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Finite sections: Stability, spectral pollution and asymptotics of condition numbers and pseudospectra

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Dedicated to Albrecht Böttcher on his 70th birthday. Herzlichen Glückwunsch, Albrecht!

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ABSTRACT

The stability of an approximating sequence (A_n) for an operator A usually requires, besides invertibility of A, the invertibility of further operators, say $B, C, \ldots$, that are well-associated to the sequence (A_n). We study this set, $\{A, B, C, \ldots\}$, of so-called *stability indicators* of (A_n) and connect it to the asymptotics of $\|A_n\|$, $\|A_n^{-1}\|$ and $\kappa(A_n) = \|A_n\|\|A_n^{-1}\|$ as well as to spectral pollution by showing that $\limsup \operatorname{Spec}_\varepsilon A_n = \operatorname{Spec}_\varepsilon A \cup \operatorname{Spec}_\varepsilon B \cup \operatorname{Spec}_\varepsilon C \cup \ldots$. We further specify, for each of $\|A_n\|$, $\|A_n^{-1}\|$, $\kappa(A_n)$ and $\operatorname{Spec}_\varepsilon A_n$, under which conditions even convergence applies.

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1. Introduction

For the study of a linear operator A on a Banach space X, one often has to resort to numerical approximation: Take a sequence $(A_n)_{n\in\mathbb{N}}$ of simpler operators approximating A in some sense, $A_n \to A$, and study the quantity of interest – say, the inverse – of A_n as $n \to \infty$ in place of A.

If the approximants A_n are of finite rank, the convergence $A_n \to A$ is, as a rule, never in operator norm as the uniform limit of A_n were compact, ruling out, for example, all invertible operators A. So $A_n \to A$ is typically weaker than uniform, making the question whether also

$$A_n^{-1} \to A^{-1} \tag{1}$$

holds largely non-trivial. If we let $\to$ refer to pointwise operator convergence on X, that is $\|A_n x - Ax\| \to 0$ for all $x \in X$, then the convergence (1) allows for asymptotically solving $Ax = b$ via $A_n x_n = b_n$ as $n \to \infty$, see e.g. [21,3,1,4,10,23,2,15,26,11].

Stability. The key to (1) is the so-called *stability* of the sequence (A_n), typically yielding

- existence of A^{-1},
- convergence of A_n^{-1},
- ... to A^{-1}.

By definition, an operator sequence $(A_n)_{n\in\mathbb{N}}$ is *stable* if all but finitely many A_n are invertible and their inverses are uniformly bounded. We can express this as follows:

$$(A_n) \text{ is stable} \quad :\Longleftrightarrow \quad \limsup \|A_n^{-1}\| < \infty, \tag{2}$$

where we put $\|B^{-1}\| := \infty$ if and only if B is not invertible.

In §6 of [11], looking at the same equivalence (2), the authors ask about quantities:

(Q1) How large is the lim sup in (2)?

(Q2) Is it possibly a limit?

(Q3) What is the asymptotics of the condition numbers, $\kappa(A_n) = \|A_n\| \cdot \|A_n^{-1}\|$?

(Q4) What is the asymptotics of the pseudospectra of A_n?

We take this as a program for our paper here.

Our results. For sequences (A_n) in the finite section algebra of band-dominated (which are bounded but generally non-normal) operators A on $X = \ell^p(\mathbb{Z})$ with $p \in (1,\infty)$, we associate a set

$$\mathrm{Stab}\big(\,(A_n)_{n\in\mathbb{N}}\,\big) \;=:\; \mathrm{Stab}(A_n)$$

of operators on X – the corresponding *stability indicators* – with each sequence (A_n), with the property that (A_n) is stable if and only if every $B \in \mathrm{Stab}(A_n)$ is invertible. We demonstrate how that same set, $\mathrm{Stab}(A_n)$, determines precise answers to each of (Q1) – (Q4):

$$\left.\begin{aligned} \limsup \|A_n\| &= \max_{B\in\mathrm{Stab}(A_n)} \|B\| \\ \limsup \|A_n^{-1}\| &= \max_{B\in\mathrm{Stab}(A_n)} \|B^{-1}\| \\ \limsup \mathrm{Spec}_\varepsilon A_n &= \bigcup_{B\in\mathrm{Stab}(A_n)} \mathrm{Spec}_\varepsilon B, \qquad \varepsilon > 0. \end{aligned}\right\} \tag{3}$$

Via certain triples (A, B, C) in $\mathrm{Stab}(A_n)$, we get a formula for $\limsup \kappa(A_n)$ and show, for each of our quantities, how also the answer to whether or not $\limsup = \lim$ is encoded in $\mathrm{Stab}(A_n)$.

A remark on spectral pollution. In an ideal world, one would hope that $A_n \to A$ generally implies $\lim \mathrm{Spec}_\varepsilon A_n = \mathrm{Spec}_\varepsilon A$. But the truth is: neither does the set sequence $\mathrm{Spec}_\varepsilon A_n$ converge in Hausdorff sense as $n \to \infty$, whence we say "lim sup" instead of "lim" in (3), nor is the result equal to $\mathrm{Spec}_\varepsilon A$. Instead it is the union of $\mathrm{Spec}_\varepsilon A$ with $\mathrm{Spec}_\varepsilon B$ for all other stability indicators B of (A_n) (note that A is always an element of $\mathrm{Stab}(A_n)$). So if, in practical computations, $\mathrm{Spec}_\varepsilon A_n$ is found to approximate points that are far away from $\mathrm{Spec}_\varepsilon A$, so-called *spectral pollution*, then the last formula in (3) exactly says who is to blame for this: some of the other $B \in \mathrm{Stab}(A_n)$ and their pseudospectra. A priori knowledge of $\mathrm{Stab}(A_n)$ may hence also help to classify and ignore spectral pollution.

The operators. Thinking of an operator A on $X := \ell^p(\mathbb{Z})$ with $p \in [1, \infty]$ as a bi-infinite matrix $(A_{ij})_{i,j\in\mathbb{Z}}$, we call A a *band operator*, $A \in \mathrm{BO}$, if its matrix is supported on finitely many diagonals only, and if A is bounded as an operator on X. Then let BDO, the set of *band-dominated operators*, denote the closure of BO in the operator norm topology.

Approximants: Pure and composed finite sections. For reasons of finite storage and computational cost, we approximate A, as $n \to \infty$, by finite matrices $A_n = (F_{ij})_{i,j=-n}^n$, each interpreted, via zero extension, as an operator on $\ell^p(\mathbb{Z})$. For the precise construction of A_n with $n \in \mathbb{N}$, let P_n denote the operator of multiplication by the characteristic function of $\{-n, \dots, n\}$ and suppose, as an example, that $A = BC + D$ with somehow simpler operators $B, C, D \in \mathrm{BDO}$. Then we either take

- *pure finite sections*: $A_n := P_n A P_n$ – cutting $(A_{ij})_{i,j=-n}^n$ out of A, extended by zeros, or
- *composed finite sections*: $A_n := (P_n B P_n)(P_n C P_n) + (P_n D P_n)$ – the sum-product of pure finite sections corresponding to the decomposition $A = BC + D$.

The latter is an illustration of something much bigger: the set $\mathcal{S}$ of all limits (A_n) of finite sum-products of pure finite section sequences $(P_n B P_n)$ with $B \in \mathrm{BDO}$. To make sense of the words "limit", "sum" and "product" in this operator sequence context, we introduce, in Section 3 below, a Banach algebra $\mathcal{F}$ of bounded operator sequences together with the two-sided ideal, $\mathcal{N}$, of null sequences in $\mathcal{F}$. We will see that, in a sense, the quotient algebra $\mathcal{F}/\mathcal{N}$ is the playground for algebraic studies of stability.

We wish to add that cutting finite sections out of an infinite matrix is much more than a toy model. See e.g. [19] for why celebrated methods like Galerkin, Ritz-Galerkin, FEM and PCE are not more than special cases of pure finite sections in $\ell^2(\mathbb{N})$.

Semi-infinite matrices and their finite sections. Our focus on bi-infinite matrices A and finite sections A_n taken from $-n$ to n is not a big restriction. If one is instead interested in a semi-infinite matrix B and its pure finite sections B_n from 1 to n then let A be the bi-infinite extension of B by c times the identity, where $c > \|B\|$, so that A_n is invertible if and only if B_n is invertible and $\|A_n^{-1}\| = \max(c^{-1}, \|B_n^{-1}\|) = \|B_n^{-1}\|$ since $c^{-1} < \|B_n\|^{-1} \leq \|B_n^{-1}\|$ by $c > \|B\| \geq \|P_nBP_n\| = \|B_n\|$. So the stability of (B_n) and the asymptotics of $\|B_n^{-1}\|$ are the same as those of (A_n) and $\|A_n^{-1}\|$, respectively. The ε-pseudospectra of A_n are of course those of B_n together with an ε-ball around c. But the latter is isolated and easily ignored if c is sufficiently large.

Banach space-valued ℓ^p over $\mathbb{Z}^d$ with $p \in [1, \infty]$. We stick to scalar-valued $\ell^p(\mathbb{Z})$ with $p \in (1, \infty)$ for much of the exposition, avoiding machinery that could easily hide the true plot of the paper. We explain how to deal with Banach space-valued $\ell^p(\mathbb{Z}^d)$ with $p \in [1, \infty]$ in Section 8.

Note that restriction to just $p = 2$ in this earlier part would be a bit too narrow as it would motivate the following elegant argument (that we learnt from [1]) which is however not generalizable to $p \neq 2$: Since $(A_n) + \mathcal{N} \mapsto \mathrm{Stab}(A_n)$ is an injective *-homomorphism between suitable C^*-algebras, preserving invertibility, it automatically preserves norms. The analysis of [11] shows norm-preservation by other means, not limited to $p = 2$.

History: operator algebraists and stability. The study of stability, naturally claimed as home territory by numerical analysts, nevertheless carried out by means of operator algebraic arguments can be traced back to at least the 1970s:

In 1974, Kozak showed in his PhD thesis [12] that a bounded sequence (A_n) is stable if and only if its coset is invertible in the quotient algebra $\mathcal{F}/\mathcal{N}$, the operator version of ℓ^∞/c_0. This opened an extremely powerful and productive back door into numerical analysis for operator algebraists, see [13,21,3,1,4,10,25,2] and many others.

A related but not quite identic approach to stability is the following: In 1971 Douglas and Howe [7] and later Gorodetski [9] realized, in their study of multidimensional discrete convolutions with homogeneous symbol, that an approximating sequence (A_n) of growing finite matrices is stable if and only if the direct sum, $\oplus A_n$, is a Fredholm operator. In fact, this approach is not too different from Kozaks: $\oplus A_n$ is a Fredholm operator if and only if its coset is invertible in the quotient algebra $L(X)/K(X)$ of bounded operators modulo compact operators on $X = \ell^p(\mathbb{Z})$ (the so-called Calkin algebra), and for $\oplus A_n$ this perfectly translates to invertibility in $\mathcal{F}/\mathcal{N}$.

Later Lange and Rabinovich [14] started a systematic study of Fredholm properties in BDO by so-called *limit operators*. (This interplay and its consequences have been strengthened and simplified in [22,23,15,6,27] and somehow finalized in [18,11]. Our approach here largely rests on these last two papers.) The main message is that $A \in \text{BDO}$ is a Fredholm operator if and only if all its limit operators – capturing the behavior of A at infinity in all kinds of directions – are invertible. Denoting the family of all limit operators of A by $\text{Lim}(A)$, one can obviously connect these Fredholm studies with Douglas, Howe and Gorodetski's $\oplus A_n$ construction, showing that an operator sequence (A_n) is stable if and only if every operator in $\text{Lim}(\oplus A_n)$ is invertible. Removing some more or less obvious redundancies from $\text{Lim}(\oplus A_n)$ if $(A_n) \in \mathcal{S}$ then leads to the set $\text{Stab}(A_n)$ that takes center stage in our paper.

More history: stability indicators. Kozaks PhD thesis [12] is a fore-runner also here. For convex polygons $\Omega \subset \mathbb{R}^2$ with vertices $v_1, \ldots, v_k \in \mathbb{Z}^2$, Kozak looked at the pure finite sections A_n with respect to $(n\Omega) \cap \mathbb{Z}^2$ of discrete convolutions A in $\ell^2(\mathbb{Z}^2)$. He showed that (A_n) is stable if and only if the operators $B_1, \ldots, B_k$ are invertible, where B_j is the compression of A to the infinite cone that fits Ω at its corner v_j. Böttcher and Silbermann [3,4] famously took the 1D version of Kozaks result to both the algebra of Toeplitz operators and their finite section algebra, ending up with two stability indicators – one for each endpoint of the interval. Moreover, they already arrived at all of the formulas (3) for their particular situation. Rabinovich, Roch and Silbermann [23] then even replaced the Toeplitz algebra by BDO. Important other work is, for example, [20], where, though limited to Hilbert space, the operator setting is even more general than BDO and [26,28], where, based on deep localization techniques of [26], all of (3) are shown in the finite section algebra $\mathcal{S}$ over BDO.

So what is different in our paper? Thanks to the quantitative results of [11], we can leave the Hilbert space / C*-algebra setting of [23,20] behind. Thanks to [18,11], we can replace [23]'s sup's by max's, drop a closure in the pseudospectral formula and hugely simplify the proofs. Based on [24] and [16], we can transfer the formulas (3) to subsequences (A_{n_k}) and thereby elegantly identify subsequences with a different $\limsup$ or, in case of their absence, conclude that the $\limsup$ is a proper limit. And finally, our approach via $\text{Lim}(\oplus A_n)$, arguably simplifying the proofs and techniques, allows to generalize the results of [26,28] from $(A_n) \in \mathcal{S}$ to the much larger sequence algebra BDS.

