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New physical parameterizations of monopole solutions in five-dimensional general relativity and the role of negative scalar field energy in vacuum solutions

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E-mail: ybalytsk@uccs.edu, detlef.hoyer@tuhh.de, apinchuk@uccs.edu and willi@konfluence.org**Keywords:** Kaluza, monopole, scalar charge, five dimensional general relativity, negative energy

Abstract

Novel parameterizations are presented for monopole solutions to the static, spherically-symmetric vacuum field equations of five-dimensional general relativity. First proposed by Kaluza, 5D general relativity unites gravity and classical electromagnetism with a scalar field. These monopoles correspond to bodies carrying mass, electric charge, and scalar charge. The new parameterizations provide physical insight into the nature of electric charge and scalar field energy. The Reissner-Nordström limit is compared with alternate physical interpretations of the solution parameters. The new parameterizations explore the role of scalar field energy and the relation of electric charge to scalar charge. The Kaluza vacuum equations imply the scalar field energy density is the negative of the electric field energy density for all known solutions, so the total electric and scalar field energy of the monopole is zero. The vanishing of the total electric and scalar field energy density for vacuum solutions seems to imply the scalar field can be understood as a negative-energy foundation on which the electric field is built.

1. Introduction

December 22 of 2021 marks the centenary of Kaluza's classical unification of gravity and electromagnetism [1]. Einstein received it from Kaluza in 1919, but delayed forwarding it for publication until 1921 as he explored its implications in a series of letters with Kaluza [2]. Kaluza found that general relativity written in five dimensions provides a perfect unification of gravity and classical electromagnetism. The 15 components of a 5D metric \tilde{g}_{ab} can be identified with the 10 components of the 4D metric $g_{\mu\nu}$, the 4 components of the electromagnetic vector potential A^μ , and a scalar field ϕ .

An assumption is made to connect to 4D physics: that no fields depend on the fifth coordinate. Then, under this condition, called the cylinder condition, the 5D Einstein equations provide the 4D Einstein equations, the Maxwell equations, and an equation for the scalar field. These equations are coupled so that the scalar field enters both the 4D gravitational field equations and the electromagnetic field equations.

In the limit that the scalar field goes to unity, the gravitational field equations reduce to the 4D Einstein equations in the presence of an electromagnetic field, and the electromagnetic field equations reduce to the Maxwell equations in curved space.

The 5D geodesic equation provides the equation of motion for a body moving under gravitational, electromagnetic, and scalar fields. There is the further intriguing identification of electric charge with motion along the fifth coordinate. In the limit that the scalar field goes to unity, the equation of motion is exactly the 4D geodesic equation modified with the Lorentz force.

Yet the original Kaluza hypothesis is purely classical, and Planck's constant does not enter. Klein [3] tried to make the theory correspond to quantum mechanics by equating electric charge Q , and its associated motion along the fifth coordinate, with a Compton-like relation $Q \propto \hbar/R$. The lengthscale R is then interpreted to be the 'radius' of the compact fifth coordinate. Klein had hoped to find the quantization of electric charge in the quantization of standing waves on the compact fifth coordinate, but the magnitudes do not correspond to the quantum of charge.

We consider here solutions to the purely classical vacuum solutions of Kaluza under the cylinder condition. We make no assumption about the nature of the fifth coordinate, and allow it to be open and macroscopic in principle.

We use tensor algebra software to evaluate solutions \tilde{g}_{ab} to the 5D vacuum Einstein equations

$$\tilde{R}_{ab} = 0 \quad (1)$$

where \tilde{R}_{ab} is the 5D Ricci tensor.

We consider in particular time-independent vacuum solutions in spherical symmetry. The radial coordinate r is the only variable for the gravitational, electromagnetic, and scalar fields.

The central object is presumed to carry mass, electric charge, and scalar charge. Each of these 3 charges corresponds to the integration constant for each of the 3 field equations: gravitational, electromagnetic, and scalar.

There is a wide body of literature regarding static, spherically-symmetric solutions to the classical vacuum Kaluza field equations (1). These solutions are variously known as '1-bodies', 'monopoles' or 'solitons'. They fall into the two classes of neutral or electrically-charged. Mass (gravitational charge) and scalar charge are expected in both cases.

These solutions invite comparison with the standard Reissner-Nordström (R-N) solution of general relativity, which gives the gravitational field of an electrically-charged object. For a neutral object, the point of comparison is the Schwarzschild solution.

Generally speaking the time-time component \tilde{g}_{tt} of the 5D metric will depend on the mass M , the component \tilde{g}_{t5} will depend on the electric charge Q , and the component \tilde{g}_{55} will depend on the scalar charge S . The other diagonal components of \tilde{g}_{ab} corresponding to angular variables are non-zero, and all other off-diagonal components are zero.

Chodos and Detweiler [4] investigated the 5D vacuum equations (1), but they also obtained a general class of exact solutions to the 5D field equations with sources, and investigated their weak-field limit. They noted that for a particular value of the scalar charge S , the scalar field effects would vanish and the Reissner-Nordström limit is recovered. Yet the R-N limit is difficult to discern in their solutions, and the separate contributions of gravitational and electromagnetic energy cannot be distinguished in their general solution.

