

# Computability Theory

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July 13, 2021



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<https://doi.org/10.15480/882.3651>  
<http://hdl.handle.net/11420/9877>  
<urn:nbn:gbv:830-882.0139748>

## Preface

- How to show that a decision problem is not algorithmically decidable?<sup>a</sup>
- Rigorous formalism initialized by the work of David Hilbert (1900) and Kurt Gödel (1931).
- Turing machine captures the notion of computability (Alan Turing, 1936)
- Several mathematical problems are known to be undecidable (halting problem, word problems, string rewriting, solving diophantine equations).

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<sup>a</sup>The problem whether two natural numbers are relatively prime is decidable by the Euclidean algorithm.

## Preface

These notes are a development of class notes for a two-hour lecture including a two-hour lab held for second-year Bachelor students of Computer Science at the Hamburg University of Technology during the last few years.

The course aims to present the basic results of computability theory, including well-known mathematical models of computability such as the Turing machine, the unlimited register machine (URM) as well as GOTO and LOOP programs, the principal classes of computational functions like the classes of primitive recursive, recursive, and partial recursive functions, the famous Ackermann function and the Ackermann class, the main theoretical concepts of computability such as Gödel numbering, universal functions, parametrization, Kleene's normal form, Kleene's recursion theorem, the theorems of Rice, undecidable and semidecidable (or recursively enumerable) sets, computable real numbers, Hilbert's tenth problem, and last but not least several important undecidable word problems including those for semi-Thue systems, semigroups, and Post correspondence systems.

The corresponding manuscript provides a more detailed representation of the topic including the necessary mathematical background on semigroups and monoids, a brief introduction to the freely available command-line program `glu` which has been developed for the interpretation of GOTO, LOOP and URM programs, and a description of `Maple`<sup>TM</sup> code for the encoding of strings of natural numbers and the Gödel numbering of SGOTO programs.

The manuscript has partly grown out of notes taken by the author during his studies at the University of Erlangen-Nuremberg. Therefore, I would like to express my thanks foremost to my teachers Martin Becker<sup>†</sup> and Volker Strehl for their wonderful and inspiring lectures in this field. Moreover, I would like to thank my colleague and friend Ralf Möller for valuable comments. I am also grateful to my collaborators Merve Nur Cakir, Mahwish Saleemi, Natalia Schmidt, and Robert Leppert for conducting the labs and Wolfgang Brandt for valuable technical support. Finally, I would like to thank my students for their attention, their stimulating questions, and their dedicated work.

The course took place in the first phase of the Corona pandemic and so we were forced to organize it as a full online course. At the end I can say that it has been a nice experience to present the topic by Zoom and explanatory videos. It appears that the participants were not fully unhappy with this situation. At the end, I'd like to add that the exam was fully organized in electronic form.

## Contents

- Unlimited register machine (URM)
- Primitive recursive functions
- Partial recursive functions
- Ackermann function
- Turing machine
- Acceptable programming systems
- Undecidability
- Word problems

\*Starred material can be safely skipped.

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# Part I

## Unlimited Register Machine

## Unlimited Register Machine – Motivation

- URM introduced by Sheperdson and Sturgis (1963) as abstract computing machine.
- URM captures the notion of computability.
- URM has infinitely many, uniquely addressable registers, each of which holds a single natural number.
- URM programming (increment, decrement, composition, iteration).

E.g., addition of two numbers via URM program  $(A1; S2)^2$ .

## Unlimited Register Machine – Contents

- States and state transformations
- Syntax of URM programs
- Semantics of URM programs
- URM computable functions

Reading: Zimmermann, Chapter 1

## States and Transformations – Contents

- Hardware
- State set and set of transformations
- Increment and decrement
- Composition
- Powers and iteration

## States and Transformations – Hardware

- URM contains an infinite number of registers named

$$R_0, R_1, R_2, R_3, \dots \quad (1)$$

- Each register stores a natural number, but only a finite number of registers can have nonzero entries.

$R$	0	1	2	3	4	5	...
$\omega$	$\omega_0$	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	

$R_0$	$\omega_0$
$R_1$	$\omega_1$
$R_2$	$\omega_2$
$R_3$	$\omega_3$

## States and Transformations – State Set

- *State set* of URM

$$\Omega = \{\omega : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \mid \omega \text{ is 0 almost everywhere}\}, \quad (2)$$

where *almost everywhere* means all but a finite number.

- *Almost everywhere* captures the notion of finite memory.
- The elements of  $\Omega$  are denoted as sequences

$$\omega = (\omega_0, \omega_1, \omega_2, \omega_3, \dots), \quad (3)$$

where  $\omega_n = \omega(n)$  denotes the content of register  $R_n$ ,  $n \in \mathbb{N}_0$ .

$R_0$	$\omega_0$
$R_1$	$\omega_1$
$R_2$	$\omega_2$

States and Transformations –  $|\Omega|$ 

The set of states  $\Omega$  of an URM is countable.

## Proof.

Denote the infinite sequence of prime numbers as

$$(p_0, p_1, p_2, p_3, p_4, \dots) = (2, 3, 5, 7, 11, \dots).$$

The mapping

$$\Omega \rightarrow \mathbb{N} : \omega \mapsto \prod_{\substack{i \\ \text{finite}}} p_i^{\omega_i} \quad (4)$$

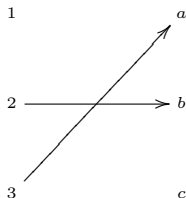
is bijective, since each natural number factors uniquely into a product of prime powers and the states are 0 almost everywhere, which means that the products are finite; e.g.,

$$(0, 5, 4, 0, 2, 0, \dots) \mapsto p_0^0 p_1^5 p_2^4 p_3^0 p_4^2 p_5^0 \dots = 3^5 5^4 11^2.$$



## States and Transformations – Partial Functions

Partial function  $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$  has source set  $\{1, 2, 3\}$ ,  $\text{dom}(f) = \{2, 3\}$  and  $\text{ran}(f) = \{a, b\}$ :



- Partial means that  $\text{dom}(f)$  is a (proper) subset of the source set.
- Total means that  $\text{dom}(f)$  equals the source set.

## States and Transformations – Partial Functions

- A *partial function*  $f : X \rightarrow Y$  is a function  $f : X' \rightarrow Y$  for some subset  $X'$  of  $X$ .

Then the *domain*

$$\text{dom}(f) = X' \quad (5)$$

is a subset of the source set  $X$  of  $f$ .

- A partial function  $f : X \rightarrow Y$  is *total* if the domain equals the source set, i.e.,  $\text{dom}(f) = X$ .

Total functions are the functions in the usual sense (see Calculus, Linear Algebra).

- E.g., square-root function

$$f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto \sqrt{x}$$

is partial, since  $\text{dom}(f) = \{0, 1, 4, 9, 16, \dots\}$  (perfect squares), e.g.,  $f(16) = 4$  but  $f(17) \notin \mathbb{N}_0$ .

States and Transformations –  $E(\Omega)$ 

- $E(\Omega) = \{f \mid f : \Omega \rightarrow \Omega, f \text{ partial}\}$  denotes the set of all partial functions on  $\Omega$ .  
 $f \in E(\Omega)$  is a potential state transformation of the URM.
- Each partial function  $f \in E(\Omega)$  has *domain*

$$\text{dom}(f) = \{\omega \in \Omega \mid f(\omega) \text{ is defined}\} \quad (6)$$

and *range*

$$\text{ran}(f) = \{\omega' \in \Omega \mid \exists \omega \in \Omega : f(\omega) = \omega'\}. \quad (7)$$

- E.g., square-root function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto \sqrt{x}$  has  $\text{dom}(f) = \{0, 1, 4, 9, 16, \dots\}$  and  $\text{ran}(f) = \mathbb{N}_0$ .

States and Transformations –  $|E(\Omega)|$ 

The set of partial functions  $E(\Omega)$  of an URM is uncountable.

## \*Proof

- The real interval  $[0, 1)$  is uncountable.
- Represent each real number  $a \in [0, 1)$  in binary format

$$a = 0.a_0a_1a_2\dots, \quad a_i \in \{0, 1\}, i \geq 0.$$

- For each such number  $a = 0.a_0a_1a_2\dots$ , define the function  $f_a : \Omega \rightarrow \Omega$  by setting  $f_a(\omega) = \omega'$ , where

$$\omega'_n = a_n \oplus \omega_n, \quad n \geq 0, \quad (8)$$

such that  $a_n \oplus \omega_n = \omega_n$  if  $a_n = 1$  and  $a_n \oplus \omega_n = 0$  if  $a_n = 0$ .

- $f_a$  is well-defined, since a state  $\omega$  with 0's almost everywhere is mapped to a state  $\omega'$  with 0's almost everywhere.
- For different real numbers  $a = 0.a_0a_1a_2\dots$  and  $b = 0.b_0b_1b_2\dots$ , the functions  $f_a$  and  $f_b$  are distinct. To see this, suppose  $a_n = 1$  and  $b_n = 0$  for some  $n \geq 0$ . Then for the state  $\omega$  which is 1 at register  $R_n$  and 0 elsewhere, we have  $f_a(\omega) = \omega$  and  $f_b(\omega) = 0$ , the zero state.
- E.g., let  $a = 0.011000\dots$  and  $\omega = (2, 3, 4, 5, 0, 0, \dots)$ . Then

$$f_a(\omega) = (0, 3, 4, 0, 0, 0, \dots).$$

## States and Transformations – C-Programs

The set of programs in any programming language over a finite alphabet is countable.

### Proof

- Each program written in any programming language over an alphabet  $\Sigma = \{0, \dots, s\}$  (keyboard alphabet) is a word in the free monoid  $\Sigma^*$ .
- The set  $\Sigma^*$  is countable, since the mapping

$$\phi : \Sigma^* \rightarrow \mathbb{N} : x_0 \dots x_{n-1} \mapsto p_0^{x_0} \cdots p_{n-1}^{x_{n-1}}$$

is a bijection, where  $(p_0, p_1, p_2, \dots) = (2, 3, 5, \dots)$  denotes the infinite sequence of primes (see (4)), e.g.,

$$(2, 0, 3, 1, 0) \mapsto p_0^2 p_1^0 p_2^3 p_3^1 p_4^0 = 2^2 5^3 7.$$



## States and Transformations – Gaps

- There is a *gap* between the set of programs in a programming language and the set of state transformations of an abstract computing machine,

$$|\Sigma^*| < |E(\Omega)|.$$

- *Continuum hypothesis* (affirmatively solved) says that there is no set whose cardinality is strictly between that of the natural numbers and the real numbers,

$$\exists X : |\mathbb{N}_0| < |X| < |\mathbb{R}|.$$

## States and Transformations – Equality

Let  $f, g \in E(\Omega)$  be partial functions.

- Function equality  $f = g$  means
  - $\text{dom}(f) = \text{dom}(g)$  and
  - for all  $\omega \in \text{dom}(f)$ ,  $f(\omega) = g(\omega)$ .
- $f \in E(\Omega)$  *total* if  $\text{dom}(f) = \Omega$  (source set).

*Total functions* are the functions in the usual sense (see Calculus, Linear Algebra).

## States and Transformations – Increment

*Increment function*  $a_k \in E(\Omega)$  for  $k$ -th register is the total function

$$a_k : \Omega \rightarrow \Omega : \omega \mapsto \omega', \quad (9)$$

where

$$\omega'_n = \begin{cases} \omega_n & \text{if } n \neq k, \\ \omega_k + 1 & \text{otherwise,} \end{cases} \quad (10)$$

$$(\omega_0, \dots, \omega_{k-1}, \omega_k, \omega_{k+1}, \dots) \mapsto (\omega_0, \dots, \omega_{k-1}, \omega_k + 1, \omega_{k+1}, \dots).$$

URM-programming primitive:  $Ak$ .

## States and Transformations – Decrement

- *Decrement function*  $s_k \in E(\Omega)$  for  $k$ -th register is the total function

$$s_k : \Omega \rightarrow \Omega : \omega \mapsto \omega', \quad (11)$$

where

$$\omega'_n = \begin{cases} \omega_n & \text{if } n \neq k, \\ \omega_k \dot{-} 1 & \text{otherwise.} \end{cases} \quad (12)$$

$$(\omega_0, \dots, \omega_{k-1}, \omega_k, \omega_{k+1}, \dots) \mapsto (\omega_0, \dots, \omega_{k-1}, \omega_k \dot{-} 1, \omega_{k+1}, \dots).$$

URM-programming primitive:  $Sk$ .

- Dyadic operation  $\dot{-} : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  denotes the *asymmetric difference*,

$$x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y, \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

e.g.,  $5 \dot{-} 3 = 2$  and  $5 \dot{-} 7 = 0$ .

## States and Transformations – \*Graph

- *Graph* of a partial function  $f \in E(\Omega)$  is given by the relation

$$R_f = \{(\omega, f(\omega)) \mid \omega \in \text{dom}(f)\}. \quad (14)$$

E.g., increment function  $a_k$ ,  $k \in \mathbb{N}_0$ , gives

$$R_{a_k} = \{(\omega, \omega') \mid \omega \in \Omega, \omega' = (w_0, \dots, w_{k-1}, w_k+1, w_{k+1}, \dots)\}.$$

- Two partial functions  $f, g \in E(\Omega)$  are *equal* if the corresponding graphs  $R_f$  and  $R_g$  are equal as sets, i.e.,

$$\text{dom}(f) = \text{dom}(g) \wedge \forall \omega \in \text{dom}(f) : f(\omega) = g(\omega). \quad (15)$$

- E.g., square-root function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto \sqrt{x}$  has the graph

$$R_f = \{(0, 0), (1, 1), (4, 2), (9, 3), (16, 4), (25, 5), \dots\}.$$

## States and Transformations – Composition

*Composition* of two partial functions  $f, g \in E(\Omega)$  is a partial function  $g \circ f$  with (inner function  $f$  and outer function  $g$ )

$$(g \circ f)(\omega) = g(f(\omega)), \quad (16)$$

where  $\omega$  belongs to the domain of  $g \circ f$ ,

$$\text{dom}(g \circ f) = \{\omega \in \Omega \mid \omega \in \text{dom}(f) \wedge f(\omega) \in \text{dom}(g)\}. \quad (17)$$

- If  $f$  and  $g$  are total, then  $g \circ f$  will be total.
- Composition is *associative*, i.e., for all  $f, g, h \in E(\Omega)$ ,

$$f \circ (g \circ h) = (f \circ g) \circ h. \quad (18)$$

## States and Transformations – Composition

Composition of increment functions  $a_k$  and  $a_l$ ,  $k, l \in \mathbb{N}_0$ .

- Case  $k = l$ :

$$(a_k \circ a_l)(\omega) = (w_0, \dots, w_k + 2, \dots).$$

- Case  $k \neq l$  with  $k < l$ :

$$(a_k \circ a_l)(\omega) = (w_0, \dots, w_k + 1, \dots, w_l + 1, \dots).$$

URM-compound statement for composition  $g \circ f$ :  $P_f; P_g$ .

## States and Transformations – Powers

Powers of partial function  $f \in E(\Omega)$  are inductively defined,

$$f^0 = \text{id}_\Omega, \quad \text{and} \quad f^{n+1} = f \circ f^n, \quad n \in \mathbb{N}_0, \quad (19)$$

where  $\text{id}_\Omega : \Omega \rightarrow \Omega : \omega \mapsto \omega$  is the *identity function* on  $\Omega$ .

In particular,

$$f^1 = f \circ \text{id}_\Omega = f.$$

The setting

$$f^{n+1} = f^n \circ f, \quad n \in \mathbb{N}_0,$$

is equally applicable, since composition is associative.

E.g.,  $n$ -th power of increment function  $a_k$ ,  $k \in \mathbb{N}_0$ ,

$$a_k^n(\omega) = (w_0, \dots, w_{k-1}, w_k + n, w_{k+1}, \dots), \quad n \in \mathbb{N}_0.$$

## States and Transformations – Iteration

- *Iteration* of partial function  $f \in E(\Omega)$  with initial state  $\omega \in \Omega$  gives the sequence of states

$$\omega, f(\omega), f^2(\omega), f^3(\omega), \dots \quad (20)$$

- If  $f$  is total, each power  $f^n$  will be total and so all states  $f^n(\omega)$  in the sequence (21) are defined.

## States and Transformations – Iteration (Tricky)

- Idea is to iterate the sequence of states

$$\omega, f(\omega), f^2(\omega), f^3(\omega), \dots \quad (21)$$

until register  $R_k$  becomes 0.

- *Iteration* of function  $f \in E(\Omega)$  w.r.t.  $k$ -th register defines a new function  $f^{*k} \in E(\Omega)$  such that
  - $f^{*k}(\omega) = f^n(\omega)$  if register  $R_k$  is 0 in state  $f^n(\omega)$  and  $n \geq 0$  is minimal.
  - $f^{*k}(\omega) = \uparrow$  (undefined) if
    - sequence (21) runs ad infinitum and  $R_k$  never becomes 0, or
    - sequence (21) aborts since  $f^j(\omega)$  is undefined before  $R_k$  would eventually be 0.

## States and Transformations – Iteration

- Implementation of iteration as `while` loop:

```
Require:  $\omega \in \Omega$   
while  $\omega_k > 0$  do  
     $w \leftarrow f(\omega)$   
end while
```

- URM-compound statement for iteration  $f^{*k}: (P_f)k$ .
- Iteration can be considered as unlimited search process.

## States and Transformations – Example

In view of increment function  $f = a_k$  and  $k$ -th register,

$$a_k^{*k}(\omega) = \begin{cases} \omega & \text{if } \omega_k = 0, \\ \uparrow & \text{otherwise.} \end{cases} \quad (22)$$

Take  $n$ -th power of increment function  $a_k$ ,

$$a_k^n(\omega) = (\omega_0, \dots, \omega_{k-1}, \omega_k + n, \omega_{k+1}, \dots), \quad n \in \mathbb{N}_0.$$

Case  $\omega_k = 0$ :

$$(\omega_0, \dots, \omega_{k-1}, 0, \omega_{k+1}, \dots) \mapsto (\omega_0, \dots, \omega_{k-1}, 0, \omega_{k+1}, \dots).$$

Case  $\omega_k \neq 0$  (runs ad infinitum):

$$\begin{aligned} & (\omega_0, \dots, \omega_{k-1}, \omega_k, \omega_{k+1}, \dots) \\ & \mapsto (\omega_0, \dots, \omega_{k-1}, \omega_k + 1, \omega_{k+1}, \dots) \\ & \mapsto (\omega_0, \dots, \omega_{k-1}, \omega_k + 2, \omega_{k+1}, \dots) \dots \end{aligned}$$

## Syntax and Semantics – Contents

- Alphabet
- Syntax of URM programs
- Semantics of URM programs

## Syntax – Alphabet

- Set of (decimal) numbers over the alphabet of digits  $\Sigma_{10} = \{0, 1, \dots, 9\}$  is defined as

$$Z = (\Sigma_{10} \setminus \{0\})\Sigma_{10}^+ \cup \Sigma_{10}, \quad (23)$$

e.g., 0, 10010, 20000, but not 0111 (no leading 0's).

- URM programs are words over the alphabet

$$\Sigma_{\text{URM}} = \{A, S, (, ), ; \} \cup Z. \quad (24)$$

## Syntax – URM Programs

Inductive definition of the set of *URM programs*  $\mathcal{P}_{\text{URM}}$ :

- *Increment*:  $A\sigma \in \mathcal{P}_{\text{URM}}$  for each  $\sigma \in Z$ .
- *Decrement*:  $S\sigma \in \mathcal{P}_{\text{URM}}$  for each  $\sigma \in Z$ .
- *Composition*: If  $P, Q \in \mathcal{P}_{\text{URM}}$ , then  $P; Q \in \mathcal{P}_{\text{URM}}$ .
- *Iteration*: If  $P \in \mathcal{P}_{\text{URM}}$  and  $\sigma \in Z$ , then  $(P)\sigma \in \mathcal{P}_{\text{URM}}$ .

$n$ -fold composition of URM program  $P \in \mathcal{P}_{\text{URM}}$ ,

$$P^n = P; P; \dots; P \quad (n \text{ times}), \quad n \geq 0. \quad (25)$$

## Syntax – Example

Well-formed URM programs:

- $A1$
- $(A1)1$
- $(A1)2$
- $A1; (A1)1$
- $(S1)1$
- $(A1; S2)2$

## Semantics

*Semantics* of URM programs is a mapping

$$|\cdot| : \mathcal{P}_{\text{URM}} \rightarrow E(\Omega)$$

defined inductively as follows:

- *Increment*:  $|A\sigma| = a_\sigma$  for each  $\sigma \in Z$ .
- *Decrement*:  $|S\sigma| = s_\sigma$  for each  $\sigma \in Z$ .
- *Composition*: If  $P, Q \in \mathcal{P}_{\text{URM}}$ , then  $|P; Q| = |Q| \circ |P|$ .
- *Iteration*: If  $P \in \mathcal{P}_{\text{URM}}$  and  $\sigma \in Z$ , then  $|(P)\sigma| = |P|^{*\sigma}$ .

Interpretation of URM programs as functions in  $E(\Omega)$ .

$|(P)\sigma|$  means that  $P$  is executed until register  $R_\sigma$  becomes zero.

## Semantics – Example

Trace of addition program  $P_+ = (A1; S2)2$ :

$R_0$	$R_1$	$R_2$	$R_3$	...	Registers
0	3	2	0	...	Init.
0	4	2	0	...	A1
0	4	1	0	...	S2
0	5	1	0	...	A1
0	5	0	0	...	S2

 $|P_+| : \Omega \rightarrow \Omega$  is given by

$$|P_+| : (\omega_0, \omega_1, \omega_2, \omega_3, \dots) \mapsto (\omega_0, \omega_1 + \omega_2, 0, \omega_3, \dots).$$

## URM Computable Functions – Contents

- URM computability
- Computation of functions
- Examples
- Normalization

## URM Computable Functions – Computability

- Partial function  $f \in E(\Omega)$  is *URM computable* if there is an URM program  $P$  such that

$$|P| = f. \quad (26)$$

- $\mathcal{P}_{\text{URM}}$  is countable, while the set  $E(\Omega)$  is not, i.e., there are partial functions in  $E(\Omega)$  that are not URM computable.
- $\mathcal{F}_{\text{URM}}$  denotes the class of all partial functions that are URM computable.
- $\mathcal{T}_{\text{URM}}$  depicts the class of all total functions which are URM computable. Then  $\mathcal{T}_{\text{URM}} \subset \mathcal{F}_{\text{URM}}$ .

## URM Computable Functions – Functions

- Calculation of function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^m$  by URM program requires to load the registers with initial values and to read out the result.
- Define load function

$$\alpha_k : \mathbb{N}_0^k \rightarrow \Omega : (x_1, \dots, x_k) \mapsto (0, x_1, \dots, x_k, 0, 0, \dots) \quad (27)$$

and result function

$$\beta_m : \Omega \rightarrow \mathbb{N}_0^m : (\omega_0, \omega_1, \omega_2, \dots) \mapsto (\omega_1, \omega_2, \dots, \omega_m). \quad (28)$$

## URM Computable Functions – Functions

- Given URM program  $P$  and integers  $k, m \in \mathbb{N}_0$ , define the partial function  $\|P\|_{k,m} : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^m$  by the composition

$$\|P\|_{k,m} = \beta_m \circ |P| \circ \alpha_k. \quad (29)$$

$f = \|P\|_{k,m}$  loads argument  $\mathbf{x} \in \mathbb{N}_0^k$ , computes  $P$ , and outputs the result  $f(\mathbf{x}) \in \mathbb{N}_0^m$ .

- A function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^m$  is *URM computable* if there is an URM program  $P$  such that

$$f = \|P\|_{k,m}. \quad (30)$$

## URM Computable Functions – Add

Addition of two natural numbers  $f_+ : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto x + y$  is URM computable.

URM program

$$P_+ = (A1; S2)2 \quad (31)$$

has the trace

$R_0$	$R_1$	$R_2$	$R_3$	$\dots$	Registers
0	$x$	$y$	0	$\dots$	Init.
0	$x + 1$	$y$	0	$\dots$	$A1$
0	$x + 1$	$y - 1$	0	$\dots$	$S2$
0	$x + 2$	$y - 1$	0	$\dots$	$A1$
0	$x + 2$	$y - 2$	0	$\dots$	$S2$
$\dots$					
0	$x + y - 1$	1	0	$\dots$	$S2$
0	$x + y$	1	0	$\dots$	$A1$
0	$x + y$	0	0	$\dots$	$S2$

## Proof (Cont'd)

Thus  $|P_+| : \Omega \rightarrow \Omega$  is given by

$$|P_+| : (\omega_0, \omega_1, \omega_2, \omega_3, \dots) \mapsto (\omega_0, \omega_1 + \omega_2, 0, \omega_3, \dots). \quad (32)$$

Hence,

$$\begin{aligned} \|P_+\|_{2,1}(x, y) &= (\beta_1 \circ |P_+| \circ \alpha_2)(x, y) & (33) \\ &= (\beta_1 \circ |P_+|)(0, x, y, 0, 0, \dots) \\ &= \beta_1(0, x + y, 0, 0, \dots) \\ &= x + y. \end{aligned}$$



## URM Computable Functions – Difference

Asymmetric difference (12) is URM computable.

URM program

$$P_{\dot{-}} = (S1; S2)2 \quad (34)$$

has the trace

$R_0$	$R_1$	$R_2$	$R_3$	$\dots$	Registers
0	$x$	$y$	0	$\dots$	Init.
$\dots$					
0	$x - y$	0	0	$\dots$	$x \geq y$
$\dots$					
0	0	0	0	$\dots$	$x < y$

and so yields the function

$$\|P_{\dot{-}}\|_{2,1}(x, y) = x \dot{-} y, \quad x, y \in \mathbb{N}_0. \quad (35)$$

## URM Computable Functions – Sign

*Sign function*  $\text{sgn} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  given by

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (36)$$

is URM computable.

Consider the trace

$R_0$	$R_1$	$R_2$	$R_3$	$R_4$	$\dots$	Registers
0	$x$	0	0	0	$\dots$	Init.
0	0	$x$	$x$	0	$\dots$	(A2; A3; S1)1
0	$x$	$x$	0	0	$\dots$	(A1; S3)3
0	$x$	$x - 1$	0	0	$\dots$	S2
0	$\text{sgn}(x)$	0	0	0	$\dots$	(S1; S2)2

## URM Computable Functions – Cosign

*Cosign function*  $\text{csg} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  given by

$$\text{csg}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (37)$$

is URM computable.

Note that

$$\text{csg}(x) = 1 \dot{-} \text{sgn}(x), \quad x \in \mathbb{N}_0. \quad (38)$$

Consider the trace

$R_0$	$R_1$	$R_2$	$R_3$	...	Registers
0	$x$	0	0	...	Init.
0	$\text{sgn}(x)$	0	0	...	pgm sgn
0	0	$\text{sgn}(x)$	0	...	(A2; S1)1
0	1	$\text{sgn}(x)$	0	...	A1
0	$1 \dot{-} \text{sgn}(x)$	0	0	...	pgm $\dot{-}$

## URM Computable Functions – Weirdos

- $P = (A1)1$  computes the monadic function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  given by (see (22))

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \uparrow & \text{otherwise.} \end{cases} \quad (39)$$

- $P = A1; (A1)1$  provides the monadic function  $f_{\uparrow} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  which is nowhere defined; i.e.,

$$f_{\uparrow}(x) = \uparrow, \quad x \in \mathbb{N}_0. \quad (40)$$

$f_{\uparrow}$  is called the *empty function*.

- $P = (S1)1$  calculates the monadic zero-function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ , i.e.,

$$f(x) = 0, \quad x \in \mathbb{N}_0. \quad (41)$$

## URM Computable Functions – Normalization

Let  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  be an URM computable function.

An URM program  $P$  with  $\|P\|_{k,1} = f$  is *normal* if for all inputs  $(x_1, \dots, x_k) \in \mathbb{N}_0^k$ ,

$$(|P| \circ \alpha_k)(x_1, \dots, x_k) = \begin{cases} (0, f(x_1, \dots, x_k), 0, 0, \dots) & \text{if } (x_1, \dots, x_k) \in \text{dom}(f), \\ \uparrow & \text{otherwise.} \end{cases} \quad (42)$$

A normal URM-program  $P$  computes a function  $f$  such that at the end of computation, all registers  $R_i$  with  $i \neq 1$  are cleared.

## URM Computable Functions – Add

URM program for the addition of two numbers

$$P_+ = (A1; S2)2$$

is already normal, since

$$\begin{aligned} (|P_+| \circ \alpha_2)(x, y) &= |P_+|(0, x, y, 0, 0, \dots) \\ &= (0, x + y, 0, 0, 0, \dots). \end{aligned}$$

## URM Computable Functions – Normalization

For each URM-computable function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  there is a normal URM-program  $P$  such that

$$\|P\|_{k,1} = f. \quad (43)$$

### Proof.

Let  $Q$  be an URM-program such that  $\|Q\|_{k,1} = f$ .

Suppose  $R_\sigma$  is the largest register that contains a non-zero value in the final state of computation. Then the corresponding normal URM-program is given by

$$P = Q; (S0)0; (S2)2; \dots; (S\sigma)\sigma. \quad (44)$$

Statement  $(Si)i$  sets the value of the  $i$ -th register to zero.  $\square$

## Skills

- Write and interpret an URM program
- Normalize an URM program

See the parser for URM programming.

## Part II

# Primitive Recursive Functions

## Primitive Recursive Functions – Motivation

- Subclass of computable total functions  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^m$  containing the arithmetic functions.
- Not contained are very fast growing total functions (superpowers).
- Inductive definition (base functions, composition, primitive recursion).
- LOOP programs as programming model for primitive recursive functions.

E.g., addition of two numbers by primitive recursion:

$$\begin{aligned}f_+(x, 0) &= x \\f_+(x, y + 1) &= (\nu \circ \pi_3^{(3)})(x, y, f_+(x, y)).\end{aligned}$$

## Primitive Recursive Functions

- Definition of primitive recursive functions
- Peano structures
- Closure properties
- Primitive recursive sets
- LOOP programs

Reading: Zimmermann, Chapter 2

## Primitive Recursive Functions

- Primitive recursive functions form an important building block on the way to an abstract formalization of computability.
- Most of the functions studied in arithmetics are primitive recursive such as addition and multiplication.
- From the programming point of view, the primitive recursive functions can be implemented via `for` loops (bounded search process).

## Primitive Recursive Functions – Contents

- Basic functions
- Composition
- Primitive Recursion
- URM computability

## Primitive Recursive Functions – Basic Functions

The class of primitive recursive functions  $\mathcal{P}$  is inductively defined.