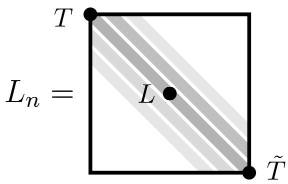
2. Two examples

Before we give the proper details, let us get into the right spirit via two examples.

Example 2.1. (Laurent operator) We start with a banded Laurent operator, that is, $L = (L_{ij})_{i,j\in\mathbb{Z}} = (a_{i-j})_{i,j\in\mathbb{Z}}$ with constant diagonals, finitely many of them nonzero, i.e. $a \in c_{00}(\mathbb{Z})$, acting boundedly on $\ell^p(\mathbb{Z})$. For the pure finite sections, $L_n = P_n L P_n$, it is easily shown that

$$\text{Stab}(L_n) = \{L, T, \tilde{T}\} \quad \text{with} \quad T = (L_{ij})_{i,j\in\mathbb{N}_0}, \quad \tilde{T} = (L_{ij})_{i,j\in-\mathbb{N}_0} \quad \text{and} \quad \mathbb{N}_0 = \mathbb{N}\cup\{0\}.$$

T is what L_n (interpreted as the finite matrix $(L_{ij})_{i,j=-n}^n$) asymptotically, as $n \to \infty$, looks like from the perspective of its top left corner, and $\tilde{T}$ is the same kind of limit when watching L_n grow from its lower right corner. L is the limit when focusing at $(0, 0)$, while $n \to \infty$. We can deal with this mix of bi-infinite and semi-infinite matrices in one set, also with T and $\tilde{T}$ facing in different directions.



So (L_n) has three stability indicators: L, T and $\tilde{T}$. Their invertibility is sufficient and necessary for the stability of (L_n). L itself is in fact redundant in this set: $\|L\| \le \|T\|$ (actually equality) and $\|L^{-1}\| \le \|T^{-1}\|$, so that invertibility of T implies that of L. We will say that L is dominated[1] by T. Also $L - \lambda I$ is dominated by $T - \lambda I$ for $\lambda \in \mathbb{C}$, whence $\mathrm{Spec}_\varepsilon L \subseteq \mathrm{Spec}_\varepsilon T$. All of these are limit operator arguments, see below.

Our formulas (3), reproducing well-known results of [1,4,10,2], show that $\|L_n\| \to \|L\|$ and

$$\limsup \|L_n^{-1}\| = \max(\|L^{-1}\|, \|T^{-1}\|,\ \|\tilde{T}^{-1}\|) \ = \ \max(\|T^{-1}\|,\ \|\tilde{T}^{-1}\|),$$
$$\limsup \kappa(L_n) = \|L\| \cdot \max(\|T^{-1}\|,\ \|\tilde{T}^{-1}\|) \ = \ \max(\kappa(T),\ \kappa(\tilde{T})),$$
$$\limsup \mathrm{Spec}_\varepsilon L_n = \mathrm{Spec}_\varepsilon L \ \cup\ \mathrm{Spec}_\varepsilon T \ \cup\ \mathrm{Spec}_\varepsilon \tilde{T} \ = \ \mathrm{Spec}_\varepsilon T \ \cup\ \mathrm{Spec}_\varepsilon \tilde{T}.$$

By Example 7.6, reproducing results of [1,4,10,2] for sequences in the algebra of finite sections of banded Laurent operators, all three lim sup's are proper limits. In case $p = 2$, $\|T^{-1}\| = \|\tilde{T}^{-1}\|$ and $\mathrm{Spec}_\varepsilon T = \mathrm{Spec}_\varepsilon \tilde{T}$ hold, leading to further simplifications. □

Example 2.2. (symmetric block-flip) Our next example, also see [11, Ex. 6.3], is

$$F \ = \ \mathrm{diag}(\cdots, B, B, \boxed{1}, B, B, \cdots), \quad \text{where } B \ = \ \begin{pmatrix} \mu & 1 \\ 1 & \mu \end{pmatrix}, \quad \mu \in [0, 1)$$
$$\text{and} \quad \boxed{1} = F_{00}.$$

The pure finite sections, $F_n = P_n F P_n$, correspond to the finite $(2n + 1) \times (2n + 1)$ matrices

[1] Between T and $\tilde{T}$, the situation is not so clear: Although, as is easily seen, $\|T\| = \|L\| = \|\tilde{T}\|$ holds, it is possible that $\|T^{-1}\| \ne \|\tilde{T}^{-1}\|$, see [2, Ex. 6.6]. This is because, after flipping both rows and columns around, $\tilde{T}$ is the transpose of T, acting on ℓ^q with $1/p + 1/q = 1$, not on ℓ^p (unless $p = q = 2$).

$$F_n = \begin{cases} \operatorname{diag}\big(B,\cdots,B,\boxed{1},B,\cdots,B\big) & \text{if } n\ge 2 \text{ is even,}\\ \operatorname{diag}\big(\mu,B,\cdots,B,\boxed{1},B,\cdots,B,\mu\big) & \text{if } n\ge 3 \text{ is odd.}\end{cases}$$

Looking at the singular values, $1-\mu$ and $1+\mu$, of B, we conclude that $\|B\| = \|B\|_2 = 1+\mu$ and $\|B^{-1}\| = \|B^{-1}\|_2 = (1-\mu)^{-1}$, whence,

if $n \ge 2$ is even then	if $n \ge 3$ is odd then
$\|F_n^{-1}\| = \|B^{-1}\| = (1-\mu)^{-1}$,	$\|F_n^{-1}\| = \max\left(\|B^{-1}\|, \mu^{-1}, 1\right) = \max\left((1-\mu)^{-1}, \mu^{-1}\right)$,
$\kappa(F_n) = (1+\mu)(1-\mu)^{-1}$,	$\kappa(F_n) = (1+\mu)\max\left((1-\mu)^{-1}, \mu^{-1}\right)$,
$\operatorname{Spec} F_n = \{\mu \pm 1,\ 1\}$,	$\operatorname{Spec} F_n = \{\mu \pm 1,\ 1,\ \mu\}$.

The sequences $\|F_n^{-1}\|$ and $\kappa(F_n)$ are convergent (in fact, constant) if and only if $\mu^{-1} \le (1-\mu)^{-1}$, i.e. if $\mu \in [\frac{1}{2}, 1)$. The pseudospectra are the ε-neighborhoods of the spectra (selfadjoint case).

In this example, (F_n) has five stability indicators; they are $\operatorname{Stab}(F_n) = \{F, C, D, \tilde{C}, \tilde{D}\}$ with

$$C = \operatorname{diag}(B, B, \dots),\ \ D = \operatorname{diag}(\mu, B, B, \dots),$$
$$\tilde{C} = \operatorname{diag}(\dots, B, B),\ \ \tilde{D} = \operatorname{diag}(\dots, B, B, \mu),$$

so that $\|F^{-1}\| = \|B^{-1}\| = \|C^{-1}\| = \|\tilde{C}^{-1}\| = (1-\mu)^{-1}$ and $\|D^{-1}\| = \|\tilde{D}^{-1}\| = (\min(1-\mu, \mu))^{-1}$. The limits C and $\tilde{C}$ arise from the perspective of the top left, resp. lower right, corner of the matrices (F_{2k}) as $k \to \infty$. D and $\tilde{D}$ are the same limits for (F_{2k+1}) as $k \to \infty$. In Section 6 below we associate the subset $\{F, C, \tilde{C}\}$ of $\operatorname{Stab}(F_n)$ with the subsequence (F_{2k}) of (F_n) and $\{F, D, \tilde{D}\}$ with the subsequence (F_{2k+1}).

Note that these subsets of stability indicators and their norms and norms of inverses help to quantify the lim sup's of $\|F_n\|$, $\|F_n^{-1}\|$ and $\kappa(F_n)$ for the corresponding subsequences and hence to tell stable from unstable subsequences (look at even vs. odd n in the case $\mu = 0$). □

3. Tools and notations

Lower norm, spectrum and pseudospectrum. Let A be a bounded linear operator on a Banach space X. A fairly convenient access to the norm of A^{-1} is by the so-called *lower norm*, the number

$$\nu(A) \ := \inf_{\|x\|=1} \|Ax\|. \tag{4}$$

Indeed, putting

$$\mu(A) := \min\{\nu(A),\, \nu(A^*)\}, \qquad \text{we have} \qquad \|A^{-1}\| \ = \ 1/\mu(A), \tag{5}$$

where A^* is the Banach space adjoint on the dual space X^* and equation (5) takes the form $\infty = 1/0$ if and only if A is not invertible.

This enables us to write the *pseudospectra* [30] of A as sublevel sets of μ,

$$\operatorname{spec}_\varepsilon A \;:=\; \{\lambda \in \mathbb{C} : \|(A-\lambda I)^{-1}\| > \tfrac{1}{\varepsilon}\} \;=\; \{\lambda \in \mathbb{C} : \mu(A-\lambda I) < \varepsilon\}, \qquad \varepsilon > 0$$

and

$$\operatorname{Spec}_\varepsilon A \;:=\; \{\lambda \in \mathbb{C} : \|(A-\lambda I)^{-1}\| \geq \tfrac{1}{\varepsilon}\} \;=\; \{\lambda \in \mathbb{C} : \mu(A-\lambda I) \leq \varepsilon\}, \qquad \varepsilon \geq 0.$$

In particular, the *spectrum* of A is part of this family:

$$\operatorname{Spec} A \;=\; \operatorname{Spec}_0 A \;\subset \operatorname{spec}_\varepsilon A \;\subset\; \operatorname{Spec}_\varepsilon A \;=\; \operatorname{clos} \operatorname{spec}_\varepsilon A, \qquad \varepsilon > 0,$$

where the last equality holds by results of Globevnik [8] and Shargorodsky [29] if $X = \ell^p(\mathbb{Z})$.

Set sequences, Hausdorff-convergence and spectral pollution.

For two bounded nonempty sets, $S, T \subset \mathbb{C}$, the expression

$$d_{\mathrm{H}}(S,T) \;:=\; \max\left(\sup_{s\in S} \operatorname{dist}(s,T)\,,\, \sup_{t\in T} \operatorname{dist}(t,S)\right), \quad \text{where} \quad \operatorname{dist}(s,T) := \inf_{t\in T} |s-t|,$$

denotes the *Hausdorff distance* of S and T. On the set of all compact subsets of $\mathbb{C}$, d_{H} is a metric; on the bounded subsets of $\mathbb{C}$ it is merely a pseudometric since $d_{\mathrm{H}}(S,T) = d_{\mathrm{H}}(\operatorname{clos} S, T)$, so that $d_{\mathrm{H}}(S,T) = 0$ iff $\operatorname{clos} S = \operatorname{clos} T$ but not necessarily $S = T$.

One says that T_n *Hausdorff-converges* to T, written $T_n \xrightarrow{\mathrm{H}} T$, if $d_{\mathrm{H}}(T_n, T) \to 0$, noting that the limit T is only unique after passing to its closure.

For a bounded sequence (T_n), Hausdorff-convergent or not, look at the sets $\limsup T_n$, the set of all partial limits of sequences (t_n) with $t_n \in T_n$, and $\liminf T_n$, the set of all limits of sequences (t_n) with $t_n \in T_n$. By the Hausdorff theorem, e.g. [10, §3.1.2], $T_n \xrightarrow{\mathrm{H}} T$ if and only if

$$\liminf T_n \;=\; \limsup T_n \;=\; \operatorname{clos} T.$$

When aiming to approximate a set T by sets T_n, the Hausdorff distance $d_{\mathrm{H}}(T_n, T)$ detects two kinds of failure: (*i*) T_n failing to approximate parts of T and (*ii*) T_n clustering at points far from T. The latter is known as *spectral pollution*.

Some helpful short notations. For $a, b \in \mathbb{Z}$ with $a \leq b$, let us write

$$a..b \;:=\; \{z \in \mathbb{Z} : a \leq z \leq b\}, \quad a.. \;:=\; \{z \in \mathbb{Z} : a \leq z\}, \quad \text{and} \quad ..b \;:=\; \{z \in \mathbb{Z} : z \leq b\}.$$

For a set $M \subset \mathbb{Z}^d$, let P_M denote the operator on $\ell^p(\mathbb{Z}^d)$ that multiplies by the characteristic function of M. In particular, for $d = 1$, put $P_+ := P_{0..}$, $P_- := P_{..0}$ and $P_n := P_{-n..n}$ with $n \in \mathbb{N}$.