Sorkin [5] obtained a solution to (1) he called the 'Kaluza-Klein monopole'. It was assumed that the fifth dimension is spacelike, microscopic, and compact, with a new ad hoc lengthscale representing the size of this dimension as proposed by Klein [3]. A 5D vacuum solution was identified by applying a procedure to a known 4D metric. It had electromagnetic behavior, but no electric field. The Sorkin solution allowed a 4D radial coordinate, instead of the usual 3D Schwarzschild-type radial coordinate, and the near-field behavior of the solution does not correspond to the R-N limit. Sorkin called his solution a 'soliton'. The mass and charge of the soliton were proposed, but it had some curious properties. One conclusion was that some theorems developed for 4D solutions to the Einstein equations may not apply to 5D solutions to (1), and therefore to the 4D projections of those solutions.

Gross and Perry [6] obtained a neutral particle solution, so $\tilde{g}_{5t} = 0$. Yet they kept an off-diagonal spatial term, $\tilde{g}_{5\phi}$, which corresponds to the azimuthal component of the magnetic vector potential, A^ϕ . This provided a radial component to the magnetic field, and was identified as a magnetic monopole solution. Since no magnetic monopoles are found in nature, we consider such solutions unphysical, and therefore expect the static Kaluza monopoles to involve electric charge and to create an electric field.

Davidson and Owen [7] obtained a neutral-body solution to (1), with a diagonal and isotropic \tilde{g}_{ab} . Their solution had a Schwarzschild near-field limit for the g_{tt} component of the 4D metric, but the spatial isotropic components did not abide the usual Schwarzschild form in isotropic coordinates. The fifth coordinate was stated to be compact and microscopic, with a characteristic radius, but in effect the classical equations under the cylinder condition were solved.

Reference [7] obtained an expression for the mass from an effective energy momentum tensor implied by the solution, under the assumption that the effective metric $g_{\mu\nu}^{eff} = \tilde{g}_{55}g_{\mu\nu}$, otherwise known as a conformal transformation of the metric. The resultant effective mass density was given as a powerlaw in the radial coordinate r , whose leading term is $\propto (2/a)\sqrt{1 - p^2/r^4}$, where a and p are free constants. The effective mass is seen to be a difference of terms, and the two terms are identified as two types of mass. Both fall off like $1/r^2$. It was speculated that the second mass was associated with electric charge, but a neutral solution was under

consideration. The most natural explanation seems to us, rather, to be that there are two contributions to the gravitating mass-energy: the central mass and its associated scalar field. That is the interpretation pursued in this article.

In a follow-up work, Davison and Owen [8] repeated their previous calculation [7], but now considered a rotation of the diagonal solution between the time coordinate and the x^5 coordinate. This introduces an off-diagonal term in the 5D metric which they identified with electric charge. However, they introduced a new free parameter into their interpretation of electric charge, so their identification is in some sense a tautology, and includes more information than just the vacuum 5D field equations.

Ferrari [9] obtained an approximate solution to (1) by postulating two lengthscales and matching terms in a power series. His solution did not provide a proper R-N limit. As we will see below, his approach missed a key lengthscale in the problem.

Liu and Wesson [10, 11] obtained a broader class of solutions that extended the earlier solutions of [6] and [8]. They consider it a 3-parameter solution because in addition to the mass and electric charge, the potentials can be raised to a variable power. However, only the exponents $a = 1$ and $b = 0$ yield a solution with a reasonable R-N limit, and the authors note they could not get a good near-field R-N limit.

In this work, we present new parameterizations of the vacuum solutions that provide insight into the nature of scalar field energy and electric charge. We also discuss the negative energy of the scalar field.

2. General analysis

2.1. Reissner-Nordström-coulomb limit

As mentioned, the entire class of 5D solutions to (1) should be compared with the R-N solution, which is an exact solution to the vacuum Einstein equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}^{EM} \quad (2)$$

where $G_{\mu\nu}$ is the 4D Einstein tensor, and $T_{\mu\nu}^{EM}$ is the electromagnetic energy-momentum tensor. This corresponds to the 5D vacuum equations (1) because the electromagnetic field emerges in the Kaluza theory from curvature in 5D. In 4D, the electromagnetic field is a source of spacetime curvature, but in 5D, both spacetime curvature and the electromagnetic field are manifestations of vacuum curvature in 5D.

For a body of electric charge Q , the electric field is obtained simultaneously with (2) from the vacuum Maxwell equations:

$$\nabla_{\mu} F^{\mu\nu} = g^{-1/2} \partial_{\mu} (g^{1/2} F^{\mu\nu}) = 0 \quad \rightarrow \quad g^{1/2} F^{rt} = g^{1/2} g^{rr} \partial_r A^t \equiv Q \quad (3)$$

where $F^{\mu\nu}$ is the electromagnetic field strength tensor, defined in terms of the electromagnetic vector potential A^{μ} as $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, g is the determinant of the metric $g_{\mu\nu}$, and the electric charge Q is identified with the integration constant. The final step follows because the only non-zero component of A^{μ} is A^t , which depends only on the r coordinate.