*Basic functions:*

- *0-adic zero function* (constant 0)

$$c_0^{(0)} : \rightarrow \mathbb{N}_0 : \mapsto 0. \quad (45)$$

- *Monadic zero function*

$$c_0^{(1)} : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto 0. \quad (46)$$

- *Successor function*

$$\nu : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto x + 1. \quad (47)$$

- *Projection functions* for  $n \geq 1$  and  $1 \leq k \leq n$ ,

$$\pi_k^{(n)} : \mathbb{N}_0^n \rightarrow \mathbb{N}_0 : (x_1, \dots, x_k, \dots, x_n) \mapsto x_k. \quad (48)$$

## Primitive Recursive Functions – Composition I

If  $g : \mathbb{N}_0^n \rightarrow \mathbb{N}_0^m$  and  $h : \mathbb{N}_0^m \rightarrow \mathbb{N}_0^k$  lie in  $\mathcal{P}$ , the *composition*

$$f = (h \circ g) : \mathbb{N}_0^n \rightarrow \mathbb{N}_0^k \quad (49)$$

lies in  $\mathcal{P}$ , where for all  $\mathbf{x} \in \mathbb{N}_0^n$ ,

$$f(\mathbf{x}) = (h \circ g)(\mathbf{x}) = h(g(\mathbf{x})). \quad (50)$$

E.g., if  $g = \pi_1^{(n)}$  and  $h = \nu$ , then

$$f(\mathbf{x}) = h(g(\mathbf{x})) = h(x_1) = x_1 + 1, \quad \mathbf{x} \in \mathbb{N}_0^n.$$

## Primitive Recursive Functions – Composition II

If  $g_1, \dots, g_m : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  and  $h : \mathbb{N}_0^m \rightarrow \mathbb{N}_0^k$  lie in  $\mathcal{P}$ , the *composition*

$$f = h(g_1, \dots, g_m) : \mathbb{N}_0^n \rightarrow \mathbb{N}_0^k \quad (51)$$

lies in  $\mathcal{P}$ , where for all  $\mathbf{x} \in \mathbb{N}_0^n$ ,

$$f(\mathbf{x}) = h(g_1(\mathbf{x}), \dots, g_m(\mathbf{x})). \quad (52)$$

If  $m = 1$ , then (see Composition I)

$$(h(g_1))(\mathbf{x}) = h(g_1(\mathbf{x})) = (h \circ g_1)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{N}_0^n. \quad (53)$$

E.g., if  $g_1 = \pi_1^{(n)}$ ,  $g_2 = \pi_2^{(n)}$  and  $h = +$ , then

$$f(\mathbf{x}) = h(g_1(\mathbf{x}), g_2(\mathbf{x})) = h(x_1, x_2) = x_1 + x_2, \quad \mathbf{x} \in \mathbb{N}_0^n, n \geq 2.$$

## Primitive Recursive Functions – Primitive Recursion

If  $g : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  and  $h : \mathbb{N}_0^{n+2} \rightarrow \mathbb{N}_0$  lie in  $\mathcal{P}$ , the *primitive recursion*

$$f = \text{pr}(g, h) : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0 \quad (54)$$

lies in  $\mathcal{P}$ , where for all  $\mathbf{x} \in \mathbb{N}_0^n$ ,  $y \in \mathbb{N}_0$ ,

$$f(\mathbf{x}, 0) = g(\mathbf{x}), \quad (55)$$

$$f(\mathbf{x}, y + 1) = h(\mathbf{x}, y, f(\mathbf{x}, y)). \quad (56)$$

E.g., if  $g = c_0^{(1)}$  and  $h = \nu \circ \pi_3^{(3)}$  ( $n = 1$ ), then

$$\begin{aligned} f(x, 0) &= g(x) = 0, \\ f(x, y + 1) &= h(x, y, f(x, y)) \\ &= (\nu \circ \pi_3^{(3)})(x, y, f(x, y)) \\ &= \nu(f(x, y)) = f(x, y) + 1. \end{aligned}$$

## Primitive Recursive Functions – Class $\mathcal{P}$

The class of *primitive recursive functions*  $\mathcal{P}$  contains the basic functions and is closed under composition and primitive recursion (Richard Dedekind, 1831-1916).

Well-definedness and uniqueness of function  $f = \text{pr}(g, h)$  defined by primitive recursion can be established via Peano structures.

## Primitive Recursive Functions – Add

Addition is primitive recursive.

The addition function

$$f_+ : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto x + y.$$

can be described by primitive recursion, i.e., for all  $x, y \in \mathbb{N}_0$ ,

$$\begin{aligned} f_+(x, 0) &= x \\ &= \pi_1^{(1)}(x), \\ f_+(x, y + 1) &= f_+(x, y) + 1 \\ &= \nu(f_+(x, y)) \\ &= (\nu \circ \pi_3^{(3)})(x, y, f_+(x, y)) \end{aligned}$$

and so  $f_+ = \text{pr}(g, h)$  with  $g = \pi_1^{(1)} = \text{id}_{\mathbb{N}_0}$  and  $h = \nu \circ \pi_3^{(3)}$ .

## Primitive Recursive Functions – Totality

Each primitive recursive function is total.

### Proof (Sketch)

- The basic functions are total.
- Let  $f = h(g_1, \dots, g_m)$ . By induction, the functions  $g_1, \dots, g_m$  and  $h$  are total. Then it follows that the composition  $f$  is total.
- Let  $f = \text{pr}(g, h)$ . By induction, the functions  $g$  and  $h$  are total. Then it follows that the function  $f$  is total. □

## Primitive Recursive Functions – URM Computability

Each primitive recursive function is URM computable.

### Proof (Sketch)

- The basic functions are URM computable.
- Let  $f = h(g_1, \dots, g_m)$ . By induction, there are normal URM programs  $P_h$  and  $P_{g_1}, \dots, P_{g_m}$  such that  $\|P_h\|_{m,k} = h$  and  $\|P_{g_i}\|_{n,1} = g_i$ ,  $1 \leq i \leq m$ .

Then there is a normal URM program  $P$  such that  $\|P\|_{n,k} = f$ .

- Let  $f = \text{pr}(g, h)$ . By induction, there are normal URM programs  $P_g$  and  $P_h$  such that  $\|P_g\|_{n,1} = g$  and  $\|P_h\|_{n+2,1} = h$ .

Then there is a normal URM program  $P$  such that  $\|P\|_{n+1,1} = f$ . □

## URM Programs for Basic Functions

- Monadic zero function  $c_0^{(1)} : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto 0$ ,

$$c_0^{(1)} = \|(S1)1\|_{1,1}.$$

- Successor function  $\nu : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto x + 1$ ,

$$\nu = \|A1\|_{1,1}.$$

- Projection functions

$$\pi_k^{(n)} : \mathbb{N}_0^n \rightarrow \mathbb{N}_0 : (x_1, \dots, x_k, \dots, x_n) \mapsto x_k,$$

$$\pi_1^{(n)} = \|(S2)2; \dots; (Sn)n\|_{n,1},$$

$$\pi_2^{(n)} = \|(S1)1; (A1; S2)2; (S3)3; \dots; (Sn)n\|_{n,1},$$

...

## Primitive Recursive Functions – Function Classes

- $\mathcal{T}_{\text{URM}}$  denotes the class of all *total* URM computable functions. Since each primitive recursive function is total and URM computable,

$$\mathcal{P} \subset \mathcal{T}_{\text{URM}} \quad (57)$$

with  $A \in \mathcal{T}_{\text{URM}} \setminus \mathcal{P}$  (Ackermann function, superpower).

- $\mathcal{F}_{\text{URM}}$  denotes the class of URM computable functions. Then

$$\mathcal{T}_{\text{URM}} \subset \mathcal{F}_{\text{URM}} \quad (58)$$

with  $f_{\uparrow} = \|\!|A1; (A1)1\!\|_{1,1} \in \mathcal{F}_{\text{URM}} \setminus \mathcal{T}_{\text{URM}}$  (nowhere defined function).

- Composition and primitive recursion can be defined for partial functions as well.

## Peano Structures – Contents

- Semi-Peano structures
- Morphisms
- Peano structures
- Fundamental Lemma
- Uniqueness of Peano structures
- Primitive recursion (addition)

## Peano Structures – Semi-Peano

A *semi-Peano structure* is a triple

$$\mathcal{A} = (A, \alpha, a),$$

where  $A$  is a non-empty set,  $\alpha : A \rightarrow A$  is an operation, and  $a \in A$  (distinguished element).

### Examples

- $(\mathbb{N}_0, \nu, x_0)$  is a semi-Peano structure, where  $\nu : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto x + 1$  (successor function) and  $x_0 \in \mathbb{N}_0$ .
- $(\Sigma^*, \circ_a, \epsilon)$  is a semi-Peano structure, where  $\Sigma$  is an alphabet,  $\circ_a : \Sigma^* \rightarrow \Sigma^* : w \mapsto wa$  is the concatenation of a word  $w$  with a fixed letter  $a \in \Sigma$ , and  $\epsilon \in \Sigma$  (empty word).

## Peano Structures – Morphisms

Let  $\mathcal{A} = (A, \alpha, a)$  and  $\mathcal{B} = (B, \beta, b)$  be semi-Peano structures.

A mapping  $\phi : A \rightarrow B$  is a *morphism* if

- $\phi$  assigns the distinguished elements

$$\phi(a) = b, \quad (59)$$

- $\phi$  commutes with the monadic operations,

$$\beta \circ \phi = \phi \circ \alpha, \quad (60)$$

i.e., the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ \downarrow \phi & & \downarrow \phi \\ B & \xrightarrow{\beta} & B \end{array}$$

## Peano Structures – Example

Consider the semi-Peano structures

$$(\mathbb{N}_0, \nu, 0) \quad \text{and} \quad (\Sigma^*, \circ_a, \epsilon), \quad a \in \Sigma,$$

where  $\epsilon$  denotes the empty word.

The mapping

$$\phi : \Sigma^* \rightarrow \mathbb{N}_0 : w \mapsto |w|$$

which assigns to each word  $w \in \Sigma^*$  its length  $|w|$  is a morphism, since

$$\phi(\epsilon) = |\epsilon| = 0$$

and for each  $w \in \Sigma^*$ ,

$$\phi(\circ_a(w)) = \phi(wa) = |wa| = |w| + |a| = |w| + 1 = \nu(\phi(w)).$$

Let  $\Sigma = \{a, b\}$ . Then  $\phi(\circ_a(baab)) = \phi(baaba) = |baaba| = |baab| + |a| = |baab| + 1 = \nu(\phi(baab))$ .

## Peano Structures

A *Peano structure* is a semi-Peano structure  $\mathcal{A} = (A, \alpha, a)$  with the following properties:

- $\alpha$  is injective,
- $a \notin \text{ran}(\alpha)$ , and
- $A$  fulfills the *induction axiom*:

*If  $T \subseteq A$  such that  $a \in T$  and  $\alpha(x) \in T$  whenever  $x \in T$ , then  $T = A$ .*

$(\mathbb{N}_0, \nu, 0)$  is a Peano structure, where  $\nu : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto x + 1$  (Guiseppe Peano, 1858-1932).

## Peano Structures

If  $\mathcal{A} = (A, \alpha, a)$  is a Peano structure, then

$$A = \{\alpha^n(a) \mid n \in \mathbb{N}_0\}. \quad (61)$$

### Proof.

Let  $T = \{\alpha^n(a) \mid n \in \mathbb{N}_0\}$ .

- We have  $a = \alpha^0(a) \in T$ .
- Let  $x \in T$ . Then  $x = \alpha^n(a)$  for some  $n \geq 0$ . Thus  $\alpha(x) = \alpha(\alpha^n(a)) = \alpha^{n+1}(a) \in T$ .

Hence by the induction axiom,  $T = A$ . □

$(\mathbb{N}_0, \nu, 0)$  is a Peano structure. Then

$$\mathbb{N}_0 = \{0, \nu(0), \nu^2(0), \nu^3(0), \dots\} = \{0, 1, 2, 3, \dots\},$$

where

$$1 := \nu(0), \quad 2 := \nu^2(0), \quad 3 := \nu^3(0), \dots$$

## Peano Structures

If  $\mathcal{A} = (A, \alpha, a)$  is a Peano structure, then for all  $m, n \in \mathbb{N}_0$ ,

$$m \neq n \implies \alpha^m(a) \neq \alpha^n(a). \quad (62)$$

### \*Proof

Define  $T$  as the set of all elements  $\alpha^m(a)$  such that  $\alpha^m(a) \neq \alpha^n(a)$  for all  $n \in \mathbb{N}_0$  with  $n > m$ .

- Suppose that  $\alpha^n(a) = \alpha^0(a) = a$  for some  $n > 0$ . Then  $a \in \text{ran}(\alpha)$  contradicting the definition. Thus  $a \in T$ .
- Let  $x \in T$ . Then  $x = \alpha^m(a)$  for some  $m \geq 0$ . Suppose that  $\alpha(x) = \alpha^{m+1}(a) \notin T$ . Then there is a number  $n > m$  such that  $\alpha^{m+1}(a) = \alpha^{n+1}(a)$ . But  $\alpha$  is injective and so  $x = \alpha^m(a) = \alpha^n(a)$  contradicting the hypothesis for  $x$ . Thus  $\alpha(x) \in T$ .

Hence by the induction axiom,  $T = A$ . □

## Peano Structures – Fundamental Lemma

If  $\mathcal{A} = (A, \alpha, a)$  is a Peano structure and  $\mathcal{B} = (B, \beta, b)$  is a semi-Peano structure, there is a unique morphism  $\phi : A \rightarrow B$ .

### \*Proof

We have  $A = \{\alpha^n(a) \mid n \in \mathbb{N}_0\}$ .

- **Existence:** Define  $\phi(\alpha^n(a)) = \beta^n(b)$  for all  $n \in \mathbb{N}_0$ . Then  $\phi(a) = \phi(\alpha^0(a)) = \beta^0(b) = b$ . Moreover, let  $x \in A$ . Then  $x = \alpha^m(a)$  for some  $m \in \mathbb{N}_0$  and so
 
$$(\phi \circ \alpha)(x) = \phi(\alpha^{m+1}(a)) = \beta^{m+1}(b) = \beta(\beta^m(b)) = \beta(\phi(\alpha^m(a))) = (\beta \circ \phi)(x).$$
 Thus  $\phi$  is a morphism.
- **Uniqueness:** Suppose there is another morphism  $\psi : A \rightarrow B$ . Define  $T = \{x \in A \mid \phi(x) = \psi(x)\}$ . Then  $\phi(a) = b = \psi(a)$  and so  $a \in T$ . Moreover, let  $x \in T$ . Then  $\phi(\alpha(x)) = (\phi \circ \alpha)(x) = (\beta \circ \phi)(x) = (\beta \circ \psi)(x) = (\psi \circ \alpha)(x) = \psi(\alpha(x))$  and so  $\alpha(x) \in T$ . Hence by the induction axiom,  $T = A$  and so  $\phi = \psi$ . □

## Peano Structures – Richard Dedekind (1831-1916)

Every Peano structure is isomorphic to  $(\mathbb{N}_0, \nu, 0)$ .

### Proof.

- $(\mathbb{N}_0, \nu, 0)$  is a Peano structure.
- Let  $\mathcal{A} = (A, \alpha, a)$  and  $\mathcal{B} = (B, \beta, b)$  be Peano structures. By the Fundamental Lemma, there are morphisms  $\phi : A \rightarrow B$  and  $\psi : B \rightarrow A$ . Since the composition of morphisms is also a morphism,  $\psi \circ \phi : A \rightarrow A$  and  $\phi \circ \psi : B \rightarrow B$  are morphisms.
- The identity mappings  $\text{id}_A : A \rightarrow A : x \mapsto x$  and  $\text{id}_B : B \rightarrow B : x \mapsto x$  are morphisms.
- By the uniqueness of morphisms,  $\psi \circ \phi = \text{id}_A$  and  $\phi \circ \psi = \text{id}_B$ .
- Hence,  $\phi$  and  $\psi$  are isomorphisms (bijective morphisms).



## Peano Structures – Addition

There is a unique total function  $f_+ : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  such that for all  $x, y \in \mathbb{N}_0$ ,

$$f_+(x, 0) = x, \quad (63)$$

$$f_+(x, y + 1) = f_+(x, y) + 1. \quad (64)$$

### Proof

- $(\mathbb{N}_0, \nu, 0)$  is a Peano structure.
- Consider the semi-Peano structure  $(\mathbb{N}_0, \nu, x)$ , where  $x \in \mathbb{N}_0$ .  
By the Fundamental Lemma, there is a *unique morphism*  $f_x : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that for all  $y \in \mathbb{N}_0$ ,

$$f_x(0) = x,$$

$$f_x(y + 1) = (f_x \circ \nu)(y) \stackrel{\text{morph}}{=} (\nu \circ f_x)(y) = f_x(y) + 1.$$

## Proof (Cont'd)

- For all  $x, y \in \mathbb{N}_0$ ,

$$\begin{aligned} f_x(0) &= x, \\ f_x(y+1) &= f_x(y) + 1. \end{aligned}$$

- Define the function  $f_+ : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  by setting for all  $x, y \in \mathbb{N}_0$ ,

$$f_+(x, y) = f_x(y).$$

Then for all  $x, y \in \mathbb{N}_0$ ,

$$\begin{aligned} f_+(x, 0) &= f_x(0) = x, \\ f_+(x, y+1) &= f_x(y+1) = f_x(y) + 1 = f_+(x, y) + 1. \end{aligned}$$

## Proof (Cont'd)

- For all  $x, y \in \mathbb{N}_0$ ,

$$f_+(x, y) = x + y.$$

Indeed, by induction, for all  $x, y \in \mathbb{N}_0$ ,

$$f_+(x, 0) = f_x(0) = x$$

and

$$f_+(x, y + 1) = f_x(y + 1) = f_x(y) + 1 = x + y + 1.$$



## Closure Properties – Contents

- Transformation of variables
- Parametrization
- Definition by cases
- Bounded sum and product
- Iteration
- Bounded minimalization

## Transformation of Variables

Given  $\phi = \begin{pmatrix} 1 & \dots & n \\ \phi(1) & \dots & \phi(n) \end{pmatrix} : [n] \rightarrow [m]$  with  $n, m \geq 1$  and total function  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$ .

*Transformation of variables* is the function

$$f^\phi : \mathbb{N}_0^m \rightarrow \mathbb{N}_0 : (x_1, \dots, x_m) \mapsto f(x_{\phi(1)}, \dots, x_{\phi(n)}). \quad (65)$$

Note that  $x_{\phi(1)}, \dots, x_{\phi(n)}$  is a list of variables from  $x_1, \dots, x_m$ .

### Example

Let  $\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 1 & 3 \end{pmatrix} : [5] \rightarrow [3]$ . Then

$$f^\phi(x_1, x_2, x_3) = f(x_2, x_3, x_1, x_1, x_3).$$

## Transformation of Variables

If  $f$  is primitive recursive, then  $f^\phi$  is primitive recursive.

**Proof.**

Transformation of variables can be described by composition,

$$f^\phi = f(\pi_{\phi(1)}^{(m)}, \dots, \pi_{\phi(n)}^{(m)}), \quad (66)$$

i.e., for all  $x_1, \dots, x_m \in \mathbb{N}_0$ ,

$$\begin{aligned} f(\pi_{\phi(1)}^{(m)}, \dots, \pi_{\phi(n)}^{(m)})(x_1, \dots, x_m) &= f(x_{\phi(1)}, \dots, x_{\phi(n)}) \\ &= f^\phi(x_1, \dots, x_m). \end{aligned}$$



## Transformation of Variables – Example

Two important special cases:

- Permutation of variables:

$$f^\phi : (x_1, x_2) \mapsto f(x_2, x_1), \quad (67)$$

where  $\phi = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} : [2] \rightarrow [2]$ .

- Identification of variables:

$$f^\phi : x_1 \mapsto f(x_1, x_1), \quad (68)$$

where  $\phi = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} : [2] \rightarrow [1]$ .

## Constant Functions

Let  $k \geq 0$ . The  $k$ -adic constant function with value  $i \in \mathbb{N}_0$  is given by

$$c_i^{(k)} : \mathbb{N}_0^k \rightarrow \mathbb{N}_0 : (x_1, \dots, x_k) \mapsto i. \quad (69)$$

$c_i^{(k)}$  is primitive recursive.

**Proof.**

- If  $k = 0$ ,  $c_i^{(0)} = \nu^i \circ c_0^{(0)}$ .
- If  $k > 0$ ,  $c_i^{(k)} = \nu^i \circ c_0^{(1)} \circ \pi_1^{(k)}$ , i.e., for all  $\mathbf{x} \in \mathbb{N}_0^k$ ,

$$\begin{aligned} c_i^{(k)}(\mathbf{x}) &= (\nu^i \circ c_0^{(1)} \circ \pi_1^{(k)})(\mathbf{x}) \\ &= (\nu^i \circ c_0^{(1)})(x_1) \\ &= \nu^i(0) = i. \end{aligned}$$



## Parametrization

Given total function  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$ , integer  $m$  with  $1 \leq m < n$  and  $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}_0^m$ .

*Parametrization* of  $f$  by  $\mathbf{a}$  is the function

$$f_{\mathbf{a}} : \mathbb{N}_0^{n-m} \rightarrow \mathbb{N}_0 \quad (70)$$

$$(x_1, \dots, x_{n-m}) \mapsto f(x_1, \dots, x_{n-m}, a_1, \dots, a_m).$$

## Example

If  $f : \mathbb{N}_0^5 \rightarrow \mathbb{N}_0$  and  $\mathbf{a} = (101, 2021)$ , then

$$f_{\mathbf{a}}(x_1, x_2, x_3) = f(x_1, x_2, x_3, 101, 2021).$$

## Parametrization

If  $f$  is primitive recursive, then  $f_{\mathbf{a}}$  is primitive recursive.

### Proof.

Parametrization can be described by composition,

$$f_{\mathbf{a}} = f(\pi_1^{(n-m)}, \dots, \pi_{n-m}^{(n-m)}, c_{a_1}^{(n-m)}, \dots, c_{a_m}^{(n-m)}), \quad (71)$$

i.e., for all  $x_1, \dots, x_{n-m} \in \mathbb{N}_0$ ,

$$\begin{aligned} & f(\pi_1^{(n-m)}, \dots, \pi_{n-m}^{(n-m)}, c_{a_1}^{(n-m)}, \dots, c_{a_m}^{(n-m)})(x_1, \dots, x_{n-m}) \\ &= f(\pi_1^{(n-m)}, \dots, \pi_{n-m}^{(n-m)}, a_1, \dots, a_m)(x_1, \dots, x_{n-m}) \\ &= f(x_1, \dots, x_{n-m}, a_1, \dots, a_m) \\ &= f_{\mathbf{a}}(x_1, \dots, x_{n-m}). \end{aligned}$$



## Definition by Cases

Define a total function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  by *case distinction*,

$$f : \mathbf{x} \mapsto \begin{cases} g_1(\mathbf{x}) & \text{if } \mathbf{x} \in A, \\ g_2(\mathbf{x}) & \text{if } \mathbf{x} \notin A, \end{cases} \quad (72)$$

where  $A \subseteq \mathbb{N}_0^k$  and  $g_1, g_2 : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  are total functions.

## Primitive Sets (Interlude)

Let  $A \subseteq \mathbb{N}_0^k$ .

- $A$  has the *characteristic function*

$$\chi_A : \mathbb{N}_0^k \rightarrow \mathbb{N}_0 : \mathbf{x} \mapsto \begin{cases} 1 & \text{if } \mathbf{x} \in A, \\ 0 & \text{otherwise,} \end{cases} \quad (73)$$

i.e., for all  $\mathbf{x} \in \mathbb{N}_0^k$ ,

$$\mathbf{x} \in A \iff \chi_A(\mathbf{x}) = 1. \quad (74)$$

- $A$  is *primitive* if  $\chi_A$  is primitive recursive.
- If  $A \subseteq \mathbb{N}_0^k$  is primitive, then  $\bar{A} = \mathbb{N}_0^k \setminus A$  is primitive, since

$$\chi_{\bar{A}} = 1 \dot{-} \chi_A \quad (75)$$

is primitive recursive.

- E.g.,  $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$  is primitive, since  $\chi_{\mathbb{N}} = \text{sgn}$  (sign function) is primitive recursive.

## Definition by Cases

If  $A \subseteq \mathbb{N}_0^k$  is primitive and  $g_1, g_2 : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  are primitive recursive, then  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  is primitive recursive.

### Proof.

Define

$$f = (\chi_A \cdot g_1) + (\chi_{\bar{A}} \cdot g_2), \quad (76)$$

i.e., for all  $\mathbf{x} \in \mathbb{N}_0^k$ ,

$$f(\mathbf{x}) = \chi_A(\mathbf{x}) \cdot g_1(\mathbf{x}) + \chi_{\bar{A}}(\mathbf{x}) \cdot g_2(\mathbf{x}).$$

If  $\mathbf{x} \in A$ , then  $\chi_A(\mathbf{x}) = 1$  and so  $f(\mathbf{x}) = g_1(\mathbf{x})$ .

Otherwise,  $\chi_{\bar{A}}(\mathbf{x}) = 1$  and so  $f(\mathbf{x}) = g_2(\mathbf{x})$ .

Since  $f$  is a composition of primitive recursive functions,  $f$  is primitive recursive. □

## Definition by Cases – Example

Consider the function

$$f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto \begin{cases} 2x & \text{if } x > 0, \\ 3 & \text{otherwise.} \end{cases}$$

Then

- $A = \mathbb{N} = \mathbb{N}_0 \setminus \{0\}$  with  $\chi_A = \text{sgn}$ ,
- $\bar{A} = \{0\}$  with  $\chi_{\bar{A}} = 1 \div \text{sgn} = \text{csg}$ ,
- $g_1 : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto 2x$ ,
- $g_2 : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto 3$ .

Thus for each  $x \in \mathbb{N}_0$ ,

$$\begin{aligned} f(x) &= \chi_A(x) \cdot g_1(x) + \chi_{\bar{A}}(x) \cdot g_2(x) \\ &= \text{sgn}(x) \cdot 2x + \text{csg}(x) \cdot 3. \end{aligned}$$

## Bounded Sum and Product

Let  $f : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0$  be a total function.

- *Bounded sum of  $f$*

$$f_{\Sigma} : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0 : (\mathbf{x}, y) \mapsto \sum_{i=0}^y f(\mathbf{x}, i). \quad (77)$$

- *Bounded product of  $f$*

$$f_{\Pi} : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0 : (\mathbf{x}, y) \mapsto \prod_{i=0}^y f(\mathbf{x}, i). \quad (78)$$

## Bounded Sum and Product

If  $f$  is primitive recursive, then  $f_\Sigma$  and  $f_\Pi$  are primitive recursive.

### Proof.

The function  $f_\Sigma$  (case  $n = 1$ ) satisfies

$$\begin{aligned} f_\Sigma(x, 0) &= f(x, 0), \\ f_\Sigma(x, y + 1) &= f_\Sigma(x, y) + f(x, y + 1). \end{aligned}$$

This is a primitive recursive scheme  $f_\Sigma = \text{pr}(g, h)$ , where

$$\begin{aligned} g(x) &= f(x, 0), \\ h(x, y, f_\Sigma(x, y)) &= +(\pi_3^{(3)}, f(\pi_1^{(3)}, \nu \circ \pi_2^{(3)}))(x, y, f_\Sigma(x, y)). \end{aligned}$$

Similar for function  $f_\Pi$ . □

## Bounded Sum – Example

- Identity function

$$\text{id}_{\mathbb{N}_0} = \pi_1^{(1)} : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto x$$

is primitive recursive.

- Bounded sum  $f_\Sigma$  (case  $n = 0$ ),

$$f_\Sigma(0) = 0,$$

$$f_\Sigma(y) = \sum_{i=0}^y f(i) = \sum_{i=0}^y i = \frac{y(y+1)}{2} = \binom{y+1}{2}, \quad y \geq 1,$$

(binomial coefficient) is primitive recursive.

## Bounded Product – Example

- The function

$$f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto \begin{cases} 1 & \text{if } x = 0, \\ x & \text{otherwise,} \end{cases}$$

is primitive recursive, since it is defined by cases,

$$f(x) = \text{csg}(x) \cdot 1 + \text{sgn}(x) \cdot x, \quad x \in \mathbb{N}_0.$$

- Bounded product  $f_{\Pi}$  (case  $n = 0$ ),

$$f_{\Pi}(y) = \prod_{i=0}^y f(i) = \prod_{i=1}^y i = y!, \quad y \in \mathbb{N}_0,$$

(factorial function) is primitive recursive.

## Iteration

Given total function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ .

- *Powers* of function  $f$ ,

$$f^0 = \text{id}_{\mathbb{N}_0} \quad \text{and} \quad f^{n+1} = f \circ f^n, \quad n \geq 0. \quad (79)$$

In particular,  $f^1 = f \circ f^0 = f \circ \text{id}_{\mathbb{N}_0} = f$ .

- *Iteration* of function  $f$ ,

$$f_{\text{it}} : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto f^y(x). \quad (80)$$

Iteration can also be defined for multivariate functions.

## Iteration

If  $f$  is primitive recursive, then  $f_{\text{it}}$  is primitive recursive.

### Proof.

Primitive recursive scheme for  $f_{\text{it}}$ ,

$$\begin{aligned} f_{\text{it}}(x, 0) &= f^0(x) = x, \\ f_{\text{it}}(x, y + 1) &= f^{y+1}(x) = (f \circ f^y)(x) = f(f_{\text{it}}(x, y)) \\ &= (f \circ \pi_3^{(3)})(x, y, f_{\text{it}}(x, y)), \end{aligned}$$

where  $f_{\text{it}} = \text{pr}(g, h)$  with  $g = \pi_1^{(1)} = \text{id}_{\mathbb{N}_0}$  and  $h = f \circ \pi_3^{(3)}$ . □

## Iteration – Example

- Consider the primitive recursive function

$$f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto 2x.$$

- Iteration of function  $f$ ,

$$f_{\text{it}} : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto f^y(x) = 2^y \cdot x,$$

since by induction

$$f_{\text{it}}(x, 0) = f^0(x) = x = 2^0 x$$

and

$$\begin{aligned} f_{\text{it}}(x, y + 1) &= f^{y+1}(x) = (f \circ f^y)(x) \\ &= f(f^y(x)) = f(2^y x) = 2^{y+1} x. \end{aligned}$$

## Bounded Minimalization

Given total function  $f : \mathbb{N}_0^{k+1} \rightarrow \mathbb{N}_0$ .

*Bounded minimalization of  $f$ ,*

$$\bar{\mu}f : \mathbb{N}_0^{k+1} \rightarrow \mathbb{N}_0 : (\mathbf{x}, y) \mapsto \bar{\mu}f(\mathbf{x}, y), \quad (81)$$

where for each  $(\mathbf{x}, y) \in \mathbb{N}_0^{k+1}$ ,

$$\begin{aligned} \bar{\mu}f(\mathbf{x}, y) &= \begin{cases} j & \text{if } j = \min\{i \mid 0 \leq i \leq y \wedge f(\mathbf{x}, i) = 0\} \text{ exists,} \\ y + 1 & \text{otherwise} \end{cases} \\ &= \text{smallest index } j \text{ with } 0 \leq j \leq y \text{ such that} \\ &\quad f(\mathbf{x}, j) = 0; \text{ otherwise } y + 1. \end{aligned}$$

Bounded search process.

## Bounded Minimalization – Bounded Search Process

Implementation of  $\bar{\mu}f$  by for loop:

**Require:**  $x \in \mathbb{N}_0^k, y \in \mathbb{N}_0$

```

for  $i \leftarrow 0, y$  do
  if  $f(x, i) = 0$  then
    return  $i$ 
  end if
end for
return  $y + 1$ 

```

Kleene notation:

$$\mu(i \leq y)[f(x, i) = 0] = \bar{\mu}f(x, y). \quad (82)$$

"Smallest  $i$  with  $i \leq y$  such that  $f(x, i) = 0$ ."

## Bounded Minimalization

If  $f$  is primitive recursive, then  $\bar{\mu}f$  is primitive recursive.

### \*Proof

By definition,

$$\bar{\mu}f(\mathbf{x}, 0) = \text{sgn}(f(\mathbf{x}, 0))$$

and

$$\bar{\mu}f(\mathbf{x}, y + 1) = \begin{cases} \bar{\mu}f(\mathbf{x}, y) & \text{if } \bar{\mu}f(\mathbf{x}, y) \leq y \text{ or } f(\mathbf{x}, y + 1) = 0, \\ y + 2 & \text{otherwise.} \end{cases}$$

This defines a primitive recursive scheme for  $\bar{\mu}f$ . □

### First Case

Two cases for  $\bar{\mu}f(\mathbf{x}, y + 1) = j \leq y + 1$ :

$j \leq y$ :  $f(\mathbf{x}, j) = 0$ , i.e.,  $\bar{\mu}f(\mathbf{x}, y) = j \leq y$ .

$j = y + 1$ :  $f(\mathbf{x}, y + 1) = 0$ , i.e.,  $\bar{\mu}(f, y) = y + 1$ .

## Bounded Minimalization – Example

Consider the asymmetric difference (13),

$$f(x, y) = x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y, \\ 0 & \text{otherwise,} \end{cases} \quad x, y \in \mathbb{N}_0.$$

Then

$$\bar{\mu}f(x, y) = \begin{cases} x & \text{if } x \leq y, \\ y + 1 & \text{otherwise,} \end{cases}$$

since the search process seeking the smallest index  $j$  such that

$$f(x, j) = x \dot{-} j = 0$$

is limited to the range  $0 \leq j \leq y$ .

### Example

$$\bar{\mu}f(3, 6) = 3 \text{ and } \bar{\mu}f(6, 3) = 4.$$

## Bounded Minimalization – \*Integral Division

For integers  $m, n$  with  $m \geq n \geq 1$ , there are unique integers  $q, r \geq 0$  such that

$$m = q \cdot n + r \quad \wedge \quad 0 \leq r \leq n - 1, \quad (83)$$

with *quotient*

$$q = \div(m, n) \quad (84)$$

and *remainder*

$$r = m \bmod n. \quad (85)$$

E.g.,  $6 = 3 \cdot 2 + 0$  and  $7 = 3 \cdot 2 + 1$ .

