \mathbf{x}', \mathbf{1} \rangle = 2(x_r - x_{r+1})$. Also $\mathbf{x}'^\top L(\Gamma')\mathbf{x}' = \mathbf{x}^\top L(\Gamma)\mathbf{x}$, and

$$\begin{aligned} \|\mathbf{x}'\|^2 - \frac{\theta^2}{n} &= \|\mathbf{x}\|^2 + (x_r + x_{r+2} - x_{r+1})^2 + x_r^2 - x_{r+1}^2 - x_{r+2}^2 - \frac{\theta^2}{n} \\ &= 1 + 2(x_r - x_{r+1}) \left(x_r + x_{r+2} - \frac{2(x_r - x_{r+1})}{n} \right) \\ &\geq 1 + 2(x_r - x_{r+1})^2 \left(1 - \frac{2}{n} \right), \end{aligned}$$

where the last inequality follows from the assumption that $x_{r+2} \geq 0$. Therefore, $\|\mathbf{x}'\|^2 - \frac{\theta^2}{n} > 1$. So by Lemma 2.1, $\mu(\Gamma') \leq \frac{\mathbf{x}'^\top L(\Gamma')\mathbf{x}'}{\|\mathbf{x}'\|^2 - \frac{\theta^2}{n}} < \mu(\Gamma)$, a contradiction.

We now assume that $x_{r+2} < 0$. Thus, we have $-x_{r+5} > \cdots > -x_{r+2} > 0$. Consider $\tilde{\Gamma}$, the mirror image of Γ, and $-\mathbf{x}$ as its Fiedler vector. In $\tilde{\Gamma}$, either the vertices with components $-x_{r+5}, \ldots, -x_{r+2}$ is isomorphic to H_1, or the vertices with components $-x_{r+5}, -x_{r+4}, -x_{r+3}$ is isomorphic to H_2. Here, $-x_{r+3}$ plays the same role as x_{r+2} in the previous argument, and since $-x_{r+3} > 0$, the proof is complete. □

A graph G is said to be *strongly path-like* if it is path-like and no two non-trivial middle blocks of G share a vertex.

Lemma 4.8. *The graph Γ is strongly path-like, and every non-trivial middle block in Γ, except for the first or last middle block, is either M_1 or M_3.*

Proof. For simplicity, let $\mu := \mu(\Gamma)$. By Theorem 4.4, Γ is path-like, and every non-trivial middle block is one of M_1, M_2, or M_3. However, Lemma 4.7 rules out the possibility of an M_2-block. To conclude that Γ is strongly path-like, we need to show that no two M_3-blocks share a common vertex. Otherwise, Γ would contain one of the subgraphs H_3 or H_4 from Figure 9. Based on the eigen-equation and Theorem 3.9, we know that Γ has a decreasing unit Fiedler vector $\mathbf{x}$ that is constant on the parts of M_3, as shown in Figure 9.

We first assume that H_3 is a subgraph of Γ. We then replace H_3 with the subgraph H_3' from Figure 9 to obtain Γ'. We define a vector $\mathbf{x}'$ on $V(\Gamma')$ such that its components on H_3' are as given in Figure 9, and on the rest of the vertices of Γ', it agrees with $\mathbf{x}$. First assume that $x_{r+2} \geq 0$. By the fact that $\mathbf{x}$ is a unit Fiedler vector of Γ, we see that

$$\theta = \langle \mathbf{x}', \mathbf{1} \rangle = x_{r+1} - x_{r+2}, \quad \|\mathbf{x}'\|^2 = 1 + x_{r+1}^2 - x_{r+2}^2, \quad \mathbf{x}'^\top L(\Gamma')\mathbf{x}' = \mu - (x_{r+2} - x_{r+3})^2.$$

The situation here is similar to that of H_1 in the proof of Lemma 4.7. Therefore, we can deduce that $\mu(\Gamma') < \mu$ which is a contradiction. Now, suppose that $x_{r+2} < 0$. Thus, we have $-x_{r+4} > -x_{r+3} > -x_{r+2} > 0$. Consider $\tilde{\Gamma}$ and $-\mathbf{x}$ as its Fiedler vector. In $\tilde{\Gamma}$, the vertices with components $-x_{r+4}, \ldots, -x_{r+1}$ is isomorphic to H_3. Here, $-x_{r+2}$ plays the same role as x_{r+2} in the previous argument, and since $-x_{r+2} > 0$, the proof is complete.