For a body of mass M , with mass lengthscale $\bar{M} \equiv GM/c^2$ and electric charge lengthscale $\bar{Q} \equiv (G/4\pi\epsilon_0 c^4)^{1/2} Q$, the solution to (2) modified by the electric field (3) is the Reissner-Nordström metric:

$$-ds^2 = -\left(1 - \frac{2\bar{M}}{r} + \frac{\bar{Q}^2}{r^2}\right) c^2 dt^2 + \left(1 - \frac{2\bar{M}}{r} + \frac{\bar{Q}^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (4)$$

where we are working in standard Schwarzschild coordinates, so that the angular components of the metric have no gravitational effects.

The mass lengthscale emerges mathematically from the field equations as an integration constant, and its identification with GM/c^2 arises from identification with Newtonian gravity at large r . In the limit that $Q \rightarrow 0$, (4) goes over to the Schwarzschild metric, and the mass parameter corresponds to the Schwarzschild mass. The electrostatic energy makes a contribution to the total energy in the field.

The solution (3) can be expanded in a Taylor series about the origin to yield:

$$\partial_r A^t = \frac{\bar{Q}}{r^2 g^{rr}} = \frac{\bar{Q}}{r^2} g_{rr} = \frac{\bar{Q}}{r^2} + \frac{2\bar{M}\bar{Q}}{r^3} - \frac{\bar{Q}(\bar{Q}^2 - 4\bar{M}^2)}{r^4} + O(r^{-5}) \quad (5)$$

where g^{rr} is the R-N component given in (4). This is the electric field in curved space. In the limit of flat space, $g^{rr} \rightarrow 1$, and we recover the usual $1/r^2$ law.

2.2. Mapping between 5D and 4D fields

The Kaluza hypothesis under the cylinder condition provides a mapping of the 4D gravitational and electromagnetic fields to components of the 5D metric tensor \tilde{g}_{ab} . There is a unique set of components of \tilde{g}_{ab}

that transform as a 4D tensor, and can therefore be identified with the 4D metric $g_{\mu\nu}$; there is likewise a unique set of components of \tilde{g}_{ab} that transform as a 4-vector, and can therefore be identified with the electromagnetic vector potential. The cylinder condition imposes a restricted class of 5D coordinate transformations that determine these identifications.

The cylinder condition implies the theory is not generally covariant in all five dimensions. The restricted class of coordinate transformations is such that the 4 spacetime coordinates are covariant, $x'^{\mu} = x^{\mu}$. The transformation on the fifth coordinate is $x'^5 = x^5 + f(x^{\mu})$. The arbitrary function of spacetime coordinates that is added to the fifth coordinate can be interpreted as the electromagnetic gauge choice. This restricted class of 5D coordinate transformations is analogous to the restricted class of coordinate transformations that hold when 4D general relativity is approximated as a small perturbation around the Minkowski metric. Just as the weak field theory loses its general covariance when the metric is restricted to be a perturbation of the Minkowski metric, the 5D theory loses its 5D covariance when the cylinder condition is imposed.

Our analysis of the solutions to (1) is based on a suite of expressions standard in the Kaluza literature that relate the 4D fields to the 5D metric. For the formulae below, we provide citations for them to expressions in [9, 11, 12].

For a 5D metric \tilde{g}_{ab} that obeys the cylinder condition, the 4D spacetime metric components $g_{\mu\nu}$ are given by⁵

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} - \frac{\tilde{g}_{\mu 5}\tilde{g}_{\nu 5}}{\tilde{g}_{55}} \quad (6)$$

where greek indices indicate the 4 spacetime coordinates, and the fifth coordinate index is 5.

For our particular case, time-independent vacuum solutions in spherical symmetry, A^{μ} has only a time component, A^t , the Coulomb potential. The time-time component and the spatial metric components of (6) are therefore given by:

$$g_{tt} = \tilde{g}_{tt} - (\tilde{g}_{t5})^2/\tilde{g}_{55}, \quad g_{ij} = \tilde{g}_{ij} \quad (7)$$

where small roman indices denote the 3 spatial components.

The Coulomb potential is given in terms of the 5D inverse metric (See footnote 5):

$$A^t = -\tilde{g}^{t5} \quad (8)$$

The scalar potential is simply the 5-5 component of the 5D metric (See footnote 5):

$$\phi^2 \equiv \tilde{g}_{55} \quad (9)$$

The nature of the scalar-modified Coulomb field (8) and the nature of the scalar field (9) emerging from the 5D vacuum equations (1) can be broadly deduced from their 4D field equations. The vacuum Maxwell equations for a radial electric field modified by the scalar field are⁶:

$$\nabla_{\mu}(\phi^3 F^{\mu\nu}) = 0 \quad \rightarrow \quad F^{rt} = \bar{Q} g^{-1/2} \phi^{-3} \quad (10)$$

where \bar{Q} is the integration constant and the indices r and t refer to the radial coordinate and the time coordinate.