## Bounded Minimalization – \*Quotient (Tricky)

Let  $m \geq n \geq 1$ . Then

$$\begin{aligned} \div(m, n) &= \chi_D(n, m) \cdot \mu(i \leq m)[m \div i \cdot n = 0] && (86) \\ &+ (\text{csg} \circ \chi_D)(n, m) \cdot \{\mu(i \leq m)[m \div i \cdot n = 0] \div 1\}, \end{aligned}$$

where  $\chi_D$  is the divisibility relation (96).

Two cases:

- If  $n$  divides  $m$ , then  $\chi_D(n, m) = 1$  and  $\mu(i \leq m)[m \div i \cdot n = 0]$  provides the quotient  $\div(m, n)$ .  
E.g., if  $m = 6$  and  $n = 2$ , then  $\mu(i \leq m)[m \div i \cdot n = 0]$  yields  $i = 3$  and so  $\div(6, 2) = 3$ .
- Otherwise,  $\chi_D(n, m) = 0$ , i.e.,  $\text{csg}(\chi_D(n, m)) = 1$ , and  $\mu(i \leq m)[m \div i \cdot n = 0] \div 1$  provides the quotient.  
E.g., if  $m = 7$  and  $n = 2$ , then  $\mu(i \leq m)[m \div i \cdot n = 0]$  yields  $i = 4$  and so  $\div(7, 2) = 4 \div 1 = 3$ .

## Primitive Recursive Sets – Contents

- Characteristic function
- Primitive recursive sets
- Set closure
- Divisibility relation
- Set of primes

## Primitive Sets

Let  $A \subseteq \mathbb{N}_0^k$ .

- $A$  has the *characteristic function*

$$\chi_A : \mathbb{N}_0^k \rightarrow \mathbb{N}_0 : \mathbf{x} \mapsto \begin{cases} 1 & \text{if } \mathbf{x} \in A, \\ 0 & \text{otherwise,} \end{cases} \quad (87)$$

i.e., for all  $\mathbf{x} \in \mathbb{N}_0^k$ ,

$$\mathbf{x} \in A \iff \chi_A(\mathbf{x}) = 1. \quad (88)$$

- $A$  is *primitive* if  $\chi_A$  is primitive recursive.

## Primitive Recursive Sets – Example

## ■ Equality relation

$$R_{=} = \{(x, y) \in \mathbb{N}_0^2 \mid x = y\} \quad (89)$$

is primitive, since  $\chi_{R_{=}}(x, y) = \text{csg}(|x - y|)$  with  $|x - y| = (x \dot{-} y) + (y \dot{-} x)$  is primitive recursive.

## ■ Inequality relation

$$R_{\neq} = \{(x, y) \in \mathbb{N}_0^2 \mid x \neq y\} \quad (90)$$

is primitive, since  $\chi_{R_{\neq}}(x, y) = 1 \dot{-} \chi_{R_{=}}(x, y)$  is primitive recursive.

## ■ Smaller relation

$$R_{<} = \{(x, y) \in \mathbb{N}_0^2 \mid x < y\} \quad (91)$$

is primitive, since  $\chi_{R_{<}}(x, y) = \text{sgn}(y \dot{-} x)$  is primitive recursive.

## Primitive Recursive Sets - Set Closure

If  $A, B \subseteq \mathbb{N}_0^k$  are primitive, then  $A \cup B$ ,  $A \cap B$ , and  $\bar{A} = \mathbb{N}_0^k \setminus A$  are primitive.

### Proof.

We have

$$\chi_{A \cup B} = \text{sgn} \circ (\chi_A + \chi_B), \quad \chi_{A \cap B} = \chi_A \cdot \chi_B, \quad \chi_{\bar{A}} = \text{csg} \circ \chi_A. \quad (92)$$

i.e., for each  $\mathbf{x} \in \mathbb{N}_0^k$ ,

$$\chi_{A \cup B}(\mathbf{x}) = \text{sgn}(\chi_A(\mathbf{x}) + \chi_B(\mathbf{x})), \quad (93)$$

$$\chi_{A \cap B}(\mathbf{x}) = \chi_A(\mathbf{x}) \cdot \chi_B(\mathbf{x}), \quad (94)$$

$$\chi_{\bar{A}}(\mathbf{x}) = \text{csg}(\chi_A(\mathbf{x})). \quad (95)$$

The functions on the right-hand sides are compositions of primitive recursive and so are also primitive recursive.  $\square$

## Primitive Recursive Sets – Divisibility

The divisibility relation

$$D = \{(x, y) \in \mathbb{N}_0^2 \mid x \text{ divides } y\} \quad (96)$$

is primitive.

### Proof

- Divisibility relation ( $x$  divides  $y$ ),

$$x \mid y \iff \exists z[0 \leq z \leq y \wedge x \cdot z = y]. \quad (97)$$

- Characteristic function of  $D$ ,

$$\chi_D(x, y) = \text{sgn}[\chi_{=(x \cdot 0, y)} + \chi_{=(x \cdot 1, y)} + \dots + \chi_{=(x \cdot y, y)}], \quad (98)$$

which is primitive recursive as a composition of primitive recursive functions. Hence,  $D$  is primitive.  $\square$

## Primitive Recursive Sets – Factorization

Consider the sequence of increasing primes

$$(p_0, p_1, p_2, p_3, p_4, \dots) = (2, 3, 5, 7, 11, \dots).$$

By the Fundamental Theorem of Arithmetic, each natural number  $x \geq 1$  can be uniquely written as a product of prime powers, i.e.,

$$x = \prod_{i=0}^{r-1} p_i^{e_i}, \quad (99)$$

where  $e_0, \dots, e_{r-1} \in \mathbb{N}_0$ .

Write  $(x)_i = e_i$  for each  $i \in \mathbb{N}_0$ , and put  $(0)_i = 0$  for all  $i \in \mathbb{N}_0$ .

E.g.,  $360 = 2^3 \cdot 3^2 \cdot 5^1$  and so  $(360)_0 = 3$ ,  $(360)_1 = 2$ ,  $(360)_2 = 1$  and  $(360)_i = 0$  for all  $i \geq 3$ .

## Primitive Recursive Sets – Set of Primes

The set of primes  $P$  is primitive.

### Proof.

By definition,

*$x \in \mathbb{N}_0$  is prime iff  $x \geq 2$  and  $i$  divides  $x$  implies  $i = 1$  or  $i = x$  for all  $i \leq x$ .*

Thus the characteristic function of  $P$  is

$$\chi_P(0) = 0, \quad \chi_P(1) = 0, \quad \chi_P(2) = 1,$$

and for all  $x \geq 3$ ,

$$\chi_P(x) = \text{csg}[\chi_D(2, x) + \chi_D(3, x) + \dots + \chi_D(x-1, x)],$$

which is a composition of primitive recursive functions. Thus  $\chi_P$  is primitive recursive by case distinction. Hence,  $P$  is primitive.  $\square$

## Primitive Recursive Sets

The function  $p : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : i \mapsto p_i$  is primitive recursive.

### Proof

- Consider the function

$$\begin{aligned} g(z, y) &= |\chi_{R_{<}}(z, y) \cdot \chi_P(y) - 1| & (100) \\ &= \begin{cases} 0 & \text{if } z < y \text{ and } y \text{ prime,} \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

$g$  is a composition of primitive recursive functions and so primitive recursive.

- Take the function

$$h(z) = \bar{\mu}g(z, z! + 1) = \mu(y \leq z! + 1)[g(z, y) = 0]. \quad (101)$$

$h$  is the bounded minimalization of a primitive recursive function and so primitive recursive.

## Proof (Cont'd)

- We have  $h(z) = \mu(y \leq z! + 1)[g(z, y) = 0]$ .
- If  $p$  is prime, then  $h(p)$  is the smallest prime  $q$  such that  $q > p$  and  $q \leq p! + 1$ .
- Theorem of Euclid:

*The  $i + 1$ -th prime  $p_{i+1}$  is bounded by the  $i$ -th prime  $p_i$  in a way that*

$$p_{i+1} \leq p_i! + 1, \quad i \geq 0.$$

- Thus  $h(p_i) = p_{i+1}$ .
- Hence, the sequence of prime numbers is given by the primitive recursive scheme

$$p_0 = 2 \quad \text{and} \quad p_{i+1} = h(p_i), \quad i \geq 0. \quad (102)$$



## Primitive Recursive Sets

The function  $\tilde{p} : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, i) \mapsto (x)_i$  is primitive recursive.

### Proof.

We have

$$\begin{aligned} (x)_i &= \mu(y \leq x)[p_i^{y+1} \not\mid x] & (103) \\ &= \mu(y \leq x)[\chi_D(p_i^{y+1}, x) = 0], \end{aligned}$$

i.e.,  $(x)_i$  is the smallest value  $y$  such that  $p_i^{y+1} \not\mid x$ , i.e.,  $p_i^y \mid x$  and  $p_i^{y+1} \not\mid x$ .  $\square$

E.g.,  $360 = 2^3 \cdot 3^2 \cdot 5^1$  and so  $(360)_0 = 3$ ,  $(360)_1 = 2$ ,  $(360)_2 = 1$  and  $(360)_i = 0$  for all  $i \geq 3$ .

## LOOP Programs – Contents

- New programming primitives
- Inductive definition
- LOOP hierarchy
- Decrementation
- Equality with  $\mathcal{P}$

## LOOP Programs

- LOOP programming model provides an implementation of the primitive recursive functions.
- LOOP programs have restricted loop variables.
- Loops are of the form  $(P; S\sigma)\sigma$ , where the loop variable  $\sigma$  does not appear in the program  $P$  (explicit control over the loop).

## LOOP Programs – New Primitives

- For each URM program  $P$  and each  $\sigma \in Z$ , the URM program  $(P; S\sigma)\sigma$  is denoted by

$$[P]\sigma. \quad (104)$$

E.g.,  $[A1]2 = (A1; S2)2$  increments  $R_1$  exactly  $R_2$  times.

- In  $[P]\sigma$ , program  $P$  is executed exactly  $R_\sigma$  times if  $\sigma$  does not appear in  $P$ .
- URM program  $(S\sigma)\sigma$  is denoted by

$$Z\sigma. \quad (105)$$

Zero setting of register  $R_\sigma$ .

LOOP Programs – Class  $\mathcal{P}_{\text{LOOP}(0)}$ 

Definition of class of LOOP-0 programs:

- Primitives:

$$A\sigma, Z\sigma \in \mathcal{P}_{\text{LOOP}(0)}, \quad \sigma \in Z. \quad (106)$$

- Copy:

$$\bar{C}(\sigma, \tau) = Z\tau; Z0; C(\sigma; \tau) \in \mathcal{P}_{\text{LOOP}(0)}, \quad (107)$$

where  $\sigma, \tau \in Z$ ,  $\sigma \neq \tau$ ,  $\sigma \neq 0 \neq \tau$ .

- Composition: If  $P, Q \in \mathcal{P}_{\text{LOOP}(0)}$ , then

$$P; Q \in \mathcal{P}_{\text{LOOP}(0)}. \quad (108)$$

For each  $\omega \in \Omega$ ,  $\bar{\omega} = \bar{C}(\sigma, \tau)(\omega)$  is given by

$$\bar{\omega}_n = \begin{cases} 0 & \text{if } n = 0, \\ \omega_\sigma & \text{if } n = \sigma \text{ or } n = \tau, \\ \omega_n & \text{otherwise,} \end{cases} \quad (109)$$

## LOOP Programs – Class $\mathcal{P}_{\text{LOOP}(n+1)}$

Suppose the class of LOOP- $n$  programs  $\mathcal{P}_{\text{LOOP}(n)}$  has already been defined.

Definition of class of LOOP- $n + 1$  programs:

- Inclusion: Each  $P \in \mathcal{P}_{\text{LOOP}(n)}$  belongs to  $\mathcal{P}_{\text{LOOP}(n+1)}$ .
- Composition: If  $P, Q \in \mathcal{P}_{\text{LOOP}(n+1)}$ , then

$$P; Q \in \mathcal{P}_{\text{LOOP}(n+1)}. \quad (110)$$

- Iteration (restricted):

$$[P]\sigma \in \mathcal{P}_{\text{LOOP}(n+1)} \quad (111)$$

where  $\sigma \in Z$  does not appear in  $P \in \mathcal{P}_{\text{LOOP}(n)}$ , e.g.,  
 $[A1]2 = (A1; S2)2 \in \mathcal{P}_{\text{LOOP}(1)}$ .

## LOOP Programs – Example

Addition of two numbers is given by the URM program

$$(A1; S2)2$$

which by (104) corresponds to the LOOP-1 program

$$[A1]2.$$

## LOOP Programs – Example

Decrementation  $S\sigma$  is not a LOOP program, but has an implementation as LOOP-1 program:

$$P_{\sigma-1} = \bar{C}(1; 3); [\bar{C}(2; 1); A2]3. \quad (112)$$

Input  $x = 0$ :

0	1	2	3	4	...	registers
0	0	0	0	0	...	init
0	0	0	0	0	...	$\bar{C}(1; 3)$
0	0	0	0	0	...	end

Input  $x \geq 1$ :

0	1	2	3	4	...	registers
0	$x$	0	0	0	...	init
0	$x$	0	$x$	0	...	$\bar{C}(1; 3)$
0	0	0	$x$	0	...	$\bar{C}(2; 1)$
0	0	1	$x$	0	...	$A2$
0	0	1	$x - 1$	0	...	$S3$
...						
0	1	2	$x - 2$	0	...	
...						
0	$x - 1$	$x$	0	0	...	end
0	$x - 1$	0	0	0	...	$Z2$

## LOOP Programs – LOOP Hierarchy

- LOOP- $n$  programs,  $n \in \mathbb{N}_0$ , form a strictly increasing sequence

$$\mathcal{P}_{\text{LOOP}(0)} \subset \mathcal{P}_{\text{LOOP}(1)} \subset \mathcal{P}_{\text{LOOP}(2)} \subset \dots, \quad (113)$$

since each LOOP- $n$  program is a LOOP- $n + 1$  program and there are LOOP- $n + 1$  programs which are not LOOP- $n$ .

- $\mathcal{P}_{\text{LOOP}}$  is the class of all *LOOP programs*,

$$\mathcal{P}_{\text{LOOP}} = \bigcup_{n \geq 0} \mathcal{P}_{\text{LOOP}(n)}. \quad (114)$$

## LOOP Programs – Computability

- Function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  is *LOOP- $n$  computable* if there is a LOOP- $n$  program  $P$  such that

$$\|P\|_{k,1} = f. \quad (115)$$

- $\mathcal{F}_{\text{LOOP}(n)}$  denotes the class of all LOOP- $n$  computable functions.
- $\mathcal{F}_{\text{LOOP}}$  denotes the class of all LOOP computable functions,

$$\mathcal{F}_{\text{LOOP}} = \bigcup_{n \geq 0} \mathcal{F}_{\text{LOOP}(n)}. \quad (116)$$

- If  $P$  is a LOOP- $n$  program and  $P'$  the corresponding normal program, then  $P'$  is a LOOP- $n$  program (see (44) and make use of  $Z\sigma$ ; see (112)).

## LOOP Programs – LOOP Hierarchy

- LOOP- $n$  programs,  $n \in \mathbb{N}_0$ , form a strictly increasing sequence

$$\mathcal{P}_{\text{LOOP}(0)} \subset \mathcal{P}_{\text{LOOP}(1)} \subset \mathcal{P}_{\text{LOOP}(2)} \subset \dots, \quad (117)$$

- Thus the LOOP- $n$  computable functions,  $n \in \mathbb{N}_0$ , form an increasing sequence

$$\mathcal{F}_{\text{LOOP}(0)} \subseteq \mathcal{F}_{\text{LOOP}(1)} \subseteq \mathcal{F}_{\text{LOOP}(2)} \subseteq \dots \quad (118)$$

We will show that this sequence is strict (small Ackermann functions).

## LOOP Programs – Main Result

The class of LOOP computable functions  $\mathcal{F}_{\text{LOOP}}$  is equal to the class of primitive recursive functions  $\mathcal{P}$ .

### Proof (Sketch)

- $\mathcal{P} \subseteq \mathcal{F}_{\text{LOOP}}$ :

Show that the basic functions are LOOP computable, and the class of LOOP computable functions is closed under composition and primitive recursion.

- $\mathcal{F}_{\text{LOOP}} \subseteq \mathcal{P}$ :

Show that each LOOP computable function is primitive recursive; use induction on the inductive definition of LOOP programs. □

## Skills

- Specify and interpret a primitive recursive function
- Define a function via Peano structure
- Establish a primitive set
- Write and interpret a LOOP program

# Part III

## Partial Recursive Functions

## Partial Recursive Functions – Motivation

- Full class of computable functions capturing the notion of computability.
- Contains the class of primitive recursive functions.
- GOTO programming as programming model for partial recursive functions.
- Church's thesis as working hypothesis for the programmer.

E.g., addition of two numbers by GOTO-2 program:

```
0 :  $T(p_2)$ 
1 : if  $x_1 = 0$  goto 9
2 :  $x_1 \leftarrow x_1 - 1$ 
3 :  $M(p_1)$ 
4 :  $T(p_2)$ 
5 : if  $x_1 = 0$  goto 8
6 :  $x_1 \leftarrow x_1 - 1$ 
7 :  $D(p_2)$ 
8 : goto 0
9 :
```

## Contents

Partial Recursive  
Functions

GOTO Programs

GOTO  
Computable  
Functions

Church's Thesis

GOTO-2  
Programs

Skills

## Partial Recursive Functions – Contents

- General definition
- GOTO programs and GOTO computable functions
- GOTO and URM programming
- GOTO-2 programming
- Church's thesis

Reading: Zimmermann, Chapter 3

## Partial Recursive Functions – Inside

- Partial recursive functions form a class of partial functions that capture the notion of computability.
- Class of partial recursive functions equals the class of URM or GOTO computable functions.
- GOTO programs will be used later for Gödelization and reduction processes.
- GOTO programming can be restricted to using two registers only.
- Computability theory centers around Church's thesis, i.e., the partial recursive functions formalize the notion of computability.

## Partial Recursive Functions – Contents

- Minimalization
- Class of partial recursive functions
- Class of recursive functions
- URM computability

## Partial Recursive Functions – Minimalization

Given partial function  $f : \mathbb{N}^{n+1} \rightarrow \mathbb{N}_0$ .

(Unbounded) minimalization of  $f$ :

$$\mu f : \mathbb{N}^n \rightarrow \mathbb{N}_0 \quad (119)$$

with

$$\mu f(\mathbf{x}) = \begin{cases} y & \text{if } f(\mathbf{x}, y) = 0, f(\mathbf{x}, i) \neq 0 \text{ defined for } 0 \leq i < y, \\ \uparrow & \text{otherwise.} \end{cases} \quad (120)$$

$\mu f$  provides an unbounded search process, where  $\mu f(\mathbf{x})$  is the smallest  $y \geq 0$  such that  $f(\mathbf{x}, y) = 0$ .

### Bounded minimalization

$\bar{\mu} f$  yields a bounded search process, where  $\bar{\mu} f(\mathbf{x}, z)$  is the smallest  $y \geq 0$  with  $y \leq z$  such that  $f(\mathbf{x}, y) = 0$ .

## Partial Recursive Functions – Minimalization

Implementation of  $\mu f$  by while loop:**Require:**  $\mathbf{x} \in \mathbb{N}_0^n$  $y \leftarrow -1$ **repeat** $y \leftarrow y + 1$  $r \leftarrow f(\mathbf{x}, y)$ **until**  $r = 0$ **return**  $y$ 

Kleene notation:

$$\mu y[f(\mathbf{x}, y) = 0] = \mu f(\mathbf{x}). \quad (121)$$

"Smallest  $y$  such that  $f(\mathbf{x}, y) = 0$ ." Compare with bounded minimalization (82).

## Partial Recursive Functions – Examples

- $\mu f$  may be partial even if  $f$  is total.

E.g., total function

$$f(x, y) = (x + y) \dot{\div} 3$$

yields partial function  $\mu f(x)$  with  $\text{dom}(\mu f) = \{0, 1, 2, 3\}$  and  $\mu f(0) = \mu f(1) = \mu f(2) = \mu f(3) = 0$ .

- $\mu f$  may be total even if  $f$  is partial.

E.g., partial function (see asymmetric difference  $\dot{\div}$ )

$$f(x, y) = \begin{cases} x - y & \text{if } y \leq x, \\ \uparrow & \text{otherwise,} \end{cases}$$

provides total function  $\mu f(x) = x$  with  $\text{dom}(\mu f) = \mathbb{N}_0$ .

## Partial Recursive Functions – Class $\mathcal{R}$

Class  $\mathcal{R}$  of *partial recursive functions* is inductively defined:

- $\mathcal{R}$  contains the base functions.
- If partial functions  $g_1, \dots, g_m : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  and  $h : \mathbb{N}_0^m \rightarrow \mathbb{N}_0$  lie to  $\mathcal{R}$ , the function given by composition

$$f = h(g_1, \dots, g_m) : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$$

lies in  $\mathcal{R}$ .

- If partial functions  $g : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  and  $h : \mathbb{N}_0^{n+2} \rightarrow \mathbb{N}_0$  are in  $\mathcal{R}$ , the function given by primitive recursion

$$f = \text{pr}(g, h) : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0$$

is in  $\mathcal{R}$ .

- If partial function  $f : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0$  is in  $\mathcal{R}$ , function given by minimalization

$$\mu f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$$

is in  $\mathcal{R}$ .

## Partial Recursive Functions – Class $\mathcal{T}$

- The *recursive functions* are the total partial recursive functions.
- $\mathcal{T}$  is the class of recursive functions.
- Each recursive function is partial recursive and so

$$\mathcal{T} \subset \mathcal{R} \quad (122)$$

with  $f_{\uparrow} \in \mathcal{R} \setminus \mathcal{T}$  (empty function), say  $f_{\uparrow} = \|\!|A1; (A1)1\|\!|_{k,1}$ .

- Each primitive recursive function is total and built up along the same lines as  $\mathcal{R}$ , and so

$$\mathcal{P} \subset \mathcal{T} \quad (123)$$

with  $A \in \mathcal{T} \setminus \mathcal{P}$  (Ackermann function).

## Partial Recursive Functions – URM Computability

Each partial recursive function is URM computable.

### Proof (Sketch)

- The basic functions are URM computable.
- Let  $f = h(g_1, \dots, g_m)$ . By induction, there are URM programs  $P_h$  and  $P_{g_1}, \dots, P_{g_m}$  such that  $\|P_h\|_{m,1} = h$  and  $\|P_{g_i}\|_{n,1} = g_i$ ,  $1 \leq i \leq m$ . Then there is an URM program  $P$  such that  $\|P\|_{n,1} = f$ .
- Let  $f = \text{pr}(g, h)$ . By induction, there are URM programs  $P_g$  and  $P_h$  such that  $\|P_g\|_{n,1} = g$  and  $\|P_h\|_{n+2,1} = h$ . Then there is an URM program  $P$  such that  $\|P\|_{n+1,1} = f$ .
- Let  $g = \mu f$ . By induction, there is an URM program  $P_f$  such that  $\|P_f\|_{n+1,1} = f$ . Then there is an URM program  $P$  such that  $\|P\|_{n,1} = g$ . □

## GOTO Programs – Contents

- Syntax
- Semantics
- Residual-step function

## Syntax – Instructions

Set of variables  $V = \{x_\sigma \mid \sigma \in \mathbb{N}\}$ .

*Instructions* of GOTO programs:

- Incrementation:

$$(l, x_\sigma \leftarrow x_\sigma + 1, m), \quad l, m \in \mathbb{N}_0, x_\sigma \in V, \quad (124)$$

$l$  label and  $m$  next label.

- Decrementation:

$$(l, x_\sigma \leftarrow x_\sigma - 1, m), \quad l, m \in \mathbb{N}_0, x_\sigma \in V, \quad (125)$$

$l$  label and  $m$  next label.

- Branching:

$$(l, \text{if } x_\sigma = 0, k, m), \quad k, l, m \in \mathbb{N}_0, x_\sigma \in V, \quad (126)$$

$l$  label and  $k, m$  bifurcation labels.

## Syntax – GOTO Programs

- Each *GOTO program* is a finite sequence of GOTO instructions

$$P = s_0; s_1; \dots; s_q, \quad (127)$$

such that there is a unique instruction  $s_i$  which has the label  $\lambda(s_i) = i$ ,  $0 \leq i \leq q$ , and different instructions have distinct labels, i.e., for all  $0 \leq i < j \leq q$ ,  $\lambda(s_i) \neq \lambda(s_j)$ .

- $\mathcal{P}_{\text{GOTO}}$  denotes the class of all GOTO programs.
- For each GOTO program  $P$ ,  $V(P)$  denotes the set of variables occurring in  $P$  and  $L(P) = \{\lambda(s_i) \mid 0 \leq i \leq q\}$  provides the set of labels in  $P$ .
- A GOTO program  $P = s_0; s_1; \dots; s_q$  is called *standard* if the  $i$ -th instruction  $s_i$  has the label  $\lambda(s_i) = i$ ,  $0 \leq i \leq q$ .

## GOTO Programs – Example

## Standard GOTO program

$$P_+ = s_0; s_1; s_2$$

for the addition of two natural numbers:

$s_0$ :	0	if $x_2 = 0$	3	1
$s_1$ :	1	$x_1 \leftarrow x_1 + 1$	2	
$s_2$ :	2	$x_2 \leftarrow x_2 - 1$	0	

$$V(P_+) = \{x_1, x_2\} \text{ and } L(P_+) = \{0, 1, 2\}.$$

## Semantics – Formalities

- *Memory*: variable  $x_\sigma$  is stored in register  $R_\sigma$ ,  $\sigma \geq 1$ .
- *Instruction counter*: register  $R_0$ .
- *Initialization*: instruction counter is set to 0 such that the execution starts with the instruction having label 0.
- *Instructions*:
  - $(l, x_\sigma \leftarrow x_\sigma + 1, m)$  increments the content of register  $R_\sigma$ .
  - $(l, x_\sigma \leftarrow x_\sigma - 1, m)$  decrements the content of register  $R_\sigma$  if it contains a number greater than zero.
  - $(l, \text{if } x_\sigma = 0, k, m)$  provides a jump to the statement with label  $k$  if the content of register  $R_\sigma$  is 0; otherwise, to the statement with label  $m$ .
- *Termination*: program halts if the instruction counter does not correspond to an instruction label.

## Semantics

Mimicking the computation of GOTO programs by partial recursive functions:

- Configuration
- Next configuration
- One-step function
- Multi-step function
- Runtime function
- Residual-step function

## Semantics – Configuration

Given GOTO program

$$P = s_0; s_1; \dots; s_q$$

with variables  $V(P) = \{x_1, \dots, x_n\}$ .

- Each  $n + 1$ -tuple  $\mathbf{z} = (z_0, z_1, \dots, z_n) \in \mathbb{N}_0^{n+1}$  is called a *configuration*.
- In each configuration  $\mathbf{z} = (z_0, z_1, \dots, z_n)$  of  $P$ ,
  - $z_0$  is the content of the instruction counter,
  - $z_i$  is the value of variable  $x_i$ ,  $1 \leq i \leq n$ .

## Semantics – Next Configuration

In program  $P$ , configuration  $\mathbf{z} = (z_0, z_1, \dots, z_n)$  transforms to configuration  $\mathbf{z}' = (z'_0, z'_1, \dots, z'_n)$  in one step, written

$$\mathbf{z} \vdash_P \mathbf{z}', \quad (128)$$

if  $z_0$  is a label in  $P$ , say  $z_0 = \lambda(s_i)$  for some  $0 \leq i \leq q$ , and

- if  $s_i = (z_0, x_\sigma \leftarrow x_\sigma + 1, m)$ ,

$$\mathbf{z}' = (m, z_1, \dots, z_{\sigma-1}, z_\sigma + 1, z_{\sigma+1}, \dots, z_n), \quad (129)$$

- if  $s_i = (z_0, x_\sigma \leftarrow x_\sigma - 1, m)$ ,

$$\mathbf{z}' = (m, z_1, \dots, z_{\sigma-1}, z_\sigma - 1, z_{\sigma+1}, \dots, z_n), \quad (130)$$

- if  $s_i = (z_0, \text{if } x_\sigma = 0, k, m)$ ,

$$\mathbf{z}' = \begin{cases} (k, z_1, \dots, z_n) & \text{if } z_\sigma = 0, \\ (m, z_1, \dots, z_n) & \text{otherwise.} \end{cases} \quad (131)$$

## Semantics – Example

Standard GOTO program  $P_+$ :

$$\begin{array}{llll}
 s_0 : & 0 & \text{if } x_2 = 0 & 3 & 1 \\
 s_1 : & 1 & x_1 \leftarrow x_1 + 1 & 2 & \\
 s_2 : & 2 & x_2 \leftarrow x_2 - 1 & 0 & 
 \end{array} \quad (132)$$

Tracing the computation with initial values  $x_1 = 3$  and  $x_2 = 2$ :

$$\begin{array}{llll}
 z_0 = (0, 3, 2) & 0 & \text{if } x_2 = 0 & 3 & 1 \\
 z_1 = (1, 3, 2) & 1 & x_1 \leftarrow x_1 + 1 & 2 & \\
 z_2 = (2, 4, 2) & 2 & x_2 \leftarrow x_2 - 1 & 0 & \\
 z_3 = (0, 4, 1) & 0 & \text{if } x_2 = 0 & 3 & 1 \\
 z_4 = (1, 4, 1) & 1 & x_1 \leftarrow x_1 + 1 & 2 & \\
 z_5 = (2, 5, 1) & 2 & x_2 \leftarrow x_2 - 1 & 0 & \\
 z_6 = (0, 5, 0) & 0 & \text{if } x_2 = 0 & 3 & 1 \\
 z_7 = (3, 5, 0) & & & & 
 \end{array} \quad (133)$$

## \*Semantics – One-Step Function

*One-step function*

$$E_P : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0^{n+1}$$

is defined as

$$E_P : z \mapsto \begin{cases} z' & \text{if } \exists z' : z \vdash_P z', \\ z & \text{otherwise.} \end{cases} \quad (134)$$

$E_P$  is primitive recursive, it has as input  $z$  and an encoding of  $P$  and outputs the next configuration (if any).

## Example

$$E_{P_+}(z_0) = z_1, \dots, E_{P_+}(z_6) = z_7, E_{P_+}(z_7) = z_7.$$

## \*Semantics – Multi-Step Function

*Multi-step function*

$$M_P : \mathbb{N}_0^{n+2} \rightarrow \mathbb{N}_0^{n+1} : (z, t) \mapsto E_P^t(z), \quad (135)$$

i.e., for all  $z \in \mathbb{N}_0^{n+1}$  and  $t \in \mathbb{N}_0$ ,

$$M_P(z, t) = \underbrace{(E_P \circ \dots \circ E_P)}_{t \text{ times}}(z).$$

$M_P$  is obtained from one-step function  $E_P$  by iteration and so is also primitive recursive.

## Example

$$M_{P_+}(z_0, 7) = E_{P_+}^7(z_0) = z_7.$$

## \*Semantics – Runtime Function I

Consider the function

$$Y_P : \mathbb{N}_0^{n+2} \rightarrow \mathbb{N}_0$$

given by

$$Y_P(\mathbf{z}, t) = \left( \chi_{L(P)} \circ \pi_1^{(n+1)} \circ E_P^t \right) (\mathbf{z}). \quad (136)$$

$Y_P$  is a composition of primitive recursive functions and so is also primitive recursive;  $\chi_{L(P)}$  is primitive recursive as  $L(P)$  is finite.

Starting from the configuration  $\mathbf{z} \in \mathbb{N}_0^n$ ,

$$E_P^t(\mathbf{z}) = (z'_0, \dots, z'_n)$$

is the configuration after  $t$  steps and the reached label is

$$\pi_1^{(n+1)}(z'_0, \dots, z'_n) = z'_0.$$

Then  $\chi_{L(P)}(z'_0) = 1$  if  $z'_0 \in L(P)$ ; otherwise, 0.