Now, assume that H_4 is a subgraph of Γ. First suppose that $x_{r+3} \geq 0$. By the eigen-equation considered on H_4, it can be seen that $x_{r+1} = p(x_r, x_{r+3})$ and $x_{r+2} = q(x_r, x_{r+3})$, where

$$\begin{aligned} p(x, y) &= \frac{(2-\mu)x + 2y}{\mu^2 - 5\mu + 4}, \\ q(x, y) &= \frac{x + (3-\mu)y}{\mu^2 - 5\mu + 4}. \end{aligned}$$

We replace H_4 with H_4' to obtain Γ', and define a vector $\mathbf{x}'$ on $V(\Gamma')$ such that its components on H_4' are as given in Figure 9, where

$$z_{r+1} := q(x_{r+3}, x_r), \quad z_{r+2} := p(x_{r+3}, x_r),$$

and on the rest of the vertices of Γ', it agrees with $\mathbf{x}$. We have $\theta = \langle \mathbf{x}', \mathbf{1} \rangle = 2z_{r+1} + z_{r+2} - x_{r+1} - 2x_{r+2}$. By substituting the values of $x_{r+1}, x_{r+2}, z_{r+1}, z_{r+2}$ in terms of x_r, x_{r+3}, μ we obtain that

$$\theta = \frac{x_r - x_{r+3}}{1 - \mu}.$$

Now, in the notation of Lemma 2.2, we have:

$$\begin{aligned} h &= (x_r - x_{r+1})^2 + 2(x_{r+1} - x_{r+2})^2 + 2(x_{r+2} - x_{r+3})^2, \\ h' &= 2(x_r - z_{r+1})^2 + 2(z_{r+1} - z_{r+2})^2 + (z_{r+2} - x_{r+3})^2. \end{aligned}$$

Moreover, since $\|\mathbf{x}\| = 1$,

$$\epsilon = \|\mathbf{x}'\|^2 - 1 - \frac{\theta^2}{n} = 2z_{r+1}^2 + z_{r+2}^2 - x_{r+1}^2 - 2x_{r+2}^2 - \frac{\theta^2}{n}.$$

It follows that

$$h' - h - \epsilon\mu = \frac{\mu(x_r - x_{r+3})}{(\mu-1)^2 n}\big((\mu - 1)n(x_r + x_{r+3}) + (x_r - x_{r+3})\big),$$

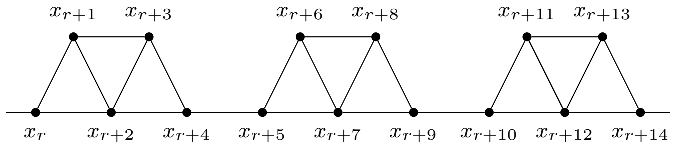

Figure 10: The subgraph H_5 consisting of three consecutive M_1.

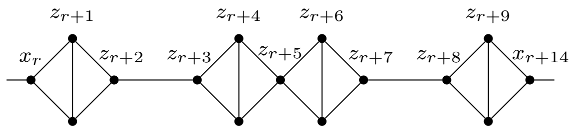

Figure 11: The subgraph H_5' and the components of $\mathbf{x}'$.

which is negative because $x_r > x_{r+3} \geq 0$. Therefore, from Lemma 2.2 it follows that $\mu(\Gamma') < \mu$, a contradiction. Now, assume that $x_{r+3} < 0$. Consider $\tilde{\Gamma}$ and $-\mathbf{x}$ as its Fiedler vector. As established earlier, $\tilde{\Gamma}$ does not contain H_3. So the vertices with components $-x_{r+6}, \ldots, -x_{r+3}$ should be isomorphic to H_4. Here, $-x_{r+3}$ plays the same role as x_{r+3}, and since $-x_{r+3} > 0$, the proof is complete. $\square$

It remains to show that no middle block in Γ can be M_1. This will be done by the following two lemmas.