Algebras of operators and matrices. Identifying operators on $X := \ell^p(\mathbb{Z}^d)$, $d \in \mathbb{N}$, with $\mathbb{Z}^d \times \mathbb{Z}^d$-matrices is almost straightforward: With an operator A on X, associate the infinite matrix $[A] = (A_{ij})_{i,j\in\mathbb{Z}^d}$ with $A_{ij} \in \mathbb{C}$ given by $P_{\{i\}}AP_{\{j\}} : \operatorname{im} P_{\{j\}} \to \operatorname{im} P_{\{i\}}$. Conversely, a bi-infinite matrix $M = (M_{ij})_{i,j\in\mathbb{Z}^d}$ induces, via matrix-vector multiplication, an operator $\operatorname{Op}(M) : (x_j)_{j\in\mathbb{Z}^d} \mapsto (\sum_{j\in\mathbb{Z}^d} M_{ij}x_j)_{i\in\mathbb{Z}^d}$, provided the sums converge. For certain classes of matrices M, $\operatorname{Op}(M)$ is a bounded operator on X, and $[\operatorname{Op}(M)] = M$. For $p < \infty$, also $\operatorname{Op}([A]) = A$, see [15, Prop. 1.31 but also Ex. 1.26 c]. Now let us put

$$\begin{aligned} \text{BO} &:= \operatorname{alg}\{S_k,\, M_b \ :\ k \in \mathbb{Z}^d,\, b \in \ell^\infty(\mathbb{Z}^d)\} \qquad \text{and} \\ \text{BDO} &:= \operatorname{clos}_{L(X)} BO(X) \ =\ \operatorname{clos}\operatorname{alg}\{S_k,\, M_b \ :\ k \in \mathbb{Z}^d,\, b \in \ell^\infty(\mathbb{Z}^d)\}, \end{aligned}$$

where S_k is the k-shift on X with $(S_kx)_{i+k} = x_i$ for $i \in \mathbb{Z}^d$, M_b is the operator of multiplication by b, $(M_bx)_i = b_ix_i$, and $\operatorname{alg}\{\dots\}$ refers to the set (in fact, the algebra) of all finite sum-products of these. We call elements of BO and BDO, respectively, *band operators* and *band-dominated operators* [23,15]. Moreover, let

$$\operatorname{prop}(A) \ :=\ \sup\{|i-j|_\infty \ :\ i,j \in \mathbb{Z}^d,\ P_{\{i\}}AP_{\{j\}} \neq 0\}$$

denote the *propagation* of $A \in \text{BO}$, a.k.a. the *bandwidth* of the corresponding matrix $[A]$.

Algebras of operator sequences. Let $\mathcal{F}$ be the set of all bounded sequences $(A_n)_{n\in\mathbb{N}}$ of bounded operators A_n on $X := \ell^p(\mathbb{Z})$, and equip $\mathcal{F}$ with elementwise operations, $\alpha(A_n) := (\alpha A_n)$, $(A_n) + (B_n) := (A_n + B_n)$ and $(A_n)(B_n) := (A_nB_n)$, and with norm $\|(A_n)\|_{\mathcal{F}} := \sup_n \|A_n\|$, turning $\mathcal{F}$ into a Banach algebra. Let $\mathcal{N}$ denote the closed ideal in $\mathcal{F}$ of all sequences $(A_n) \in \mathcal{F}$ with $\|A_n\| \to 0$. Ultimately, we want the A_n to be finite matrices growing with n, so let us put

$$\begin{aligned} \mathcal{F}_\blacktriangledown &:= \{(P_nA_nP_n)_{n\in\mathbb{N}} \ :\ (A_n) \in \mathcal{F}\}, \\ \mathcal{N}_\blacktriangledown &:= \{(A_n) \in \mathcal{F}_\blacktriangledown \ :\ \|A_n\| \to 0\}, \end{aligned}$$

where now A_n is interpreted as operator on $\ell^p(-n..n)$ when it comes to invertibility and inverses. In analogy to the operator algebras BO and BDO, now let

$$\begin{aligned} \text{BS} &:= \operatorname{alg}\{(S_k)_{n\in\mathbb{N}},\ (M_{b^{(n)}})_{n\in\mathbb{N}} \ :\ k \in \mathbb{Z},\ b^{(n)} \in \ell^\infty(\mathbb{Z}),\ \sup \|b^{(n)}\|_\infty < \infty\} \ \cap\ \mathcal{F}_\blacktriangledown, \\ \text{BDS} &:= \operatorname{clos}_{\mathcal{F}} \text{BS} \ =\ \operatorname{clos}\operatorname{alg}\{(S_k),\ (M_{b^{(n)}}) \ :\ \dots\} \ \cap\ \mathcal{F}_\blacktriangledown \end{aligned}$$

refer to the algebras of all *band sequences* and all *band-dominated sequences*, respectively. Being finite sum-products of $(S_k)_n$ and $(M_{b^{(n)}})_n$, band sequences $(A_n) \in \text{BS}$ are exactly those bounded sequences of band operators $A_n \in \text{BO}$ with the property that $\sup_n \operatorname{prop}(A_n) < \infty$. Finally, let

$$\mathcal{S} \ :=\ \operatorname{clos}\operatorname{alg}\{(P_nBP_n) \ :\ B \in \text{BDO}\}$$

denote the so-called *finite section algebra* announced in the introduction. $(A_n) \in \mathcal{S}$ means that

$$(A_n)_{n\in\mathbb{N}} \;=\; \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_n A^{(i,j,k)} P_n)_{n\in\mathbb{N}} \tag{6}$$

with finite sets $J_i, K_i \subseteq \mathbb{N}$ for each $i \in \mathbb{N}$, with topology, sum and product of $\mathcal{F}$ and with all $A^{(i,j,k)} \in \mathrm{BDO}$. For each individual A_n that means

$$A_n \;=\; \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_n A^{(i,j,k)} P_n), \qquad n \in \mathbb{N}, \tag{7}$$

now with topology, sum and product of $L(X)$ and with the convergence $\lim_{i\to\infty}$ uniform in n. Then $A_n \to A := \lim_i \sum_j \prod_k A^{(i,j,k)}$, independently of the representation of (A_n).

Lemma 3.1. *It holds that* $\mathcal{S} \subset \mathrm{BDS}$.

Proof. By the definitions of $\mathcal{S}$, $\mathcal{F}_\blacktriangledown$ and BDO, $\mathcal{S} \subset \mathcal{F}_\blacktriangledown$ holds as well as

$$\begin{aligned} \mathcal{S} &\subset \operatorname{clos\,alg}\{(P_n)_n, (S_k)_n, (M_b)_n \;:\; k \in \mathbb{Z},\; b \in \ell^\infty(\mathbb{Z})\} \\ &\subset \operatorname{clos\,alg}\{(S_k)_n, (M_{b^{(n)}})_n \;:\; k \in \mathbb{Z},\; b^{(n)} \in \ell^\infty(\mathbb{Z}),\, \sup \|b^{(n)}\|_\infty < \infty\}. \qquad \blacksquare \end{aligned}$$

Fredholm operators, Calkin algebra and limit operators. Recall that a bounded linear operator A on a Banach space X is a *Fredholm operator* if its coset, $A + K(X)$, modulo compact operators $K(X)$, is invertible in the so-called *Calkin algebra* $L(X)/K(X)$. This holds if and only if the nullspace of A has finite dimension and the range of A has finite codimension in X. In particular, Fredholm operators have a closed range.

Since the coset $A+K(X)$ cannot be affected by changing finitely many matrix entries, its study takes place "at infinity". This is where limit operators [22,23,15,18] come in:

Definition 3.2. For $A \in \mathrm{BDO}$ on $X = \ell^p(\mathbb{Z}^d)$ with $p \in (1,\infty)$ and $d \in \mathbb{N}$, we look at all its translates $S_{-k}AS_k$ with $k \in \mathbb{Z}^d$ and speak of a *limit operator*, A_h, on $\ell^p(\mathbb{Z}^d)$ if, for a particular sequence $h = (h_n)$ in $\mathbb{Z}^d$ with $|h_n| \to \infty$, the corresponding sequence of translates converges pointwise to A_h, that is, $S_{-h_n}AS_{h_n} \to A_h$ as $n \to \infty$.

Moreover, let $\mathrm{Lim}(A)$ denote the set of all limit operators of A, together with the following local versions: For a sequence $h = (h_n)$ in $\mathbb{Z}^d$ with $|h_n| \to \infty$, we put

$$\mathrm{Lim}_h(A) \;:=\; \{A_g \;:\; g \text{ is a subsequence of } h\} \;\subseteq\; \mathrm{Lim}(A).$$

For $d = 1$ we fix the special cases $\mathrm{Lim}_+(A) := \mathrm{Lim}_{(1,2,\ldots)}(A)$ and $\mathrm{Lim}_-(A) := \mathrm{Lim}_{(-1,-2,\ldots)}(A)$.

By repeated application of the Bolzano-Weierstraß theorem, it is shown [22,23,15] that $\mathrm{Lim}_h(A) \neq \varnothing$ for all $A \in \mathrm{BDO}$ and all sequences $h = (h_n)$ in $\mathbb{Z}^d$ with $|h_n| \to \infty$

if X is a scalar-valued ℓ^p space. For Banach space-valued ℓ^p spaces, this introduces an additional condition on A, see Section 8 below.

For $A \in \mathrm{BDO}$, the identification between a coset $A + K(X)$ in the Calkin algebra and the set $\mathrm{Lim}(A)$ preserves algebra operations, invertibility and inverses, hence spectra, but also norms, hence pseudospectra. See [14,22,23,15,6,27,18,11] for the key steps.

The stacked operator $\oplus A_n$. Let $(A_n) \in \mathcal{F}_\blacktriangledown$. There are different ways to assemble a sequence of growing finite square matrices A_n via a direct sum. One way is the classical block diagonal matrix, acting on $\ell^p(\mathbb{N})$ or $\ell^p(\mathbb{Z})$. Following [23,15,11], we take an alternative approach, where each A_n remains an operator on $\ell^p(\mathbb{Z})$: Take $u = (u_{m,n})_{m,n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}^2)$ and let A_n act on $(u_{\cdot,n}) \in \ell^p(\mathbb{Z})$ for each $n \in \mathbb{N}$. Because that way, each A_n acts on its own invariant subspace of $\ell^p(\mathbb{Z}^2)$, the sequence (A_n) acts as a direct sum:

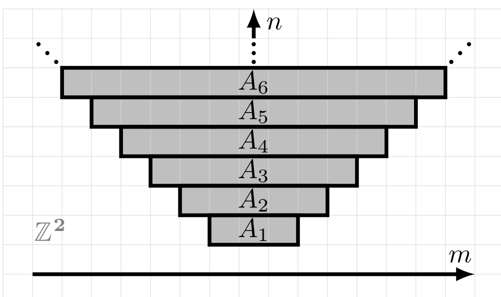
$$(\oplus A_n\, u)_{m,n} := (A_n(u_{\cdot,n}))_m\,, \quad m \in \mathbb{Z},\ n \in \mathbb{N}.$$

We refer to A_n as the n-th *layer* of the operator $\oplus A_n$ defined above. For completeness, put $A_n := 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}$ and extend the construction of $\oplus A_n$ to all $n \in \mathbb{Z}$. It then follows that

$$\| \oplus A_n\| \;=\; \sup_n \|A_n\| \;=\; \|(A_n)\|_{\mathcal{F}}. \tag{8}$$

For studies of invertibility, we will instead extend our construction by a suitable multiple of the identity, see Remark 5.3 a).

By $(A_n) \in \mathcal{F}_\blacktriangledown$, we get the following support pattern for $\oplus A_n$:



Let $\mathrm{Lay}(\oplus A_n)$ denote the set of all layers of $\oplus A_n$, i.e. $\{A_n : n \in \mathbb{N}\}$, and carry the notation over to sets such as $\mathrm{Lay}(\mathrm{Lim}(\oplus A_n)) := \cup\, \mathrm{Lay}(\oplus B_n)$, the union taken over all $\oplus B_n \in \mathrm{Lim}(\oplus A_n)$.

Lemma 3.3. *Let $(A_n) \in \mathcal{F}_\blacktriangledown$ and $Y = \ell^p(\mathbb{Z}^2)$. Then $\oplus A_n \in \mathrm{BDO}(Y)$ if and only if $(A_n) \in \mathrm{BDS}$.*

Proof. First note that $\oplus A_n \in \mathrm{BO}(Y)$ iff $\infty > \mathrm{prop}(\oplus A_n) = \sup_n \mathrm{prop}(A_n)$, i.e. $(A_n) \in \mathrm{BS}$. Then, recalling (8), pass to the closure on both sides. ∎

Proposition 3.4. *For $(A_n) \in \mathrm{BDS}$, the following are equivalent:*

(i) (A_n) *is stable;*
(ii) $(A_n) + \mathcal{N}_\blacktriangledown$ *is invertible in* $\mathcal{F}_\blacktriangledown / \mathcal{N}_\blacktriangledown$*;*
(iii) $\oplus A_n$ *(with each* A_n *extended to* X *by an according multiple of the identity rather than zero, see (6.4) in [11] and Remark 5.3 a) below) is a Fredholm operator;*
(iv) *all operators in* $\mathrm{Lim}(\oplus A_n)$*, i.e., all limit operators of* $\oplus A_n$*, are invertible;*
(v) *all operators in* $\mathrm{Lay}(\mathrm{Lim}(\oplus A_n))$*, i.e., all layers of all limit operators of* $\oplus A_n$*, are invertible.*

Proof. The equivalence of (*iii*) and (*iv*) is by Lemma 3.3 and [18, Thm. 11]. The uniform boundedness condition of the inverses in (*v*) is redundant, see footnote 16 of [11]. The rest of the proof is as in §6 of [11]. ■

Consequently, $\mathrm{Lim}(\oplus A_n)$ and $\mathrm{Lay}(\mathrm{Lim}(\oplus A_n))$ could both act as the set of stability indicators.

4. Establishing our lim sup formulas in the case $(A_n) \in \mathrm{BDS}$

We start in the context that we think is as general as possible: $(A_n) \in \mathrm{BDS}$. The price is that we have to keep the set of stability indicators rather large and unspecific. This changes when we specialize to $(A_n) \in \mathcal{S}$. The following propositions cover both situations.

To this end, given $(A_n) \in \mathrm{BDS}$, put

$$\mathcal{B}(A_n) \;:=\; \begin{cases} \mathrm{Stab}(A_n) & \text{if } (A_n) \in \mathcal{S}, \\ \mathrm{Lay}(\mathrm{Lim}(\oplus A_n)) & \text{otherwise} \end{cases} \tag{9}$$

with $\mathrm{Stab}(A_n)$ from Definition 5.4.