## \*Semantics – Runtime Function II

*Runtime function*

$$Z_P : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0$$

is given by

$$Z_P(\mathbf{z}) = \mu t [Y_P(\mathbf{z}, t) = 0]. \quad (137)$$

$Z_P$  is the minimalization of a primitive recursive function and so partial recursive.

If  $E_P^t(\mathbf{z}) = (z'_0, \dots, z'_n)$ , then

$$Z_P(\mathbf{z}) = \begin{cases} t & t \text{ minimal with } z'_0 \notin L(P), \\ \uparrow & \text{otherwise.} \end{cases}$$

That is,  $Z_P(\mathbf{z})$  is the smallest number of steps required such that the program  $P$  terminates when started with configuration  $\mathbf{z}$ .

## Semantics – Example

Standard GOTO program  $P_+$ :

$s_0$ :	0	if $x_2 = 0$	3	1
$s_1$ :	1	$x_1 \leftarrow x_1 + 1$	2	
$s_2$ :	2	$x_2 \leftarrow x_2 - 1$	0	

Label set  $L(P_+) = \{0, 1, 2\}$ .Tracing the computation with initial values  $x_1 = 3$  and  $x_2 = 2$ :

$\mathbf{z}_0 = (0, 3, 2)$	0	if $x_2 = 0$	3	1
$\mathbf{z}_1 = E_{P_+}(\mathbf{z}_0) = (1, 3, 2)$	1	$x_1 \leftarrow x_1 + 1$	2	
$\mathbf{z}_2 = E_{P_+}(\mathbf{z}_1) = (2, 4, 2)$	2	$x_2 \leftarrow x_2 - 1$	0	
$\mathbf{z}_3 = E_{P_+}(\mathbf{z}_2) = (0, 4, 1)$	0	if $x_2 = 0$	3	1
$\mathbf{z}_4 = E_{P_+}(\mathbf{z}_3) = (1, 4, 1)$	1	$x_1 \leftarrow x_1 + 1$	2	
$\mathbf{z}_5 = E_{P_+}(\mathbf{z}_4) = (2, 5, 1)$	2	$x_2 \leftarrow x_2 - 1$	0	
$\mathbf{z}_6 = E_{P_+}(\mathbf{z}_5) = (0, 5, 0)$	0	if $x_2 = 0$	3	1
$\mathbf{z}_7 = E_{P_+}(\mathbf{z}_6) = (3, 5, 0)$				

$M_{P_+}((0, 3, 2), 7) = E_{P_+}^7(0, 3, 2) = (3, 5, 0)$  with  $3 \notin L(P_+)$  and  
 $Z_{P_+}(0, 3, 2) = 7$ .

## Semantics – Example

Standard GOTO program  $P_{+1}$  (URM program (A1)1):

$$\begin{array}{llll} s_0 : & 0 & \text{if } x_1 = 0 & 2 & 1 \\ s_1 : & 1 & x_1 \leftarrow x_1 + 1 & 0 & \end{array} \quad (138)$$

$L(P_{+1}) = \{0, 1\}$ . Tracing computation with initial value  $x_1 = 0$ :

$$\begin{array}{llll} z_0 = (0, 0) & 0 & \text{if } x_1 = 0 & 2 & 1 \\ z_1 = (2, 0) & & & & \end{array}$$

Then  $Z_{P_{+1}}(z_0) = 1$ .

Tracing computation with initial value  $x_1 > 0$ :

$$\begin{array}{llll} z_0 = (0, x_1) & 0 & \text{if } x_1 = 0 & 2 & 1 \\ z_1 = (1, x_1) & 1 & \text{if } x_1 \leftarrow x_1 + 1 & 0 & \\ z_2 = (0, x_1 + 1) & 0 & \text{if } x_1 = 0 & 2 & 1 \\ z_3 = (1, x_1 + 1) & 1 & \text{if } x_1 \leftarrow x_1 + 1 & 0 & \end{array}$$

Then  $Z_{P_{+1}}(z_0) = \uparrow$ .

## \*Semantics – Residual-Step Function

*Residual-step function*

$$R_P : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0^{n+1}$$

given by

$$R_P(\mathbf{z}) = M_P(\mathbf{z}, Z_P(\mathbf{z})) = \begin{cases} M_P(\mathbf{z}, Z_P(\mathbf{z})) & \text{if } \mathbf{z} \in \text{dom}(Z_P), \\ \uparrow & \text{otherwise,} \end{cases} \quad (139)$$

is partial recursive.

If  $\mathbf{z} \in \text{dom}(Z_P)$ , then  $Z_P(\mathbf{z}) = t$  provides the number of steps  $P$  performs until it terminates and then  $M_P(\mathbf{z}, t) = E_P^t(\mathbf{z})$  is the **final configuration** reached.

Residual-step function  $R_P$  mimicks the execution of GOTO program  $P$ .

## Semantics – Example

Standard GOTO program  $P_+$ :

$s_0$ :	0	if $x_2 = 0$	3	1
$s_1$ :	1	$x_1 \leftarrow x_1 + 1$	2	
$s_2$ :	2	$x_2 \leftarrow x_2 - 1$	0	

Tracing the computation with initial values  $x_1 = 3$  and  $x_2 = 2$ :

$z_0 = (0, 3, 2)$	0	if $x_2 = 0$	3	1
$z_1 = E_{P_+}(z_0) = (1, 3, 2)$	1	$x_1 \leftarrow x_1 + 1$	2	
$z_2 = E_{P_+}(z_1) = (2, 4, 2)$	2	$x_2 \leftarrow x_2 - 1$	0	
$z_3 = E_{P_+}(z_2) = (0, 4, 1)$	0	if $x_2 = 0$	3	1
$z_4 = E_{P_+}(z_3) = (1, 4, 1)$	1	$x_1 \leftarrow x_1 + 1$	2	
$z_5 = E_{P_+}(z_4) = (2, 5, 1)$	2	$x_2 \leftarrow x_2 - 1$	0	
$z_6 = E_{P_+}(z_5) = (0, 5, 0)$	0	if $x_2 = 0$	3	1
$z_7 = E_{P_+}(z_6) = (3, 5, 0)$				

$$R_{P_+}(0, 3, 2) = M_{P_+}((0, 3, 2), Z_{P_+}(0, 3, 2)) = M_{P_+}((0, 3, 2), 7) = E_{P_+}^7(0, 3, 2) = (3, 5, 0).$$

## Semantics – Computable Functions

For each GOTO program  $P$  and  $k \geq 0$ , define the partial function

$$\|P\|_{k,1} = \beta_1 \circ R_P \circ \alpha_k, \quad (140)$$

where  $\alpha_k : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^{n+1}$  loads the registers with arguments,

$$\alpha_k : (x_1, x_2, \dots, x_k) \mapsto (0, x_1, x_2, \dots, x_k, 0, \dots, 0) \quad (141)$$

$R_P$  is the residual-step function

$$R_P : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0^{n+1} \quad (142)$$

and  $\beta_1 : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0$  reads out the result,

$$\beta_1 : (x_0, x_1, \dots, x_n) \mapsto x_1. \quad (143)$$

Here  $n$  must be large enough to provide sufficient workspace for the computation of  $P$ ,

$$n \geq \max\{k, \max\{\sigma \mid x_\sigma \in V(P)\}\}. \quad (144)$$

## Semantics – Example

Standard GOTO program  $P_+$ :

$s_0$ :	0	if $x_2 = 0$	3	1
$s_1$ :	1	$x_1 \leftarrow x_1 + 1$	2	
$s_2$ :	2	$x_2 \leftarrow x_2 - 1$	0	

Computation of  $P_+$  ( $n = 2$ ):

$$\begin{aligned}
 \|P_+\|_{2,1}(3, 2) &= (\beta_1 \circ R_{P_+} \circ \alpha_2)(3, 2) && (145) \\
 &= (\beta_1 \circ R_{P_+})(0, 3, 2) \\
 &= \beta_1(3, 5, 0), \quad \text{by (133),} \\
 &= 5.
 \end{aligned}$$

## GOTO Computable Functions – Contents

- GOTO computable functions
- URM computability implies GOTO computability

## GOTO Computable Functions

- Partial function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  is *GOTO computable* if there is a GOTO program  $P$  that computes  $f$  in the sense that by (140)

$$f = \llbracket P \rrbracket_{k,1} = \beta_1 \circ R_P \circ \alpha_k. \quad (146)$$

- $\mathcal{F}_{\text{GOTO}}$  denotes the class of all GOTO computable functions.
- $\mathcal{T}_{\text{GOTO}}$  depicts the class of all total GOTO computable functions.

## GOTO Computable Functions

Each GOTO computable function is partial recursive.

### Proof.

Let  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  be GOTO computable. Then by (146), there is a GOTO program  $P$  such that

$$f = \|P\|_{k,1} = \beta_1 \circ R_P \circ \alpha_k.$$

$f$  is the composition of primitive recursive functions  $\beta_1, \alpha_k$  and partial recursive function  $R_P$  and so is also partial recursive.  $\square$

## GOTO Computable Functions – Example

Standard GOTO program  $P_+$ :

0	if $x_2 = 0$	3	1
1	$x_1 \leftarrow x_1 + 1$	2	
2	$x_2 \leftarrow x_2 - 1$	0	

We have

$$\|P_+\|_{2,1}(x_1, x_2) = x_1 + x_2.$$

## URM and GOTO Computable Functions

Each URM computable function is GOTO computable.

### Proof

The idea is to translate each URM program  $P$  into a GOTO program  $\phi(P)$  such that both compute the same function,

$$\|P\|_{k,1} = \|\phi(P)\|_{k,1}, \quad k \in \mathbb{N}_0.$$

Let  $P$  be an URM program which does not make use of register  $R_0$ . Write the program  $P$  as a string

$$P = \tau_0 \tau_1 \dots \tau_q,$$

where each substring (token)  $\tau_i$  is of the form

$$"A\sigma", "S\sigma", "(" \text{ or } ")"\sigma" \text{ with } \sigma \in Z \setminus \{0\}.$$

Note that each opening parenthesis "(" has a unique closing parenthesis ")"

## Proof (Cont'd)

Replace each token  $\tau_i$  by a GOTO instruction  $s_i$ :

- If  $\tau_i = "A\sigma"$ , put

$$s_i = (i, x_\sigma \leftarrow x_\sigma + 1, i + 1),$$

- if  $\tau_i = "S\sigma"$ , set

$$s_i = (i, x_\sigma \leftarrow x_\sigma - 1, i + 1),$$

- if  $\tau_i = "("$  and  $\tau_j = ")"\sigma"$  is the corresponding closing parenthesis, define

$$s_i = (i, \text{if } x_\sigma = 0, j + 1, i + 1),$$

- if  $\tau_i = ")\sigma"$  and  $\tau_j = "("$  is the associated opening parenthesis, put

$$s_i = (i, \text{if } x_\sigma = 0, i + 1, j + 1).$$

## Proof (Cont'd)

In this way, from the URM program (written as a sequence of tokens)

$$P = \tau_0 \tau_1 \dots \tau_q$$

one obtains the GOTO program

$$\phi(P) = s_0; s_1; \dots; s_q$$

which has the required property

$$\|P\|_{k,1} = \|\phi(P)\|_{k,1}.$$



## URM and GOTO Computable Functions – Example

URM program

$$P_+ = (A1; S2)2$$

is given by the sequence of tokens

$$P_+ = \tau_0\tau_1\tau_2\tau_3,$$

where

$$\tau_0 = "(", \tau_1 = "A1", \tau_2 = "S2", \tau_3 = ")2".$$

Translation yields the GOTO program

$\tau_0 :$	$s_0 :$	0	if $x_2 = 0$	4	1
$\tau_1 :$	$s_1 :$	1	$x_1 \leftarrow x_1 + 1$	2	
$\tau_2 :$	$s_2 :$	2	$x_2 \leftarrow x_2 - 1$	3	
$\tau_3 :$	$s_3 :$	3	if $x_2 = 0$	4	1

Compare this program with the GOTO program for addition (132) which has only three instructions.

## Church's Thesis – Contents

- Big picture
- Church's thesis

## Comparison of Function Classes - Big Picture

$$\mathcal{R} = \mathcal{F}_{\text{URM}} = \mathcal{F}_{\text{GOTO}} = \mathcal{F}_{\text{GOTO2}}$$

$$\mathcal{T} = \mathcal{T}_{\text{URM}} = \mathcal{T}_{\text{GOTO}} = \mathcal{T}_{\text{GOTO2}}$$

$$\mathcal{P} = \mathcal{F}_{\text{LOOP}}$$

## Proof.

We have explicitly proved

$$\mathcal{F}_{\text{URM}} \subseteq \mathcal{F}_{\text{GOTO}}, \quad \mathcal{F}_{\text{GOTO}} \subseteq \mathcal{R}, \quad \mathcal{R} \subseteq \mathcal{F}_{\text{URM}}.$$

By ring closure, all three classes are equal. □

## Church's Thesis

Each computable partial function in the intuitive sense is partial recursive (Alonso Church, 1903-1995).

- What does *intuitive* mean?  
Just sit down and write an algorithm in your favorite programming language. Suppose it provides a function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$ . Then by Church's thesis, this function is partial recursive.
- Church's thesis characterizes the nature of computation and cannot be formally proved. Nevertheless, it has reached universal acceptance.
- If the thesis could be disproved, the construction of a *hyper-computer* would be possible which would be superior to say GOTO computation.
- Church's thesis is often practically used in the sense that if a partial function is intuitively computable, it will be partial recursive. In this way, the thesis can lead to more intuitive and less rigorous proofs.

## GOTO-2 Programs – Contents

- Encoding of URM states
- Basic GOTO-2 programs
- URM computability implies GOTO-2 computability
- Use of flow charts (diagrammatic representation of algorithm)

## GOTO-2 Programs – Inside

- GOTO-2 programs are GOTO programs with two variables  $x_1$  and  $x_2$ .
- Goal is to simulate URM programs by GOTO-2 programs.

## GOTO-2 Programs - Encoding of URM States

- URM has countable state set

$$\Omega = \{\omega \mid \omega = (\omega_i)_{i \in \mathbb{N}_0} \text{ almost all entries } 0\}$$

by the bijection  $G : \Omega \rightarrow \mathbb{N}$  given by

$$G(\omega) = \prod_{i \text{ finite}} p_i^{\omega_i}, \quad (147)$$

where  $(p_0, p_1, p_2, \dots)$  is the sequence of primes.

- $G$  is primitive recursive, since multiplication is primitive recursive.
- E.g., if  $\omega = (0, 2, 3, 0, 5, 0, 0, \dots)$ , then  $G(\omega) = p_1^2 p_2^3 p_4^5$ .

## GOTO-2 Programs - Encoding of URM States

- Collection of functions

$$G_i : \mathbb{N}_0 \rightarrow \mathbb{N}_0, \quad i \in \mathbb{N}_0,$$

given by

$$G_i(x) = (x)_i, \quad (148)$$

where  $(x)_i$  is the exponent of  $p_i$  in the (unique) prime factorization of  $x > 0$ .

- In particular, set  $G_i(0) = 0$  for all  $i \in \mathbb{N}_0$ .
- By (103),  $G_i$  is primitive recursive,  $i \in \mathbb{N}_0$ .
- E.g., if  $\omega = (0, 2, 3, 0, 5, 0, 0, \dots)$ , then  $x = G(\omega) = p_1^2 p_2^3 p_4^5$  and so  $(x)_1 = 2$ ,  $(x)_2 = 3$ ,  $(x)_4 = 5$ ; otherwise,  $(x)_i = 0$ .

## GOTO-2 Programs – Basic Programs

Consider three basis GOTO-2 programs:

$$|M(k)|(0, x) = (0, k \cdot x), \quad (149)$$

$$|D(k)|(0, k \cdot x) = (0, x), \quad (150)$$

$$|T(k)|(0, x) = \begin{cases} (1, x) & \text{if } k \text{ divides } x, \\ (0, x) & \text{otherwise.} \end{cases} \quad (151)$$

## Usage

$$|M(p_i)| : (0, p_0^{\omega_0} \cdots p_i^{\omega_i} \cdots) \mapsto (0, p_0^{\omega_0} \cdots p_i^{\omega_i+1} \cdots),$$

$$|D(p_i)| : (0, p_0^{\omega_0} \cdots p_i^{\omega_i+1} \cdots) \mapsto (0, p_0^{\omega_0} \cdots p_i^{\omega_i} \cdots),$$

$$|T(p_i)| : (0, p_0^{\omega_0} \cdots p_i^{\omega_i} \cdots) \mapsto \begin{cases} (1, p_0^{\omega_0} \cdots p_i^{\omega_i} \cdots) & \text{if } \omega_i > 0, \\ (0, p_0^{\omega_0} \cdots p_i^0, \dots) & \text{otherwise.} \end{cases}$$

\*GOTO-2 Program  $M(k)$ 

We have  $|M(k)|(0, x) = (0, k \cdot x)$ .

Implementation of  $M(k)$  by two consecutive loops:

- Initially,  $x_1 = 0$  and  $x_2 = x$ .
- 1st loop:  $x_2$  is decremented, while in each decrementation step,  $x_1$  is incremented  $k$  times. After this loop,  $x_1 = k \cdot x$  and  $x_2 = 0$ :

$$(A1^k; S2)2$$

- 2nd loop:  $x_2$  is incremented and  $x_1$  is decremented such that upon termination,  $x_1 = 0$  and  $x_2 = k \cdot x$  (swapping states):

$$(S1; A2)1$$

\*GOTO-2 Program  $D(k)$ 

We have  $|D(k)|(0, k \cdot x) = (0, x)$ .

Implementation of  $D(k)$  by two consecutive loops:

- Initially,  $x_1 = 0$  and  $x_2 = k \cdot x$ .
- 1st loop:  $x_1$  is incremented, while  $x_2$  is decremented  $k$  times.  
After this loop,  $x_1 = x$  and  $x_2 = 0$ :

$$(A1; S2^k)2$$

- 2nd loop:  $x_2$  is incremented and  $x_1$  is decremented such that upon termination,  $x_1 = 0$  and  $x_2 = x$  (swapping states):

$$(S1; A2)1$$

## GOTO-2 Program – Main Result

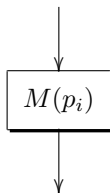
For each URM program  $P$ , there exists a GOTO-2 program  $\bar{P}$  with the same semantics, i.e., for all states  $\omega, \omega' \in \Omega$ ,

$$|P|(\omega) = \omega' \iff |\bar{P}|(0, G(\omega)) = (0, G(\omega')). \quad (152)$$

The assignment  $P \mapsto \bar{P}$  will be established by making use of the inductive definition of URM programs and flow diagrams to simplify notation.

## GOTO-2 Program – Increment

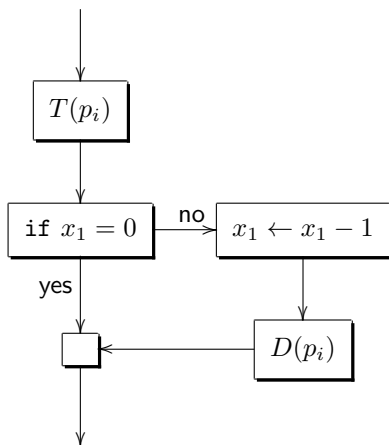
Program  $\overline{A_i}$  can be realized by the flow chart



$$(\omega_0, \dots, \omega_i, \dots) \mapsto (\omega_0, \dots, \omega_i + 1, \dots).$$

## GOTO-2 Programs – Decrement

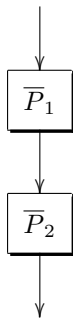
Program  $\overline{S_i}$  is given by the flow diagram



$$(\omega_0, \dots, \omega_i, \dots) \mapsto (\omega_0, \dots, \omega_i \div 1, \dots).$$

## GOTO-2 Programs – Composition

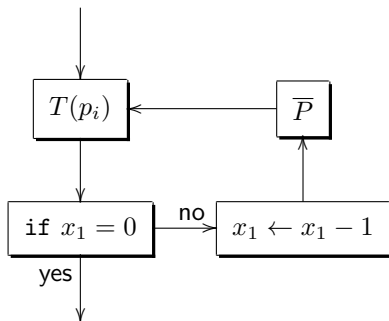
Program  $\overline{P_1; P_2}$  can be depicted by the flow chart



Concatenation of GOTO-2 programs  $\overline{P_1}$  and  $\overline{P_2}$ .

## GOTO-2 Programs – Iteration

Program  $\overline{(P)}_i$  can be represented by the flow diagram

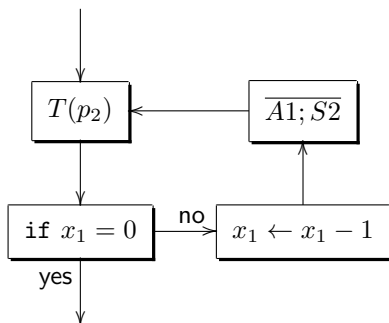


If  $T(p_i) = (0, x)$ , then  $\omega_i = 0$  and exit.

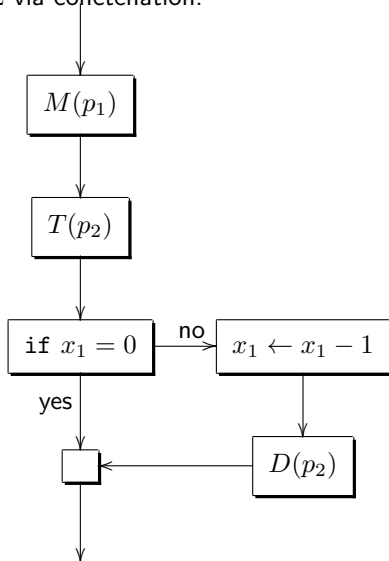
Otherwise, set  $x_1 = 0$ , execute  $\overline{P}$  and start anew.

## GOTO-2 Programs – Example

URM program  $P_+ = (A1; S2)^2$  translates into flow chart via iteration:



## Example (Cont'd)

Program  $\overline{A1; S2}$  via concatenation:

## Example (Cont'd)

URM program  $P_+ = (A1; S2)2$  translates into GOTO-2 program  $\bar{P}_+$ :

```

0 :  $T(p_2)$ 
1 : if  $x_1 = 0$  goto 9
2 :  $x_1 \leftarrow x_1 - 1$ 
3 :  $M(p_1)$ 
4 :  $T(p_2)$ 
5 : if  $x_1 = 0$  goto 8
6 :  $x_1 \leftarrow x_1 - 1$ 
7 :  $D(p_2)$ 
8 : goto 0
9 :
```

## GOTO-2 Programs – Proof (End)

- By inductive definition of URM programs, the assignment  $P \mapsto \bar{P}$  is well-defined.
- The computation of a function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^m$  by a GOTO-2 program requires to load the registers with the initial values and to identify the result.
- Define primitive recursive functions  $\hat{\alpha}_k : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^2$ ,

$$\hat{\alpha}_k : (x_1, \dots, x_k) \mapsto (0, G(0, x_1, \dots, x_k, 0, \dots)) \quad (153)$$

and  $\hat{\beta}_m : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0^m$ ,

$$(a, b) \mapsto (G_1(b), \dots, G_m(b)). \quad (154)$$

## GOTO-2 Programs – Proof (End)

Each URM program  $P$  is (semantically) equivalent to the associated GOTO-2 program  $\bar{P}$  in the sense that for all  $k, m \in \mathbb{N}_0$ :

$$\|P\|_{k,m} \stackrel{!}{=} \|\bar{P}\|_{k,m} \stackrel{\text{def}}{=} \hat{\beta}_m \circ R_{\bar{P}} \circ \hat{\alpha}_k, \quad (155)$$

where  $R_{\bar{P}}$  is the residual-step function of GOTO-2 program  $\bar{P}$ .



## GOTO-2 Programs – Example

URM program  $P_+ = (A1; S2)2$  with

$$\|P_+\|_{2,1}(x_1, x_2) = x_1 + x_2.$$

Note that by (145),  $R_{P_+}(0, x_1, x_2) = (3, x_1 + x_2, 0)$ .

Computation of GOTO-2 program  $\overline{P}_+$ :

$$\begin{aligned} \|\overline{P}_+\|_{2,1}(x_1, x_2) &= (\hat{\beta}_1 \circ R_{\overline{P}_+} \circ \hat{\alpha}_2)(x_1, x_2) \\ &= (\hat{\beta}_1 \circ R_{\overline{P}_+})(0, p_1^{x_1} \cdot p_2^{x_2}) \\ &= \hat{\beta}_1(0, p_0^3 \cdot p_1^{x_1+x_2}), \quad \text{by (133), (152),} \\ &= x_1 + x_2. \end{aligned}$$

## Skills

- Determine  $\mu f$  from partial recursive function  $f$
- Write and interpret a GOTO program
- Compile URM program into GOTO program
- Compile URM program into GOTO-2 program
- Explicate Church's thesis

# Part IV

# Ackermann Function

## Ackermann Function – Motivation

- Class of small Ackermann functions  $B_n : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ ,  $n \geq 0$ .
- Small Ackermann functions provide bound on the complexity of LOOP programs and establish the LOOP hierarchy.
- Definition of Ackermann function based on small Ackermann functions.
- Ackermann function belongs to the class of fast-growing functions (superpowers).
- E.g., tetration:

$$A(4, n) = \underbrace{2^{2^{\dots^2}}}_{n+3} - 3.$$

## Ackermann Function – Contents

- Small Ackermann functions
- Complexity of LOOP programs
- Ackermann function
- Superpowers

Reading: Zimmermann, Chapter 4

## Ackermann Function – Inside

- David Hilbert (1926) posed the problem whether all total computable functions are primitive recursive or not.
- Wilhelm Ackermann (1928) introduced a function, which is recursive but not primitive recursive.

## Tetration

$$\begin{aligned}
 A(4, n) &= 2 \uparrow^2 (n + 3) - 3, \quad n \geq 0, \\
 A(4, 0) &= 2 \uparrow^2 3 - 3 = 2^4 - 3 = 13, \\
 A(4, 1) &= 2 \uparrow^2 4 - 3 = 2^{2^4} - 3 = 65533, \\
 A(4, 2) &= 2 \uparrow^2 5 - 3 = 2^{65536} - 3, \\
 A(4, 3) &= 2 \uparrow^2 6 - 3 = 2^{2^{65536}} - 3, \\
 A(4, 4) &= 2 \uparrow^2 7 - 3 = 2^{2^{2^{65536}}} - 3.
 \end{aligned}$$

## Small Ackermann Functions – Contents

- General definition
- Primitive recursiveness
- Properties
- LOOP computability

## Small Ackermann Functions – Definition

Sequence  $(B_n)_{n \in \mathbb{N}_0}$  of *small Ackermann functions*

$$B_n : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \quad (156)$$

is defined inductively:

$$B_0 : x \mapsto \begin{cases} 1 & \text{if } x = 0, \\ 2 & \text{if } x = 1, \\ x + 2 & \text{otherwise,} \end{cases} \quad (157)$$

and for each  $n \geq 0$ ,

$$B_{n+1} : x \mapsto B_n^x(1), \quad x \in \mathbb{N}_0, \quad (158)$$

where  $B_n^x = \underbrace{B_n \circ \dots \circ B_n}_x$  is the  $x$ -fold iteration of  $B_n$ .

## Small Ackermann Functions – Primitive Recursiveness

Small Ackermann function  $B_n$ ,  $n \in \mathbb{N}_0$ , is primitive recursive.

### Proof

The function  $B_0$  is defined by primitive case distinction:

$$B_0(x) = \begin{cases} \nu(x) & \text{if } \text{sgn}(x) = 0, \\ \nu(x) & \text{if } \text{sgn}(x) = 1 \text{ and } \text{sgn}(x \dot{-} 1) = 0, \\ (\nu \circ \nu)(x) & \text{if } \text{sgn}(x \dot{-} 1) = 1. \end{cases} \quad (159)$$

Then

$$B_0(x) = \nu(x) \cdot \text{csg}(x) + \nu(x) \cdot [\text{sgn}(x) \cdot \text{csg}(x \dot{-} 1)] + \nu^2(x) \cdot \text{sgn}(x \dot{-} 1).$$

## Proof (Cont'd)

If  $n \geq 1$ ,  $B_n$  is given by the primitive recursive scheme

$$B_n(0) = 1, \quad (160)$$

$$B_n(x+1) = B_{n-1}(B_n(x)), \quad x \in \mathbb{N}_0. \quad (161)$$

More precisely,

$$B_n(0) = B_{n-1}^0(1) = \text{id}_{\mathbb{N}_0}(1) = 1$$

and

$$B_n(x+1) = B_{n-1}^{x+1}(1) = B_{n-1}(B_{n-1}^x(1)) = B_{n-1}(B_n(x)).$$

□

## Small Ackermann Functions – \*Properties

For all  $x, n, p \in \mathbb{N}_0$ :

$$B_1(x) = \begin{cases} 1 & \text{if } x = 0, \\ 2x & \text{otherwise,} \end{cases} \quad (162)$$

$$B_2(x) = 2^x, \quad (163)$$

$$B_3(x) = \begin{cases} 1 & \text{if } x = 0, \\ 2^{B_3(x-1)} & \text{otherwise,} \end{cases} \quad (164)$$

$$x < B_n(x), \quad (165)$$

$$B_n(x) < B_n(x+1), \quad (166)$$

$$B_0^p(x) \leq B_1^p(x), \quad (167)$$

$$B_n(x) \leq B_{n+1}(x), \quad (168)$$

$$B_n^p(x) < B_n^p(x+1), \quad (169)$$

$$B_n^p(x) < B_n^{p+1}(x), \quad (170)$$

$$B_n^p(x) \leq B_{n+1}^p(x), \quad (171)$$

$$2^{p+1}x \leq B_1^{p+1}(x), \quad (172)$$

$$2B_n^p(x) \leq B_n^{p+1}(x), \quad n \geq 1. \quad (173)$$

## Small Ackermann Functions – Growth

$B_{n+1}$  grows faster than any power of  $B_n$ , i.e., for all  $n$  and  $p$ , there exists  $x_0 \in \mathbb{N}_0$  such that for all  $x \geq x_0$ ,

$$B_n^p(x) < B_{n+1}(x). \quad (174)$$

### \*Proof

- Let  $n = 0$ . For each number  $x \geq 2$ ,  $B_0^p(x) = x + 2p$ .

On the other hand, by (162), for each  $x \geq 1$ ,  $B_1(x) = 2x$ .

Put  $x_0 = 2p + 1$ . Then for each  $x \geq x_0$ ,

$$B_0^p(x) = x + 2p \leq 2x = B_1(x).$$

- Let  $n > 0$ . First, let  $p = 0$ . Then for each  $x \geq 0$ ,

$$B_n^0(x) = x < B_n(x) \leq B_{n+1}(x)$$

by (165) and (168).

## Proof (Cont'd)

Second, let  $p > 0$  and assume that  $B_n^p(x) < B_{n+1}(x)$  holds for all  $x \geq x'_0$ . Put  $x_0 = x'_0 + 5$ . Then

$$\begin{aligned}
 B_n^{p+1}(x) &< B_n^{p+1}(2 \cdot (x \dot{-} 2)), & x \geq 5, \text{ by (169),} \\
 &= B_n^{p+1}(B_1(x \dot{-} 2)) \\
 &\leq B_n^{p+1}(B_n(x \dot{-} 2)), & \text{ by (171),} \\
 &= B_n^{p+2}(x \dot{-} 2) \\
 &= B_n^2(B_n^p(x \dot{-} 2)) \\
 &< B_n^2(B_{n+1}(x \dot{-} 2)), & \text{ by induction, } x \geq x'_0 + 2, \\
 &= B_n^2(B_n^{x \dot{-} 2}(1)) \\
 &= B_n^x(1) \\
 &= B_{n+1}(x), & x \geq 2.
 \end{aligned}$$



## Small Ackermann Functions – LOOP Computability

For each  $n \geq 1$ ,  $B_n$  is LOOP- $n$  computable.

## Proof

First, define the LOOP-1 program

$$P_1 = \bar{C}(1; 2); \bar{C}(1; 3); Z1; A1; [Z1]3; [A1; A1]2. \quad (175)$$

For each input  $x$ , the program evaluates as follows:

0	1	2	3	4	...	registers
0	$x$	0	0	0	...	init
0	$x$	$x$	0	0	...	
0	$x$	$x$	$x$	0	...	
0	0	$x$	$x$	0	...	
0	1	$x$	$x$	0	...	
0	1	0	0	0	...	$x = 0$ end
0	$2x$	0	0	0	...	$x \neq 0$ end

By (162), the program satisfies  $\|P_1\|_{1,1} = B_1$ .