Lemma 4.9. *The graph Γ does not contain three consecutive middle blocks M_1.*

Proof. We let $\mu := \mu(\Gamma)$. For a contradiction, assume that Γ contains three consecutive middle blocks M_1. This gives rise to the subgraph H_5 of Figure 10. Applying the eigen-equation on the vertices of H_5 in Γ, we can write $x_{r+1}, \ldots, x_{r+13}$ in terms of x_r, x_{r+14}, and μ. In fact by the symmetry of H_5, we have

$$x_{r+i} = a_i x_r + b_i x_{r+14}, \quad x_{r+14-i} = b_i x_r + a_i x_{r+14}, \quad \text{for } i = 1, \ldots, 7,$$

in which a_i and b_i satisfy the following equations:

$$\begin{aligned}
(3-\mu)\, a_1 &= 1 + a_2 + a_3, & (3-\mu)\, b_1 &= b_2 + b_3,\\
(4-\mu)\, a_2 &= 1 + a_1 + a_3 + a_4, & (4-\mu)\, b_2 &= b_1 + b_3 + b_4,\\
(3-\mu)\, a_3 &= a_1 + a_2 + a_4, & (3-\mu)\, b_3 &= b_1 + b_2 + b_4,\\
(3-\mu)\, a_4 &= a_2 + a_3 + a_5, & (3-\mu)\, b_4 &= b_2 + b_3 + b_5,\\
(3-\mu)\, a_5 &= a_4 + a_6 + a_7, & (3-\mu)\, b_5 &= b_4 + b_6 + b_7,\\
(3-\mu)\, a_6 &= a_5 + a_7 + b_6, & (3-\mu)\, b_6 &= b_5 + b_7 + a_6,\\
(4-\mu)\, a_7 &= a_5 + a_6 + b_5 + b_6, & (4-\mu)\, b_7 &= b_5 + b_6 + a_5 + a_6.
\end{aligned}$$

It turns out that[2] a_i and b_i can be obtained as rational functions in terms of μ. We now replace H_5 by the subgraph H_5' depicted in Figure 11 to obtain Γ'. We also define a vector $\mathbf{x}'$ on $V(\Gamma')$ such that its components on H_5' are as given in Figure 11 and on the rest of vertices it agrees with $\mathbf{x}$. We set

$$z_{r+i} := c_i x_r + d_i x_{r+14}, \quad z_{r+10-i} := d_i x_r + c_i x_{r+14}, \quad \text{for } i = 1, \ldots, 5,$$

in which

$$\begin{aligned}
d_1 &= -8\alpha\beta, & c_1 &= d_1 + (-\mu^3 + 8\mu^2 - 18\mu + 10)\beta,\\
d_2 &= 8(\mu - 2)\alpha\beta, & c_2 &= d_2 + (2\mu^2 - 10\mu + 8)\beta,\\
d_3 &= -8(\mu^2 - 5\mu + 4)\alpha\beta, & c_3 &= d_3 + (-2\mu + 4)\beta,\\
d_4 &= 4(\mu^3 - 8\mu^2 + 18\mu - 10)\alpha\beta, & c_4 &= d_4 + 2\beta,\\
d_5 &= -4\alpha, & c_5 &= -4\alpha,
\end{aligned}$$