Equation (10) means that the Coulomb field will be modified by the scalar field as well as by the metric, so the effects of ϕ and g^{rr} could be difficult to distinguish in Coulomb experiments. Therefore in our results we also provide the Taylor series expansions of g_{tt} and A^t , to compare with the Reissner-Nordström and Coulomb limits (4) and (5).

2.3. Zero field energy in 5D vacuum

It has not been previously recognized that the solutions to (1) imply the combined energy density of the scalar and electromagnetic field vanishes. To see how that is so, let us introduce the scalar field and electromagnetic field energy-momentum tensors, $T_{\mu\nu}^{\phi}$ and $T_{\mu\nu}^{EM}$. They enter the Kaluza-modified vacuum Einstein equations as follows⁷:

$$G_{\mu\nu} = \frac{1}{\phi} T_{\mu\nu}^{\phi} + \phi^2 T_{\mu\nu}^{EM} \quad (11)$$

where $G_{\mu\nu}$ is the Einstein tensor, and where the gravitational and electrical constants multiplying the electromagnetic energy-momentum in (2) are absorbed into the units of electric charge \bar{Q} . The scalar field energy-momentum, on the other hand, has no coupling constant in the Kaluza theory; it is a pure curvature term like $G_{\mu\nu}$, and can be considered gravitational in nature, and emerging from 5D curvature like the other fields.

⁵ See e.g., (2) in [9], (A2) in [11], (2) in [12].

⁶ See e.g., (13) in [4], (6c) in [9], (A12b) in [11], (25) in [12].

⁷ See e.g., (6a) in [9], (A11) in [11], (21) in [12].

The electromagnetic energy-momentum tensor is given by the usual formula:

$$T_{\mu\nu}^{EM} = g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \quad (12)$$

The scalar field energy-momentum tensor is given by:

$$T_{\mu\nu}^{\phi} = \nabla_{\mu} \nabla_{\nu} \phi - g_{\mu\nu} \square \phi \quad (13)$$

where $g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} \equiv \square$.

The only non-zero components of the electromagnetic field tensor are $F_{tr} = -F_{rt}$, so $F^{\alpha\beta} F_{\alpha\beta} = 2g^{tt}g^{rr}F_{rt}^2$. There is a factor of ϕ^2 multiplying the electromagnetic energy-momentum in (11), so the total, scalar-modified electromagnetic energy density T_{tt}^{EM} of the Coulomb field is

$$\phi^2 T_{tt}^{EM} = \frac{1}{2} g^{rr} \phi^2 F_{rt}^2 \quad (14)$$

where the last step uses (10) and that $g = |g_{tt}g_{rr}r^4 \sin^2\theta|$.

The vacuum field equation for the scalar field is given by⁸:

$$\frac{3}{4} \phi^2 F^{\alpha\beta} F_{\alpha\beta} = R = 3 \frac{\square \phi}{\phi} \quad (15)$$

where the last step follows from taking the trace of (11).

The only non-zero gradient of the scalar field is $\partial_r \phi$ $\partial_t \phi$. Therefore the scalar field energy density T_{tt}^{ϕ} is given by

$$\frac{1}{\phi} T_{tt}^{\phi} = -\frac{1}{\phi} g_{tt} \square \phi = -\frac{1}{4} g_{tt} \phi^2 F^{\alpha\beta} F_{\alpha\beta} = -\frac{1}{2} g^{rr} \phi^2 F_{rt}^2 = -\phi^2 T_{tt}^{EM} \quad (16)$$

where the successive equalities follow from (13), (15), and (14). The scalar and electromagnetic energy densities in (11) are equal and opposite, and therefore these are zero-mass solutions.

The nature of electrically-charged singularities has been investigated by many authors. The R-N solution (4) allows a solution with $\bar{M} = 0$, corresponding to a massless electric charge. However, the energy in the electric field will look like an effective mass at infinity [13–17]. Indeed, the electric field acts to stabilize the energy of a point particle [15]. Yet the Kaluza vacuum equations for zero mass have also zero total scalar plus electric field energy, and therefore zero effective mass at infinity.

Therefore we can conclude that the vacuum 5D field equations (1) are all zero-mass solutions. This is a result new to the literature. For the 5D vacuum solutions, it appears the electric field energy stands on a foundation of negative scalar field energy.

It is well-known that scalar fields are associated with negative energy [18–20]. The form (13) of the scalar field energy-momentum does not produce positive-definite energy, because of the second derivative [20]. For example, if we consider scalar waves $\phi(\omega t)$ of angular frequency ω , massless and moving at the velocity of light, then the second term in (13) is zero. In flat space, then, $T_{tt}^{\phi} = \partial_t \partial_t \phi = -\omega^2 \phi$, which is not positive definite if ϕ is oscillatory.