Remark 4.1. a) Since the original operator A and its approximants A_n all act on $\ell^p(\mathbb{Z})$ or subspaces thereof, it is desirable to also have the stability indicators of (A_n) in that setting and not acting on $\ell^p(\mathbb{Z}^2)$, even though we pass that space on our way – hence the Lay operation.

b) The following propositions hold with $\mathcal{B}(A_n)$ equal to $\mathrm{Lay}(\mathrm{Lim}(\oplus A_n))$. However, it were sufficient to just have the maximizers B of $\|B\|$ and C of $\|C^{-1}\|$ in $\mathcal{B}(A_n)$, making sure that

$$\|\mathcal{B}(A_n)\|_\infty \;=\; \|\mathrm{Lim}(\oplus A_n)\|_\infty \qquad \text{and} \qquad \|\big(\mathcal{B}(A_n)\big)^{-1}\|_\infty \;=\; \|\big(\mathrm{Lim}(\oplus A_n)\big)^{-1}\|_\infty.$$

For general $(A_n) \in \mathrm{BDS}$, however, these maximizers could be anywhere in $\mathrm{Lay}(\mathrm{Lim}(\oplus A_n))$, whence, without further information on (A_n), we keep $\mathcal{B}(A_n)$ this large. But for $(A_n) \in \mathcal{S}$, it is possible to say beforehand in which directions limit operators of $\oplus A_n$ stand a chance of their layers maximizing $\|B\|$ or $\|C^{-1}\|$, while other directions can be disregarded without looking at the particular example. This study is done in Section 5, first for sequences (A_n) of pure finite sections and then for composed finite sections, $(A_n) \in \mathcal{S}$, both leading to the same economic version of $\mathcal{B}(A_n)$ termed $\mathrm{Stab}(A_n)$ in Definition 5.4 below. □

4.1. The lim sup *of* $\|A_n\|$

Proposition 4.2. *For* $(A_n) \in \mathrm{BDS}$ *and* $\mathcal{B}(A_n)$ *from* (9)*, it holds that*

$$\limsup \|A_n\| \;=\; \max_{B\in\mathcal{B}(A_n)} \|B\| \;=:\; \|\mathcal{B}(A_n)\|_\infty \,.$$

Proof. Steps ①–③ in the following argument only require $(A_n) \in \mathrm{BDS}$. For step ④, let $(A_n) \in \mathcal{S} \subset \mathrm{BDS}$. Then, following §6 of [11], we argue as follows:

$$\limsup \|A_n\| \overset{①}{=} \|(A_n)+\mathcal{N}\|_{\mathcal{F}/\mathcal{N}} \overset{②}{=} \|\oplus A_n+\mathcal{K}\|_{\mathcal{L}/\mathcal{K}} \overset{③}{=} \|\mathrm{Lim}(\oplus A_n)\|_\infty \overset{④}{=} \|\mathrm{Stab}(A_n)\|_\infty \tag{10}$$

① Technically this is a first semester exercise but conceptually it is an extremely crucial piece of the puzzle that we trace back to [1].

② Here $\mathcal{L} := L(X)$ and $\mathcal{K} := K(X)$ abbreviate the sets of bounded, resp. compact, operators on X. The key is (8) and that $\oplus A_n$ is compact if and only if (A_n) is a null-sequence. We know this from [7] and [9]. See [23,15,11] for generalizations and extensions.

③ Lemma 3.3 implies that $\oplus A_n \in \mathrm{BDO}$, so that the limit operator approach [14,22,23,15,6,27,18,11] applies. [11] shows that $\|.\|_\infty$ is always attained as a maximum.

④ This step requires $(A_n) \in \mathcal{S}$, see Remark 4.1 b), Propositions 5.2 and 5.9 and [11, §6]. ■

For pure finite sections, $A_n = P_n A P_n$ with $A \in \mathrm{BDO}$, we show in Section 4.5 below that $\|A_n\| \to \|A\|$. For composed finite sections $(A_n) \in \mathcal{S}$, neither do the norms $\|A_n\|$ generally converge, nor is $\limsup \|A_n\|$ generally equal to $\|A\|$. See Example 4.7 below.

4.2. The lim sup *of* $\|A_n^{-1}\|$

We now come to the sequence of the inverses, A_n^{-1}, where, of course, every A_n is inverted as an operator on $\ell^p(-n..n)$, not $\ell^p(\mathbb{Z})$.

Proposition 4.3. *For* $(A_n) \in \mathrm{BDS}$ *and* $\mathcal{B}(A_n)$ *from* (9)*, it holds that*

$$\limsup \|A_n^{-1}\| \;=\; \max_{B\in\mathcal{B}(A_n)} \|B^{-1}\| \;=:\; \|\big(\mathcal{B}(A_n)\big)^{-1}\|_\infty \,.$$

In particular, the lim sup *is finite if and only if the maximum is finite, i.e.* (A_n) *is stable if and only if every* $B \in \mathcal{B}(A_n)$ *is invertible.*

Proof. First, let $(A_n) \in \text{BDS}$ be stable, so that almost all A_n are invertible and the inverses are uniformly bounded. Let us check the following equalities one by one:

$$\begin{aligned} \limsup \|A_n^{-1}\| &\overset{①}{=} \|(A_n^{-1}) + \mathcal{N}\|_{\mathcal{F}/\mathcal{N}} \overset{②}{=} \|\oplus A_n^{-1} + \mathcal{K}\|_{\mathcal{L}/\mathcal{K}} \overset{⑤}{=} \|R + \mathcal{K}\|_{\mathcal{L}/\mathcal{K}} \\ &\overset{③}{=} \|\text{Lim}(R)\|_\infty \overset{⑥}{=} \|\big(\text{Lim}(\oplus A_n)\big)^{-1}\|_\infty \overset{④}{=} \|\big(\text{Stab}(A_n)\big)^{-1}\|_\infty \end{aligned} \quad (11)$$

①-② See above. Note that the finitely many non-existent A_n^{-1} can be ignored in the sequence (A_n^{-1}) modulo $\mathcal{N}$ and in the operator $\oplus A_n^{-1}$ modulo $\mathcal{K}$.

⑤ To be more precise, put $R := \oplus R_n$ with $R_n = A_n^{-1}$ where existent and 0 otherwise. Then $R(\oplus A_n) - \oplus P_n \in \mathcal{K}$ and $(\oplus A_n)R - \oplus P_n \in \mathcal{K}$, so that R is a Fredholm regularizer of $\oplus A_n$.

③ As above, now with $R \in \text{BDO}$, by Lemma 3.3 and [27, Thm. 21]. Again note that $\|\cdot\|_\infty$ is attained as a maximum, by Theorem 3.2 in [11].

⑥ Here we use the fact that, by Theorem 16 of [27], the limit operators of a Fredholm regularizer of B are the inverses of the limit operators of B.

④ This step requires $(A_n) \in \mathcal{S}$, see Remark 4.1 b), Propositions 5.2 and 5.9 and [11, §6].

If $(A_n) \in \text{BDS}$ is not stable then, by Proposition 3.4, one operator $B \in \text{Lim}(\oplus A_n)$ is not invertible. If $(A_n) \in \mathcal{S}$ and $\mathcal{B}(A_n) = \text{Stab}(A_n)$ then, by Propositions 5.2 and 5.9, the layers of that B are in $\text{Stab}(A_n)$, and one of them is not invertible, by Proposition 3.4. ∎

4.3. The lim sup *of pseudospectra,* $\text{spec}_\varepsilon A_n$ *and* $\text{Spec}_\varepsilon A_n$

Here is the translation of Proposition 4.3 into the language of $\mu(\cdot)$ from (5): If $(A_n) \in \text{BDS}$ then

$$\liminf \mu(A_n) \;=\; \min_{B \in \mathcal{B}(A_n)} \mu(B). \quad (12)$$

Further, notice this standard lemma:

Lemma 4.4. *For arbitrary sets* T_i *with* i *in any index set* $\mathcal{I}$, *one has*

$$\text{clos} \bigcup_{i \in \mathcal{I}} \text{clos}\, T_i \;=\; \text{clos} \bigcup_{i \in \mathcal{I}} T_i.$$

Proof. $\boxed{\subseteq}$ Start with $T_j \subseteq \cup_{i\in\mathcal{I}} T_i$ for any $j \in \mathcal{I}$, take the closure on both sides, then the union $\cup_{j\in\mathcal{I}}$ on the left. Then again take the closure on both sides. Direction $\boxed{\supseteq}$ is obvious. ■

Then here is our result on the lim sup of $\operatorname{spec}_\varepsilon A_n$ and $\operatorname{Spec}_\varepsilon A_n$:

Proposition 4.5. *For a sequence* $(A_n) \in \mathrm{BDS}$ *and* $\mathcal{B}(A_n)$ *from* (9), *one has*

$$\limsup \operatorname{spec}_\varepsilon A_n = \operatorname{clos} \bigcup_{B\in\mathcal{B}(A_n)} \operatorname{spec}_\varepsilon B \;=\; \bigcup_{B\in\mathcal{B}(A_n)} \operatorname{clos}\operatorname{spec}_\varepsilon B \qquad \textit{and} \tag{13}$$

$$\limsup \operatorname{Spec}_\varepsilon A_n = \bigcup_{B\in\mathcal{B}(A_n)} \operatorname{Spec}_\varepsilon B. \tag{14}$$

Moreover, the two lim sup *sets in* (13) *and* (14) *are equal.*

Proof. Equality of (13) and (14) follows from $\operatorname{clos}\operatorname{spec}_\varepsilon B = \operatorname{Spec}_\varepsilon B$, by Globevnik-Shargorodsky [8,29]. (14) follows from (13) by passing to the closure under the lim sup, which does not change the result, see e.g. [10, Prop. 3.5] in combination with Lemma 4.4. So it remains to prove (13). We start with

$$\bigcup_{B\in\mathcal{B}(A_n)} \operatorname{spec}_\varepsilon B \;\subseteq\; \limsup \operatorname{spec}_\varepsilon A_n. \tag{15}$$

If $\lambda \in \cup_B \operatorname{spec}_\varepsilon B$ then $\exists C \in \mathcal{B}(A_n)$ with $\lambda \in \operatorname{spec}_\varepsilon C$, i.e.

$$\varepsilon \;>\; \mu(C-\lambda I) \;\geq\; \min_{B\in\mathcal{B}(A_n)} \mu(B-\lambda I) \overset{P.\,5.7}{=} \min_{D\in\mathcal{B}(A_n-\lambda I)} \mu(D) \overset{(12)}{=} \liminf \mu(A_n-\lambda I_n),$$

so that $\mu(A_n - \lambda I_n) < \varepsilon$, i.e. $\lambda \in \operatorname{spec}_\varepsilon A_n$, holds for infinitely many $n \in \mathbb{N}$. So clearly, $\lambda \in \limsup \operatorname{spec}_\varepsilon A_n$, and we have (15). Taking the closure on both sides of (15), noting that lim sup is already closed [10, Prop. 3.2], gives

$$\operatorname{clos} \bigcup_{B\in\mathcal{B}(A_n)} \operatorname{spec}_\varepsilon B \;\subseteq\; \limsup \operatorname{spec}_\varepsilon A_n. \tag{16}$$

Next we show

$$\limsup \operatorname{spec}_\varepsilon A_n \;\subseteq\; \bigcup_{B\in\mathcal{B}(A_n)} \operatorname{Spec}_\varepsilon B. \tag{17}$$

If $\lambda \in \limsup \operatorname{spec}_\varepsilon A_n$ then λ is a partial limit, i.e. $\lambda = \lim \lambda_{n_k}$, of a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in \operatorname{spec}_\varepsilon A_n$, i.e. $\mu(A_n - \lambda_n I_n) < \varepsilon$, for all $n \in \mathbb{N}$. By Lipschitz continuity of μ, e.g. [17, Lemma 2.1], it follows that

$$\mu(A_{n_k} - \lambda I_{n_k}) \;\le\; \mu(A_{n_k} - \lambda_{n_k} I_{n_k}) + |\lambda - \lambda_{n_k}| \;<\; \varepsilon + |\lambda - \lambda_{n_k}|,$$

so that

$$\varepsilon \;\ge\; \liminf \mu(A_{n_k} - \lambda I_{n_k}) \;\ge\; \liminf \mu(A_n - \lambda I_n) \overset{(12)}{=} \min_{B\in\mathcal{B}(A_n)} \mu(B - \lambda I),$$

whence $\lambda \in \mathrm{Spec}_\varepsilon B$ for some $B \in \mathcal{B}(A_n)$, and we have (17).

It remains to puzzle the pieces together, where (GS) is by Globevnik-Shargorodsky [8,29]:

$$\begin{aligned}\limsup \mathrm{spec}_\varepsilon A_n &\overset{(17)}{\subseteq} \bigcup_{B\in\mathcal{B}(A_n)} \mathrm{Spec}_\varepsilon B \overset{(GS)}{=} \bigcup_{B\in\mathcal{B}(A_n)} \mathrm{clos}\,\mathrm{spec}_\varepsilon B \\ &\subseteq \mathrm{clos} \bigcup_{B\in\mathrm{Stab}(A_n)} \mathrm{clos}\,\mathrm{spec}_\varepsilon B \overset{L.\,4.4}{=} \mathrm{clos} \bigcup_{B\in\mathcal{B}(A_n)} \mathrm{spec}_\varepsilon B \\ &\overset{(16)}{\subseteq} \limsup \mathrm{spec}_\varepsilon A_n.\end{aligned}$$

This shows equality of all sets in this chain of inclusions and hence proves (13). ■

Whether and when the two lim sup in Proposition 4.5 are proper (Hausdorff) limits is also answered in Section 7.