where $\alpha = (\mu^5 - 14\mu^4 + 68\mu^3 - 132\mu^2 + 84\mu - 8)^{-1}$, and $\beta = (\mu^4 - 10\mu^3 + 32\mu^2 - 36\mu + 12)^{-1}$. We have $\theta = \langle \mathbf{x}, \mathbf{1} \rangle = \sum_{i=1}^{9} z_{r+i} + z_{r+1} + z_{r+4} + z_{r+6} + z_{r+9} - \sum_{i=1}^{13} x_{r+i}$. Now, in the notation of Lemma 2.2, we have

$$\begin{aligned}
h &= \sum_{i=r}^{r+13} (x_i - x_{i+1})^2 + \sum_{i=0}^{2} \sum_{j=r}^{r+2} (x_{j+5i} - x_{j+5i+2})^2,\\
h' &= 2(x_r - z_{r+1})^2 + 2 \sum_{i=r+1}^{r+8} (z_i - z_{i+1})^2 - (z_{r+2} - z_{r+3})^2 - (z_{r+7} - z_{r+8})^2 + 2(z_{r+9} - x_{r+14})^2.
\end{aligned}$$

Moreover, using the assumption that $n \geq 26$, we obtain that

$$\epsilon \geq \|\mathbf{x}'\|^2 - 1 - \frac{\theta^2}{26} = \sum_{i=1}^{9} z_{r+i}^2 + z_{r+1}^2 + z_{r+4}^2 + z_{r+6}^2 + z_{r+9}^2 - \sum_{i=1}^{13} x_{r+i}^2 - \frac{\theta^2}{26}. \tag{5}$$

Given that h, h', and the above upper bound for ϵ can be written in terms of x_r, x_{r+14}, and μ, using the bound $\mu < 0.059$, we obtain that $h' - h - \epsilon\mu \leq g(\mu)(x_r - x_{r+14})^2$, where

$$\begin{aligned}
g(\mu) = {}& 89840239867146\,\mu^7 - 27583516871090\,\mu^6 + 6145217469087\,\mu^5 - 945809659518\,\mu^4\\
&+ 94690418508\,\mu^3 - 5793343920\,\mu^2 + 207213552\,\mu - 3852576.
\end{aligned}$$

Again, as $\mu < 0.059$, we have $g(\mu) < 0$. Therefore, $h' - h - \epsilon\mu < 0$ and by Lemma 2.2, $\mu(\Gamma') < \mu$, a contradiction. □

[2] To help the reader with computations, we provide a Maple code available at https://wp.kntu.ac.ir/ghorbani/ComputFiles/MapleCodeMinDeg3.txt that can be used to derive the functions appeared in the proofs of Lemmas 4.9 and 4.10.

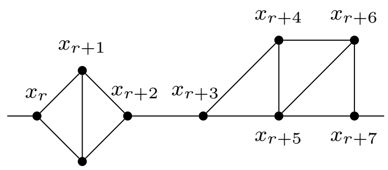

Figure 12: The subgraph H_6 and the components of $\mathbf{x}$.

The next lemma shows that if Γ would contain a middle block M_1, then we could move it to either left or right end of graphs, to obtain a graph with smaller μ. This in view Lemma 4.9 implies that Γ may contain at most three M_1 block at either ends.

The next lemma shows that if Γ contains a middle block M_1, it is possible to move it to either the left or right end of the graph, resulting in a graph with a smaller μ. Then Lemma 4.9 implies that Γ may contain at most three M_1 blocks only at either end.

Lemma 4.10. *The graph Γ does not contain M_1, except possibly in the first or last three non-trivial middle blocks.*