We shall comment on the negative-energy aspect of the vacuum solution scalar field further when we consider the individual solutions.

When matter sources are introduced, the scalar field energy need not be negative. In a subsequent paper we will investigate solutions with matter sources, such as considered by [4] in their original paper.

3. Results

3.1. Methodology

We consider new physical parameterizations of known solutions to (1) that allow us to identify energy in the scalar field, and to relate electric charge to scalar charge and mass. We start by establishing existing baseline solutions known in the literature, and enumerating their characteristics. Then we introduce our new parameterizations, and discuss their physical implications.

3.2. Baseline electrically-neutral solution (Davidson & Owen)

Let us establish a baseline solution for an electrically-neutral body against which to compare our parameterizations. Our neutral-body baseline is the Davidson & Owen solution [7], but written in standard, Schwarzschild-like coordinates

⁸ See e.g., (6b) in [9], (A12a) in [11], (26) in [12].

$$\tilde{g}_{tt}^{do} = \left(1 - \frac{2M}{r}\right)^\alpha, \quad \tilde{g}_{55}^{do} = -\left(1 - \frac{2M}{r}\right)^\beta, \quad \tilde{g}_{rr}^{do} = -\left(1 - \frac{2M}{r}\right)^{-\alpha-\beta} \quad (17)$$

where $\alpha^2 + \beta^2 + \alpha\beta = 1$

Since the metric is diagonal, $\tilde{g}_{\mu\nu} = g_{\mu\nu}$, we can read off the 4D metric components directly, and write the leading terms of their Taylor expansions:

$$g_{tt}^{do} = 1 - \frac{2\alpha M}{r} + O(r^{-2}), \quad \tilde{g}_{55}^{do} = -1 + \frac{2\beta M}{r} + O(r^{-2}) \quad (18)$$

The baseline neutral solution (17) has a Schwarzschild limit when $\alpha = 1$ and $\beta = 0$. But in this limit, the scalar field vanishes.

If α and β are allowed to vary from the Schwarzschild limit, then at large distances r , there is a gravitational charge $\propto \alpha M$ and a scalar charge $\propto \beta M$. Therefore we expect there to be a scalar mass or scalar charge which is proportional to the energy in the scalar field. Yet for the Schwarzschild limit of our baseline neutral solution (17), the scalar field energy vanishes.

We can conclude that this solution apportions a fixed fraction of the total mass-energy M into the scalar and gravitational fields. To recover the pure Schwarzschild solution requires putting all the energy into the gravitational field, and setting scalar field energy to zero.

We can also conclude that this neutral-body solution does not suffer from the zero-energy field condition mentioned above. In this solution, there is always energy in the field. The reason this energy is captured in the neutral-body scalar field equation is that the solution to (15) is a particular solution, and the homogeneous solution is implicitly excluded. The neutral-body solutions are implicitly including a delta-function mass source, as in the Schwarzschild solution.

Let us summarize the key properties of the Davidson & Owen neutral-body baseline solution for comparison with the new parameterizations.

- the fifth coordinate is spacelike
- the sign of the scalar potential is positive for positive β
- the scalar field vanishes in the Schwarzschild limit
- at large r , separate scalar and gravitational charges can be identified, but both are proportional to M
- the total mass-energy is fixed at M , and the solutions vary the fraction allocated to the gravitational and scalar fields

3.3. Baseline electrically-charged solution (Liu & Wesson)

Let us establish a baseline solution for an electrically-charged body against which to compare our alternative parameterizations. Our electrically-charged baseline solution is from Liu & Wesson [10]. This solution has viable R-N and Coulomb limits at large distances.

$$\tilde{g}_{tt}^{lw} = 1 - \frac{2\bar{M}}{r}, \quad \tilde{g}_{t5}^{lw} = \frac{\bar{Q}}{r}, \quad \tilde{g}_{55}^{lw} = -1 - \frac{\bar{Q}^2}{2\bar{M}r}, \quad \tilde{g}_{rr}^{lw} = -\left(1 - \frac{2\bar{M}}{r} + \frac{\bar{Q}^2}{2\bar{M}r}\right)^{-1} \quad (19)$$

The Taylor series for the 4D components are given by

$$g_{tt}^{lw} = 1 - \frac{2\bar{M}}{r} + \frac{\bar{Q}^2}{r^2} - \frac{\bar{Q}^4}{2\bar{M}r^3} + O(r^{-4}) \quad (20)$$

$$A_{lw}^t = \frac{\bar{Q}}{r} + \frac{\bar{Q}}{r^2} \left(2\bar{M} - \frac{\bar{Q}^2}{2\bar{M}}\right) + O(r^{-3}) \quad (21)$$

This solution has the correct Reissner-Nordström and Coulomb behavior at leading order. However, we know its total field energy is zero, so that explains the higher order terms in g_{tt} that are not in the R-N limit. We see explicitly that if the electric charge $\bar{Q} \rightarrow 0$, then the scalar charge goes to zero along with it. However, this solution does not have a well-behaved zero-mass solution like the R-N does. In the limit that $\bar{M} \rightarrow 0$, the scalar field and the Coulomb field diverge.