4.4. *The* lim sup *of the condition numbers,* $\kappa(A_n) = \|A_n\| \cdot \|A_n^{-1}\|$

The situation is less satisfactory for the condition numbers. Their lim sup is between the largest product, $\|B\|\|B^{-1}\|$, and the product of the largest factors, $\|B\|$ and $\|C^{-1}\|$, with $B, C \in \mathcal{B}(A_n)$.

Proposition 4.6. *For* $(A_n) \in \mathrm{BDS}$ *with* $\mathcal{B}(A_n)$ *from* (9)*, one has*

$$\begin{aligned}\sup_{B\in\mathcal{B}(A_n)} \kappa(B) \;\le\; \limsup \kappa(A_n) &\le \limsup \|A_n\| \cdot \limsup \|A_n^{-1}\| \\ &= \|\mathcal{B}(A_n)\|_\infty \cdot \|\big(\mathcal{B}(A_n)\big)^{-1}\|_\infty.\end{aligned}$$

Sketch of proof. For instable (A_n), the statement is $\infty \le \infty \le \infty$; for stable (A_n), it follows by Polski's theorem [4, Prop. 2.2] and Lemma 4.8, bounding $\|B\|$ and $\|B^{-1}\|$ from above. ■

An equality for $\limsup \kappa(A_n)$ in case $(A_n) \in \mathcal{S}$ is derived in Proposition 7.7 below. The inequalities in Proposition 4.6 can be strict, see Example 4.7. We do not know whether $\sup \kappa(B)$ is also a maximum. When exactly lim sup = lim holds for any of the three occurrences in Proposition 4.6 is the topic of Section 7. If one of $\limsup \|A_n\|$ and

$\limsup \|A_n^{-1}\|$ is a limit then the second "$\le$" sign is an equality. If even both are limits then also $\limsup \kappa(A_n)$ is a limit.

Example 4.7. a) Let $A = BC$ and $A_n = (P_n B P_n)(P_n C P_n)$ for $n \in \mathbb{N}$ with

$$B = \operatorname{diag}(\dots, D, D, \boxed{1}, D, D, \dots) \qquad \text{and} \qquad C = \operatorname{diag}(\dots, E, E, \boxed{1}, E, E, \dots),$$

where $D = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1/2 \end{smallmatrix}\right)$, $E = \left(\begin{smallmatrix} 2 & 0 \\ -1 & 1/2 \end{smallmatrix}\right)$ and $B_{00} = \boxed{1} = C_{00}$. Then, for $k \in \mathbb{N}$,

$$A_{2k} = \operatorname{diag}(DE, \dots DE, \boxed{1}, DE, \dots, DE) \quad \text{and} \quad A_{2k+1} = \operatorname{diag}(1/4, A_{2k}, 4).$$

Both $DE = \left(\begin{smallmatrix} 3 & 1/2 \\ -1/2 & 1/4 \end{smallmatrix}\right)$ and $(DE)^{-1} = \left(\begin{smallmatrix} 1/4 & -1/2 \\ 1/2 & 3 \end{smallmatrix}\right)$ have norm[2] $N := \frac{1}{8}(11 + \sqrt{185}) \approx 3.075$. So

$$\begin{array}{lllll} & \|A_n\| = N, & \|A_n^{-1}\| = N, & \kappa(A_n) = N^2 & \text{if } n \ge 2 \text{ is even} \\ \text{and} & \|A_n\| = 4, & \|A_n^{-1}\| = 4, & \kappa(A_n) = 16 & \text{if } n \ge 3 \text{ is odd,} \end{array}$$

whence $\limsup \kappa(A_n) = 16$. The set $\mathcal{B}(A) := \operatorname{Stab}(A_n)$ consists of A, F, G, H, J with

$$F = \operatorname{diag}(DE, \dots), \quad G = \operatorname{diag}(\dots, DE), \quad H = \operatorname{diag}(1/4, DE, \dots),$$
$$J = \operatorname{diag}(\dots, DE, 4),$$

so that the first "$\le$" sign in Proposition 4.6 is "$<$" since $\|J\| = 4$ maximizes the norm, $\|H^{-1}\| = 4$ maximizes the norm of the inverse and $\kappa(H) = \kappa(J) = 4N$ maximize the condition number.

b) Instead, putting $A_n = (P_n B P_n)(P_n C P_n)$ with

$$B = \operatorname{diag}(\dots, D, D, D, \dots) \qquad \text{and} \qquad C = \operatorname{diag}(\dots, E, E, E, \dots),$$

both A_{2k} and A_{2k+1} cut through a block on the left or right endpoint, leading to

$$\begin{array}{lllll} & \|A_n\| = 4, & \|A_n^{-1}\| = N, & \kappa(A_n) = 4N & \text{if } n \ge 2 \text{ is even} \\ \text{and} & \|A_n\| = N, & \|A_n^{-1}\| = 4, & \kappa(A_n) = 4N & \text{if } n \ge 3 \text{ is odd,} \end{array}$$

so that $\|A_n\|$ and $\|A_n^{-1}\|$ are both alternating (oppositely) between 4 and N. Even though $\limsup \kappa(A_n) = 4N$ is a limit now, the second "$\le$" sign in Proposition 4.6 is a "$<$" here. $\square$

[2] Meaning the induced operator norm for $p = 2$. The tedious computation of the largest singular value can be replaced by noting that $\|DE\|_F = \|(DE)^{-1}\|_F = \sqrt{9 + \frac{1}{4} + \frac{1}{4} + \frac{1}{16}} = \frac{1}{4}\sqrt{153} \in (3, \frac{13}{4})$ for the Frobenius norm and recalling that $\frac{1}{\sqrt{2}}\|M\|_F \le \|M\| \le \|M\|_F$ for 2×2 matrices M, so that $2 < \frac{3}{\sqrt{2}} < \|DE\| = \|(DE)^{-1}\| < \frac{13}{4} < 4$.

4.5. Pure finite sections: improved asymptotic results of $\|A_n\|$ and $\kappa(A_n)$

First a standard result that is sometimes [4,10,2] stated as an add-on to Banach-Steinhaus:

Lemma 4.8. *If $A \in L(X)$ and $\|A_n x\| \to \|Ax\|$ for all $x \in X$ then $\|A\| \le \liminf \|A_n\|$.*

Proof. Let $\varepsilon > 0$ and $x \in X$ with $\|x\| = 1$ and $\|Ax\| \overset{\varepsilon/2}{\approx} \|A\|$, where $a \overset{\delta}{\approx} b$ means $|a - b| < \delta$. For sufficiently large $n \in \mathbb{N}$, by assumption, $\|A_n x\| \overset{\varepsilon/2}{\approx} \|Ax\|$, so that $\|A\| \overset{\varepsilon}{\approx} \|A_n x\| \le \|A_n\|$. ■

As a consequence, for pure finite sections, one can improve Propositions 4.2 and 4.6 as follows (this is a special case of results from e.g. [26,28]):

Proposition 4.9. *For the pure finite sections, $A_n = P_n A P_n$, of an operator $A \in \mathrm{BDO}$,*

$$\lim \|A_n\| = \|A\|,$$
$$\limsup \|A_n^{-1}\| = \|\big(\mathrm{Stab}(A_n)\big)^{-1}\|_\infty \qquad \textit{and}$$
$$\limsup \kappa(A_n) = \|A\| \cdot \|\big(\mathrm{Stab}(A_n)\big)^{-1}\|_\infty \,.$$

Proof. By $A_n \to A$ and Lemma 4.8, it follows that $\|A\| \le \liminf \|A_n\|$. Moreover, by $\|P_n\| = 1$, we have $\|A_n\| = \|P_n A P_n\| \le \|A\|$. It follows that $\|A\| \le \liminf \|A_n\| \le \limsup \|A_n\| \le \|A\|$, so that $\lim \|A_n\|$ exists and equals $\|A\|$. The rest is by Proposition 4.3 and $\kappa(A_n) = \|A_n\|\|A_n^{-1}\|$. ■

Note that $\limsup \kappa(A_n)$ is now given by an equality and that it is a limit if and only if $\limsup \|A_n^{-1}\|$ is a limit. When exactly that is the case is answered in Section 7 below.

5. Dominant directions of $\oplus A_n$ and the definition of $\mathrm{Stab}(A_n)$

The propositions in Section 4 ask for the limit operators of $\oplus A_n$. Let us study this set and identify its maximal elements in terms of $\|B\|$ and $\|C^{-1}\|$. Our analysis of the latter is limited to the algebra $\mathcal{S}$. We start with pure finite sections and then proceed to the composed case.

Definition 5.1. Given two operators, A and B, we will say that A *dominates* B (or that B *is dominated by* A) if $\|A\| \ge \|B\|$ and $\|A^{-1}\| \ge \|B^{-1}\|$, while keeping the convention of putting $\|A^{-1}\| = \infty$ in case of non-invertibility.

In particular, if A dominates B then also invertibility of A implies that of B. So if, among all $B_n \in \mathrm{Lay}(\mathrm{Lim}(\oplus A_n))$, we ignore the B_n that are dominated by others from the set, the remaining set still captures stability of (A_n) and the lim sup's of Section 4.

5.1. *First: pure finite sections, $A_n = P_nAP_n$*

For $A_n = P_nAP_n$ with $A \in \mathrm{BDO}$, shift $\oplus A_n$ along a sequence $h = (h_k) = (\,\binom{i_k}{j_k}\,)$ in $\mathbb{Z}^2$:

$$(\oplus_n A_n)_h \leftarrow S_{-\binom{i_k}{j_k}}(\oplus_n A_n)S_{\binom{i_k}{j_k}} \;=\; \oplus_n\, S_{-i_k}A_{n+j_k}S_{i_k} \;=\; \oplus_n\, S_{-i_k}P_{n+j_k}AP_{n+j_k}S_{i_k}$$
$$= \oplus_n\,(S_{-i_k}P_{n+j_k}S_{i_k})(S_{-i_k}AS_{i_k})(S_{-i_k}P_{n+j_k}S_{i_k}) \;\to\; \oplus_n\, B_n \tag{18}$$

as $k \to \infty$ if the pointwise limit exists. By uniqueness of the limit, $(\oplus A_n)_h = \oplus B_n$.

Proposition 5.2. *Let (A_n) be the sequence of pure finite sections, $A_n = P_nAP_n$, of an operator $A \in \mathrm{BDO}$ and let $\oplus B_n$ be a limit operator of $\oplus A_n$ with respect to a sequence h in $\mathbb{Z}^2$.*

a) Depending on the direction of h in $\mathbb{Z}^2$, the layers B_n are of the form

(a)  *the operator A itself,*
(b)  *a limit operator of A,*
(c)  *$P_-A_gP_-$ with $A_g \in \mathrm{Lim}_+(A)$,*
(d)  *$P_+A_gP_+$ with $A_g \in \mathrm{Lim}_-(A)$,*
(e) *0 or*
(f) translates, $B_n = S_{-c}B'_nS_c$, of any of the operators B'_n in (a)–(e).

b) Case (b) is dominated by (a) and case (f) is dominated by the previous cases.

Remark 5.3. a) Since every A_n is only of interest as an operator on $\ell^p(-n..n)$, we can ignore the case (e) and we have to understand the operators (c) as operators on $\mathrm{im}\,P_- = \ell^p(..0)$ and the operators (d) as operators on $\mathrm{im}\,P_+ = \ell^p(0..)$. Alternatively, extend every A_n to $\ell^p(\mathbb{Z})$ by c times the identity before forming $\oplus A_n$. Choose $c \in \mathbb{R}$ such that it does not change the property of interest, e.g. put $c = 0$ in (10) and $c > \sup \|A_n\|$ in (11). This way was followed in [11, §6]. In the current paper we ignore the nth layer outside of $-n..n$ whence, in the limit, we have to be a bit flexible and tolerant about having operators on $\ell^p(\mathbb{Z}), \ell^p(0..)$ and $\ell^p(..0)$ in the same set.

b) To start with, a sequence h need not belong to any of the “directions” (a)–(f) and still produce a limit operator $(\oplus A_n)_h$. But there is always a subsequence of h belonging to a particular case, (a)–(f), leaving the limit operator unchanged. We pass to such a subsequence.

c) It is tempting to classify the cases (a)–(f) via the angle[3] that h asymptotically encloses with the n-axis: (a) is for $0°$, (b) for $(0°, 45°)$, (c) and (d) are for $\pm 45°$, respec-

[3] Indeed, by compactness of the unit circle, $h_k/|h_k|$ has a convergent subsequence that we can pass to without changing the limit operator $(\oplus A_n)_h$, hence giving h an angle, asymptotically.

tively, (e) is for $\notin [-45°, 45°]$, and (f) is a finite shift of a sequence in one of (a)–(e). While this is roughly what happens, it is incorrect since (b) and (e) also reach 0° and ±45°. E.g., $h_k = \binom{k}{k^2}$ has angle 0° with the n-axis but is in case (b), not (a)+(f), or $\binom{k^2 \pm k}{k^2}$ has angle 45° and is (b), resp. (e), not (c)+(f). So we stick with the following clumsy way of distinguishing (a)–(f). □

Proof of Proposition 5.2. a) Let $A_n = P_n A P_n$ and let the sequence $h = (h_k) = (\binom{i_k}{j_k})$ in $\mathbb{Z}^2$ be such that $|h_k| \to \infty$ and that the limit operator $(\oplus A_n)_h =: \oplus B_n$ exists. By (18), for each $n \in \mathbb{Z}$,

$$B_n \leftarrow (S_{-i_k} P_{n+j_k} S_{i_k})(S_{-i_k} A S_{i_k})(S_{-i_k} P_{n+j_k} S_{i_k}) \quad \text{as} \quad k \to \infty. \tag{19}$$

Without loss, we can focus on B_0 since every other layer, say B_m, is the 0th layer, C_0, of

$$\oplus C_n = \oplus B_{n+m} = S_{-\binom{0}{m}} (\oplus B_n) S_{\binom{0}{m}} = (\oplus A_n)_{h+\binom{0}{m}},$$

which is just another limit operator of $\oplus A_n$.