## Proof (Cont'd)

Let  $n \geq 1$ . Suppose there is a normal LOOP- $n$  program  $P_n$  that computes  $B_n$ ; that is,  $\|P_n\|_{1,1} = B_n$ . Put  $m = n(P_n) + 1^a$  and consider the LOOP- $n + 1$  program

$$P_{n+1} = [Am]1; A1; [P_n]m. \quad (176)$$

Note that register  $R_m$  is unused in program  $P_n$ . Program  $P_{n+1}$  computes  $B_{n+1}(x) = B_n^x(1)$  as follows:

0	1	2	3	4	...	$m$	...	registers
0	$x$	0	0	0	...	0	...	init
0	0	0	0	0	...	$x$	...	
0	1	0	0	0	...	$x$	...	
0	$B_n^x(1)$	0	0	0	...	0	...	end

Since  $B_{n+1}(x) = B_n^x(1)$ , we have  $\|P_{n+1}\|_{1,1} = B_{n+1}$ . □

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$^a n(P)$  is the largest register used in program  $P$ .

## Complexity of LOOP Programs– Contents

- Complexity
- Bound on LOOP computable functions
- Proper LOOP hierarchy

## Complexity of LOOP Programs

Function  $\lambda : \mathcal{P}_{\text{LOOP}} \rightarrow \mathbb{N}_0$  assigns to each LOOP program a measure of *complexity* inductively as follows:

$$\lambda(A\sigma) = 1, \quad \sigma \in \mathbb{N}_0, \quad (177)$$

$$\lambda(Z\sigma) = 1, \quad \sigma \in \mathbb{N}_0, \quad (178)$$

$$\lambda(\bar{C}(\sigma; \tau)) = 1, \quad \sigma \neq \tau, \sigma, \tau \in \mathbb{N}_0, \quad (179)$$

$$\lambda(P; Q) = \lambda(P) + \lambda(Q), \quad P, Q \in \mathcal{P}_{\text{LOOP}}, \quad (180)$$

$$\lambda([P]\sigma) = \lambda(P) + 2, \quad P \in \mathcal{P}_{\text{LOOP}}, \sigma \in \mathbb{N}_0. \quad (181)$$

For each LOOP program  $P$ , the *complexity measure*  $\lambda(P)$  is

$$\lambda(P) = x_0 + 2x_1, \quad (182)$$

where  $x_0$  is the number of LOOP-0 subprograms of  $P$  and  $x_1$  is the number of iterations of  $P$ .

## Complexity of LOOP Programs – Example

LOOP program

$$P = \bar{C}(1; 3); [\bar{C}(2; 1); A2]3$$

has the complexity

$$\begin{aligned} \lambda(P) &= \lambda(\bar{C}(1; 3)) + \lambda([\bar{C}(2; 1); A2]3) \\ &= 1 + \lambda(\bar{C}(2; 1); A2) + 2 \\ &= 3 + \lambda(\bar{C}(2; 1)) + \lambda(A2) \\ &= 5. \end{aligned}$$

## Complexity of LOOP Programs – Boundedness

A  $k$ -adic total function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0$  is said to be *bounded* by a monadic total function  $g : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  if for each  $\mathbf{x} \in \mathbb{N}_0^k$ ,

$$f(\mathbf{x}) \leq g(\max(\mathbf{x})), \quad (183)$$

where  $\max(\mathbf{x}) = \max\{x_1, \dots, x_k\}$  for  $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{N}_0^k$ .

## Complexity of LOOP Programs – Example

## ■ Addition function

$$f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto x + y$$

is bounded by the function

$$g : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto 2x,$$

since  $x + y \leq 2 \max\{x, y\}$  for all  $x, y \in \mathbb{N}_0$ .

## ■ Multiplication function

$$f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto xy$$

is bounded by the function

$$g : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto x^2,$$

since  $xy \leq \max\{x, y\}^2$  for all  $x, y \in \mathbb{N}_0$ .

## Complexity of LOOP Programs – Main Result

For each LOOP- $n$  program  $P$  and each input  $\mathbf{x} \in \mathbb{N}_0^k$  with  $k \geq n(P)$ ,<sup>a</sup>

$$\max(\|P\|_{k,k}(\mathbf{x})) \leq B_n^{\lambda(P)}(\max(\mathbf{x})), \quad (184)$$

i.e., the maximum value of  $P$  computed in the registers  $R_1, \dots, R_k$  is bounded by the  $\lambda(P)$ -th power of the  $n$ -th small Ackermann function.

---

<sup>a</sup> $n(P)$  is the largest register used in program  $P$ .

## \*Proof

First, let  $P$  be a LOOP-0 program; that is,  $P = P_1; P_2; \dots; P_m$ , where  $P_i$ ,  $1 \leq i \leq m$ , is a LOOP-0 operation:  $A\sigma$ ,  $Z\sigma$  or  $\bar{C}(\sigma, \tau)$ . Then

$$\begin{aligned} \max(\|P\|_{k,k}(\mathbf{x})) &\leq m + \max(\mathbf{x}) \\ &= \lambda(P) + \max(\mathbf{x}) \\ &\leq B_0^{\lambda(P)}(\max(\mathbf{x})), \end{aligned}$$

since the "worst-case" operation is incrementation (of the register with the maximal initial value).

## Proof (Cont'd)

Suppose the assertion holds for LOOP- $n$  programs.

Let  $P$  be a LOOP- $n + 1$  program of the form  $P = Q; R$ , where  $Q$  and  $R$  are LOOP- $n + 1$  programs.

Then for  $k \geq \max\{n(Q), n(R)\}$ ,

$$\begin{aligned}
 \max(\|Q; R\|_{k,k}(\mathbf{x})) &= \max(\|R\|_{k,k}(\|Q\|_{k,k}(\mathbf{x}))) \\
 &\leq B_{n+1}^{\lambda(R)}(\max(\|Q\|_{k,k}(\mathbf{x}))), \text{ by induction,} \\
 &\leq B_{n+1}^{\lambda(R)}(B_{n+1}^{\lambda(Q)}(\max(\mathbf{x}))), \text{ by induction,} \\
 &= B_{n+1}^{\lambda(R)+\lambda(Q)}(\max(\mathbf{x})) \\
 &= B_{n+1}^{\lambda(P)}(\max(\mathbf{x})).
 \end{aligned}$$

## Proof (Cont'd)

Let  $P$  be a LOOP- $n + 1$  program of the form  $P = [Q]\sigma$ , where  $Q$  is a LOOP- $n$  program.

Then for  $k \geq \max\{n(Q), \sigma\}$ ,

$$\begin{aligned}
 \max(\|[Q]\sigma\|_{k,k}(\mathbf{x})) &= \max(\|Q; S\sigma\|_{k,k}^{x_\sigma}(\mathbf{x})) \\
 &\leq \max(\|Q\|_{k,k}^{x_\sigma}(\mathbf{x})) && (185) \\
 &\leq B_n^{x_\sigma \cdot \lambda(Q)}(\max(\mathbf{x})), \text{ by induction,} \\
 &\leq B_{n+1}^{\lambda(Q)+2}(\max(\mathbf{x})), \text{ see (186),} \\
 &= B_{n+1}^{\lambda(P)}(\max(\mathbf{x})).
 \end{aligned}$$

The last inequality follows from the assertion

$$B_n^{x \cdot l}(y) \leq B_{n+1}^{l+2}(y), \quad x \leq y, \quad l \geq 0, \quad (186)$$

which can be proved by using the above properties of the small Ackermann functions. □

## LOOP Hierarchy – Main Result

For each  $n \geq 0$ , the class of LOOP- $n$  functions is a proper subclass of the class of LOOP- $n + 1$  functions.

### Proof.

- The  $n$ -th small Ackermann function  $B_n$  is LOOP- $n$  computable.
- Suppose  $B_{n+1}$  is LOOP- $n$  computable for some  $n \geq 0$ . Then by (184), there is an integer  $\ell \geq 0$  such that for all  $x \in \mathbb{N}_0$ ,

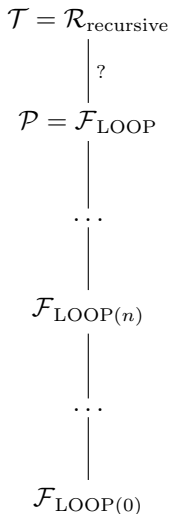
$$B_{n+1}(x) \leq B_n^\ell(x).$$

However, by (174),  $B_{n+1}$  grows faster than any power of  $B_n$ .  
Contradiction!



## Computable Total Functions

Hierarchy of total computable functions (all inclusions are strict):



## Ackermann Function – Contents

- Definition and properties
- Hierarchy of parametrized functions
- Redefinition using small Ackermann functions

## Ackermann Function – Definition

*Ackermann function*  $A : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  is defined as

$$A(0, y) = y + 1, \quad (187)$$

$$A(x + 1, 0) = A(x, 1), \quad (188)$$

$$A(x + 1, y + 1) = A(x, A(x + 1, y)). \quad (189)$$

Original definition by Wilhelm Ackermann (1928) had three arguments; above definition by Rózsa Péter (1935).

## Ackermann Function – Totality

Ackermann function is a total function.

## Proof.

- Lexicographic ordering on  $\mathbb{N}_0^2$ :

$$(x, y) \succ_{\text{lex}} (x', y') \quad :\iff \quad x > x' \vee (x = x' \wedge y > y').$$

- The evaluation of  $A(x, y)$  yields in each step (188-189) a unique expression  $A(x', y')$  which is evaluated next.
- Here  $(x, y)$  is lexicographically larger than  $(x', y')$ , i.e.,  $(x + 1, 0) \succ_{\text{lex}} (x, 1)$  and  $(x + 1, y + 1) \succ_{\text{lex}} (x + 1, y)$ .
- Thus the derivation provides a strictly decreasing chain of pairs in  $\mathbb{N}_0^2$ .
- The lexicographic ordering on  $\mathbb{N}_0^2$  is *well-founded*, i.e., there is no infinitely decreasing chain. Hence, each derivation must terminate. □

## Ackermann Function – Example

$$\begin{aligned}
 A(1, 2) &= A(0, A(1, 1)) \\
 &= A(0, A(0, A(1, 0))) \\
 &= A(0, A(0, A(0, 1))) \\
 &= A(0, A(0, 2)) \\
 &= A(0, 3) \\
 &= 4.
 \end{aligned}$$

Here

$$(1, 2) \succ_{\text{lex}} (1, 1) \succ_{\text{lex}} (1, 0) \succ_{\text{lex}} (0, 1).$$

## Ackermann Function – Partial Recursiveness

Ackermann function is partial recursive.

### Proof.

- Ackermann function is a computable function in the intuitive sense.
- Intuitive means that we have not given a definition in terms of the construction of partial recursive functions (starting from the base functions and using a finite number of operations: composition, primitive recursion, minimalization).
- By Church's thesis, the Ackermann function is partial recursive (recursive, since total). □

## Ackermann Function – Partial Recursiveness

Ackermann function is not primitive recursive.

## Proof [Sketch] – Diagonalization

For each primitive recursive function  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$ , there is a constant  $c \geq 0$  such that for all  $x_1, \dots, x_n \in \mathbb{N}_0$ ,

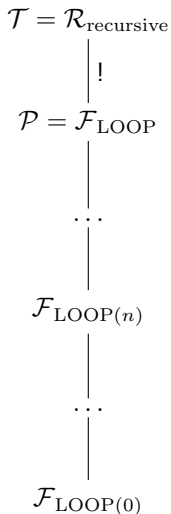
$$f(x_1, \dots, x_n) < A(c, x_1 + \dots + x_n). \quad (190)$$

This can be proved by induction on the construction of primitive recursive functions. In case of  $n = 0$ ,  $f$  is a constant and then  $f \leq A(0, f) = f + 1$ .

Suppose, Ackermann function  $A$  would be primitive recursive. Then the monadic function  $f(x) = A(x, x)$  is primitive recursive by (68). Thus there is a constant  $c$  such that  $f(x) < A(c, x)$  for all  $x \in \mathbb{N}_0$ . This particularly holds for  $x = c$ . Hence,  $f(c) < A(c, c) = f(c)$ . Contradiction! □

## Computable Total Functions

Hierarchy of total computable functions (all inclusions are strict):



## Ackermann Function – Hierarchy of Operations

Ackermann function gives rise to a collection of monadic functions

$$A_x : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : y \mapsto A(x, y), \quad x \in \mathbb{N}_0. \quad (191)$$

Initial cases:

- Successor:  $A_0(y) = y + 1 = \nu(y + 3) - 3,$
- Addition:  $A_1(y) = y + 2 = f_{\text{add}}(2, y + 3) - 3,$
- Multiplication:  $A_2(y) = 2y + 3 = f_{\text{mult}}(2, y + 3) - 3,$
- Exponentiation:  $A_3(y) = 2^{y+3} - 3 = f_{\text{exp}}(2, y + 3) - 3,$
- Tetration:  $A_4(y) = \underbrace{2^{2^{\dots^2}}}_{y+3} - 3 = f_{\text{tet}}(2, y + 3) - 3.$

**\*Proof**

- $A_0(y) = A(0, y) = y + 1.$
- $A_1(0) = A(1, 0) = A(0, 1) = 2.$  Suppose  $A_1(y) = y + 2.$   
Then  $A_1(y + 1) = A(1, y + 1) = A(0, A(1, y)) =$   
 $A(1, y) + 1 = (y + 1) + 2.$
- $A_2(0) = A(2, 0) = A(1, 1) = 3.$  Suppose  $A_2(y) = 2y + 3.$   
Then  $A_2(y + 1) = A(2, y + 1) = A(1, A(2, y)) =$   
 $A(2, y) + 2 = (2y + 3) + 2 = 2(y + 1) + 3.$
- $A_3(0) = A(3, 0) = A(2, 1) = 5.$  Suppose  $A_3(y) = 2^{y+3} - 3.$   
Then  $A_3(y + 1) = A(3, y + 1) = A(2, A(3, y)) =$   
 $2 \cdot (2^{y+3} - 3) + 3 = 2^{(y+1)+3} - 3.$

## \*Proof (Cont'd)

- $A_4(0) = A(4, 0) = A(3, 1) = 2^4 - 3 = 2^{2^2} - 3$ . Suppose that

$$A_4(y) = \underbrace{2^{2^{\dots^2}}}_{y+3} - 3.$$

Then

$$\begin{aligned} A_4(y+1) &= A(4, y+1) = A(3, A(4, y)) = 2^{A(4, y)+3} - 3 \\ &= \underbrace{2^{2^{\dots^2}}}_{(y+1)-3} - 3. \end{aligned}$$



## Pointless Large Number Stuff?

Ackermann function can be used as a benchmark of a compiler's ability to optimize recursion (Ygne Sundblad, 1971).

- Computation of  $A(x, y)$  requires deep recursion, which can provoke stack overflow;  $A$  is largely immune against compiler optimization.

- Function

$$f(y) = A(3, y)$$

has  $A(3, y) + 12^y - 2$  function calls.<sup>a</sup>

- How large can  $y$  be taken such that the compiler is able to compute  $f(y)$ ?
- My Maple<sup>TM</sup> implementation works until  $y = 8$ . But  $f(9)$  yields already too many levels of recursion.

Compare this with the direct computation of  $A(3, y) = 2^{y+3} - 3$  by a CAS.

---

<sup>a</sup>Wikipedia, 2020

## Ackermann Function – Redefinition

Combination of the small Ackermann functions into the dyadic function  $A : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ ,

$$A(x, y) = B_x(y), \quad x, y \in \mathbb{N}_0. \quad (192)$$

By the properties of the small Ackermann functions,  $A$  is a total computable function.

## Ackermann Function – Redefinition

Function  $A$  in (192) is given by the equations

$$A(0, y) = B_0(y), \quad (193)$$

$$A(x + 1, 0) = 1, \quad (194)$$

$$A(x + 1, y + 1) = A(x, A(x + 1, y)). \quad (195)$$

## Proof.

We have

$$A(0, y) = B_0(y),$$

$$A(x + 1, 0) = B_{x+1}(0) = B_x^0(1) = 1,$$

$$\begin{aligned} A(x + 1, y + 1) &= B_{x+1}(y + 1) = B_x^{y+1}(1) = B_x(B_x^y(1)) \\ &= B_x(B_{x+1}(y)) = B_x(A(x + 1, y)) \\ &= A(x, A(x + 1, y)). \end{aligned}$$



## Ackermann Function

Ackermann function  $A$  is recursive but not primitive recursive.

### Proof.

Suppose  $A$  is primitive recursive. Then  $A$  is LOOP- $n$  computable for some  $n \geq 0$ . By (184), there is an integer  $\ell \geq 0$  such that for all  $x, y \in \mathbb{N}_0$ ,

$$A(x, y) \leq B_n^\ell(\max\{x, y\}). \quad (196)$$

By (174), there is a number  $y_0 \geq 0$  such that for all  $y \geq y_0$ ,

$$B_n^\ell(y) < B_{n+1}(y). \quad (197)$$

Taking  $y_0 \geq n + 1$ ,  $x = n + 1$  and  $y \geq y_0$  leads to a contradiction:

$$\begin{aligned} A(n + 1, y) &\leq B_n^\ell(\max\{n + 1, y\}) = B_n^\ell(y) & (198) \\ &< B_{n+1}(y) = A(n + 1, y). \end{aligned}$$



## Superpowers – Contents

- Sequence of dyadic operations
- General definition
- Tetration and pentation
- \*Infinite tetration
- Ackermann class
- Representation of Ackermann function

## Superpowers – Inside

- Ackermann function is a classical example of a recursive function, which is not primitive recursive.
- Ackermann function belongs to the class of superpowers.
- Superpowers were introduced by Donald Knuth (1976).
- Knuth's superpower notation is based on the infinite sequence of operators

$$+, \cdot, \uparrow, \uparrow^2, \uparrow^3, \dots,$$

where  $\uparrow$  denotes exponentiation.

Tetration

$$a \uparrow^2 n = \underbrace{a^{a^{\dots^a}}}_n$$

## Superpowers – Sequence

Consider the following sequence of dyadic functions,

$$a + n = a + 1 + \dots + 1, \quad (n \text{ 1's}),$$

$$a \cdot n = a + a + \dots + a, \quad (n \text{ a's}),$$

$$a \uparrow n = a \cdot a \cdot \dots \cdot a, \quad (n \text{ a's}),$$

$$a \uparrow^2 n = a \uparrow a \uparrow \dots \uparrow a, \quad (n \text{ a's}),$$

$$a \uparrow^3 n = a \uparrow^2 a \uparrow^2 \dots \uparrow^2 a, \quad (n \text{ a's}),$$

$$\vdots$$

where all operations  $*$  are assumed to be *right associative*; i.e.,

$$a * b * c = a * (b * c).$$

Exponentiation is not associative, since

$$2 \uparrow (1 \uparrow 2) = 2 \uparrow 1 = 2^1 = 2 \text{ and } (2 \uparrow 1) \uparrow 2 = 2 \uparrow 2 = 2^2 = 4.$$

## Superpowers – Definition

- Superpowers  $a \uparrow^m n$  are defined inductively,

$$a \uparrow^1 n = a \uparrow n \quad (199)$$

and for  $m \geq 2$ ,

$$a \uparrow^m n = \underbrace{a \uparrow \uparrow \dots \uparrow}_m n \quad (200)$$

$$= \underbrace{a \uparrow \dots \uparrow}_{m-1} \underbrace{a \uparrow \dots \uparrow}_{m-1} \underbrace{a \dots \uparrow \dots \uparrow}_{m-1} a$$

$a$  appears  $n$ -times

$$= a \uparrow^{m-1} a \uparrow^{m-1} \dots \uparrow^{m-1} a, \quad (n \text{ a's}).$$

- Evaluation by right associativity

$$a \uparrow^m n = \quad (201)$$

$$a \uparrow^{m-1} (a \uparrow^{m-1} (\dots (a \uparrow^{m-1} a) \dots)), \quad n \text{ a's.}$$

## Superpowers – Definition

- Definition of  $a \uparrow^m n$  can be extended to include  $m \in \{-2, -1, 0\}$  by setting

$$a \uparrow^{-2} n = n + 1, \quad (202)$$

$$a \uparrow^{-1} n = a + n, \quad (203)$$

$$a \uparrow^0 n = a \cdot n. \quad (204)$$

- Definition (200) is only valid for  $m \geq 0$ , since e.g.

$$a \uparrow^{-1} 3 = a + 3$$

and

$$a \uparrow^{-1} 3 = a \uparrow^{-2} (a \uparrow^{-2} a) = a \uparrow^{-2} (a + 1) = a + 2.$$

But

$$a \uparrow^{-2} a \uparrow^{-1} 2 = a \uparrow^{-2} (a + 2) = a + 3.$$

## Superpowers – Recursion

For all numbers  $m \geq -1$  and  $n \geq 2$ ,

$$a \uparrow^m n = a \uparrow^{m-1} a \uparrow^m (n-1). \quad (205)$$

### Proof.

We have

$$\begin{aligned} a \uparrow^m n &= a \uparrow^{m-1} a \uparrow^{m-1} \dots \uparrow^{m-1} a, \quad (n \text{ a's}), \\ &= a \uparrow^{m-1} (a \uparrow^{m-1} \dots \uparrow^{m-1} a) \\ &= a \uparrow^{m-1} a \uparrow^m (n-1). \end{aligned}$$



## Superpowers – Example

For all  $m \geq -1$ .

$$2 \uparrow^m 2 = 4. \quad (206)$$

## Proof.

We have  $2 \uparrow^{-1} 2 = 2 + 2 = 4$ .

Suppose  $2 \uparrow^m 2 = 4$  holds for some  $m \geq -1$ . Then by (205),

$$2 \uparrow^{m+1} 2 = 2 \uparrow^m (2 \uparrow^{m+1} 1).$$

But  $2 \uparrow^{m+1} 1 = 2$  and so by induction,

$$2 \uparrow^{m+1} 2 = 2 \uparrow^m 2 = 4.$$



## Tetration

*Tetration* is the dyadic function

$$a \uparrow^2 n = \underbrace{a^{a^{\dots^a}}}_n. \quad (207)$$

We have  $a \uparrow^2 0 = 1$ ,  $a \uparrow^2 1 = a$ , and by (205),

$$a \uparrow^2 (n + 1) = a \uparrow (a \uparrow^2 n). \quad (208)$$

For  $a = 2$ , the first few values are

$$\begin{aligned} 2 \uparrow^2 1 &= 2, \\ 2 \uparrow^2 2 &= 2 \uparrow 2 = 4, \\ 2 \uparrow^2 3 &= 2 \uparrow (2 \uparrow 2) = 2 \uparrow 4 = 16, \\ 2 \uparrow^2 4 &= 2 \uparrow (2 \uparrow (2 \uparrow 2)) = 2 \uparrow 16 = 65536, \\ 2 \uparrow^2 5 &= 2 \uparrow 65536 \approx 2 \cdot 10^{19728}. \end{aligned}$$

## Pentation

*Pentation* is the expression

$$a \uparrow^3 n, \quad (209)$$

which is a power tower with  $n - 1$  power towers given by tetration.

We have

$$2 \uparrow^3 3 = 2 \uparrow^2 (2 \uparrow^2 2) = 2^{16} = 65,536,$$

$$\begin{aligned} 3 \uparrow^3 3 &= 3 \uparrow^2 (3 \uparrow^2 3) \\ &= 3 \uparrow^2 7,625,597,848,987, \end{aligned}$$

where

$$\begin{aligned} 3 \uparrow^2 3 &= 3 \uparrow (3 \uparrow 3) = 3 \uparrow 27 \\ &= 7,625,597,848,987 \approx 7,6 \cdot 10^{12}. \end{aligned}$$

The number  $3 \uparrow^3 3$  is called *tritritri* and is a power tower with 7,625,597,848,987 3's.

## \*Infinite Tetration

Consider the infinite tetration of real numbers

$$x^{x^{x^{\dots}}} = a \quad (210)$$

for  $x > 0$  and its convergence to a number  $a$ .

Let  $x > 0$ . The tetration limit

$$x^{x^{x^{\dots}}}$$

exists for  $x \in [\frac{1}{e^e}, e^{\frac{1}{e}}]$ .

## \*Infinite Tetration – Proof

The expression (210) has the form  $a = x^a$ . For this, we consider the real-valued function  $f(t) = x^t$  for  $x > 0$ . The convergence  $\lim_{t \rightarrow a} f(t) = f(a)$  means that

$$|f(t) - f(a)| < |t - a|$$

with  $f(a) = a$  and  $t$  sufficiently close to  $a$ . Thus

$$\frac{|f(t) - f(a)|}{|t - a|} < 1.$$

By the mean value theorem,

$$|f'(\xi)| < 1$$

for some  $\xi$  between  $t$  and  $a$ . Since  $f'$  is continuous, we must have

$$|f'(a)| \leq 1.$$

Moreover,  $f(t) = e^{\ln(x^t)} = e^{t \ln(x)}$  and so  $f'(t) = e^{t \ln(x)} \ln(x) = f(t) \ln(x)$ . Thus

$$f'(a) = x^a \ln(x) = a \ln(x) = \ln(x^a) = \ln(a)$$

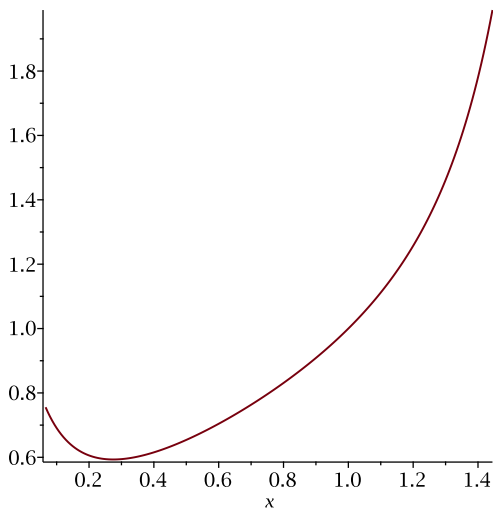
and therefore

$$|\ln(a)| \leq 1.$$

This leads to  $a \in [1/e, e]$ . But  $a = x^a$  and so  $x = a^{1/a}$ . Hence  $x \in [1/e^e, e^{1/e}]$  as claimed.

## \*Tetration (Maple™)

```
> plot( x^(x^(x^x)), x = 1/e^e .. e^(1/e));
```



## Ackermann Functional and Class

- *Ackermann functional*  $\Gamma$  is defined on the set of dyadic functions  $f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  as follows,

$$\Gamma(f(m, n)) = f(m - 1, f(m, n - 1)), \quad m, n \in \mathbb{N}. \quad (211)$$

- *Ackermann class*  $\mathcal{A}$  is the set of all dyadic functions  $f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  which are fixed points of the Ackermann functional  $\Gamma$  for large enough values of  $m$  and  $n$ ; i.e., there are  $m_0$  and  $n_0$  such that for all  $m \geq m_0$  and  $n \geq n_0$ ,

$$f(m, n) = \Gamma(f(m, n)). \quad (212)$$

By (189), the Ackermann function belongs to the class  $\mathcal{A}$ :

$$A(m + 1, n + 1) = A(m, A(m + 1, n)), \quad m, n \geq 0,$$

i.e.,

$$A(m, n) = A(m - 1, A(m, n - 1)), \quad m, n \geq 1.$$

## Ackermann Class – Boundaries

- Definition of function in Ackermann class requires *boundary conditions*.
- Boundary conditions for the Ackermann function,

$$A(0, n) = n + 1, \quad n \geq 0, \quad (213)$$

$$A(m, 0) = A(m - 1, 1), \quad m \geq 1. \quad (214)$$

## Ackermann Class – Superpowers

Consider the superpowers

$$f(a, m, n) = a \uparrow^m n. \quad (215)$$

Then by (205),

$$\begin{aligned} f(a, m, n) &= a \uparrow^m n = a \uparrow^{m-1} (a \uparrow^m (n-1)) & (216) \\ &= f(a, m-1, f(a, m, n-1)). \end{aligned}$$

For fixed  $a \in \mathbb{N}_0$ , define the dyadic function  $f_a : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  as

$$f_a(m, n) = f(a, m, n). \quad (217)$$

Then by (216),

$$f_a(m, n) = f_a(m-1, f_a(m, n-1)). \quad (218)$$

Hence,  $f_a = f(a, \cdot, \cdot)$  belongs to the Ackermann class.

## Ackermann Class – Main Result

Let  $a$ ,  $\alpha$ ,  $k$ ,  $m$  and  $n$  be numbers with  $n + k \geq 2$  and  $m + \alpha \geq 1$ .  
Then the dyadic function

$$f(m, n) = a \uparrow^{m+\alpha} (n + k) - k \quad (219)$$

belongs to the Ackermann class for fixed values of  $a$ ,  $\alpha$ , and  $k$ .

### Proof.

We have

$$\begin{aligned} f(m, n) &= a \uparrow^{m+\alpha} (n + k) - k \\ &= a \uparrow^{m+\alpha-1} (a \uparrow^{m+\alpha} (n + k - 1)) - k, \quad \text{by (205),} \\ &= a \uparrow^{m-1+\alpha} (a \uparrow^{m+\alpha} (n - 1 + k) - k + k) - k \\ &= a \uparrow^{m-1+\alpha} (f(m, n - 1) + k) - k, \quad \text{by (219),} \\ &= f(m - 1, f(m, n - 1)), \quad \text{by (219).} \end{aligned}$$



## Ackermann Class – Main Result

## Ackermann function

$$A(m, n) = 2 \uparrow^{m-2} (n + 3) - 3, \quad m, n \in \mathbb{N}_0. \quad (220)$$

## Proof

- By (219), the function  $A(m, n)$  with  $a = 2$ ,  $\alpha = -2$ , and  $k = 3$  belongs to the Ackermann class.
- Boundary condition  $A(0, n) = n + 1$  for all  $n \geq 0$  is satisfied, since by definition

$$A(0, n) = 2 \uparrow^{-2} (n + 3) - 3 = (n + 4) - 3 = n + 1.$$

## Proof (Cont'd)

- Boundary condition  $A(m, 0) = A(m - 1, 1)$  for all  $m \geq 1$  is also fulfilled. To see this, the left-hand side gives by (205),

$$2 \uparrow^{m-2} 3 - 3 = 2 \uparrow^{m-3} (2 \uparrow^{m-2} 2) - 3,$$

while the right-hand side gives

$$2 \uparrow^{m-3} 4 - 3.$$

Moreover, by (206),  $2 \uparrow^{m-2} 2 = 4$  for each  $m \geq 1$  and hence the boundary condition holds.



## Ackermann Function – Example

By (220),

$$A(0, n) = 2 \uparrow^{-2} (n + 3) - 3 = n + 3 + 1 - 3 = n + 1,$$

$$A(1, n) = 2 \uparrow^{-1} (n + 3) - 3 = n + 3 + 2 - 3 = n + 2,$$

$$A(2, n) = 2 \uparrow^0 (n + 3) - 3 = 2(n + 3) - 3 = 2n + 3,$$

$$A(3, n) = 2 \uparrow^1 (n + 3) - 3 = 2^{n+3} - 3,$$

$$A(4, n) = 2 \uparrow^2 (n + 3) - 3 = \underbrace{a^{a^{\dots^a}}}_{n+3} - 3.$$

## Ackermann Function – Example

## Tetration

$$A(4, n) = 2 \uparrow^2 (n + 3) - 3, \quad n \geq 0,$$

$$A(4, 0) = 2 \uparrow^2 3 - 3 = 2 \uparrow (2 \uparrow 2) - 3 = 2^4 - 3 = 13,$$

$$A(4, 1) = 2 \uparrow^2 4 - 3 = 2 \uparrow (2 \uparrow (2 \uparrow 2)) - 3 = 2^{2^4} - 3 = 65533,$$

$$A(4, 2) = 2 \uparrow^2 5 - 3 = 2 \uparrow (2 \uparrow (2 \uparrow (2 \uparrow 2))) - 3 = 2^{65536} - 3,$$

$$A(4, 3) = 2 \uparrow^2 6 - 3 = 2 \uparrow (2 \uparrow (2 \uparrow (2 \uparrow (2 \uparrow 2)))) - 3$$

$$= 2^{2^{65536}} - 3,$$

$$A(4, 4) = 2 \uparrow^2 7 - 3 = 2 \uparrow (2 \uparrow (2 \uparrow (2 \uparrow (2 \uparrow (2 \uparrow 2)))))) - 3$$

$$= 2^{2^{2^{65536}}} - 3.$$

The computation of  $A(4, 1)$  may already abort due to too many levels of recursion.