Proof. Assume, for contradiction, that Γ contains an M_1-block, except for the first and last three non-trivial middle blocks. Lemmas 4.8 and 4.9 imply that an M_1-block must come together with an M_3-block, which means that Γ contains a subgraph isomorphic to H_6 from Figure 12. Applying the eigen-equation on the vertices of H_6 in Γ, we see that

$$x_{r+1} = g(x_r, x_{r+7}), \quad x_{r+2} = k(x_r, x_{r+7}), \quad x_{r+3} = l(x_r, x_{r+7}),$$
$$x_{r+4} = m(x_r, x_{r+7}), \quad x_{r+5} = p(x_r, x_{r+7}), \quad x_{r+6} = q(x_r, x_{r+7}),$$

where

$$\begin{aligned}
g(x, y) &= \big((-\mu^5 + 16\,\mu^4 - 96\,\mu^3 + 263\,\mu^2 - 312\,\mu + 111)x + (\mu^2 - 9\,\mu + 21)y\big)\,\frac{\omega}{\mu - 4},\\
k(x, y) &= \big((2\,\mu^4 - 26\,\mu^3 + 116\,\mu^2 - 198\,\mu + 90)x + (-\mu^3 + 11\,\mu^2 - 39\,\mu + 42)y\big)\,\frac{\omega}{\mu - 4},\\
l(x, y) &= \big((-2\,\mu^2 + 12\,\mu - 12)x + (\mu^3 - 10\,\mu^2 + 30\,\mu - 21)y\big)\,\omega,\\
m(x, y) &= \big((2\,\mu^2 - 16\,\mu + 30)x + (2\,\mu^4 - 26\,\mu^3 + 116\,\mu^2 - 201\,\mu + 102)y\big)\,\frac{\omega}{\mu - 4},\\
p(x, y) &= \big(2(\mu - 3)x + \big(-\mu^3 + 8\,\mu^2 - 17\,\mu + 9\big)\,(\mu - 3)\,y\big)\,\omega,\\
q(x, y) &= \big((-4\,\mu + 18)x + (-\mu^5 + 16\,\mu^4 - 95\,\mu^3 + 255\,\mu^2 - 297\,\mu + 114)y\big)\,\frac{\omega}{\mu - 4},
\end{aligned}$$

and $\omega = (\mu^5 - 14\,\mu^4 + 70\,\mu^3 - 149\,\mu^2 + 126\,\mu - 33)^{-1}$. We replace H_6 by H_6' to obtain Γ' and define a vector $\mathbf{x}'$ on $V(\Gamma')$ such that its components on H_6' are as given in Figure 13 and on the rest of vertices agree with $\mathbf{x}$.

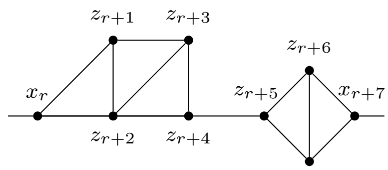

Figure 13: The subgraph H_6' and the components of $\mathbf{x}'$.

We set

$$\begin{aligned} z_{r+1} &:= q(x_{r+7}, x_r), \quad z_{r+2} := p(x_{r+7}, x_r), \quad z_{r+3} := m(x_{r+7}, x_r), \\ z_{r+4} &:= l(x_{r+7}, x_r), \quad z_{r+5} := k(x_{r+7}, x_r), \quad z_{r+6} := g(x_{r+7}, x_r). \end{aligned}$$

We have $\theta = \langle \mathbf{x}', \mathbf{1} \rangle = \sum_{i=1}^{6} z_{r+i} + z_{r+6} - x_{r+1} - \sum_{i=1}^{6} x_{r+i}$. By substituting the values of $x_{r+1}, \ldots, x_{r+6}$, and $z_{r+1}, \ldots, z_{r+6}$ in terms of x_r, x_{r+7}, μ we obtain that

$$\theta = (3 - \mu)(\mu - 5)(x_r - x_{r+7})\omega.$$

Now, in the notation of Lemma 2.2, we have

$$\begin{aligned} h &= (x_r - x_{r+1})^2 + (x_{r+1} - x_{r+2})^2 + \sum_{i=r}^{r+6} (x_i - x_{i+1})^2 + \sum_{i=r+3}^{r+5} (x_i - x_{i+2})^2, \\ h' &= (x_r - z_{r+1})^2 + (x_r - z_{r+2})^2 + \sum_{i=r+1}^{r+5} (z_i - z_{i+1})^2 + \sum_{i=r+1}^{r+2} (z_i - z_{i+2})^2 \\ &\quad + (z_{r+5} - z_{r+6})^2 + 2(z_{r+6} - x_{r+7})^2. \end{aligned}$$