(a) If h is a subsequence of $\mathbb{N} \cdot \binom{0}{1}$, i.e. $h_k = \binom{0}{j_k}$ with $j_k \to +\infty$ then $S_{-i_k} = I = S_{i_k}$, so that, by (19) with $n = 0$, $B_0 \leftarrow P_{j_k} A P_{j_k} \to A$ as $k \to \infty$, whence $B_0 = A$.

(b) If $|i_k| \to \infty$ but $|i_k| - j_k \to -\infty$, so that $i_k - j_k \to -\infty$ and $-i_k - j_k \to -\infty$ then

$$S_{-i_k} P_{j_k} S_{i_k} = P_{-j_k - i_k .. j_k - i_k} \to I \quad \text{as} \quad k \to \infty, \quad \text{so that, by (19),} \quad B_0 = A_i,$$

the limit operator of A with respect to the sequence $i = (i_k)_k$.

(c) If h is a subsequence of $\mathbb{N} \cdot \binom{1}{1}$, i.e. $h_k = \binom{j_k}{j_k}$ with $j_k \to +\infty$ then

$$S_{-i_k} P_{j_k} S_{i_k} = P_{-2j_k..0} \to P_{..0} = P_- \text{ as } k \to \infty, \quad \text{so that, by (19),}$$
$$B_0 = P_- A_{j'} P_-,$$

where $A_{j'} \in \mathrm{Lim}_j(A)$ is a limit operator of A w.r.t. a subsequence of $j = (j_k)_k \to +\infty$.

(d) If h is a subsequence of $\mathbb{N} \cdot \binom{-1}{1}$, i.e. $h_k = \binom{-j_k}{j_k}$ with $j_k \to +\infty$ then

$$S_{-i_k} P_{j_k} S_{i_k} = P_{0..2j_k} \to P_{0..} = P_+ \text{ as } k \to \infty, \quad \text{so that, by (19),}$$
$$B_0 = P_+ A_{-j'} P_+,$$

where $A_{-j'}$ is a limit operator of A w.r.t. a subsequence of $-j = (-j_k)_k \to -\infty$.

(e) If $|i_k| - j_k \to +\infty$, i.e. $i_k - j_k \to +\infty$ or $-i_k - j_k \to +\infty$ then

$$S_{-i_k} P_{j_k} S_{i_k} = P_{-j_k - i_k .. j_k - i_k} \to 0 \quad \text{as} \quad k \to \infty, \quad \text{so that} \quad B_0 = 0.$$

(f) If h differs from one of (a)–(e) by a sequence g with a bounded subsequence then pass to a subsequence of h, where g is constant, say $g \equiv c$, (Bolzano-Weierstraß in $\mathbb{Z}^2$), not changing the limit operator, $(\oplus A_n)_h = \oplus B_n$. So $B_0 = S_{-c}B_0' S_c$ with B_0' from one of (a)–(e).

b) For (b), it remains to note that $\|A\| \geq \|A_g\|$ for every limit operator A_g of A and that $(A_g)^{-1} = (A^{-1})_g$ if A is invertible, see e.g. [22], whence also $\|A^{-1}\| \geq \|(A^{-1})_g\| = \|(A_g)^{-1}\|$.

For (f), note that $\|S_{-c}BS_c\| = \|B\|$ and $\|(S_{-c}BS_c)^{-1}\| = \|S_{-c}B^{-1}S_c\| = \|B^{-1}\|$. ■

After Proposition 5.2 we are left with the layers B_n of type (a), (c) and (d), and this is what we will collect in $\operatorname{Stab}(A_n)$. In the case of pure finite sections, $A_n = P_nAP_n$, checking these three directions is enough, by Proposition 5.2, to find, among $\operatorname{Lay}(\operatorname{Lim}(\oplus A_n))$,

- a maximizer of $\|B_n\|$,
- a maximizer of $\|B_n^{-1}\|$ and
- a subset of operators B_n whose invertibility implies that of all the others.

In Proposition 5.9 below we show that the same three directions, $\binom{0}{1}$, $\binom{1}{1}$ and $\binom{-1}{1}$, are sufficient for the same purpose also for composed finite sections, $(A_n) \in \mathcal{S}$. Wrapping up, we evaluate

$$\oplus B_n \leftarrow S_{-\binom{i_k}{j_k}}(\oplus A_n)S_{\binom{i_k}{j_k}} = \oplus\, S_{-i_k}A_{n+j_k}S_{i_k}, \quad \text{so that} \quad B_0 \leftarrow S_{-i_k}A_{j_k}S_{i_k} \text{ as } k \to \infty, \tag{20}$$

for $j_k \to +\infty$ and (a) $i_k = 0$, (c) $i_k = j_k$ and (d) $i_k = -j_k$. So here is the definition:

Definition 5.4. For a sequence $(A_n) \in \mathcal{S}$, let $\operatorname{Stab}(A_n)$ denote the set consisting of the pointwise limits of all subsequences of

$$A_n, \qquad S_{-n}A_nS_n \qquad \text{and} \qquad S_nA_nS_{-n}.$$

We call the elements of $\operatorname{Stab}(A_n)$ the *stability indicators* of (A_n).

Remark 5.5. a) Every sequence $(A_n) \in \mathcal{S}$ converges pointwise. So, for the first sequence in Definition 5.4, there is actually no need to pass to subsequences. The pointwise limit is the operator A that was to be approximated in the first place. In particular, $A \in \operatorname{Stab}(A_n)$ holds.

b) In cases, where also $S_{-n}A_nS_n$ and $S_nA_nS_{-n}$ converge, e.g. for $(A_n) \in \operatorname{clos\,alg}\{P_nLP_n : L \text{ banded Laurent, see Example 2.1}\}$, $\operatorname{Stab}(A_n)$ consists of just three elements and all lim sup's in Section 4 are limits, by the results of Section 7 below.

c) That at least $\|A_n\|$ converges in situation b) can also be shown by applying Lemma 4.8 to the limits A, B, C of A_n, $S_{-n}A_nS_n$ and $S_nA_nS_{-n}$, using that all S_k are isometries. Indeed,

$$\|A\|,\|B\|,\|C\| \overset{L.\,4.8}{\leq} \liminf \|A_n\| \;\leq\; \limsup \|A_n\| \overset{P.\,4.2}{=} \|\mathrm{Stab}(A_n)\|_\infty$$
$$= \max\{\|A\|,\|B\|,\|C\|\},$$

whence $\liminf \|A_n\| = \limsup \|A_n\| = \lim \|A_n\| = \max\{\|A\|,\|B\|,\|C\|\}$. □

Proposition 5.6. *For pure finite sections, $A_n = P_nAP_n$ with $A \in \mathrm{BDO}$, we have*

$$\mathrm{Stab}(A_n) \;=\; \Big\{\; A,\quad P_-A_fP_-,\quad P_+A_gP_+ \quad:\quad A_f \in \mathrm{Lim}_+(A),\;\; A_g \in \mathrm{Lim}_-(A) \;\Big\}.$$

Proposition 5.7. *In either case,* ***a)*** $(A_n) \in \mathcal{S}$ *and* $\mathcal{B}(A_n) := \mathrm{Stab}(A_n)$ *or* ***b)*** $(A_n) \in \mathrm{BDS}$ *and* $\mathcal{B}(A_n) := \mathrm{Lay}(\mathrm{Lim}(\oplus A_n))$*, we have*

$$\mathcal{B}(A_n - \lambda I_n) \;=\; \mathcal{B}(A_n) - \lambda I \;:=\; \{B - \lambda I : B \in \mathcal{B}(A_n)\}, \qquad \lambda \in \mathbb{C}.$$

Remark 5.8. Recall that $B \in \mathcal{B}(A_n)$ can be an operator on $\ell^p(\mathbb{Z})$ (bi-infinite matrix) or $\ell^p(k..)$ or $\ell^p(..k)$ with $k \in \mathbb{Z}$ (different semi-infinite matrices). When writing $B - \lambda I$ then I shall denote the identity on the corresponding space. □

Proof of Proposition 5.7. **a)** By Definition 5.4, $\mathrm{Stab}(A_n - \lambda I_n)$ consists of the pointwise limits of subsequences of

$$A_n - \lambda I_n, \quad S_{-n}(A_n - \lambda I_n)S_n \qquad \text{and} \qquad S_n(A_n - \lambda I_n)S_{-n}$$

as $n \to \infty$. The conclusion follows from $I_n \to I$, $S_{-n}I_nS_n \to P_-$ and $S_nI_nS_{-n} \to P_+$.

b) Every $\oplus B_n \in \mathrm{Lim}(\oplus(A_n - \lambda I_n)) = \mathrm{Lim}(\oplus A_n - \lambda \oplus I_n)$ is of the form $\oplus C_n - \lambda \oplus D_n$, where $\oplus C_n \in \mathrm{Lim}(\oplus A_n)$ and $D_n = I$ or $P_{k-n..}$ or $P_{..k+n}$ with $k \in \mathbb{Z}$. After decomposition in layers, $B_n = C_n - \lambda D_n$, the claim follows. ■

5.2. Now: composed finite sections, $(A_n) \in \mathcal{S}$

Let us check that still $\binom{0}{1}$, $\binom{1}{1}$ and $\binom{-1}{1}$ are the dominant directions in $\oplus A_n$ when $(A_n) \in \mathcal{S}$, say (7) holds. Shifting $\oplus A_n$ along the integer sequence $h = (h_l) = (\binom{\alpha_l}{\beta_l})$ in $\mathbb{Z}^2$, leads to

$$\begin{aligned}(\oplus_n A_n)_h \leftarrow S_{-\binom{\alpha_l}{\beta_l}}(\oplus_n A_n)S_{\binom{\alpha_l}{\beta_l}} \;&=\; \oplus_n S_{-\alpha_l}A_{n+\beta_l}S_{\alpha_l}\\ &= \oplus_n S_{-\alpha_l}\Big(\lim_i \sum_j \prod_k (P_{n+\beta_l}A^{(i,j,k)}P_{n+\beta_l})\Big)S_{\alpha_l}\\ &= \oplus_n \lim_i \sum_j \prod_k (S_{-\alpha_l}P_{n+\beta_l}S_{\alpha_l})(S_{-\alpha_l}A^{(i,j,k)}S_{\alpha_l})(S_{-\alpha_l}P_{n+\beta_l}S_{\alpha_l})\\ &\to\; \oplus_n B_n \end{aligned} \tag{21}$$

as $l \to \infty$ if the limits exist. With very much analogy to Proposition 5.2, we have:

Proposition 5.9. *For $(A_n) \in \mathcal{S}$, let $\oplus B_n$ be a limit operator of $\oplus A_n$ w.r.t. a sequence h in $\mathbb{Z}^2$.*

a) Depending on the direction of h in $\mathbb{Z}^2$, the operators B_n are of the form

- *(a)*  *the pointwise limit of A_n, that is $\lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} A^{(i,j,k)} =: A$,*
- *(b)*  *a limit operator of A,*
- *(c)*  $\lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_- A_g^{(i,j,k)} P_-)$ *with* $A_g^{(i,j,k)} \in \mathrm{Lim}_+(A^{(i,j,k)})$,
- *(d)*  $\lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_+ A_g^{(i,j,k)} P_+)$ *with* $A_g^{(i,j,k)} \in \mathrm{Lim}_-(A^{(i,j,k)})m$
- *(e)* 0 *or*
- *(f) translates, $B_n = S_{-c} B'_n S_c$, of any of the operators B'_n in (a)–(e).*

b) Case (b) is dominated by (a) and case (f) is dominated by the previous cases.

Again, Remark 5.3 applies.

Proof. **a)** Let the sequence $h = (h_l) = \left(\binom{\alpha_l}{\beta_l}\right)$ in $\mathbb{Z}^2$ be such that $|h_l| \to \infty$ and that the limit operator $(\oplus A_n)_h =: \oplus B_n$ exists. As in the proof of Proposition 5.2, w.l.o.g., let $n = 0$. By (21),

$$B_0 \leftarrow \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (S_{-\alpha_l} P_{\beta_l} S_{\alpha_l})(S_{-\alpha_l} A^{(i,j,k)} S_{\alpha_l})(S_{-\alpha_l} P_{\beta_l} S_{\alpha_l}) \quad \text{as} \quad l \to \infty. \tag{22}$$

(a) If h is a subsequence of $\mathbb{N} \cdot \binom{0}{1}$, i.e. $h_l = \binom{0}{\beta_l}$ with $\beta_l \to +\infty$ then $S_{-\alpha_l} = I = S_{\alpha_l}$, so that, by (22), $B_0 \leftarrow \lim_i \sum_j \prod_k P_{\beta_l} A^{(i,j,k)} P_{\beta_l} \to A$ as $l \to \infty$, whence $B_0 = A$.

(b) If $|\alpha_l| \to \infty$ but $|\alpha_l| - \beta_l \to -\infty$, so that $\alpha_l - \beta_l \to -\infty$ and $-\alpha_l - \beta_l \to -\infty$ then

$$S_{-\alpha_l} P_{\beta_l} S_{\alpha_l} \;=\; P_{-\beta_l-\alpha_l..\beta_l-\alpha_l} \;\to\; I \quad \text{as} \quad l \to \infty,$$

so that, by (22) and by the standard properties of limit operators, e.g. [15, Prop. 3.4],

$$B_0 \;=\; \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} A_\alpha^{(i,j,k)} \;=\; \left(\lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} A^{(i,j,k)}\right)_\alpha \;=\; A_\alpha,$$

meaning the limit operator of A with respect to the sequence $\alpha = (\alpha_l)_l$ in $\mathbb{Z}$.