The scalar field in this solution goes to zero when the electric charge goes to zero. Yet the neutral baseline (17) has a non-zero scalar field. So in spite of the provision of a good R-N limit by this solution, it does not seem to provide a good neutral body limit as in the baseline neutral solution.

This solution has a spacelike fifth dimension, and the scalar potential is negative, like the gravitational potential. There is no contribution by the scalar field to g_{tt}^{lw} .

Let us summarize the key properties of the Liu & Wesson electrically-charged baseline solution.

- the fifth coordinate is spacelike
- the sign of the scalar potential is negative, like gravity
- the scalar charge is proportional to \bar{Q}^2/\bar{M} ; it vanishes when $\bar{Q} \rightarrow 0$, but diverges when $\bar{M} \rightarrow 0$. There is no $\bar{Q} = 0$ scalar field limit for this solution.
- there is no contribution of scalar field energy to g_{tt} at leading order
- the total scalar and electromagnetic field energy is zero per (16), so the emergence of R-N-like terms quadratic in \bar{Q}^2 at leading order is at the ‘expense’ of the scalar field

3.4. New solution parameterization class A

We consider a reparameterization of the baseline 2-parameter solution \tilde{g}_{ab}^A with viable R-N and Coulomb limits for large r , which we call here ‘Class A’. Liu & Wesson considered their solution a ‘3-parameter’ solution compared to the ‘2-parameter’ solutions of Gross & Perry, where their third parameter was an exponent. However, only the exponent equal to one yields viable R-N and Coulomb limits. Yet a finite exponent can yield approximate R-N behavior so long as r is sufficiently large and the potentials sufficiently small. We constrain ourselves to exponents of one to pursue the possibility of an exact R-N limit.

Consider a reparameterization of the baseline solution for a monopole with mass, electric charge, and scalar charge.

$$\tilde{g}_{tt}^A = 1 - \frac{a}{r}, \quad \tilde{g}_{t5}^A = \frac{\sqrt{ab}}{r}, \quad \tilde{g}_{55}^A = -1 - \frac{b}{r}, \quad \tilde{g}_{rr}^A = -\left(1 - \frac{a}{r} + \frac{b}{r}\right)^{-1} \quad (22)$$

where a and b are constants corresponding to gravitational charge and scalar charge. Electric charge emerges from their product. Under this parameterization, electric charge is seen as an auxiliary quantity to the scalar charge and the mass. Under the standard identification of the baseline solution, electric charge is fixed, and scalar charge is determined from electric charge and mass, leading to the divergences discussed previously.

The Taylor series for the 4D components are given by

$$g_{tt}^A = 1 - \frac{a}{r} + \frac{ab}{r^2} - \frac{ab^2}{r^3} + O(r^{-4}) \quad (23)$$

$$A_A^t = \frac{\sqrt{ab}}{r} + \frac{\sqrt{ab}}{r^2}(a - b) + O(r^{-3}) \quad (24)$$

Since the Class A solution form is a reparameterization of the baseline solution, we also refer to the baseline as solution A0. The Class A parameterizations allow us to isolate the gravitational charge a and the scalar charge b .

The R-N limit provides a direct constraint on the signature of the fifth dimension. When $\tilde{g}_{t5} = \bar{Q}/r$, as is the case in the baseline electrically-charged solution (19) and in many other solutions, then \tilde{g}_{55} must be spacelike in order to reproduce the \bar{Q}^2 term of (4). If we flip the sign on b in (22), then the electric charge becomes imaginary. Therefore, the R-N limit appears to argue for a spacelike fifth coordinate. Otherwise, it would imply the fifth dimension is imaginary, and this may be allowable as well but has not been investigated.

The total scalar and electric field energies are constrained to zero by (16), but in such a way that electric charge is multiplicative in gravitational charge (mass) and scalar charge. If either the mass or scalar charge goes to zero, then the electric charge goes to zero. Yet a Schwarzschild-like limit emerges if the scalar charge is set to zero.

The Class A solution forms reported here allow for either sign for a and b , and either sign for \tilde{g}_{tt} and \tilde{g}_{55} . However, the \tilde{g}_{55} must be spacelike to yield the correct R-N and Coulomb limits.

Let us summarize the key properties of the Class A solution forms.