Moreover,

$$\epsilon = \|\mathbf{x}'\|^2 - 1 - \frac{\theta^2}{n} = \sum_{i=1}^{6} z_{r+i}^2 + z_{r+6}^2 - x_{r+1}^2 - \sum_{i=1}^{6} x_{r+i}^2 - \frac{\theta^2}{n}.$$

It follows that

$$h' - h - \epsilon\mu = \frac{\omega^2}{n} \mu(\mu - 3)(\mu - 5)(x_r - x_{r+7}) \left(\frac{n}{\omega}(x_r + x_{r+7}) + (\mu - 3)(\mu - 5)(x_r - x_{r+7}) \right).$$

We know that $x_r > \cdots > x_{r+7}$. If all the vertices of the block M_1 in H_6 have the same sign, with no loos of generality we can assume that $x_{r+7} \geq 0$. Then it is easy to check that $h' - h - \epsilon\mu < 0$. Now, suppose that the vertices of M_1 do not have the same sign. So, we may assume that $x_{r+3} \geq 0$ and and $x_{r+7} < 0$. Note that $l(x_r, x_{r+7}) = x_{r+3} \geq 0$ which means $(-2\mu^2 + 12\mu - 12)x_r \leq -(\mu^3 - 10\mu^2 + 30\mu - 21)x_{r+7}$. Using $\mu < 0.059$, we see that $(-2\mu^2 + 12\mu - 12)x_r \geq -12x_r$ and $-(\mu^3 - 10\mu^2 + 30\mu - 21)x_{r+7} \leq 17x_{r+7}$. Hence $17x_{r+7} \geq -12x_r$ and thus $x_r + x_{r+7} \geq 0$ and $26(x_r + x_{r+7}) > \frac{3}{5}(x_r - x_{r+7})$. Since $n > 26$, $\mu < 0.059$ and $\omega < 0$, we have $\frac{3}{5} > -(\mu - 3)(\mu - 5)\omega$, it follows that $h' - h - \epsilon\mu < 0$. Theretofore, Lemma 2.2 implies that $\mu(\Gamma') < \mu$, a contradiction. □

The following theorem, which is a reformulation of our main result, Theorem 1.4, now follows from Lemmas 4.8 and 4.10.

Theorem 4.11. *Any μ-minimal graph with $\delta = 3$ is strongly path-like and except for the first and last three non-trivial middle blocks, every of its non-trivial middle blocks is M_3.*

Remark 4.12. As demonstrated in the last part of the proof of Theorem 4.4, the diameter of an end block of Γ is at most 4. Furthermore, by Lemma 4.1, we have $|S_0| \leq 3$, $|S_1|, |S_k| \leq 4$, and $|S_j| \leq 2$ in all other cases. Therefore, the number of vertices in an end block cannot exceed 12. By Lemma 4.7, M_2 can be the first middle block with a common vertex with the end block. According to Lemma 4.9, Γ does not contain three consecutive middle blocks M_1. However, we can assume that the second and third non-trivial middle blocks are M_1, as stated in Theorem 4.11. Therefore, the first and last three non-trivial middle blocks have at most 16 vertices. Consequently, these four exceptional blocks contain at most 27 vertices.

5 Graphs with asymptotic maximum diameter

In [9], it was proved that for $d \geq 3$ and $n \geq 2d + 4$, the maximum diameter of a graph with order n and $\delta = d$ is $3\lfloor \frac{n}{d+1} \rfloor - \ell$ for some $\ell \in \{1, 2, 3\}$. In [1], we have investigated graphs with order n, $\delta = d$, and asymptotic maximum diameter, that is $3n/(d+1) + O(1)$. We determined their structure and their algebraic connectivity. To state these results, some notation is in order.