(c) If h is a subsequence of $\mathbb{N} \cdot \binom{1}{1}$, i.e. $h_l = \binom{\beta_l}{\beta_l}$ with $\beta_l \to +\infty$ then

$$S_{-\alpha_l} P_{\beta_l} S_{\alpha_l} \;=\; P_{-2\beta_l..0} \;\to\; P_{..0} \;=\; P_- \quad \text{as} \quad l \to \infty,$$

so that, by (22),

$$B_0 \;=\; \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_- A_{\beta'}^{(i,j,k)} P_-)$$

with a subsequence β' of $\beta = (\beta_l)_l \to +\infty$ for which all limit operators $A_{\beta'}^{(i,j,k)}$ exist. (Such a subsequence β' exists by a standard diagonal argument, see e.g. the proof of Corollary 7.4 below.)

(d) This is again in analogy to (c).

(e, f) Both are as in Proposition 5.2.

b) See the proof of Proposition 5.2. ■

... again leaving us with (a), (c) and (d) and confirming our Definition 5.4 of $\mathrm{Stab}(A_n)$.

Proposition 5.10. *If (A_n) is in the finite section algebra $\mathcal{S}$, say (6) holds, we have*

$$\begin{aligned}\mathrm{Stab}(A_n) \;=\; \Big\{ A,\; \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_- A_f^{(i,j,k)} P_-), \quad \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_+ A_g^{(i,j,k)} P_+) \\ : \quad A_f^{(i,j,k)} \in \mathrm{Lim}_+(A^{(i,j,k)}), \quad A_g^{(i,j,k)} \in \mathrm{Lim}_-(A^{(i,j,k)}) \Big\}\end{aligned}$$

with $A := \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} A^{(i,j,k)}$ denoting the pointwise limit of A_n.

6. Subsequence versions of our lim sup formulas

As a preparation for Section 7, we prove subsequence versions of Propositions 4.2 and 4.3, implying subsequence versions of Propositions 4.5 and 4.6. The goal is to show, for $(A_n) \in \mathrm{BDS}$,

$$\limsup \|A_{h_n}\| = \|\mathcal{B}_h(A_n)\|_\infty\,, \tag{23}$$

$$\limsup \|A_{h_n}^{-1}\| = \|\big(\mathcal{B}_h(A_n)\big)^{-1}\|_\infty\,, \tag{24}$$

$$\sup_{B\in\mathcal{B}_h(A_n)} \kappa(B) \;\le\; \limsup \kappa(A_{h_n}) \le \|\mathcal{B}_h(A_n)\|_\infty \cdot \|\big(\mathcal{B}_h(A_n)\big)^{-1}\|_\infty\,, \tag{25}$$

$$\limsup \mathrm{Spec}_\varepsilon A_{h_n} \;=\; \bigcup_{B\in\mathcal{B}_h(A_n)} \mathrm{Spec}_\varepsilon B \tag{26}$$

for arbitrary monotonic sequences $h = (h_n)$ in $\mathbb{N}$.

For $(A_n) \in \mathrm{BDS}$ and a monotonic sequence $h = (h_n)$ in $\mathbb{N}$, put

$$\mathcal{B}_h(A_n) \;:=\; \begin{cases} \mathrm{Stab}_h(A_n) & \text{if } (A_n) \in \mathcal{S}, \\ \mathrm{Lay}(\mathrm{Lim}_{\binom{*}{h}}(\oplus A_n)) & \text{otherwise}, \end{cases} \tag{27}$$

where $\mathrm{Stab}_h(A_n)$ is given by Definition 6.1 below and $\mathrm{Lim}_{\binom{*}{h}}(\oplus A_n)$ is the set of all limit operators $(\oplus A_n)_g$ with sequences $g = (\binom{i_k}{j_k})_k$ in $\mathbb{Z}^2$ with $i_k \in \mathbb{Z}$ and $j = (j_k)$ a subsequence of h.

Definition 6.1. For a sequence $(A_n) \in \mathcal{S}$ and a monotonic sequence $h = (h_n)$ in $\mathbb{N}$, let $\mathrm{Stab}_h(A_n)$ denote the set consisting of the pointwise limits of all subsequences of

$$A_{h_n}, \qquad S_{-h_n} A_{h_n} S_{h_n} \qquad \text{and} \qquad S_{h_n} A_{h_n} S_{-h_n}.$$

Proposition 6.2. *If $h = (h_n)$ is a monotonic sequence in $\mathbb{N}$ and $(A_n) \in \mathcal{S}$, say (6) holds, then*

$$\mathrm{Stab}_h(A_n) \;=\; \Big\{ A,\ \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_- A_f^{(i,j,k)} P_-), \qquad \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_+ A_g^{(i,j,k)} P_+) \;:\; A_f^{(i,j,k)} \in \mathrm{Lim}_h(A^{(i,j,k)}), \quad A_g^{(i,j,k)} \in \mathrm{Lim}_{-h}(A^{(i,j,k)}) \Big\}.$$

For $A_n = P_n A P_n$, this simplifies to $\{A,\ P_- A_f P_-,\ P_+ A_g P_+ \ :\ A_f \in \mathrm{Lim}_h(A),\ A_g \in \mathrm{Lim}_{-h}(A)\}$.

For the study of the stability of subsequences (A_{h_n}) in [16], the "unwanted" layers of $\oplus A_n$ have been removed, destroying its triangular pattern. Because this pattern and its dominant directions play an important role in our paper, we instead replace the unwanted layers, A_m with $m \notin \operatorname{im} h$, by cP_m with $c \in \mathbb{R}$ chosen to make these layers irrelevant for the current purpose.

So fix $c \in \mathbb{R}$ and put

$$A'_{n,c} \;:=\; \begin{cases} A_n, & n \in \operatorname{im} h, \\ cP_n, & n \notin \operatorname{im} h. \end{cases}$$

Then

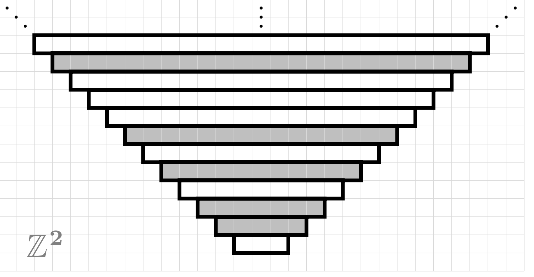
$$\mathcal{B}(A'_{n,c}) \;=\; \mathcal{B}_h(A_n) \cup \{cI\} \tag{28}$$

with identity I as in Remark 5.8 and both

$$\limsup \|A'_{n,0}\| = \limsup \|A_{h_n}\| \tag{29}$$

$$\limsup \|A'^{-1}_{n,c}\| = \limsup \|A^{-1}_{h_n}\| \tag{30}$$

holding for large enough c, say $c > \sup \|A_n\|$, since then



Pattern of $\oplus A'_{n,c}$ with $h = (2, 3, 5, 7, 11, \dots)$.
Grey layers are original layers A_n, unchanged,
and white layers are cP_n with a fixed $c \in \mathbb{R}$.

$$c^{-1} < (\sup \|A_n\|)^{-1} = \inf \|A_n\|^{-1} \le \inf \|A_n^{-1}\| \le \inf \|A_{h_n}^{-1}\| \le \limsup \|A_{h_n}^{-1}\|. \tag{31}$$

Proof of (23), (24), (25) and (26). It is enough to show (23) and (24). The rest follows like Propositions 4.5 and 4.6 follow from 4.2 and 4.3. For $c > \sup \|A_n\|$,

$$\limsup \|A_{h_n}\| \overset{(29)}{=} \limsup \|A'_{n,0}\| \overset{⑦}{=} \|\mathrm{Lim}(\oplus A'_{n,0})\|_\infty \overset{⑧}{=} \|\mathcal{B}_h(A_n)\|_\infty$$

$$\text{and} \quad \limsup \|A_{h_n}^{-1}\| \overset{(30)}{=} \limsup \|A'^{-1}_{n,c}\| \overset{⑦}{=} \|\big(\mathrm{Lim}(\oplus A'_{n,c})\big)^{-1}\|_\infty \overset{⑨}{=} \|\big(\mathcal{B}_h(A_n)\big)^{-1}\|_\infty .$$

⑦ This is by steps ①, ②, ⑤, ③ and ⑥ in the proofs of Propositions 4.2 and 4.3. Note that these steps hold for all sequences in BDS.

⑧ Let $\oplus B_n \in \mathrm{Lim}(\oplus A'_{n,0})$. W.l.o.g, just consider B_0. By (20), $B_0 \leftarrow S_{-i_k} A'_{j_k,0} S_{i_k}$ as $k \to \infty$.

a) If $j_k \in \operatorname{im} h$ eventually then $B_0 \in \mathrm{Lay}(\mathrm{Lim}_{\binom{*}{h}}(\oplus A_n))$, by (27). If $(A_n) \in \mathcal{S}$, further restrict to directions $\binom{0}{1}$ and $\binom{\pm 1}{1}$, i.e. to $\mathrm{Stab}_h(A_n)$, without missing any maximizers.

b) If $j_k \notin \operatorname{im} h$ eventually then $B_0 = 0$, clearly dispensable when maximizing $\|B_n\|$.

c) Also if j_k keeps alternating between $\operatorname{im} h$ and its complement and the limit B_0 exists then $B_0 = 0$, as we see by passing to the subsequence $\notin \operatorname{im} h$.

⑨ The argument is as in ⑧, only that now case b) and c) are of the form $B_0 = cI$, which does not contribute to maximizing $\|B_n^{-1}\|$, by (31). ■

7. When is lim sup a limit?

Convention 7.1. *For $s \in \{\|\cdot\|, \|(\cdot)^{-1}\|, \mathrm{Spec}_\varepsilon\}$, we interpret $s(A_n)$ as $\|A_n\|$ or $\|A_n^{-1}\|$ or $\mathrm{Spec}_\varepsilon A_n$ and, correspondingly, $s(\mathcal{B}(A_n))$ as $\|\mathcal{B}(A_n)\|_\infty$ or $\|\big(\mathcal{B}(A_n)\big)^{-1}\|_\infty$*

or $\cup_{B\in\mathcal{B}(A_n)} \mathrm{Spec}_\varepsilon B$*. Depending on the context, we interpret the signs* $\prec, \preceq$ *either as* $<, \leq$ *or* $\subsetneq, \subseteq$.

Then, for monotonic sequences $h = (h_n)$ in $\mathbb{N}$, (23), (24) and (26) can be summarized as

$$\limsup s(A_{h_n}) \;=\; s(\mathcal{B}_h(A_n)), \qquad (A_n) \in \mathrm{BDS}. \tag{32}$$

For the full sequence, $h = (h_n) = (n)$, we recover Propositions 4.2, 4.3 and 4.5.

So let $(A_n) \in \mathrm{BDS}$ and $s \in \{\|\cdot\|, \|(\cdot)^{-1}\|, \mathrm{Spec}_\varepsilon\}$, see Convention 7.1. Now, when is the lim sup in (32) a limit? The answer is simple: If a subsequence $h = (h_n)$ of the naturals exists, where $\limsup s(A_{h_n})$ differs from $\limsup s(A_n)$ then $s(A_n)$ is not convergent; otherwise it is. So let us search for monotonic sequences h in $\mathbb{N}$ with

$$s(\mathcal{B}_h(A_n)) \overset{(32)}{=} \limsup s(A_{h_n}) \;\prec\; \limsup s(A_n) \overset{(32)}{=} s(\mathcal{B}(A_n)). \tag{33}$$

The search for promising subsequences h of $\mathbb{N}$ is not as hopeless as it first seems. For a monotonic sequence g in $\mathbb{N}$, write $h \subseteq g$ if h is a subsequence of g and note that, by (27),

$$h \subseteq g \quad \text{implies} \quad \mathcal{B}_h(A_n) \;\subseteq\; \mathcal{B}_g(A_n), \quad \text{whence also} \quad s(\mathcal{B}_h(A_n)) \;\preceq\; s(\mathcal{B}_g(A_n)). \tag{34}$$

This monotonicity, (34), is guiding our search for sequences h with (33).

Definition 7.2. We say that a monotonic sequence g in $\mathbb{N}$ is *minimizing* for the operator sequence $(A_n) \in \mathrm{BDS}$ and write $g \in m(A_n)$ if $\mathcal{B}_h(A_n) = \mathcal{B}_g(A_n)$ for all subsequences $h \subseteq g$.