## Skills

- Evaluation of small Ackermann function
- Computation of complexity of LOOP program
- Evaluation of Ackermann function for small values
- Evaluation and interpretation of superpowers for small values

# Part V

# Programming Systems

## Programming Systems – Motivation

- Gödel numbering of partial recursive functions.
- Entry point into the theory of computability.
- Main theorems (parametrization, universal computation, normalization of minimalization).

Addition program  $P_+$

$s_0$ :	0	if $x_2 = 0$	3	1
$s_1$ :	1	$x_1 \leftarrow x_1 + 1$	2	
$s_2$ :	2	$x_2 \leftarrow x_2 - 1$	0	

has Gödel number

$$\rho(P_+) = 1, 269, 420, 373, 033, 475, 160, 642, 134.$$

## Programming Systems – Contents

- Gödel numbering of GOTO programs and partial recursive functions
- Kleene's smn theorem (parametrization)
- Universal functions
- Kleene's normal form

Reading: Zimmermann, Section 2.3 (pairing functions), Chapter 5

## Gödel Numbering – Contents

- Cantor's pairing function
- Gödel numbering of  $\mathbb{N}_0^*$
- Gödel numbering of GOTO programs
- Gödel numbering of partial recursive functions

In mathematical logic, *Gödel numbering* refers to a function that assigns to each well-formed formula of some formal language a unique natural number called its *Gödel number* (Kurt Gödel, 1906-1978).

Maple™ demo software: `cc-J-proc.mw` (clickable worksheet)

## Cantor's Pairing Function – Definition

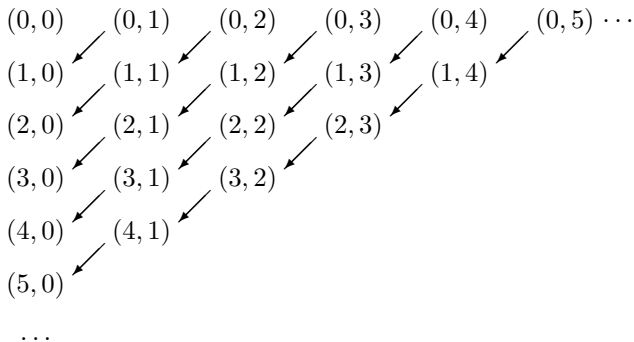
- Each primitive recursive bijection  $f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  is called a *pairing function*.
- *Cantor's pairing function*  $\sigma_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  is defined as

$$\begin{aligned}\sigma_2(m, n) &= \binom{m+n+1}{2} + m & (221) \\ &= \frac{(m+n)(m+n+1)}{2} + m.\end{aligned}$$

As a consequence, Cartesian product  $\mathbb{N}_0^2$  is denumerable since  $\mathbb{N}_0$  is denumerable.

## Proof.

- List the elements of  $\mathbb{N}_0^2$  in form of a table:



- $\sigma_2$  provides a numbering of the elements  $\mathbb{N}_0^2$  by running successively through the anti-diagonals starting with  $(0, 0)$ . Hence,  $\sigma_2$  is bijective.



## Cantor's Pairing Function

Cantor's pairing function  $\sigma_2$  is primitive recursive.

### Proof

- By (86), integral division function  $\div$  is primitive recursive by bounded minimalization:
- By the representation of the binomial coefficient,

$$\binom{n}{2} = \frac{(n-1)n}{2} = \div((n-1) \cdot n, 2), \quad n \geq 2,$$

we have

$$\sigma_2(m, n) = \div((m+n) \cdot (m+n+1), 2) + m.$$

Thus  $\sigma_2$  is a composition of primitive recursive function and hence also primitive recursive. □

## Cantor's Pairing Function – Inverse

There are the coordinate functions  $\kappa_2, \tau_2 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that for all  $m, n \in \mathbb{N}_0$ :

$$\kappa_2(\sigma_2(m, n)) = m, \quad (222)$$

$$\tau_2(\sigma_2(m, n)) = n, \quad (223)$$

$$\sigma_2(\kappa_2(n), \tau_2(n)) = n. \quad (224)$$

## Proof

Let  $n \in \mathbb{N}_0$ .

- There exists a number  $t \geq 0$  such that

$$\frac{1}{2}t(t+1) \leq n < \frac{1}{2}(t+1)(t+2). \quad (225)$$

- Put

$$m = n - \frac{1}{2}t(t+1). \quad (226)$$

## Proof (Cont'd)

Put  $m = n - \frac{1}{2}t(t+1) = n - \binom{t+1}{2}$ .

■ Then

$$\begin{aligned} m &= n - \frac{1}{2}t(t+1) \\ &\leq \left[ \frac{1}{2}(t+1)(t+2) - 1 \right] - \left[ \frac{1}{2}t(t+1) \right] = t. \end{aligned}$$

■ Put

$$\kappa_2(n) = m \quad \text{and} \quad \tau_2(n) = t - m. \quad (227)$$

■ Then

$$\begin{aligned} \sigma_2(\kappa_2(n), \tau_2(n)) &= \sigma_2(m, t - m) \\ &= \binom{m + (t - m) + 1}{2} + m = n. \quad \square \end{aligned}$$

## Cantor's Pairing Function – Example

Let  $n = 17$ .

- Then

$$\frac{1}{2}t(t+1) \leq 17 < \frac{1}{2}(t+1)(t+2)$$

is satisfied by  $t = 5$ , since  $15 \leq 17 < 21$ .

- Put  $m = n - \frac{1}{2}t(t+1) = 2$ .
- Then  $\kappa_2(17) = m = 2$  and  $\tau_2(17) = t - m = 3$ .
- Check:

$$\sigma_2(2, 3) = \binom{2+3+1}{2} + 2 = \binom{6}{2} + 2 = 17. \quad (228)$$

## Cantor's Pairing Function

Coordinate functions  $\kappa_2$  and  $\tau_2$  are primitive recursive.

**Proof.**

Let  $n \in \mathbb{N}_0$ .

- The number  $t$  in (225) can be obtained by the primitive recursive bounded minimalization:

$$t = \bar{\mu}(t \leq n) \left[ \frac{1}{2}(t+1)(t+2) \div n > 0 \right].$$

- Thus  $\kappa_2(n) = n - \frac{1}{2}t(t+1)$  and  $\tau_2(n) = t - \kappa_2(n)$  are compositions of primitive recursive functions and hence primitive recursive.



## Gödel Numbering of $\mathbb{N}_0^*$

$\mathbb{N}_0^*$  denotes the union of all Cartesian products  $\mathbb{N}_0^\ell$ ,  $\ell \geq 0$ , i.e.,

$$\mathbb{N}_0^* = \bigcup_{\ell \geq 0} \mathbb{N}_0^\ell. \quad (229)$$

In particular,  $\mathbb{N}_0^0 = \{\epsilon\}$ , where  $\epsilon$  is the empty string, and  $\mathbb{N}_0^1 = \mathbb{N}_0$ .

Gödel Numbering of  $\mathbb{N}_0^*$  – Definition

Define encoding

$$\sigma : \mathbb{N}_0^* \rightarrow \mathbb{N}_0 \quad (230)$$

using Cantor's pairing function  $\sigma_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ , i.e., for all  $x \in \mathbb{N}_0^*$  and  $y \in \mathbb{N}_0$ ,

$$\sigma(\epsilon) = 0, \quad (231)$$

$$\sigma(x) = \sigma_2(0, x) + 1, \quad (232)$$

$$\sigma(x, y) = \sigma_2(\sigma(x), y) + 1. \quad (233)$$

Second equation (232) is a special case of the third one (233), since for each  $y \in \mathbb{N}_0$ ,

$$\sigma(\epsilon, y) = \sigma_2(\sigma(\epsilon), y) + 1 = \sigma_2(0, y) + 1 = \sigma(y). \quad (234)$$

$\sigma(x)$  is called the *Gödel number* of the string  $x \in \mathbb{N}_0^*$ .

Gödel Numbering of  $\mathbb{N}_0^*$  – Example

We have

$$\begin{aligned}\sigma(1, 3) &= \sigma_2(\sigma(1), 3) + 1 && (235) \\ &= \sigma_2(\sigma_2(0, 1) + 1, 3) + 1 \\ &= \sigma_2(2, 3) + 1 \\ &= 17 + 1 = 18.\end{aligned}$$

## Gödel Numbering of $\mathbb{N}_0^*$

Encoding function  $\sigma$  is well-defined.

### Proof.

In each step of the evaluation of  $\sigma$ , the length of the string decreases by 1.

More concretely, by (233), the evaluation of  $\sigma(x, y)$  is replaced by the evaluation of  $\sigma(x)$ , where  $x \in \mathbb{N}_0^*$  and  $y \in \mathbb{N}_0$ .  $\square$

Gödel Numbering of  $\mathbb{N}_0^*$ 

Encoding function  $\sigma$  is primitive recursive.

**Proof.**

- For strings of length  $\leq 1$ , by (231)-(232),  $\sigma$  is primitive recursive, since  $\sigma_2$  is primitive recursive.
- Suppose  $\sigma$  is primitive recursive for strings of length  $\leq \ell$ , where  $\ell \geq 1$ .

For strings of length  $\ell + 1$ , by (233),  $\sigma$  can be written as composition of primitive recursive functions:<sup>a</sup>

$$\sigma_{|\mathbb{N}_0^{\ell+1}} = \nu \circ \sigma_2(\sigma_{|\mathbb{N}_0^\ell}(\pi_1^{(\ell+1)}, \dots, \pi_\ell^{(\ell+1)}), \pi_{\ell+1}^{(\ell+1)})_{|\mathbb{N}_0^{\ell+1}}. \quad (236)$$

By induction hypothesis,  $\sigma$  is primitive recursive for strings of length  $\leq \ell$  and so the right-hand side is a composition of primitive recursive functions and hence also primitive recursive. □

---

<sup>a</sup> $\sigma(\mathbf{x}, y) = \sigma_2(\sigma(\mathbf{x}), y) + 1.$

Gödel Numbering of  $\mathbb{N}_0^*$ 

Encoding function  $\sigma$  is bijective.

**\*Proof**

Let  $A$  be the set of numbers  $n \in \mathbb{N}_0$  such that there is a unique  $\mathbf{x} \in \mathbb{N}_0^*$  with  $\sigma(\mathbf{x}) = n$ . Claim that  $A = \mathbb{N}_0$ .

- 0 lies in  $A$  since  $\sigma(\epsilon) = 0$  and  $\sigma(\mathbf{x}) > 0$  for all  $\mathbf{x} \neq \epsilon$ .
- Let  $n > 0$ . Define

$$u = \kappa_2(n - 1) \quad \text{and} \quad v = \tau_2(n - 1). \quad (237)$$

Then by (224),  $\sigma_2(u, v) = \sigma_2(\kappa_2(n - 1), \tau_2(n - 1)) = n - 1$ .

By construction,  $\kappa_2(z) \leq z$  and  $\tau_2(z) \leq z$  for all  $z \in \mathbb{N}_0$ .

Thus  $u = \kappa_2(n - 1) < n$  and hence  $u \in A$ , i.e., there is exactly one string  $\mathbf{x} \in \mathbb{N}_0^*$  such that  $\sigma(\mathbf{x}) = u$ .

Then  $\sigma(\mathbf{x}, v) = \sigma_2(\sigma(\mathbf{x}), v) + 1 = \sigma_2(u, v) + 1 = n$ .

### \*Proof (Cont'd)

Suppose for some  $\mathbf{x}, \mathbf{y} \in \mathbb{N}_0^*$  and  $v, w \in \mathbb{N}_0$ ,

$$\sigma(\mathbf{x}, v) = n = \sigma(\mathbf{y}, w).$$

Then by (233),

$$\sigma_2(\sigma(\mathbf{x}), v) = \sigma_2(\sigma(\mathbf{y}), w).$$

But  $\sigma_2$  is bijective and so  $\sigma(\mathbf{x}) = \sigma(\mathbf{y})$  and  $v = w$ .

Since  $\sigma(\mathbf{x}) = u < n$ , by induction  $\mathbf{x} = \mathbf{y}$ .

Thus  $n \in A$  and hence by the induction axiom,  $A = \mathbb{N}_0$ . □

Gödel Numbering of  $\mathbb{N}_0^*$  – Functions  $\kappa, \tau$ 

Define the functions  $\kappa, \tau : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that for all  $n \in \mathbb{N}_0$ ,

$$\kappa(n) = \kappa_2(n \dot{-} 1) \quad \text{and} \quad \tau(n) = \tau_2(n \dot{-} 1), \quad (238)$$

where  $\kappa_2, \tau_2$  form the inverse functions of  $\sigma_2$ , i.e.,

$$\sigma_2(\kappa_2(n), \tau_2(n)) = n, \quad n \in \mathbb{N}_0. \quad (239)$$

Marginal conditions:

$$\kappa(1) = \kappa(0) = 0 \quad \text{and} \quad \tau(1) = \tau(0) = 0. \quad (240)$$

Gödel Numbering of  $\mathbb{N}_0^*$  – Functions  $\kappa, \tau$ 

Functions  $\kappa, \tau$  are primitive recursive, and for each  $n \geq 1$  there are unique  $\mathbf{x} \in \mathbb{N}_0^*$  and  $y \in \mathbb{N}_0$  with  $\sigma(\mathbf{x}, y) = n$  such that

$$\kappa(n) = \sigma(\mathbf{x}) \quad \text{and} \quad \tau(n) = y. \quad (241)$$

## Proof

- $\kappa$  and  $\tau$  are compositions of primitive recursive functions in (238) and thus are also primitive recursive.
- Let  $n \geq 1$ . Since  $\sigma$  is bijective, there are unique  $\mathbf{x} \in \mathbb{N}_0^*$  and  $y \in \mathbb{N}_0$  such that  $\sigma(\mathbf{x}, y) = n$ .

Thus by (233),  $\sigma_2(\sigma(\mathbf{x}), y) = \sigma(\mathbf{x}, y) - 1 = n - 1$ .

But by (224),  $\sigma_2(\kappa_2(n - 1), \tau_2(n - 1)) = n - 1$ .

Since  $\sigma_2$  is bijective,  $\kappa_2(n - 1) = \sigma(\mathbf{x})$  and  $\tau_2(n - 1) = y$ .

Hence  $\kappa(n) = \sigma(\mathbf{x})$  and  $\tau(n) = y$ . □

Gödel Numbering of  $\mathbb{N}_0^*$  – String Length

A string  $\mathbf{x} \in \mathbb{N}_0^*$  has length  $\ell \geq 0$  if  $\ell$  is the smallest number such that

$$\kappa^\ell(\sigma(\mathbf{x})) = 0. \quad (242)$$

## Proof

- For the empty string,  $\kappa^0(\sigma(\epsilon)) = \sigma(\epsilon) = 0$ , since  $\kappa^0 = \text{id}_{\mathbb{N}_0}$ .
- Let  $\mathbf{x} = x_1 \dots x_\ell \in \mathbb{N}_0^\ell$  with  $\ell \geq 1$ .

We have  $\sigma(\mathbf{x}) = n$  for some  $n \geq 1$ .

By (241),  $\sigma(x_1 \dots x_{\ell-1}) = \kappa(n)$ , and by induction,  $\kappa^{\ell-1}(\sigma(x_1 \dots x_{\ell-1})) = 0$ , where  $\ell - 1$  is minimal with this property. Thus

$$\begin{aligned} \kappa^\ell(\sigma(\mathbf{x})) &= \kappa^\ell(n) = \kappa^{\ell-1}(\kappa(n)) \\ &= \kappa^{\ell-1}(\sigma(x_1 \dots x_{\ell-1})) = 0 \end{aligned}$$

and  $\ell$  is minimal with this property. □

## Gödel Numbering of $\mathbb{N}_0^*$ – Example

Take a number  $x \in \mathbb{N}_0$ , i.e., a string of length 1.

Then by definition  $\sigma(x) = n \geq 1$ . By (234),

$$\sigma(x) = \sigma(\epsilon, x)$$

and therefore by (241),

$$\kappa^1(n) = \sigma(\epsilon) = 0 \quad \text{and} \quad \tau(n) = x.$$

Gödel Numbering of  $\mathbb{N}_0^*$  – Length

The *length function*  $\lg : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  given by

$$\lg(n) = \mu \ell (\kappa^\ell(n) = 0) \quad (243)$$

is primitive recursive.

**Proof.**

Since  $\kappa$  is primitive recursive,  $\kappa^\ell$  is primitive recursive.

Minimalization is restricted as we have  $\ell \leq n$ ; i.e., the string is not longer than the number it encodes.

In view of minimalization, the correct definition would be

$f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (n, \ell) \mapsto \kappa^\ell(n)$  and then

$$\lg(n) = \mu \ell [f(n, \ell) = 0].$$



By (242),  $\lg$  assigns to each Gödel number  $n$  the length  $\ell$  of the string  $x$  with  $\sigma(x) = n$ .

## Gödel Numbering of $\mathbb{N}_0^*$ – Inverse

The inverse value of  $n \geq 1$  is given by

$$\sigma^{-1}(n) = (\tau \circ \kappa^{\ell-1}(n), \dots, \tau \circ \kappa(n), \tau(n)) \in \mathbb{N}_0^\ell, \quad (244)$$

where  $\lg(n) = \ell$ .

### Proof

Let  $x \in \mathbb{N}_0$ .

- Then  $\sigma(x) = n \geq 1$  and by (234),  $\sigma(x) = \sigma(\epsilon, x)$ .
- Thus by (241),

$$\kappa(n) = \sigma(\epsilon) = 0 \quad \text{and} \quad \tau(n) = x.$$

- Hence,  $\sigma^{-1}(n) = x = \tau(n)$ .

## Proof (Cont'd)

Let  $\mathbf{x} \in \mathbb{N}_0^\ell$ ,  $\ell \geq 1$ , and  $y \in \mathbb{N}_0$ .

- Then  $\sigma(\mathbf{x}, y) = n \geq 1$ .

- By (241),

$$\kappa(n) = \sigma(\mathbf{x}) \quad \text{and} \quad \tau(n) = y.$$

Since  $\kappa(n) < n$ ,  $\mathbf{x} = \sigma^{-1}(\kappa(n))$  is by induction given by

$$\begin{aligned} \mathbf{x} &= \sigma^{-1}(\kappa(n)) \\ &= ((\tau \circ \kappa^{\ell-1})(\kappa(n)), \dots, (\tau \circ \kappa)(\kappa(n)), \tau(\kappa(n))) \\ &= (\tau \circ \kappa^\ell(n), \dots, \tau \circ \kappa^2(n), \tau \circ \kappa(n)). \end{aligned}$$

- Hence,  $(\mathbf{x}, y)$  has the form

$$(\mathbf{x}, y) = \sigma^{-1}(n) = (\tau \circ \kappa^\ell(n), \dots, \tau \circ \kappa(n), \tau(n))$$

as required. □

Gödel Numbering of  $\mathbb{N}_0^*$  – Example

Let  $n = 18$ .

- Computation of length:

$$\kappa^2(18) = \kappa(\kappa(18)) = \kappa(\kappa_2(17)) = \kappa(2) = \kappa_2(1) = 0,$$

since by (228),  $\sigma_2(2, 3) = 17 [= \sigma_2(\kappa_2(17), \tau_2(17))]$  and so  $\kappa_2(17) = 2$ .

Thus by (243),  $\lg(18) = 2$ .

- Computation of string:

$$\tau(18) = \tau_2(17) = 3,$$

since  $\sigma_2(2, 3) = 17 [= \sigma_2(\kappa_2(17), \tau_2(17))]$  and so  $\tau_2(17) = 3$ ,  
and

$$\tau(\kappa(18)) = \tau(\kappa_2(17)) = \tau(2) = \tau_2(1) = 1.$$

Hence,  $\sigma(1, 3) = 18$ , see (235).

## Gödel Numbering of GOTO Instructions – Definition

Let  $P = s_0; s_1; \dots; s_m$  be a standard GOTO (SGOTO) program.

The  $l$ -th instruction  $s_l$  has label  $l$ ,  $0 \leq l \leq m$ , and is encoded by

$$\sigma'(s_l) = \begin{cases} 3 \cdot \sigma(i, k) & \text{if } s_l = (l, x_i \leftarrow x_i + 1, k), \\ 3 \cdot \sigma(i, k) + 1 & \text{if } s_l = (l, x_i \leftarrow x_i - 1, k), \\ 3 \cdot \sigma(i, k, m) + 2 & \text{if } s_l = (l, \text{if } x_i = 0, k, m). \end{cases} \quad (245)$$

$\sigma'(s_l)$  is called the *Gödel number* of instruction  $s_l$ ,  $0 \leq l \leq m$ .

Function  $\sigma'$  is primitive recursive, since it is defined by primitive case distinction and the functions involved are primitive recursive.

## Gödel Numbering of GOTO Instructions – Inverse

Given the Gödel number  $e = \sigma'(s_l) \in \mathbb{N}_0$  of  $l$ -th instruction  $s_l$ .

- By division with remainder, write

$$e = 3n + t, \quad (246)$$

where  $n = \div(e, 3)$  and  $t = \text{mod}(e, 3)$ .

- By (244),  $\sigma^{-1}(n) = (\tau(\kappa(n)), \tau(n))$  for increment and decrement, and  $\sigma^{-1}(n) = (\tau(\kappa^2(n)), \tau(\kappa(n)), \tau(n))$  for branching. Then by (245),

$$s_l = \begin{cases} (l, x_{\tau(\kappa(n))} \leftarrow x_{\tau(\kappa(n))} + 1, \tau(n)) & \text{if } t = 0, \\ (l, x_{\tau(\kappa(n))} \leftarrow x_{\tau(\kappa(n))} - 1, \tau(n)) & \text{if } t = 1, \\ (l, \text{if } x_{\tau(\kappa^2(n))} = 0, \tau(\kappa(n)), \tau(n)) & \text{if } t = 2. \end{cases} \quad (247)$$

## Gödel Numbering of GOTO Instructions – Example

Suppose  $e = 55$  is the Gödel number of  $l$ -th instruction  $s_l$ .

- Write

$$55 = 3 \cdot 18 + 1$$

and so  $n = 18$  and  $t = 1$  (decrement).

- By (235),

$$\sigma(1, 3) = 18$$

and so

$$s_l = (l, x_1 \leftarrow x_1 - 1, 3).$$

## Gödel Numbering of GOTO Programs – Definition

SGOTO program  $P = s_0; s_1; \dots; s_m$  has the *Gödel number*

$$\rho(P) = \sigma(\sigma'(s_0), \sigma'(s_1), \dots, \sigma'(s_m)). \quad (248)$$

$\rho : \mathcal{P}_{\text{SGOTO}} \rightarrow \mathbb{N}_0$  is bijective and primitive recursive.

## Proof.

- Function  $\rho$  is bijective since  $\sigma$  is bijective and the instructions encoded by  $\sigma'$  are uniquely determined.
- Function  $\rho$  is a composition of primitive recursive functions and thus also primitive recursive.



## Gödel Numbering of GOTO Programs – Example

Consider the SGOTO program  $P_+$  given by

$s_0$ :	0	if $x_2 = 0$	3	1
$s_1$ :	1	$x_1 \leftarrow x_1 + 1$	2	
$s_2$ :	2	$x_2 \leftarrow x_2 - 1$	0	

Encoding of instructions:

$$\sigma'(s_0) = 3 \cdot \sigma(2, 3, 1) + 2 = 1889,$$

$$\sigma'(s_1) = 3 \cdot \sigma(1, 2) = 39,$$

$$\sigma'(s_2) = 3 \cdot \sigma(2, 0) + 1 = 46.$$

Encoding of program:

$$\begin{aligned} \rho(P_+) &= \sigma(1889, 39, 46) \\ &= 1, 269, 420, 373, 033, 475, 160, 642, 134. \end{aligned}$$

## Gödel Numbering of GOTO Programs – List

- $P_e$  denotes the SGOTO program  $P$  with Gödel number  $e$ .
- Gödel numbering provides a list of all SGOTO programs

$$P_0, P_1, P_2, \dots \quad (249)$$

- Conversely, each  $e \in \mathbb{N}_0$  can be uniquely assigned an SGOTO program  $P$  such that  $\rho(P) = e$  (by the above procedure).
- Gödel number  $e$  can be considered as a building plan of the SGOTO program  $P_e$ .

## Gödel Numbering of $\mathcal{R}$ – List

Let  $n \geq 0$ ,

- The  $n$ -adic partial recursive function computing the SGOTO program  $P_e$  is

$$\phi_e^{(n)} = \|P_e\|_{n,1}. \quad (250)$$

Then  $e$  is the *Gödel number* or *index* of  $\phi_e^{(n)}$ .

- The list of all SGOTO programs in (249) yields a list of all  $n$ -adic partial recursive functions:

$$\phi_0^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}, \dots \quad (251)$$

This list contains many repetitions, since each  $n$ -adic partial recursive function has infinitely many indices; e.g., add zero to the result.

## Parametrization

- Parametrization theorem (smn theorem) is a cornerstone of computability theory (Kleene, 1943)
- Refers to computable functions in which some arguments are considered as parameters.
- Applications in reduction steps and recursion theory.

## Parametrization – Dyadic Case

For each dyadic partial recursive function  $f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ , there is a monadic primitive recursive function  $g$  such that for all  $x \in \mathbb{N}_0$ ,

$$f(x, \cdot) = \phi_{g(x)}^{(1)}, \quad (252)$$

i.e., for all  $x, y \in \mathbb{N}_0$ ,

$$f(x, y) = \phi_{g(x)}^{(1)}(y).$$

## Proof

- Given SGOTO program  $P_e = s_0; s_1; \dots; s_m$  such that

$$f = \|P_e\|_{2,1}.$$

- For each  $x \in \mathbb{N}_0$ , consider the following SGOTO program  $Q_x$ :

0	if $x_1 = 0$	3	1
1	$x_2 \leftarrow x_2 + 1$	2	
2	$x_1 \leftarrow x_1 - 1$	0	
3	$x_1 \leftarrow x_1 + 1$	4	
4	$x_1 \leftarrow x_1 + 1$	5	
	⋮		
$2 + x$	$x_1 \leftarrow x_1 + 1$	$3 + x$	
	$s'_0$		
	$s'_1$		
	⋮		
	$s'_m$		

where  $P'_e = s'_0; s'_1; \dots; s'_m$  is the "shifted" SGOTO program that is derived from  $P_e$  by replacing each occurring label  $j$  with  $j + 3 + x$ ,  $0 \leq j \leq q$ .

## Proof (Cont'd)

- Milestones of the computation of  $Q_x$ :

0	$y$	0	0	0	...	init
0	0	$y$	0	0	...	step 3
0	$x$	$y$	0	0	...	step $3 + x$
0	$f(x, y)$	.	.	.	...	end

- Thus

$$\|Q_x\|_{1,1}(y) = f(x, y), \quad x, y \in \mathbb{N}_0.$$

- Take the function  $g : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  defined by

$$g(x) = \rho(Q_x), \quad x \in \mathbb{N}_0,$$

i.e.,  $g(x)$  is the Gödel number of the program  $Q_x$ . By (248), this function is primitive recursive.

- Hence,  $\phi_{g(x)}^{(1)} \stackrel{\text{def}}{=} \|Q_x\|_{1,1} = f(x, \cdot)$  for each  $x \in \mathbb{N}_0$ . □

## Parametrization – Example

Consider the dyadic addition

$$f_+ : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto x + y$$

and the corresponding SGOTO program  $P_+$

$$\begin{array}{llll} s_0 : & 0 & \text{if } x_2 = 0 & 3 \quad 1 \\ s_1 : & 1 & x_1 \leftarrow x_1 + 1 & 2 \\ s_2 : & 2 & x_2 \leftarrow x_2 - 1 & 0 \end{array}$$

with

$$f_+ = \|P_+\|_{2,1}.$$

Fixing the first argument to  $x = 3$  yields the monadic function

$$f_+(3, \cdot) : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : y \mapsto 3 + y.$$

## Parametrization – Example (Cont'd)

SGOTO program  $Q_3$  computing  $f_+(3, \cdot)$ :

0	<b>if</b> $x_1 = 0$	3	1
1	$x_2 \leftarrow x_2 + 1$	2	
2	$x_1 \leftarrow x_1 - 1$	0	
3	$x_1 \leftarrow x_1 + 1$	4	
4	$x_1 \leftarrow x_1 + 1$	5	
5	$x_1 \leftarrow x_1 + 1$	6	
6	<b>if</b> $x_2 = 0$	9	7
7	$x_1 \leftarrow x_1 + 1$	8	
8	$x_2 \leftarrow x_2 - 1$	6	

The last three statements provide the "shifted" program  $P_+$ .

## Parametrization – Example (Cont'd)

Milestones of the computation of  $Q_3$ :

0	$y$	0	0	0	...	init
0	0	$y$	0	0	...	step 3
0	3	$y$	0	0	...	step 6
0	$3 + y$	.	.	.	...	end

Let  $g(3) = \rho(Q_3)$  denote the Gödel number of  $Q_3$ . Then

$$\phi_{g(3)}^{(1)} \stackrel{\text{def}}{=} \|Q_3\|_{1,1} = f_+(3, \cdot).$$

## Parametrization – Example

Consider the dyadic addition

$$f_+ : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto x + y$$

and the corresponding URM program

$$P_+ = (A1; S2)2,$$

i.e.,

$$\|P_+\|_{2,1} = f_+.$$

Fixing the first argument to  $x = 3$  yields the monadic function

$$f_+(3, \cdot) : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : y \mapsto 3 + y.$$

Consider the URM program

$$P_{+3} = (S1; A2)1; A1; A1; A1; P_+.$$

## Parametrization – Example (Cont'd)

Consider the URM program

$$P_{+3} = (S1; A2)1; A1; A1; A1; P_+.$$

Milestones of the computation of  $P_{+3}$ :

0	$y$	0	0	0	...	init
0	0	$y$	0	0	...	reload $y$
0	3	$y$	0	0	...	generate $x = 3$
0	$3 + y$	.	.	.	...	addition

Gödel number of  $P_{+3}$  is obtained by converting into SGOTO program (by tokenization) and then using (248).

## Parametrization – General Case

For each pair  $m, n \geq 1$ , there is an  $m + 1$ -adic primitive recursive function  $s_{m,n}$  such that for all  $e \in \mathbb{N}_0$  and  $\mathbf{x} \in \mathbb{N}_0^m$ ,

$$\phi_e^{(m+n)}(\mathbf{x}, \cdot) = \phi_{s_{m,n}(e, \mathbf{x})}^{(n)}, \quad (253)$$

i.e., for all  $e \in \mathbb{N}_0$ ,  $\mathbf{x} \in \mathbb{N}_0^m$  and  $\mathbf{y} \in \mathbb{N}_0^n$ ,

$$\phi_e^{(m+n)}(\mathbf{x}, \mathbf{y}) = \phi_{s_{m,n}(e, \mathbf{x})}^{(n)}(\mathbf{y}).$$

## Proof (Similar to dyadic case)

- Take an SGOTO program  $P_e = s_0; s_1; \dots; s_m$  calculating the function  $\phi_e^{(m+n)}$ , i.e.,

$$\phi_e^{(m+n)} = \|P_e\|_{m+n,1}.$$

- For each  $\mathbf{x} \in \mathbb{N}_0^m$ , extend  $P_e$  to SGOTO program  $Q_{e,\mathbf{x}}$ :

0	$\mathbf{y}$	0	0	0	...	init
0	$\mathbf{0}$	$\mathbf{y}$	0	0	...	reload $\mathbf{y}$
0	$\mathbf{x}$	$\mathbf{y}$	0	0	...	generate parameter $\mathbf{x}$
0	$\phi_e^{(m+n)}(\mathbf{x}, \mathbf{y})$	.	.	.	...	apply $P_e$

- Then

$$\|Q_{e,\mathbf{x}}\|_{n,1}(\mathbf{y}) = \phi_e^{(m+n)}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \mathbb{N}_0^m, \mathbf{y} \in \mathbb{N}_0^n.$$

## Proof (Cont'd)

- We have

$$\|Q_{e,\mathbf{x}}\|_{n,1}(\mathbf{y}) = \phi_e^{(m+n)}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \mathbb{N}_0^m, \mathbf{y} \in \mathbb{N}_0^n.$$

- Consider the function  $s_{m,n} : \mathbb{N}_0^{m+1} \rightarrow \mathbb{N}_0$  defined by

$$s_{m,n}(e, \mathbf{x}) = \rho(Q_{e,\mathbf{x}}),$$

i.e.,  $s_{m,n}(e, \mathbf{x})$  is the Gödel number of program  $Q_{e,\mathbf{x}}$ .  
By (248), this function is primitive recursive.