- the fifth coordinate is spacelike
- the sign of the scalar potential is negative or positive
- the electric charge is the product of scalar charge and gravitational charge (mass)
- the electric charge goes to zero when the scalar charge b goes to zero, and in that case, the total mass is just the gravitational charge a

3.4.1. A particular Class A solution form with a neutral scalar field

We can use the freedom of this solution to posit a scalar charge that depends both on mass and electric charge by setting $b = \bar{Q}^2/2\bar{M} + \alpha\bar{M}$, where α is a constant, and setting gravitational charge $a = 2\bar{M}$. Let us call this

solution form A1. Then the 4D component expansions become:

$$g_{tt}^{A1} = 1 - \frac{2\bar{M}}{r} + \frac{\bar{Q}^2 + 2\alpha\bar{M}^2}{r^2} + O(r^{-3}) \quad (25)$$

$$A_{A1}^t = \frac{\sqrt{\bar{Q}^2 + 2\alpha\bar{M}^2}}{r} + O(r^{-2}) \quad (26)$$

with

$$\tilde{g}_{55}^{A1} = -1 - \left(\frac{\bar{Q}^2/2\bar{M} + \alpha\bar{M}}{r} \right) \quad (27)$$

The A1 solution form provides a scalar field in the limit that $\bar{Q} \rightarrow 0$. It also provides viable R-N and Coulomb limits to leading order. The R-N limit is modified now to include the energy from the scalar field. However, the total scalar and electromagnetic energies are zero, and the net mass is just the gravitational charge $a = 2\bar{M}$. Yet this solution suggests the intriguing possibility that the electric field exists on a foundation of negative scalar field energy that goes otherwise unseen. And although the electric field is technically not contributing to the mass seen at infinity, it is so small in practice that the difference would be undetectable.

The solution form A1 suggests a new effect, that mass alone can give rise to electric charge. This suggests testing for weak electric fields from neutral bodies. But such tests are very difficult to perform. A weak electric field from mass alone could feasibly go undetected.

3.5. New solution parameterization class B

Here we introduce another two-parameter solution class parameterization that can also be obtained from the baseline solution, which we call ‘Class B’. It provides a viable R-N limit, and treats the scalar field in a novel way not described before.

$$\tilde{g}_{tt}^B = 1 - \frac{b}{r} + \frac{a}{r}, \quad \tilde{g}_{r5}^B = \frac{\sqrt{ab}}{r}, \quad \tilde{g}_{55}^B = -1 + \frac{ab}{r(a-b)}, \quad \tilde{g}_{rr}^B = \frac{r(b-a)}{a^2 + a(r-3b) + b(b-r)} \quad (28)$$

The 4D components are given by

$$g_{tt}^B = 1 - \frac{(a-b)^2}{a(b-r) + br} = 1 + \frac{a-b}{r} + \frac{ab}{r^2} + \frac{a^2b^2}{r^3(a-b)} + O(r^{-4}) \quad (29)$$

$$A_B^t = \frac{(a-b)\sqrt{ab}}{a^2 + a(r-3b) + b(b-r)} = \frac{\sqrt{ab}}{r} - \frac{\sqrt{ab}(a^2 - 3ab + b^2)}{r^2(a-b)} + O(r^{-3}) \quad (30)$$

The Class B solution forms (28) improve on Class A (22) by providing two independent pieces to the \tilde{g}_{tt} , allowing separate contributions to gravitational charge for mass and for scalar field energy. It also allows for a positive scalar field, if $a > b$. That constraint is also necessary for \tilde{g}_{rr} to be spacelike.

The Class B solution forms do not necessarily have a coordinate singularity at values of $r > 0$, like the Schwarzschild and Reissner-Nordström metrics. The condition for the singularity to exist is $a^2 + b^2 = 3ab$.

Let us summarize the key properties of the Class B solution forms.

- $a > b$ for \tilde{g}_{rr} to be spacelike
- the fifth coordinate is spacelike
- the sign of the scalar potential is positive
- there are two independent gravitational charges, and the electric and scalar charges are obtained from their product
- there is a contribution of scalar field energy to g_{tt} at leading order, in such a way that the scalar field energy enters like the mass. This only includes the scalar field accruing from electric charge.
- there is no coordinate singularity at $r > 0$ unless $a^2 + b^2 = 3ab$.

3.5.1. A particular Class B solution form with scalar field energy

We can consider a specific Class B solution form with $a = \bar{Q}^2/2\bar{M}$ and $b = 2\bar{M}$. Let us call this solution B0. The 4D component expansions are:

$$g_{tt}^{B0} = 1 - \frac{2\bar{M}}{r} + \frac{\bar{Q}^2}{2\bar{M}r} + \frac{\bar{Q}^2}{r^2} + O(r^{-3}) \quad (31)$$

Table 1. Comparison of select 5D solutions \tilde{g}_{ab} in standard Schwarzschild coordinates. The corresponding 4D metric components $g_{\mu\nu}$ are given, along with the Coulomb potential A^t . The angular metric components are $g_{\theta\theta} = r^2$ and $g_{\varphi\varphi} = r^2 \sin^2 \theta$ for all solutions except the baseline neutral. For economy of notation, the overbars on the mass and charge lengthscales are dropped. ‘NA’ means not applicable, $g_{tt}^S \equiv 1 - 2M/r$, $g_{tt}^{RN} \equiv g_{tt}^S + Q^2/r^2$, and $\varepsilon_r^{(2,3)}$ indicates additional small terms of order $r^{-2,-3}$. Unique features of the new solution forms are highlighted. Solution form A1 provides a non-zero scalar field as $Q \rightarrow 0$. Solution form B0 provides a positive sign of the scalar field, and accounts for scalar field energy in g_{tt} . Expansions of A^t for A0, A1, and B0 are shown only to order ε_r^2 to fit in the table; see the relevant sections for higher order expressions. In general the A^t solutions vary at order ε_r^2 , even as they agree at leading order. Electrically-charged solutions A0, A1, B0 tie together the scalar and electric field so their total energy is zero.