The *sequential join* of vertex-disjoint graphs $G_1, G_2, \ldots, G_k$ is formed from $G_1 \cup G_2 \cup \cdots \cup G_k$ by adding edges joining each vertex of G_i with each vertex of G_{i+1} for $i = 1, \ldots, k-1$. We denote the sequential join of the sequence of $3m$ complete graphs

$$K_a, K_b, K_c, K_a, K_b, K_c, \ldots, K_a, K_b, K_c$$

by $H(a, b, c; m)$. For instance, Figure 14 depicts the graph $H(2, 3, 4; 3)$.

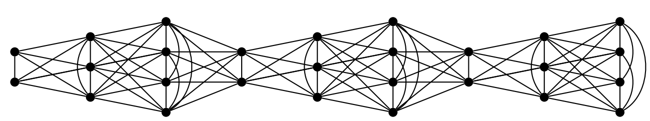

Figure 14: The graph $H(2, 3, 4; 3)$.

Let $d \geq 2$, $t \geq 1$, and $a_1, b_1, c_1, \ldots, a_t, b_t, c_t$ be positive integers such that for each $i = 1, \ldots, t$ we have $a_i + b_i + c_i = d + 1$. Let $G_i := H(a_i, b_i, c_i; m_i)$ which has $n_i = (d+1)m_i$ vertices. We define the family $\mathscr{F}_{n,d,C}$ for an absolute C as follows.

Definition 5.1. Given positive integers n, d and a constant C, a graph G belongs to $\mathscr{F}_{n,d,C}$ if:

(i) there exist positive integers $t \leq C$, $m_1, \ldots, m_t$, and $a_1, b_1, c_1, \ldots, a_t, b_t, c_t$, and graphs $H_0, \ldots, H_t$ with $\sum_{i=0}^{t} |V(H_i)| \leq C$, such that $a_i + b_i + c_i = d + 1$ for $i = 1, \ldots, t$, and $\sum_{i=1}^{t} m_i(d+1) + \sum_{i=0}^{t} |V(H_i)| = n$,

(ii) G is connected and obtained form $H_0 \cup G_1 \cup H_1 \cup G_2 \cup \cdots \cup H_{t-1} \cup G_t \cup H_t$, where $G_i := H(a_i, b_i, c_i; m_i)$, by connecting arbitrary vertices from the first (resp. last) cell of G_i to arbitrary vertices of H_{i-1} (resp. H_i).

The graphs $G_1, \ldots, G_t$ are called *major subgraphs* of G.

Theorem 5.2 (Abdi and Ghorbani [1])**.** *Let G be a graph of order n and $\delta = d$. If the dimaeter of G is $\frac{3n}{d+1} + O(1)$, then for some constant C, G belongs to the family $\mathscr{F}_{n,d,C}$.*

Theorem 5.3 (Abdi and Ghorbani [1])**.** *Let $G \in \mathscr{F}_{n,d,C}$ such that its major subgraphs are all $H(a, b, c; m)$. Then $\mu(G) = (1 + o(1))\frac{abc\pi^2}{n^2}$.*

Building upon the above results as well as Theorem 1.5, we are able to provide a positive answer to Problem 1.6.

Theorem 5.4. *Graphs with $\delta = 3$ and asymptotic maximum diameter have asymptotically minimum algebraic connectivity.*

Proof. Suppose G is a graph with $\delta = 3$ and an asymptotic maximum diameter, that is, $3n/4 + O(1)$. By Theorem 5.2, $G \in \mathscr{F}_{n,3,C}$ for some constant C. Major subgraphs $H(a, b, c; m)$ of G are such that $a, b, c \in \{1, 2\}$ and $a + b + c = 4$, making $abc = 2$ in any case. Thus, from Theorem 5.3, we have $\mu(G) = (1 + o(1))\frac{2\pi^2}{n^2}$. By Theorem 1.5, this represents the asymptotically minimum algebraic connectivity in $\mathcal{G}_{n,3}$. □

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