Example 7.3. For pure finite sections, $A_n = P_nAP_n$ with $A \in \mathrm{BDO}$, and a monotonic sequence f in $\mathbb{N}$, by $\mathrm{Lim}_f(A) \neq \varnothing$, first pass to a subsequence $g \subseteq f$ such that the limit operator A_g exists and then to a subsequence $h \subseteq g \subseteq f$ such that also A_{-h} exists. By Proposition 6.2,

$$\mathcal{B}_h(A_n) \;=\; \mathrm{Stab}_h(A_n) \;=\; \{A,\; P_-A_hP_-,\; P_+A_{-h}P_+\}$$

cannot get smaller by passing to subsequences of h. So h is a minimizing sequence for (A_n). □

This was a good practice, here is the general statement for $(A_n) \in \mathcal{S}$:

Corollary 7.4. *For* $(A_n) \in \mathcal{S}$*, every monotonic sequence in* $\mathbb{N}$ *has a minimizing subsequence.*

Proof. For $(A_n) \in \mathcal{S}$, say (6) holds, enumerate the set of all involved $A^{(i,j,k)}$ by $A^{(1)}, A^{(2)}, \dots$. For a monotonic sequence f in $\mathbb{N}$, first pass to $g^{(1)} \subseteq f$ such that $A^{(1)}_{g^{(1)}}$ and $A^{(1)}_{-g^{(1)}}$ exist, then to a subsequence $g^{(2)} \subseteq g^{(1)} \subseteq f$ such that $A^{(2)}_{g^{(2)}}$ and $A^{(2)}_{-g^{(2)}}$ exist, and so on. Then take $h = (h_n) \subseteq f$ with $h_n = g^{(n)}_n$ for $n \in \mathbb{N}$ and note that all limit operators $A^{(i,j,k)}_h$ and $A^{(i,j,k)}_{-h}$ exist, whence

$$\mathcal{B}_h(A_n) = \mathrm{Stab}_h(A_n) = \Big\{A,\ \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_- A^{(i,j,k)}_h P_-),\ \lim_{i\to\infty} \sum_{j\in J_i} \prod_{k\in K_i} (P_+ A^{(i,j,k)}_{-h} P_+)\Big\} \tag{35}$$

cannot get any smaller via subsequences of h, so that h is a minimizing sequence for (A_n). ■

Proposition 7.5. *If g is a minimizing sequence for $(A_n) \in \mathrm{BDS}$ then $s(A_{g_n})$ is convergent.*

Proof. By Definition 7.2 and (32), all subsequences of $s(A_{g_n})$ have the same lim sup. ■

Example 7.6. For $(A_n) \in \operatorname{clos\,alg}\{P_n L P_n : L \text{ banded Laurent}\}$, see Example 2.1, already $\mathrm{Stab}(A_n)$ is of the minimal form (35) with limit operators $L_g = L$. So $h = (h_n) = (n)$ is minimizing and all three spectral quantities $s(A_n)$ converge, by Proposition 7.5. □

Proposition 7.7. ***a)*** *For $(A_n) \in \mathrm{BDS}$ and $s \in \{\|\cdot\|, \|(\cdot)^{-1}\|, \mathrm{Spec}_\varepsilon\}$, the quantity $s(A_n)$ is convergent if and only if $\limsup s(A_n) = \limsup s(A_{g_n})$ for all minimizing sequences g of (A_n).*

b) *If $(A_n) \in \mathcal{S}$, we have*

$$\begin{aligned}
\limsup \|A_n\| &= \max_{g\in m(A_n)} \|\mathrm{Stab}_g(A_n)\|_\infty\,, \\
\limsup \|A_n^{-1}\| &= \max_{g\in m(A_n)} \|\big(\mathrm{Stab}_g(A_n)\big)^{-1}\|_\infty\,, \\
\limsup \kappa(A_n) &= \Big(\max_{g\in m(A_n)} \|\mathrm{Stab}_g(A_n)\|_\infty\Big)\cdot\Big(\max_{g\in m(A_n)} \|\big(\mathrm{Stab}_g(A_n)\big)^{-1}\|_\infty\Big)\,, \\
\limsup \mathrm{Spec}_\varepsilon A_n &= \bigcup_{g\in m(A_n)} \bigcup_{B\in \mathrm{Stab}_g(A_n)} \mathrm{Spec}_\varepsilon B.
\end{aligned}$$

For the corresponding lim inf, *replace* $\max_g$ *by* $\min_g$ *and* $\cup_g$ *by* $\cap_g$.

Proof. a) The implication $\boxed{\Rightarrow}$ is obvious and $\boxed{\Leftarrow}$ holds by monotonicity, (34).

b) For $s \in \{\|\cdot\|, \|(\cdot)^{-1}\|, \kappa, \mathrm{Spec}_\varepsilon\}$ and $(A_n) \in \mathcal{S}$, every partial limit S of $s(A_n)$ is the limit of a subsequence $s(A_{h_n})$. By Corollary 7.4, take a minimizing sequence $g \subseteq h$ for (A_n) and note that $S = \lim s(A_{h_n}) = \lim s(A_{g_n})$. For $s = \kappa$, the latter is $\lim \|A_{g_n}\| \cdot \lim \|A_{g_n}^{-1}\|$ since all limits exist, by Proposition 7.5. Now apply (32) and recall that $\limsup s(A_n)$ is, by definition, the largest such partial limit S, in the sense of the maximum or the set union, respectively. ■

Corollary 7.8. *So the following are equivalent for* $(A_n) \in \text{BDS}$ *and* $s \in \{\|\cdot\|, \|(\cdot)^{-1}\|, \text{Spec}_\varepsilon\}$*:*

(i) $s(A_n)$ *is divergent,*
(ii) there exists a minimizing sequence g *of* (A_n) *with* $\limsup s(A_{g_n}) \prec \limsup s(A_n)$*,*
(iii) there exists a minimizing sequence g *of* (A_n) *with* $s(\mathcal{B}_g(A_n)) \prec s(\mathcal{B}(A_n))$

If $(A_n) \in \mathcal{S}$*, say* (6) *holds, then all are equivalent to (iv): there is a monotonic sequence* g *in* $\mathbb{N}$*, for which all limit operators* $A_g^{(i,j,k)}$ *and* $A_{-g}^{(i,j,k)}$ *exist and* $s(\text{Stab}_g(A_n)) \prec s(\text{Stab}(A_n))$*.*

Remark 7.9. a) Note that $\text{Stab}_g(A_n)$ in Proposition 7.7 b) and Corollary 7.8 (*iv*) is of the simple form (35) with only three elements, say $\text{Stab}_g(A_n) = \{A, B, C\}$, so that $s(\text{Stab}_g(A_n))$ is either $\max\{\|A\|, \|B\|, \|C\|\}$, $\max\{\|A^{-1}\|, \|B^{-1}\|, \|C^{-1}\|\}$ or $\text{Spec}_\varepsilon A \cup \text{Spec}_\varepsilon B \cup \text{Spec}_\varepsilon C$.

b) If $s(\text{Stab}(A_n)) = s(A)$ then convergence of $s(A_n)$ is clear without looking at subsequences, limit operators, etc. Recall, e.g., $s = \|\cdot\|$ for pure finite sections, $A_n = P_nAP_n$.

c) Convergence of $\text{Spec}_\varepsilon A_n$ implies that of $\|A_n^{-1}\|$. Indeed, putting $f_n(\lambda) := \mu(A_n - \lambda I_n)$ for $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, convergence of $\text{Spec}_\varepsilon A_n$ for all $\varepsilon > 0$ is equivalent to pointwise convergence of f_n, by [17], while convergence of $\|A_n^{-1}\|$ is just convergence of $f_n(0)$, by (5).

d) The reverse implication of c) is not true, see the following example. □

Example 7.10. Recall the symmetric block-flip F from Example 2.2 with $\mu = 0$ and put $A := F + 2I$, which is selfadjoint. Then, with $F_n = P_nFP_n$ and $A_n = P_nAP_n$, $\text{Spec}\, A_n = \text{Spec}\, F_n + 2$, which is $\{1, 3\}$ if n is even and $\{1, 2, 3\}$ if n is odd.

Consequently, $\|A_n^{-1}\| = \mu(A_n)^{-1} = \text{dist}(0, \text{Spec}\, A_n)^{-1} = 1$ for all n, which is constant (hence convergent), while $\text{Spec}_\varepsilon A_n = \text{Spec}\, A_n + \{z \in \mathbb{C} : |z| \le \varepsilon\}$ differs between even and odd n and does not Hausdorff-converge as $n \to \infty$. □

8. $\ell^p(\mathbb{Z}^d, Y)$: Banach space-valued ℓ^p over $\mathbb{Z}^d$ with $d \in \mathbb{N}$, $p \in [1, \infty]$

8.1. How to pass to $p \in \{1, \infty\}$

A serious problem is that $P_n \not\to I$ pointwise if $p = \infty$, affecting the identification between operators and their matrix, the convergence $\|(P_n - I)T\| \to 0$ for compact operators T on ℓ^∞, the somehow dual statement $\|T(P_n - I)\| \to 0$ on ℓ^1, the whole interplay between matrix decay and compact operators with consequences reaching far into Fredholmness and limit operators for ℓ^1 and ℓ^∞.

The solution is to put the sequence (P_n) in the first place and to change the notion of $\to$ such that $P_n \to I$ does hold, then change the set of compact operators T such that $\|(P_n - I)T\| \to 0$ and $\|T(P_n - I)\| \to 0$ for all such T and finally adapt the notion of

Fredholmness to it. This major refurbishment is called $\mathcal{P}$-theory [23], and it received its ultimate polishing in [27]. For $p \in (1, \infty)$, everything coincides with the classic theory, which is why we present that case above.

8.2. Banach space-valued ℓ^p spaces

Identifications like that of $L^p(\mathbb{R})$ with the $L^p([0,1])$-valued $\ell^p(\mathbb{Z})$ motivate to study spaces

$$\ell^p(\mathbb{Z}, Y) \;:=\; \{(y_i)_{i\in\mathbb{Z}} \;:\; y_i \in Y,\; (\|y_i\|_Y) \in \ell^p(\mathbb{Z})\} \tag{36}$$

with a Banach space Y. Operators A on such a space have matrices $(A_{ij})_{i,j\in\mathbb{Z}}$ with operator entries A_{ij} on Y. Again, compactness of A is only loosely related with decay properties of the matrix, as already one non-compact entry A_{ij} can change everything. $\mathcal{P}$-theory [27] also comes to rescue here. Its redefinition of compact operators, convergence and Fredholmness is such that also $\ell^p(\mathbb{Z}, Y)$ smoothly integrates into the classic theory.

Exchanging $\ell^p(\mathbb{Z})$ for $\ell^p(\mathbb{Z}, Y)$ in the previous sections requires two additional assumptions:

- For many arguments we rely on the fact that $\mathrm{Lim}_h(A) \neq \varnothing$ for $A \in \mathrm{BDO}$ and a monotonic sequence h in $\mathbb{N}$. The proof uses Bolzano-Weierstraß for the entries A_{ij} and that requires scalar (or at least finite-dimensional) matrix entries. Indeed, $\mathrm{Lim}_h(A) = \varnothing$ can happen when $\dim Y = \infty$. To avoid this, restrict consideration to so-called *rich* band-dominated operators A, literally imposing that $\mathrm{Lim}_h(A) \neq \varnothing$ for all h. Equivalently, $A \in \mathrm{BDO}$ is rich if and only if the set of all its entries A_{ij} is relatively compact in $L(Y)$.
- Another property that comes automatic with $\dim Y < \infty$, see [8,29], but has to be assumed additionally when $\dim Y = \infty$ is the continuity of the map $\varepsilon \mapsto \mathrm{Spec}_\varepsilon A$ for bounded operators A on $X = \ell^p(\mathbb{Z}, Y)$. We rely on this property in Section 4.3. It is shown to hold [8,29] for bounded linear operators A on a Banach space X that is finite-dimensional or complex uniformly convex or has a complex uniformly convex dual space. If we assume that Y is uniformly G-convex in the sense of [5] (and hence also complex uniformly convex) then, by [5], our $X = \ell^p(\mathbb{Z}, Y)$ inherits that property and we are back at continuity of $\varepsilon \mapsto \mathrm{Spec}_\varepsilon A$.

The last result, [5], is unfortunately only known to us for $p \in [1, \infty)$. So for our results on the asymptotics of $\mathrm{spec}_\varepsilon A_n$ and $\mathrm{Spec}_\varepsilon A_n$, we can (so far) either pass to $p = \infty$ or to $\dim Y = \infty$. For the rest of the paper both can happen at the same time.

8.3. ℓ^p-sequences over $\mathbb{Z}^d$ with $d \in \mathbb{N}$

All our arguments of equalities ①-⑨ extend to ℓ^p-spaces over $\mathbb{Z}^d$. The stacked operator $\oplus A_n$ then acts on $\ell^p(\mathbb{Z}, \ell^p(\mathbb{Z}^d)) \cong \ell^p(\mathbb{Z}^{d+1})$.

As P_n one could, for example, use multiplication by the characteristic function of the cube $(-n..n)^d$. For $d = 2$, the support pattern of $\oplus A_n$ is then an infinite upside down pyramid with square shaped layers. Besides A, the stability indicators (c) and (d) are limits along sequences $h = \binom{g_k}{|g_k|_\infty}$ with $g_k \to \infty$ in $\mathbb{Z}^d$; that is, sequences h on the surface of the pyramid in $\mathbb{Z}^{d+1}$. The counterparts of cases $(b), (e), (f)$ in Propositions 5.2 and 5.9 are again redundant.

We can still study the concept of minimal sequences g but the corresponding set $\mathrm{Stab}_g(A_n)$ generally has uncountably many (instead of three) elements. This increase is not due to the method being inappropriate; instead, this growth is necessary to capture stability in higher dimensions. Already for $d = 2$, a convolution operator A and truncations P_n to $(n\Omega) \cap \mathbb{Z}^2$ with a convex polygon Ω with corners $v_1, \ldots, v_k \in \mathbb{Z}^2$, the stability of the sequence $(P_n A P_n)$ is equivalent to invertibility of all compressions $P_U A P_U$ of A with the set U running through the k limits, as $n \to \infty$, of $n(\Omega - v_j)$ for $j = 1, \ldots, k$, bringing us back to Kozak [12]. For general $A \in$ BDO [22] and general convex geometries Ω [16], one already has compressions $P_U A_h P_U$ of all kinds of limit operators A_h of A with corresponding boundary geometries U that are no longer dominated by finitely many of these operators $P_U A_h P_U$. For example, for a disk Ω, one cannot avoid looking at infinitely many compressions $P_U A_h P_U$ with half planes U.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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