Potentials	Schwarz.	Baseline-neutral (17)	R-N-Coulomb (4), (5)	Baseline-charged(A0) (19)	A1 section 3.3.1	B0 section 3.4.1
\tilde{g}_{tt}	NA	$\left(1 - \frac{2M}{r}\right)^\alpha$	NA	$1 - \frac{2M}{r}$	$1 - \frac{2M}{r}$	$1 - \frac{2M}{r}$
g_{tt}	g_{tt}^S	$1 - \frac{2\alpha M}{r} + \varepsilon_r^2$	g_{tt}^{RN}	$g_{tt}^{RN} + \varepsilon_r^3$	$g_{tt}^S + \frac{Q^2 + 2\alpha M^2}{r^2} + \varepsilon_r^3$	$g_{tt}^{RN} + \frac{Q^2}{2Mr} + \varepsilon_r^3$
$-(\tilde{g}_{rr} = g_{rr})$	$(g_{tt}^S)^{-1}$	$(g_{tt}^S)^{-\alpha-\beta}$	$\left(g_{tt}^S + \frac{Q^2}{r^2}\right)^{-1}$	$\left(g_{tt}^S + \frac{Q^2}{2Mr}\right)^{-1}$	$\left(1 - \frac{M(2-\alpha)}{r} + \frac{Q^2}{2Mr}\right)^{-1}$	see (28)
\tilde{g}_{55}	NA	$(g_{tt}^S)^\beta$	NA	$-1 - \frac{Q^2}{2Mr}$	$-1 - \frac{Q^2/2M + \alpha M}{r}$	$-1 + \frac{Q^2/r}{Q^2/2M - 2M}$
\tilde{g}_{r5}	NA	NA	NA	Q/r	$\frac{\sqrt{Q^2 + 2\alpha M^2}}{r}$	Q/r
A^t	NA	NA	$\frac{Q}{r} + \frac{MQ}{r^2} + \varepsilon_r^3$	$Q/r + \varepsilon_r^2$	$\sqrt{Q^2 + 2\alpha M^2}/r + \varepsilon_r^2$	$Q/r + \varepsilon_r^2$

$$A_{B0}^t = \frac{\bar{Q}}{r} - \frac{\bar{Q} (4\bar{M}^2 - 3\bar{Q}^2 + \bar{Q}^4/4\bar{M}^2)}{r^2 (\bar{Q}^2/2\bar{M} - 2\bar{M})} + O(r^{-3}) \quad (32)$$

$$\tilde{g}_{55}^{B0} = -1 + \frac{\bar{Q}^2/r}{(\bar{Q}^2/2\bar{M} - 2\bar{M})} \quad (33)$$

The B0 solution form allows for a contribution from scalar field energy to g_{tt} , but otherwise preserves the R-N limit at leading order in the mass and electric field terms

As in the previous solutions, the apparent scalar field energy sums to zero with the electric field energy, so that the total mass is just the gravitational charge.

Like the baseline solution, the B0 solution scalar field vanishes when $\bar{Q} = 0$. Also, the scalar field energy term does not include a contribution for neutral bodies. And it shares the same singularity at zero mass as the baseline solution A0.

This solution allows the scalar potential to be positive, while the fifth coordinate remains spacelike.

4. Conclusions

Two new classes of physical solution forms, A and B, have been investigated. We provided specific parameter choices A1 and B0 for class A and B solutions, geared toward a reasonable R-N-Coulomb limit and a non-zero scalar field.

We showed that the vacuum Kaluza equations have zero total field energy. The energy apparent in the electric field comes at the expense of negative energy in the scalar field. In this way, the scalar field acts as a sort of ‘foundation’ for the electric field. Therefore the total energy at infinity is just the neutral mass, not the neutral mass plus electrostatic energy that is implied by the R-N solution.

We selected from among the families of new solution forms 2 with meaningful physical interpretations. The new solution forms presented here have several attractive features:

- separate scalar charge and mass contributions to the gravitational charge
- positive or negative scalar potential, but always a spacelike fifth coordinate in order to match the R-N limit
- electric charge as the product of gravitational charge (mass) and scalar charge
- scalar and electric charge as the product of 2 gravitational charges
- a charged solution with a scalar field in the neutral limit, implying weak electric fields from neutral masses
- viable Reissner-Nordstrom, Schwarzschild, and Coulomb limits for large r , with the understanding that the net scalar and electromagnetic energy densities sum to zero, so that they do not contribute to the mass seen at infinity

Our new solution forms are compared to the baseline solutions and 4D limits in table 1.

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Data availability statement

No new data were created or analysed in this study.

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