- Hence,  $\phi_{s_{m,n}(e,\mathbf{x})}^{(n)} = \|Q_{e,\mathbf{x}}\|_{n,1}$ . □

## Universal Function

Another basic result of computability theory is the existence of a computable function called universal function (interpreter) that is capable of computing any other computable function (of the same arity).

Let  $n \geq 1$ . A *universal function* for  $n$ -adic partial recursive functions is an  $n + 1$ -adic function  $\psi_{\text{univ}}^{(n)} : \mathbb{N}_0^{n+1} \rightarrow \mathbb{N}_0$  such that for all  $e \in \mathbb{N}_0$ ,

$$\psi_{\text{univ}}^{(n)}(e, \cdot) = \phi_e^{(n)}, \quad (254)$$

i.e., for all  $e \in \mathbb{N}_0$  and  $\mathbf{x} \in \mathbb{N}_0^n$ ,

$$\psi_{\text{univ}}^{(n)}(e, \mathbf{x}) = \phi_e^{(n)}(\mathbf{x}). \quad (255)$$

Universal function  $\psi_{\text{univ}}^{(n)}$  takes as input Gödel number  $e$  and argument  $\mathbf{x}$  in order to compute  $\phi_e^{(n)}(\mathbf{x})$ .

## Main Result

For each arity  $n \geq 1$ , the universal function  $\psi_{\text{univ}}^{(n)}$  exists and is partial recursive.

## Proof[Sketch]

Let  $e$  be a Gödel number.

- Find the SGOTO program  $P_e$  by using the inverse of the bijection (248).
- Mimick the computation of  $P_e$  by universal function (similar to mimicking of GOTO-programs by partial recursive function in Part III).
- Residual-step function  $R_{P_e}$  as in (139) mimicks the execution of  $P_e$ .
- Then by (140),  $\psi_{\text{univ}}^{(n)}(e, \cdot) = \beta_1 \circ R_{P_e} \circ \alpha_n = \|P_e\|_{n,1}$  exists and is partial recursive. □

## Universal Function – Example

Consider the dyadic addition function

$$f_+ : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto x + y$$

and the corresponding SGOTO program  $P_+$ :

$$\begin{array}{llll} s_0 : & 0 & \text{if } x_2 = 0 & 3 \quad 1 \\ s_1 : & 1 & x_1 \leftarrow x_1 + 1 & 2 \\ s_2 : & 2 & x_2 \leftarrow x_2 - 1 & 0 \end{array}$$

- Associated Gödel number

$$\begin{aligned} e &= \rho(P_+) = \sigma(\sigma'(s_0), \sigma'(s_1), \sigma'(s_2)) \\ &= 1, 269, 420, 373, 033, 475, 160, 642, 134. \end{aligned}$$

## Universal Function – Example (Cont'd)

- Universal function  $\phi_{\text{univ}}^{(2)}(e, \cdot)$  mimicks the computation of  $P_e = P_+$  by multi-step function (135).
- Runtime function (137) minimalizes the multi-step function (i.e., number of steps) until a label  $\ell$  is found which is not a program label (here  $\ell = 3$ ).
- Residual-step function  $R_{P_e}$  (139) then takes the number of steps as given by the runtime function and  $(\beta_1 \circ R_{P_e} \circ \alpha_n)(\mathbf{x})$  outputs in register  $R_1$  the result of the program execution.

## Normal Form

- Kleene set  $T$  (1943) tells whether an SGOTO program will halt when run with an input.
- Kleene's normal form implies that any partial recursive function can be defined by using a single instance of (unbounded) minimalization applied to a primitive recursive function.
- In the context of programming, any program can be written using a single `while` loop.

## Extended Kleene Set

Let  $n \geq 1$ . *Extended Kleene set*  $S_n \subseteq \mathbb{N}_0^{n+3}$  is given as

$$(e, \mathbf{x}, z, t) \in S_n \quad :\iff \quad (256)$$

$$\chi_{L(P_e)}(\pi_1^{(k)}(M(e, \omega_{\mathbf{x}}, t))) = 0 \quad \wedge \quad \pi_2^{(k)}(M(e, \omega_{\mathbf{x}}, t)) = z,$$

where

- $\omega_{\mathbf{x}} = (0, x_1, \dots, x_n, 0, \dots) \in \mathbb{N}_0^k$  initial configuration given by starting label  $\ell = 0$  and input  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{N}_0^n$ .
- $M(e, \omega_{\mathbf{x}}, t)$  is the multi-step function executing  $P_e$  with initial configuration  $\omega_{\mathbf{x}}$  for  $t$  steps, gives configuration after  $t$  steps.
- $\pi_1^{(k)}(\ell, y_1, y_2, \dots) = \ell$  is the label of the current configuration.
- $\chi_{L(P_e)}(\ell) = 1$  if  $\ell \in L(P_e)$  is a label of  $P_e$  (pgm runs) and 0 otherwise (pgm aborts).
- $\pi_2^{(k)}(\ell, y_1, y_2, \dots) = y_1$  is the value of (result) register  $R_1$ .

## Extended Kleene Set

$(e, \mathbf{x}, z, t) \in S_n$  iff program  $P_e$  with input  $\mathbf{x} \in \mathbb{N}_0^n$  terminates after  $t$  steps and the result of computation is  $z$ .

For each  $n \geq 1$ , the set  $S_n$  is primitive.

### Proof.

Characteristic function of  $S_n$  is by (256) given as

$$\chi_{S_n}(e, \mathbf{x}, z, t) = \text{csg} \left( \chi_{L(P_e)}(\pi_1^{(k)}(M(e, \omega_{\mathbf{x}}, t))) \right) \cdot \text{csg} \left| \pi_2^{(k)}(M(e, \omega_{\mathbf{x}}, t)) - z \right|.$$

The right-hand side is a composition of primitive recursive functions; in particular, the characteristic function of  $L(P_e)$  is primitive recursive since the label set  $L(P_e)$  is finite (Part VII). Thus  $\chi_{S_n}$  is primitive recursive and hence  $S_n$  is primitive.  $\square$

## Extended Kleene Set – Example

Consider the dyadic addition function

$$f_+ : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto x + y$$

and the corresponding SGOTO program  $P_+$ :

$$\begin{array}{llll} s_0 : & 0 & \text{if } x_2 = 0 & 3 \quad 1 \\ s_1 : & 1 & x_1 \leftarrow x_1 + 1 & 2 \\ s_2 : & 2 & x_2 \leftarrow x_2 - 1 & 0 \end{array}$$

- Associated Gödel number

$$\begin{aligned} e &= \rho(P_+) = \sigma(\sigma'(s_0), \sigma'(s_1), \sigma'(s_2)) \\ &= 1, 269, 420, 373, 033, 475, 160, 642, 134. \end{aligned}$$

- Initial configuration ( $k = 3$ )

$$\omega = (0, x, y) \in \mathbb{N}_0^3.$$

## Extended Kleene Set – Example (Cont'd)

Extended Kleene set  $S_2$  ( $n = 2$ ) consists of all quadruples

$$(e, (x, y), x + y, 3y + 1), \quad x, y \in \mathbb{N}_0,$$

where

- $e$  is the Gödel number,
- $(x, y)$  is the input,
- $z = x + y$  is the result of computation, and
- $t = 3y + 1$  is the number of computational steps.

## Kleene Set

Let  $n \geq 1$ . *Kleene set*  $T_n \subseteq \mathbb{N}_0^{n+2}$  is given by

$$(e, \mathbf{x}, y) \in T_n \quad :\iff \quad (e, \mathbf{x}, \kappa_2(y), \tau_2(y)) \in S_n. \quad (257)$$

- Cantor pairing function  $\sigma_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  has inverse function pair  $\kappa_2, \tau_2 : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ , where

$$\sigma_2(\kappa_2(y), \tau_2(y)) = y, \quad y \in \mathbb{N}_0, \quad (258)$$

or equivalently,

$$\sigma_2^{-1}(y) = (\kappa_2(y), \tau_2(y)), \quad y \in \mathbb{N}_0. \quad (259)$$

- Component  $y$  encodes both, the result of computation  $z = \kappa_2(y)$  and the number of steps  $t = \tau_2(y)$ .

## Kleene Set

For each  $n \geq 1$ , the set  $T_n$  is primitive.

### Proof.

Characteristic function of  $T_n$  is given by

$$\chi_{T_n}(e, \mathbf{x}, y) = \chi_{S_n}(e, \mathbf{x}, \kappa_2(y), \tau_2(y)).$$

The right-hand side is a composition of primitive recursive functions as  $\chi_{S_n}$  is primitive recursive and  $\kappa_2, \tau_2$  are primitive recursive.

Thus  $\chi_{T_n}$  is primitive recursive and hence  $T_n$  is primitive.  $\square$

## Kleene Set – Example

Consider the dyadic addition function

$$f_+ : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto x + y$$

and the corresponding SGOTO program  $P_+$ :

$$\begin{array}{llll} s_0 : & 0 & \text{if } x_2 = 0 & 3 \quad 1 \\ s_1 : & 1 & x_1 \leftarrow x_1 + 1 & 2 \\ s_2 : & 2 & x_2 \leftarrow x_2 - 1 & 0 \end{array}$$

- Associated Gödel number

$$\begin{aligned} e &= \rho(P_+) = \sigma(\sigma'(s_0), \sigma'(s_1), \sigma'(s_2)) \\ &= 1, 269, 420, 373, 033, 475, 160, 642, 134. \end{aligned}$$

- Initial configuration ( $k = 3$ )

$$\omega = (0, x, y) \in \mathbb{N}_0^3.$$

## Kleene Set – Example (Cont'd)

Kleene set  $T_2$  ( $n = 2$ ) consists of all triples

$$(e, (x, y), w), \quad x, y \in \mathbb{N}_0,$$

where

- $e$  is the Gödel number,
- $(x, y)$  is the input,
- $\sigma_2^{-1}(w) = (z, t)$ ,
- $z = x + y$  is the result of computation, and
- $t = 3y + 1$  is the number of computational steps.

## Normal Form Theorem

Let  $n \geq 1$ . For all  $e \in \mathbb{N}_0$  and  $\mathbf{x} \in \mathbb{N}_0^n$ ,

$$\begin{aligned}\phi_e^{(n)}(\mathbf{x}) &= \kappa_2(\mu y [|\chi_{T_n}(e, \mathbf{x}, y) - 1|]) \\ &= \kappa_2(\mu y [\chi_{T_n}(e, \mathbf{x}, y) = 1]).\end{aligned}\tag{260}$$

- $\chi_{T_n}(e, \mathbf{x}, y) = 1$  iff  $(e, \mathbf{x}, y) \in T_n$ , i.e., program  $P_e$  with input  $\mathbf{x}$  yields output  $\kappa_2(y)$  in  $\tau_2(y)$  steps.
- Unbounded minimalization  $\mu y[\chi_{T_n}(e, \mathbf{x}, y) = 1]$  finds the smallest  $y \geq 0$  (if any) such that  $(e, \mathbf{x}, y) \in T_n$ .
- If so,  $\kappa_2(y) = z$  yields the result of computation.

## Consequence

By (260), any partial recursive function can be defined by using a single instance of (unbounded) minimalization applied to a primitive recursive function (since  $\chi_{T_n}$  is primitive recursive).

## Skills

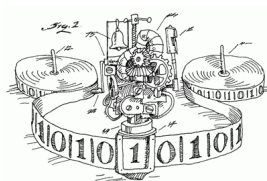
- Computation of functions  $\sigma_2$ ,  $\kappa_2$  and  $\tau_2$
- Computation of functions  $\sigma$ ,  $\kappa$  and  $\tau$
- Computation of Gödel number of GOTO program
- Specification of GOTO program  $Q_{e,x}$  (parametrization)
- Finding the Kleene sets  $S_n$  and  $T_n$

# Part VI

## Turing Machine

## Turing Machine – Motivation

- Turing machine as first universal computation machine.
- Understanding the behavior of a Turing machine.
- Busy beaver problem and its relation to superpowers.



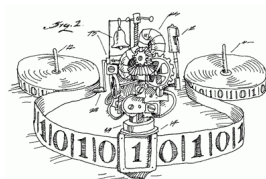
## Turing Machine – Contents

- Machinery
- \*Post-Turing machine
- \*Turing computable functions
- Busy beaver

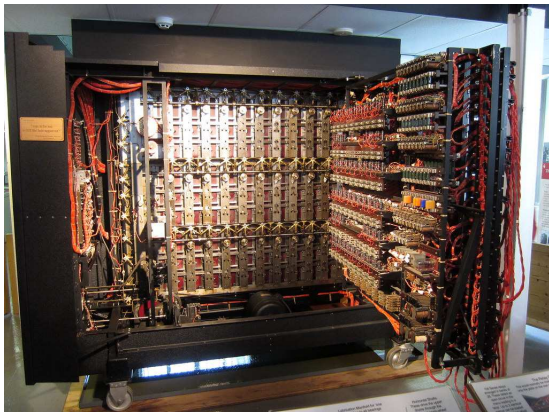
Reading: Zimmermann, Chapter 6

## Turing Machine – Inside

- Turing machine has been described as a thought experiment representing a computing machine (Alan Turing, 1912-1954).
- Turing machine is a theoretical device that manipulates symbols on a strip of tape according to a set of rules.
- Turing machine is a device of universal computation.
- Busy beaver problem is a prominent example of recreational mathematics.



## Turing machine at Bletchley Park (WW II)



"The Imitation Game" is a 2014 American historical drama based on the biography of Alan Turing.



## Machinery – Overall Behaviour

- Turing machine consists of infinite tape and associated read/write (R/W) head connected to a control mechanism.
- Tape is divided into denumerably many cells, each of which containing a symbol from the tape alphabet.
- Tape alphabet contains special symbol " $b$ " signifying that a cell is blank or empty; all cells are blank up to finitely many (compare this to the URM states).
- Cells are scanned, one at a time, by R/W head which is able to move in both directions or remain fixed.
- At any time instant, the machine will be in one of a finite number of states.
- Behaviour of R/W head and change of machine's state are governed by present state and symbol in cell under scan.

## Machinery – Formal Definition

*Turing machine* is a quintuple  $M = (\Sigma, Q, \delta, q_0, q_F)$  consisting of

- finite alphabet  $\Sigma$ , called *tape alphabet*, contains distinguished *blank* symbol  $b$ , subset  $\Sigma_I = \Sigma \setminus \{b\}$  is called *input alphabet*,
- finite set  $Q$  of *states*,
- partial function  $\delta : Q \times \Sigma \rightarrow Q \times \Sigma \times \{L, R, \Lambda\}$  called *transition function*,
- *start state*  $q_0 \in Q$ ,
- *halt state*  $q_F \in Q$  such that  $\delta(q_F, \sigma)$  is undefined for all  $\sigma \in \Sigma$  in  $\Sigma$ .

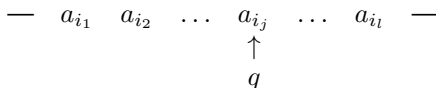
## Machinery – Tape

- Symbols  $L$ ,  $R$ , and  $\Lambda$  are interpreted as *left move*, *right move*, and *no move*, respectively.
- Tape cells can be numbered by the set of integers  $\mathbb{Z}$ .
- Tape content can be considered as mapping  $\tau : \mathbb{Z} \rightarrow \Sigma$ , where  $\tau(z)$  is the content of cell  $z \in \mathbb{Z}$ .
- Tape contains blanks almost everywhere, i.e., only a finite portion of the tape contains symbols from the input alphabet.

$z$	...	-3	-2	-1	0	1	2	3	...
$\tau(z)$	...	$b$	1	1	$b$	1	1	$b$	...

## Machinery – Configurations

- *Configuration* of Turing machine  $M$  consists of
  - content of the tape,
  - cell controlled by R/W head,
  - state of the machine.
- Picture



Current state  $q \in Q$ ; arrow marking cell controlled by R/W head; all cells to the left of  $a_{i_1}$  and to the right of  $a_{i_l}$  contain the blank symbol.

- Shorthand notation of configuration

$$(a_{i_1} \dots a_{i_{j-1}}, q, a_{i_j} \dots a_{i_l}).$$

## Machinery – Transitions

Transition  $\delta(q, a) = (q', a', D)$  means

- machine is in state  $q \in Q$  and reads the symbol  $a \in \Sigma$  (in cell under R/W head),
- replaces  $a \in \Sigma$  by  $a' \in \Sigma$ ,
- moves left ( $D = L$ ), moves right ( $D = R$ ) or stands still ( $D = \Lambda$ ),
- enters the state  $q' \in Q$ .

The R/W head is shifted and not the tape of cells.

## Machinery – Move Right

Configuration  $(uc, q, adv)$  with  $a, c, d \in \Sigma$ ,  $u, v \in \Sigma^*$ ,  $q \in Q$ ,  
and  $q \neq q_F$ :

$$\begin{array}{ccccccccc} - & u & c & a & d & v & - \\ & & & \uparrow & & & \\ & & & q & & & \end{array}$$

Move right:  $\delta(q, a) = (q', a', R)$

$$\begin{array}{ccccccccc} - & u & c & a' & d & v & - \\ & & & & \uparrow & & \\ & & & & q' & & \end{array}$$

New configuration  $(uca', q', dv)$ .

## Machinery – Move Left

Configuration  $(uc, q, adv)$  with  $a, c, d \in \Sigma$ ,  $u, v \in \Sigma^*$ ,  $q \in Q$ ,  
and  $q \neq q_F$ :

$$\begin{array}{ccccccccc} - & u & c & a & d & v & - \\ & & & \uparrow & & & \\ & & & q & & & \end{array}$$

Move left:  $\delta(q, a) = (q', a', L)$

$$\begin{array}{ccccccccc} - & u & c & a' & d & v & - \\ & & & \uparrow & & & \\ & & & q' & & & \end{array}$$

New configuration  $(u, q', ca'dv)$ .

## Machinery – No Move

Configuration  $(uc, q, adv)$  with  $a, c, d \in \Sigma$ ,  $u, v \in \Sigma^*$ ,  $q \in Q$ ,  
and  $q \neq q_F$ :

$$\begin{array}{ccccccccc} - & u & c & a & d & v & - \\ & & & \uparrow & & & \\ & & & q & & & \end{array}$$

No move:  $\delta(q, a) = (q', a', \Lambda)$

$$\begin{array}{ccccccccc} - & u & c & a' & d & v & - \\ & & & \uparrow & & & \\ & & & q' & & & \end{array}$$

New configuration  $(uc, q', a'dv)$ .

## Machinery – One-State Transition

Configuration  $(uc, q, av)$  with  $a, c \in \Sigma$ ,  $u, v \in \Sigma^*$ ,  $q \in Q$ ,  
and  $q \neq q_F$ .

$$(u', q', v') = \begin{cases} (uca', q', v) & \text{if } \delta(q, a) = (q', a', R), \\ (u, q', ca'v) & \text{if } \delta(q, a) = (q', a', L), \\ (uc, q', a'v) & \text{if } \delta(q, a) = (q', a', \Lambda). \end{cases} \quad (261)$$

is the configuration *reached in one step*. Write

$$(uc, q, av) \vdash (u', q', v').$$

## Machinery – Computation

- Input word  $\mathbf{x} = x_1x_2 \dots x_n \in \Sigma_I^*$ .
- *Initial configuration*

$$\begin{array}{ccccccc} \text{—} & x_1 & x_2 & \dots & x_n & \text{—} \\ & \uparrow & & & & \\ & q_0 & & & & \end{array}$$

- *Computation* is a sequence of configurations

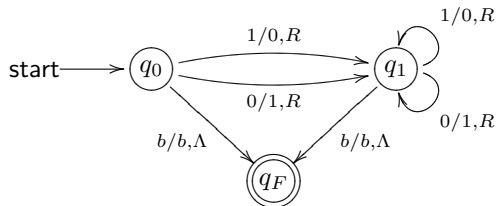
$$(b, q_0, \mathbf{x}) \vdash (u_1, q_1, v_1) \vdash (u_2, q_2, v_2) \vdash \dots$$

started with initial configuration.

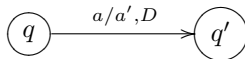
- Computation *terminates* if the machine reaches the stop state  $q_F$ ; output is given by the collection of symbols on the tape.

## Machinery – Example

Turing machine  $M = (\Sigma, Q, \delta, q_0, q_F)$  with tape alphabet  $\Sigma = \{0, 1, b\}$ , state set  $Q = \{q_0, q_1, q_F\}$ , and transition function  $\delta$  given by automaton graph:



Transition  $\delta(q, a) = (q', a', D)$  is indicated by

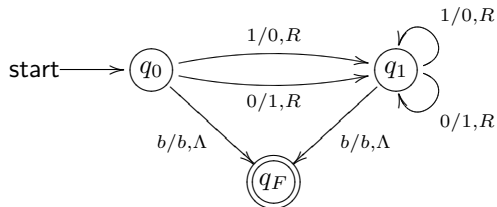


## Machinery – Example

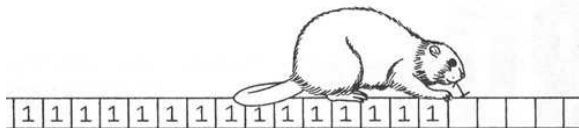
Computation provides bitwise complement of binary input word 0011:

$$\begin{aligned}
 (b, q_0, 0011) &\vdash (1, q_1, 011) \\
 &\vdash (11, q_1, 11) \\
 &\vdash (110, q_1, 1) \\
 &\vdash (1100, q_1, b) \\
 &\vdash (1100, q_F, b).
 \end{aligned}$$

Turing automaton graph:



## Busy Beaver



## Busy Beaver – Contents

- Problem statement
- Busy Beaver function
- Small Busy Beavers
- Correlation with superpowers
- Rado's result

## Busy Beaver Problem

- *Busy beaver problem:*

*Construct a Turing machine with binary input alphabet  $\Sigma = \{1, b\}$  which halts after writing the most 1's on the tape when started from the blank tape.*

- *Busy beaver* is a Turing machine which solves the Busy beaver problem.
- *Busy beaver function*  $\Upsilon : \mathbb{N} \rightarrow \mathbb{N}$ , where  $\Upsilon(n)$  is the maximum number of 1's finally on the tape among all halting two-symbol  $n$ -state Turing machines when started with the blank tape (final state  $q_F$  not counted).
- *Maximum shifts function*  $S : \mathbb{N} \rightarrow \mathbb{N}$ , where  $S(n)$  is the maximum number of moves (L or R) that can be made by any halting two-symbol  $n$ -state Turing machine.

## Busy Beaver

For small values of  $n$ , the busy beaver function is known:

$$\Upsilon(1) = 1,$$

$$\Upsilon(2) = 4,$$

$$\Upsilon(3) = 6,$$

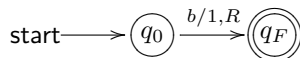
$$\Upsilon(4) = 13.$$

In all other cases only lower bounds have been established.

## Busy Beaver with One State

Busy beaver  $M_1 = (\{1, b\}, \{q_0, q_F\}, \delta, q_0, q_F)$  computes

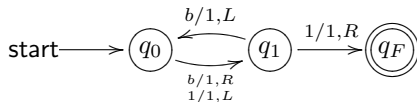
$$(b, q_0, b) \vdash (b1, q_F, b).$$



## Busy Beaver with Two States

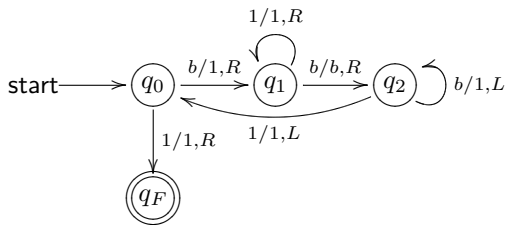
The busy beaver  $M_2 = (\{1, b\}, \{q_0, q_1, q_F\}, \delta, q_0, q_F)$  computes

$$\begin{aligned} (b, q_0, b) &\vdash (b1, q_1, b) \vdash (b, q_0, 11b) \\ &\vdash (b, q_1, b11b) \vdash (b, q_0, b111b) \\ &\vdash (b1, q_1, 111b) \vdash (b11, q_F, 11b). \end{aligned}$$



## Busy Beaver with Three States

Busy beaver  $M_3 = (\{1, b\}, \{q_0, q_1, q_2, q_F\}, \delta, q_0, q_F)$  computes

$$\begin{aligned} (b, q_0, b) &\vdash (b1, q_1, b) \vdash (b1b, q_2, b) \vdash (b1, q_2, b1b) \\ &\vdash (b, q_2, 111b) \vdash (b, q_2, b111b) \vdash (b, q_0, b1111b) \\ &\vdash (b1, q_1, 111b) \vdash (b11, q_1, 11b) \vdash (b111, q_1, 1b) \\ &\vdash (b1111, q_1, b) \vdash (b1111b, q_2, b) \vdash (b1111, q_2, b1b) \\ &\vdash (b111, q_2, 111b) \vdash (b11, q_0, 1111b) \vdash (b111, q_F, 111b). \end{aligned}$$


## Busy Beaver Function – Superpowers

(Milton Green, 1964) For each number  $k \geq 2$ ,

$$\Upsilon(2k) > 3 \uparrow^{k-2} 3. \quad (262)$$

In particular,

$$\Upsilon(4) > 3 \uparrow^0 3 = 3 \cdot 3 = 9,$$

$$\Upsilon(6) > 3 \uparrow^1 3 = 3^3 = 27,$$

$$\Upsilon(8) > 3 \uparrow^2 3 = 3 \uparrow (3 \uparrow 3) = 3^{3^3} = 7\,625\,597\,484\,987,$$

$$\Upsilon(10) > 3 \uparrow^3 3 = 3 \uparrow^2 (3 \uparrow^2 3) = 3 \uparrow^2 3^{3^3} \quad (\text{tritri})$$

$$\Upsilon(12) > 3 \uparrow^4 3 = 3 \uparrow^3 (3 \uparrow^3 3) \quad (\text{1st Graham number}).$$

## Busy Beaver Function – Rado's Result

The busy beaver function  $\Upsilon$  is not partial recursive (Tibor Rado, 1962).

### Proof Idea

For any computable function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ ,  $\Upsilon(n) > f(n)$  for all sufficiently large  $n$  and hence  $\Upsilon$  is not computable.

The maximum shifts function  $S$  is not partial recursive.

### Proof.

There is a two-symbol  $n$ -state Turing machine  $M$  (don't allow no moves) that writes  $\Upsilon(n)$  1's on the tape and then halts. This Turing machine makes at least  $\Upsilon(n)$  moves. Thus  $S(n) \geq \Upsilon(n)$ . Hence  $S$  is not computable.  $\square$

## Skills

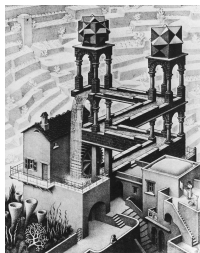
- Writing and interpreting a Turing program

# Part VII

# Undecidability

## Undecidability – Motivation

- In computability theory, decision problems are yes-or-no questions on an input set.
- Undecidable problem does not allow to construct a general algorithm that always leads to a correct yes-or-no answer.
- Most prominent examples: halting problem, word problems in formal language theory and algebra, Hilbert's tenth problem.



Self-reference (M.C. Escher, 1898-1972)

## Undecidability – Contents

- Undecidable sets
- Semidecidable sets
- Recursively enumerable sets
- Theorems of Rice and Rice-Shapiro
- Recursion theory

Reading: Zimmermann, Chapter 7

## Decidable and Undecidable Sets

- Decidable sets
- Prototypical undecidable set
- Reduction
- Halting problem
- Undecidable sets

## Decidable and Undecidable Sets

Let  $A \subseteq \mathbb{N}_0^n$ .

- $A$  is *decidable* if characteristic function  $\chi_A$  is recursive, where for all  $\mathbf{x} \in \mathbb{N}_0^n$ ,

$$\chi_A(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in A, \\ 0 & \text{otherwise,} \end{cases} \quad (263)$$

i.e., characteristic function  $\chi_A$  is total and computable.

- If  $A$  is decidable, the question " $\mathbf{x} \in A$ " can be decided by computing  $\chi_A(\mathbf{x})$ .
- Algorithm for computing  $\chi_A$  is a *decision procedure* for  $A$ .
- $A$  is *undecidable* if  $A$  is not decidable (i.e.,  $\chi_A$  is not computable).

## Decidable Sets – Closure Properties

If  $A, B \subseteq \mathbb{N}_0^n$  are decidable, then decidable are

$$\bar{A}, A \cup B, A \cap B, \text{ and } A \setminus B = A \cap \bar{B}. \quad (264)$$

## Proof.

For each  $x \in \mathbb{N}_0^n$ ,

$$\chi_{\bar{A}}(\mathbf{x}) = \text{csg} \circ \chi_A(\mathbf{x}), \quad (265)$$

$$\chi_{A \cup B}(\mathbf{x}) = \text{sgn} \circ (\chi_A(\mathbf{x}) + \chi_B(\mathbf{x})), \quad (266)$$

$$\chi_{A \cap B}(\mathbf{x}) = \chi_A(\mathbf{x}) \cdot \chi_B(\mathbf{x}), \quad (267)$$

$$\chi_{A \setminus B}(\mathbf{x}) = \chi_A \cdot \chi_{\bar{B}}(\mathbf{x}). \quad (268)$$

The functions on the right-hand sides are compositions of recursive functions and so are also recursive.  $\square$

## Decidable Sets – Examples

- Each finite set  $A \subseteq \mathbb{N}_0$  is decidable, since

$$\chi_A(x) = \text{sgn} \left( \sum_{a \in A} \chi_{=(a, x)} \right), \quad x \in \mathbb{N}_0, \quad (269)$$

is a composition of primitive recursive functions and thus is also primitive recursive.

E.g., if  $A = \{1, 3, 5\}$ , then for each  $x \in \mathbb{N}_0$ ,

$$\chi_A(x) = \text{sgn} [\chi_{=(1, x)} + \chi_{=(3, x)} + \chi_{=(5, x)}].$$

- Empty set  $\emptyset$  is finite and so is decidable;  $\chi_{\emptyset}(x) = 0$  for all  $x \in \mathbb{N}_0$ .
- $\mathbb{N}_0$  is decidable, since  $\mathbb{N}_0 = \bar{\emptyset}$ ,  $\emptyset$  is decidable and decidable sets are closed under complement (264);  $\chi_{\mathbb{N}_0}(x) = 1$  for all  $x \in \mathbb{N}_0$ .
- Set of prime numbers  $P$  is decidable, since  $\chi_P$  is primitive recursive (chapter 2).

## Prototypical Undecidable Set

- Consider the sequence of all monadic partial recursive functions

$$\phi_0^{(1)}, \phi_1^{(1)}, \phi_2^{(1)}, \dots \quad (270)$$

where  $\phi_e^{(1)} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  denotes the partial recursive function computed by GOTO program  $P_e$ ,  $e \in \mathbb{N}_0$ .

- The set

$$K = \left\{ x \in \mathbb{N}_0 \mid x \in \text{dom } \phi_x^{(1)} \right\} \quad (271)$$

is undecidable, where  $x \in \text{dom } \phi_x^{(1)}$  means that program  $P_x$  with input  $x$  halts.

## Proof

Suppose  $K$  would be decidable; i.e.,  $\chi_K$  would be recursive.

- Function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  given by

$$f(x) = \begin{cases} 0 & \text{if } x \notin K, \\ \uparrow & \text{otherwise,} \end{cases} \quad (272)$$

is partial recursive.

Indeed, function  $\chi_K$  is computable and so there is an an URM program  $P$  for  $\chi_K$  computing it. Consider the URM program

$$P; (A1)1. \quad (273)$$

Recall the program  $(A1)1$  in (39).

## Proof (Cont'd)

- Consider the program  $P; (A1)1$ :

$$0 \quad x \quad 0 \quad 0 \quad \dots \quad x \in K$$

$$0 \quad 1 \quad 0 \quad 0 \quad \dots \quad P$$

$$0 \quad \uparrow \quad 0 \quad 0 \quad \dots \quad (A1)1$$

and

$$0 \quad x \quad 0 \quad 0 \quad \dots \quad x \notin K$$

$$0 \quad 0 \quad 0 \quad 0 \quad \dots \quad P$$

$$0 \quad 0 \quad 0 \quad 0 \quad \dots \quad (A1)1$$

- Thus  $P; (A1)1$  is an URM program for  $f$ .
- Hence, function  $f$  is partial recursive and so it has an index  $e \in \mathbb{N}_0$  with  $\phi_e^{(1)} = f$ .
- Diagonalization* argument:

$$e \in K \Leftrightarrow e \in \text{dom } \phi_e^{(1)} \Leftrightarrow e \in \text{dom } (f) \Leftrightarrow e \notin K.$$

Contradiction. □

## Computable Function I

Function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  in (272) is not partial recursive.

- Function  $h : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  defined by exchanging the cases in  $f$ ,

$$h(x) = \begin{cases} 0 & \text{if } x \in K, \\ \uparrow & \text{otherwise,} \end{cases} \quad (274)$$

is partial recursive, since for each  $x \in \mathbb{N}_0$ ,

$$\begin{aligned} h(x) &= 0 \cdot \phi_x^{(1)}(x) \\ &= 0 \cdot \psi_{\text{univ}}^{(1)}(x, x). \end{aligned} \quad (275)$$

## Computable Function II

Function  $h : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  in (274) is partial recursive.

- Function  $h' : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  related to  $h$  and given by

$$h'(x) = \begin{cases} x & \text{if } x \in K, \\ \uparrow & \text{otherwise,} \end{cases} \quad (276)$$

is partial recursive, since for each  $x \in \mathbb{N}_0$ ,

$$\begin{aligned} h'(x) &= x \cdot \text{sgn}(\phi_x^{(1)}(x) + 1) \\ &= x \cdot \text{sgn}(\psi_{\text{univ}}^{(1)}(x, x) + 1). \end{aligned} \quad (277)$$

- Domain and range of partial recursive function  $h'$  are undecidable sets, since

$$\text{dom}(h') = K = \text{ran}(h'). \quad (278)$$

## Diagonalization and Reduction

- Now one undecidable set  $K$  is at hand (proved by diagonalization).
- Any other set can be proved to be undecidable by using reduction.
- Let  $A \subseteq \mathbb{N}_0^k$  and  $B \subseteq \mathbb{N}_0^l$ .

$A$  is *reducible* to  $B$  if there is a recursive function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^l$  such that

$$\mathbf{x} \in A \iff f(\mathbf{x}) \in B, \quad \mathbf{x} \in \mathbb{N}_0^k. \quad (279)$$

Equivalently,

$$\chi_A(\mathbf{x}) = \chi_B(f(\mathbf{x})), \quad \mathbf{x} \in \mathbb{N}_0^k, \quad (280)$$

i.e., if  $B$  is decidable, then  $A$  is decidable; or by contraposition, if  $A$  is undecidable, then  $B$  is undecidable.

## Multidimensional Functions

A function  $f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^l$  is partial recursive (recursive, primitive recursive) if each coordinate function

$$f_i = \pi_i^{(l)} \circ f : \mathbb{N}_0^k \rightarrow \mathbb{N}_0, \quad 1 \leq i \leq l, \quad (281)$$

is partial recursive (recursive, primitive recursive).

### Example

Function

$$f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0^3 : (x_1, x_2) \mapsto (x_1^2, x_1x_2, x_2^2)$$

is primitive recursive, since the coordinate functions are primitive recursive:

$$f_1 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x_1, x_2) \mapsto x_1^2,$$

$$f_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x_1, x_2) \mapsto x_1x_2,$$

$$f_3 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x_1, x_2) \mapsto x_2^2.$$

## Halting Problem

- *Halting problem* is one of the famous undecidability results.
- Interpretation: Given program  $P$  and input  $x$  to  $P$ , it cannot be decided generally whether  $P$  with input  $x$  terminates or continues to run forever.
- First proof for Turing machine by Alan Turing (1936).
- By Church's thesis, halting problem is undecidable not only for Turing machines but for any formalism capturing the notion of computability.

## Halting Problem

The set

$$H = \left\{ (x, y) \in \mathbb{N}_0^2 \mid y \in \text{dom } \phi_x^{(1)} \right\} \quad (282)$$

is undecidable, where  $y \in \text{dom } \phi_x^{(1)}$  means that program  $P_x$  with input  $y$  halts.

**Proof.**

- Function

$$f : \mathbb{N}_0 \rightarrow \mathbb{N}_0^2 : x \mapsto (x, x)$$

is primitive recursive, since the coordinate functions are  $f_1(x) = x$  and  $f_2(x) = x$  (identity functions).

- Set  $K$  can be reduced to set  $H$  by the function  $f$ .

Indeed, for each  $x \in \mathbb{N}_0$ ,

$$\chi_K(x) = \chi_H(x, x) = \chi_H(f(x)).$$

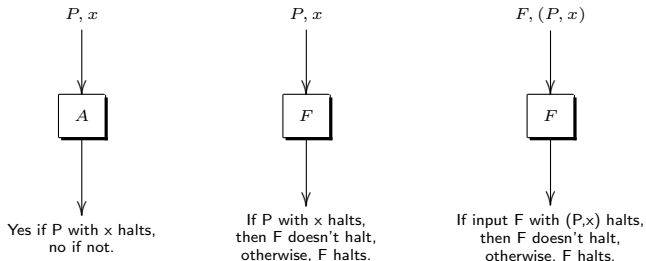
Since  $K$  is undecidable,  $H$  is undecidable. □

## Halting Problem (Revisited)

- Suppose there is an *almighty* GOTO program  $A$  solving the Halting problem:  
 $A$  with input  $(P, x)$  outputs "yes" if  $P$  with input  $x$  halts; otherwise, output is "no".
- Change GOTO program  $A$  into *nasty* GOTO program  $F$ :  
 $F$  with input  $(P, x)$  doesn't halt if  $P$  with input  $x$  halts; otherwise,  $F$  halts; see pgm (273).
- Self-reference (diagonalization): Program  $F$  is fed with input  $F$  which has input  $(P, x)$ :  
 $F$  with input  $F, (P, x)$  doesn't halt if input  $F$  with  $P, x$  halts, and  $F$  with input  $F, (P, x)$  halts if input  $F$  with  $P, x$  doesn't halts. Contradiction!

By Church's thesis,  $A$  and  $F$  can be realized by GOTO programs.

## Halting Problem (Revisited)



## Undecidable Sets

Let  $a \in \mathbb{N}_0$ . The following sets are undecidable:

$$C = \left\{ x \in \mathbb{N}_0 \mid \phi_x^{(1)} = c_0^{(1)} \right\}, \quad (283)$$

$$E = \left\{ (x, y) \in \mathbb{N}_0^2 \mid \phi_x^{(1)} = \phi_y^{(1)} \right\}, \quad (284)$$

$$T = \left\{ x \in \mathbb{N}_0 \mid \phi_x^{(1)} \text{ is total} \right\}, \quad (285)$$

$$I_a = \left\{ x \in \mathbb{N}_0 \mid a \in \text{dom } \phi_x^{(1)} \right\}, \quad (286)$$

$$O_a = \left\{ x \in \mathbb{N}_0 \mid a \in \text{ran } \phi_x^{(1)} \right\}, \quad (287)$$

where  $c_0^{(1)}$  is the monadic zero function.

Proof by Theorem of Rice.

## Undecidable Sets

Practical implications using the formalism of GOTO programs:

- The problem whether a GOTO program halts with a given input is undecidable (set  $H$ ).
- The problem whether a GOTO program computes a specific function (here  $c_0^{(1)}$ ) is undecidable (set  $C$ ).
- The problem whether two GOTO programs are semantically equivalent, i.e., exhibit the same input-output behaviour, is undecidable (set  $E$ ).
- The problem whether a GOTO program always halts is undecidable (set  $T$ ).
- The problem whether a GOTO program halts for a specific input is undecidable (set  $I_a$ ).
- The problem whether a GOTO program halts and computes a specific output is undecidable (set  $O_a$ ).

## Semidecidable Sets

- Semidecidability
- Partial decision procedure
- Existential quantification
- Function graph

## Semidecidable Sets

Let  $A \subseteq \mathbb{N}_0^n$ .

- Informally,  $A$  is semidecidable if there is an algorithm such that the set of inputs for which the algorithm halts is exactly the set of elements in  $A$ .
- $A$  is *semidecidable* if the function  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  defined by

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in A, \\ \uparrow & \text{otherwise,} \end{cases} \quad (288)$$

is partial recursive;  $f$  is the *semicharacteristic function* of  $A$ .

- Thus if  $A$  is semidecidable, there is a partial recursive function  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  such that  $A = \text{dom}(f)$ .

## Semidecidable Sets

Let  $A \subseteq \mathbb{N}_0^n$ .  $A$  is semidecidable iff there is an  $n$ -adic partial recursive function  $f$  such that

$$\text{dom}(f) = A. \quad (289)$$

### Proof

- Let  $A$  be semidecidable. Then the corresponding function  $f$  given in (288) has the property that  $\text{dom}(f) = A$ .
- Let  $A$  be a subset of  $\mathbb{N}_0^n$  for which there is a partial recursive function  $h : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  such that  $\text{dom}(h) = A$ . Then the function  $f = \nu \circ c_0^{(1)} \circ h : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  satisfies for each  $\mathbf{x} \in \mathbb{N}_0^n$ :

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in A, \\ \uparrow & \text{otherwise.} \end{cases} \quad (290)$$

$f$  is partial recursive as it is a composition of partial recursive functions. Thus  $f$  is the semicharacteristic function of  $A$ . Hence,  $A$  is semidecidable.  $\square$

## Gödel Numbering of Domains

Let  $n \geq 1$ .

- Gödel numbering of GOTO programs yields an enumeration of all  $n$ -adic partial recursive functions

$$\phi_0^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}, \dots, \quad (291)$$

where partial recursive function  $\phi_e^{(n)} : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  is computed by GOTO program  $P_e$ , i.e.,  $\phi_e^{(n)} = \|P_e\|_{n,1}$  for each  $e \in \mathbb{N}_0$ .

- By (289), we obtain an enumeration of all semidecidable subsets of  $\mathbb{N}_0^n$ :

$$D_0^{(n)}, D_1^{(n)}, D_2^{(n)}, \dots, \quad (292)$$

where  $D_e^{(n)} = \text{dom } \phi_e^{(n)}$ ,  $e \in \mathbb{N}_0$ .

Semidecidable Sets –  $K$ 

$K = \{x \mid x \in \text{dom } \phi_x^{(1)}\}$  is semidecidable.

## Proof.

Function  $h : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  in (274) defined by

$$h(x) = \begin{cases} 0 & \text{if } x \in K, \\ \uparrow & \text{otherwise} \end{cases} \quad (293)$$

is partial recursive, since

$$h(x) = 0 \cdot \phi_x^{(1)}(x) = 0 \cdot \psi_{\text{univ}}^{(1)}(x, x).$$

Since  $K = \text{dom}(h)$ , it follows from (289) that  $K$  is semidecidable. □

## Semidecidable Sets – Halting Problem

$H = \{(x, y) \mid y \in \text{dom } \phi_x^{(1)}\}$  is semidecidable.

**Proof.**

Universal function  $\psi_{\text{univ}}^{(1)}$  has the property

$$\psi_{\text{univ}}^{(1)}(x, y) = \begin{cases} \phi_x^{(1)}(y) & \text{if } y \in \text{dom } \phi_x^{(1)}, \\ \uparrow & \text{otherwise.} \end{cases} \quad (294)$$

Since  $H = \text{dom } \psi_{\text{univ}}^{(1)}$  and  $\psi_{\text{univ}}^{(1)}$  is partial recursive, it follows from (289) that  $H$  is semidecidable. □

## Decidable Sets

Each decidable set is semidecidable.

### Proof

Let  $A \subseteq \mathbb{N}_0^n$  be decidable, i.e., its characteristic function  $\chi_A$  is recursive.

- Function  $g : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  defined by

$$g(\mathbf{x}) = (\text{csg} \circ \chi_A)(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in A, \\ 1 & \text{otherwise,} \end{cases}$$

is recursive, since it is a composition of recursive functions.

## Proof (Cont'd)

- Let  $P$  be an URM program computing  $g$ , i.e.,  $g = \|P\|_{n,1}$ . Then the URM program

$$P; (A1)1$$

computes the function (recall the program  $(A1)1$  in (39))

$$h(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in A, \\ \uparrow & \text{otherwise.} \end{cases}$$

- Function  $h$  is partial recursive, since it can be computed by an URM program.
- Since  $A = \text{dom}(h)$ , it follows from (289) that  $A$  is semidecidable. □

## Partial Decision Procedure

Let  $A \subseteq \mathbb{N}_0^n$  be a semidecidable set.

- There is a partial recursive function  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  such that by (289),  $A = \text{dom}(f)$ .
- Let  $P$  be a GOTO program computing  $f$ , i.e.,  $f = \|P\|_{n,1}$ .
- *Partial decision procedure* for  $A$ :
  - Start the program  $P$  with input  $\mathbf{x} \in \mathbb{N}_0^n$ .
  - If  $P$  halts,  $\mathbf{x} \in A$ . Otherwise,  $P$  will continue to run forever (undefined case).

## Decidability vs. Semidecidability

Let  $A \subseteq \mathbb{N}_0^n$ .

$$A \text{ is decidable} \iff A \text{ and } \bar{A} \text{ semidecidable.} \quad (295)$$

### Proof.

- Let  $A$  be decidable.

Then by (264),  $\bar{A}$  is also decidable. But each decidable set is semidecidable and so  $A$  and  $\bar{A}$  are semidecidable.

- Let  $A$  and  $\bar{A}$  be semidecidable.

Run the partial decision procedures for  $A$  and  $\bar{A}$  in parallel:  
For each input  $x \in \mathbb{N}_0^n$ , one of these procedures will halt in a finite number of steps giving an answer to the decision procedure for  $A$ .



## Complement of Halting Problem

The complement of the halting problem  $H$  given by the set

$$\bar{H} = \left\{ (x, y) \in \mathbb{N}_0^2 \mid y \notin \text{dom } \phi_x^{(1)} \right\} \quad (296)$$

is not semidecidable.

### Proof.

Suppose  $\bar{H}$  would be semidecidable. Since  $H$  is semidecidable, it follows by (295) that  $H$  is decidable but it is not.  $\square$

## Semidecidable Sets – Existential Quantification

A set  $A \subseteq \mathbb{N}_0^n$  is semidecidable iff there is a decidable set  $B \subseteq \mathbb{N}_0^{n+1}$  such that

$$A = \exists y[(\mathbf{x}, y) \in B] := \{\mathbf{x} \mid \exists y : (\mathbf{x}, y) \in B\}. \quad (297)$$

### Proof

Let  $B$  be decidable and  $A = \exists y[(\mathbf{x}, y) \in B]$ .

- Consider the function  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  given by

$$\begin{aligned} f(\mathbf{x}) &= \mu(\text{csg} \circ \chi_B)(\mathbf{x}) \\ &= \mu y[(\text{csg} \circ \chi_B)(\mathbf{x}, y) = 0] \\ &= \begin{cases} y & \text{if } (\mathbf{x}, y) \in B \text{ for some } y \in \mathbb{N}_0 \text{ (minimal),} \\ \uparrow & \text{otherwise.} \end{cases} \end{aligned}$$

- This function is (by minimalization) partial recursive with  $\text{dom}(f) = A$ . By (289),  $A$  is semidecidable.

## Proof (Cont'd)

Let  $A \subseteq \mathbb{N}_0^n$  be semidecidable.

- There is an index  $e \in \mathbb{N}_0$  such that  $\text{dom } \phi_e^{(n)} = A$ .
- Consider the Kleene set  $T_n$ , i.e.,  $(e, \mathbf{x}, y) \in T_n$  iff GOTO  $P_e$  with input  $\mathbf{x}$  halts with output  $\kappa_2(y)$  in  $\tau_2(y)$  steps; alternatively consider  $\phi_e^{(n)}(\mathbf{x})$  instead of  $P_e$  with input  $\mathbf{x}$ .
- $T_n$  is primitive (decidable).
- For each  $\mathbf{x} \in \mathbb{N}_0^n$ , we have

$$\mathbf{x} \in A \iff \mathbf{x} \in \exists y[(e, \mathbf{x}, y) \in T_n],$$

where  $e$  is kept fixed by parametrization (70). Hence,  
 $A = \exists y[(e, \mathbf{x}, y) \in T_n]$ . □

## Semi-decidable Sets – Representation

Let  $A \subseteq \mathbb{N}_0^n$  be semidecidable.

- There is an index  $e \in \mathbb{N}_0$  such that

$$A = \text{dom } \phi_e^{(n)}. \quad (298)$$

- By the above proof, the Kleene set  $T_n$  satisfies

$$A = \exists y[(e, \mathbf{x}, y) \in T_n]. \quad (299)$$

## Existential Quantification – Example

Consider the Kleene set  $T_1$ , i.e.,  $(e, x, y) \in T_1$  iff GOTO  $P_e$  with input  $x$  halts with output  $\kappa_2(y)$  in  $\tau_2(y)$  steps; alternatively consider  $\phi_e^{(1)}(x)$  instead of  $P_e$  with input  $x$ .

- $T_1$  is primitive (decidable).
- The semidecidable set  $K = \{x \mid x \in \text{dom } \phi_x^{(1)}\}$  fulfills

$$K = \exists y[(x, x, y) \in T_1]. \quad (300)$$

- The semidecidable set  $H = \{(x, y) \mid y \in \text{dom } \phi_x^{(1)}\}$  satisfies

$$H = \exists z[(x, y, z) \in T_1]. \quad (301)$$

## Existential Quantification – Example

The set  $Z_0 = \{x \mid \phi_x^{(1)}(x) = 0\}$  is semidecidable.

- Consider the extended Kleene set  $S_1$ , i.e.,  $(e, x, y, t) \in S_1$  iff GOTO  $P_e$  with input  $x$  halts with output  $y$  in  $t$  steps; alternatively consider  $\phi_e^{(1)}(x)$  instead of  $P_e$  with input  $x$ .
- $S_1$  is primitive (decidable).
- The set  $Z_0$  fulfills

$$Z_0 = \exists t[(x, x, 0, t) \in S_1]$$

and so is semidecidable by (297).

- $Z_0$  is not decidable (Theorem of Rice).
- Complementary set  $\bar{Z}_0 = \{x \mid \phi_x^{(1)}(x) \neq 0\}$  cannot be semidecidable by (295).

## Existential Quantification – Closure Property

If  $B \subseteq \mathbb{N}_0^{n+1}$  is semidecidable, then

$$A = \exists y[(\mathbf{x}, y) \in B] \subseteq \mathbb{N}_0^n \quad (302)$$

is semidecidable.

## Proof

Let  $B \subseteq \mathbb{N}_0^{n+1}$  be semidecidable.

- As shown, there is a decidable set  $C \subseteq \mathbb{N}_0^{n+2}$  such that  $B = \exists z[(\mathbf{x}, y, z) \in C]$ .
- The search for a pair  $(y, z)$  with  $(\mathbf{x}, y, z) \in C$  can be replaced by the search for a number  $u$  such that  $(\mathbf{x}, \kappa_2(u), \tau_2(u)) \in C$  by using the inverse pair  $\kappa_2, \tau_2$  of the Cantor pairing  $\sigma_2$ .
- Thus  $A = \exists u[(\mathbf{x}, \kappa_2(u), \tau_2(u)) \in C]$  and hence  $A$  is semidecidable by (297). □

## Existential Quantification – Example

In view of Fermat's Last Theorem (anno 1637), consider the set

$$F = \exists x \exists y \exists z [xyz \neq 0 \wedge x^n + y^n = z^n] \subseteq \mathbb{N}_0,$$

which can be written in the form

$$F = \exists t [\sigma_{31}(t)\sigma_{32}(t)\sigma_{33}(t) \neq 0 \wedge \sigma_{31}(t)^n + \sigma_{32}(t)^n = \sigma_{33}(t)^n],$$

where  $\sigma_3 : \mathbb{N}_0^3 \rightarrow \mathbb{N}_0$  is bijective and  $\sigma_{31}, \sigma_{32}, \sigma_{33} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  are the corresponding coordinate functions (305,306) with

$$\begin{aligned}\sigma_3(x, y, z) &= \sigma_2(\sigma_2(x, y), z), \\ \sigma_{31}(u) &= \kappa_2(\kappa_2(u)), \\ \sigma_{32}(u) &= \tau_2(\kappa_2(u)), \\ \sigma_{33}(u) &= \tau_2(u).\end{aligned}$$

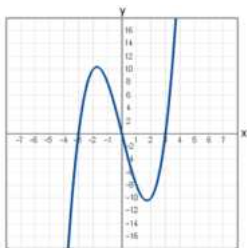
It is well-known (1994) that  $F = \{1, 2\}$  and so  $F$  is decidable.

## Function Graph

Each partial function  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  has a *graph* given by the  $n + 1$ -adic relation

$$\text{graph}(f) = \{(\mathbf{x}, f(\mathbf{x})) \in \mathbb{N}_0^{n+1} \mid \mathbf{x} \in \mathbb{N}_0^n\}. \quad (303)$$

If  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  is partial recursive, then  $\text{graph}(f)$  is semidecidable.



## Proof

Let  $f$  be partially recursive.

- Then there is an index  $e \in \mathbb{N}_0$  such that

$$f = \phi_e^{(n)}.$$

- By the extended Kleene set  $S_n$ , define the function

$$g(\mathbf{x}, y) = \mu t (\chi_{S_n}(e, \mathbf{x}, y, t) = 1), \quad \mathbf{x} \in \mathbb{N}_0^n, y \in \mathbb{N}_0, \quad (304)$$

where  $(e, \mathbf{x}, y, t) \in S_n$  iff program  $P_e$  with input  $\mathbf{x}$  (or  $\phi_e^{(n)}(\mathbf{x})$ ) terminates after  $t$  steps with output  $y$ , and  $e$  is kept fixed by parametrization (70);  $g(\mathbf{x}, y)$  is the smallest number of steps.

- Function  $g$  is partial recursive as it is given by minimalization of the primitive recursive function  $\chi_{S_n}$  with parameter  $e$ .
- Since  $\text{dom}(g) = \text{graph}(f)$ , it follows from (289) that  $\text{graph}(f)$  is semidecidable. □

## Function Graph – Example

Let  $e \in \mathbb{N}_0$ . The function  $\phi_e^{(1)} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  is partial recursive and so its function graph

$$\text{graph}(\phi_e^{(1)}) = \{(x, \phi_e^{(1)}(x)) \mid x \in \mathbb{N}_0\}$$

is semidecidable.

## Recursively Enumerable Sets

- Semidecidable sets can equivalently be described by recursively enumerable (r.e.) sets.
- An r.e. set can be enumerated by a recursive function.
- Restriction to the case  $\mathbb{N}_0^n$  with  $n = 1$  is no loss of generality.

Encoding of  $\mathbb{N}_0^n$ 

For each  $n \geq 1$ , define primitive recursive bijection  $\sigma_n : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$ .

- For  $n = 1$ ,

$$\sigma_1 = \text{id}_{\mathbb{N}_0}. \quad (305)$$

- For  $n > 1$ ,

$$\sigma_n(x_1, \dots, x_n) = \sigma_2(\sigma_{n-1}(x_1, \dots, x_{n-1}), x_n), \quad (306)$$

where  $\sigma_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  is the Cantor pairing (primitive recursive bijection) given in (221).

E.g., by (228),

$$\begin{aligned} \sigma_3(2, 3, 4) &= \sigma_2(\sigma_2(2, 3), 4) = \sigma_2(17, 4) \\ &= \binom{17 + 4 + 1}{2} + 17 = 248. \end{aligned}$$

## Encoding of $\mathbb{N}_0^n$

Let  $u = \sigma_n(x_1, \dots, x_n)$ . Then by (222-224),

- $\sigma_{n,n} = \tau_2(u) = x_n$ ,
- $\sigma_{n,n-l} = \tau_2 \kappa_2^l(u) = x_{n-l}$ ,  $1 \leq l \leq n-2$ ,
- $\sigma_{n,1} = \kappa_2^{n-1}(u) = x_1$ .

E.g.,  $\sigma_3(2, 3, 4) = \sigma_2(\sigma_2(2, 3), 4) = 248$  and so

$$\tau_2(248) = 4, \tau_2 \kappa_2(248) = 3, \text{ and } \kappa_2^2(248) = 2.$$

## Semidecidable Sets

Let  $A \subseteq \mathbb{N}_0^n$  and  $\sigma_n : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  be a recursive bijection.

$$A \text{ semidecidable} \iff \sigma_n(A) \text{ semidecidable.} \quad (307)$$

### Proof

Let  $A \subseteq \mathbb{N}_0^n$  be semidecidable, i.e., there is a partial recursive function  $f : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  such that  $\text{dom}(f) = A$ . Then  $g = f \circ \sigma_n^{-1} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  is partial recursive with  $\text{dom}(g) = \sigma_n(A)$ :

$$g : \mathbb{N}_0 \xrightarrow{\sigma_n^{-1}} \mathbb{N}_0^n \xrightarrow{f} \mathbb{N}_0.$$

Hence,  $\sigma_n(A)$  is semidecidable. Indeed, by (17),

$$\begin{aligned} x \in \text{dom}(g) &\iff x \in \text{dom}(\sigma_n^{-1}) \wedge \sigma_n^{-1}(x) \in \text{dom}(f) \\ &\iff \sigma_n^{-1}(x) \in \text{dom}(f), \quad \sigma_n \text{ bijective, total,} \\ &\iff x \in \sigma_n(\text{dom}(f)) \\ &\iff x \in \sigma_n(A). \end{aligned}$$

## Proof (Cont'd)

Let  $\sigma_n(A) \subseteq \mathbb{N}_0$  be semidecidable, i.e., there is a partial recursive function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that  $\text{dom}(f) = \sigma_n(A)$ . Then  $g = f \circ \sigma_n : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$  is partial recursive with  $\text{dom}(g) = A$ :

$$g : \mathbb{N}_0^n \xrightarrow{\sigma_n} \mathbb{N}_0 \xrightarrow{f} \mathbb{N}_0.$$

Hence,  $A$  is semidecidable. Indeed, by (17),

$$\begin{aligned} \mathbf{x} \in \text{dom}(g) &\iff \mathbf{x} \in \text{dom}(\sigma_n) \wedge \sigma_n(\mathbf{x}) \in \text{dom}(f) \\ &\iff \sigma_n(\mathbf{x}) \in \text{dom}(f), \quad \sigma_n \text{ total,} \\ &\iff \sigma_n(\mathbf{x}) \in \sigma_n(A) \\ &\iff \mathbf{x} \in A, \quad \sigma_n \text{ bijective.} \end{aligned}$$



## Semidecidable Sets – Example

Halting problem  $H = \{(x, y) \mid y \in \text{dom } \phi_x^{(1)}\}$  is semidecidable with

$$H = \text{dom } \psi_{\text{univ}}^{(1)}.$$

By the Cantor pairing  $\sigma_2 : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ , the function

$$g = \psi_{\text{univ}}^{(1)} \circ \sigma_2^{-1} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$$

is partial recursive with  $\text{dom}(g) = \sigma_2(H)$ . Hence,  $\sigma_2(H)$  is semidecidable. Indeed, by (17),

$$\begin{aligned} x \in \text{dom}(g) &\iff x \in \text{dom}(\sigma_2^{-1}) \wedge \sigma_2^{-1}(x) \in \text{dom}(\psi_{\text{univ}}^{(1)}) \\ &\iff \sigma_2^{-1}(x) \in \text{dom}(\psi_{\text{univ}}^{(1)}), \quad \sigma_2 \text{ bijective, total,} \\ &\iff x \in \sigma_2 \left( \text{dom}(\psi_{\text{univ}}^{(1)}) \right) \\ &\iff x \in \sigma_2(H). \end{aligned}$$

## Recursive Enumeration

Let  $A \subseteq \mathbb{N}_0$ .

$A$  is *recursively enumerable* (r.e.) if  $A$  is empty or there is a recursive function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that

$$A = \text{ran}(f). \quad (308)$$

Such a function  $f$  is an *enumerator* of  $A$ , i.e.,

$$A = \{f(0), f(1), f(2), \dots\}. \quad (309)$$

Note that the empty set is decidable and so semi-decidable.

## Recursive Enumeration

Let  $A \subseteq \mathbb{N}_0$  be a nonempty r.e. set. There is an enumerator  $f$  of  $A$  such that

$$f(0) < f(1) < f(2) < \dots$$

### Proof

Define the monadic function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  by minimalization and primitive recursion:

$$f(0) = \mu y[y \in A], \quad (310)$$

$$f(n+1) = \mu y[y \in A \wedge y > f(n)], \quad n \geq 0. \quad (311)$$

Clearly,  $f$  is recursive (since  $A$  is infinite), strictly monotonous and satisfies  $\text{ran}(f) = A$ .  $\square$

## Main Theorem

For each set  $A \subseteq \mathbb{N}_0$  and  $n \geq 1$ , the following are equivalent:

- $A$  is semidecidable.
- $A$  is r.e.
- There is an  $n$ -adic partial recursive function  $g$  with  $A = \text{ran}(g)$ .

Proof will be established by ring closure.

## Proof

Let  $A \neq \emptyset$  be semidecidable.

- By (292),  $A = \text{dom } \phi_e^{(1)}$  for some index  $e \in \mathbb{N}_0$ .
- Fix  $a \in A$  and use the Kleene set  $T_1$  to define the monadic function (enumerator of  $A$ )

$$f : x \mapsto \begin{cases} \kappa_2(x) & \text{if } (e, \kappa_2(x), \tau_2(x)) \in T_1, \\ a & \text{otherwise,} \end{cases} \quad (312)$$

where  $(e, y, z) \in T_1$  iff program  $P_e$  with input  $y$  yields the output  $\kappa_2(z)$  after  $\tau_2(z)$  steps.

- Function  $f$  is primitive recursive as defined by primitive case distinction and parametrization fixing  $e$  (70);  $\kappa_2$ ,  $\tau_2$  and  $\chi_{T_1}$  are primitive recursive.
- Since the program  $P_e$  computes  $\phi_e^{(1)}$  and  $A = \text{dom } \phi_e^{(1)}$ , it follows that  $A = \text{ran}(f)$ , i.e., in the first case,

$$z \in \text{ran}(f) \iff \exists x : z = \kappa_2(x) \iff z \in \text{dom } \phi_e^{(1)}.$$

## Proof (Cont'd)

Let  $A$  be r.e., i.e.,  $A = \emptyset$  or  $A = \text{ran}(f)$  for some monadic recursive function  $f$ .

- Define the partial recursive function  $g : \mathbb{N}_0^n \rightarrow \mathbb{N}_0$ , where for all  $\mathbf{x} \in \mathbb{N}_0^n$ ,

$$g(\mathbf{x}) = \begin{cases} \uparrow & \text{if } A = \emptyset, \\ f(\sigma_n(\mathbf{x})) & \text{otherwise.} \end{cases}$$

- If  $A = \emptyset$ , then  $\text{ran}(g) = \emptyset$ .
- If  $A \neq \emptyset$ , then

$$\begin{aligned} y \in \text{ran}(g) &\iff \exists \mathbf{x} \in \mathbb{N}_0^n : y = f(\sigma_n(\mathbf{x})) \\ &\iff x' \in \mathbb{N}_0 : y = f(x'), \quad \sigma_n \text{ bijective,} \\ &\iff y \in \text{ran}(f). \end{aligned}$$

## Proof (Cont'd)

Let  $g$  be an  $n$ -adic partial recursive function with  $A = \text{ran}(g)$ .

- By (303), the graph of  $g$ ,

$$G = \{(\mathbf{x}, g(\mathbf{x})) \in \mathbb{N}_0^{n+1} \mid \mathbf{x} \in \mathbb{N}_0^n\},$$

is semidecidable.

- Since semidecidable sets are closed under existential quantification (302), the set

$$A = \exists x_1 \dots \exists x_n [(x_1, \dots, x_n, y) \in G]$$

is semidecidable. □

Recursive Enumeration –  $K$ 

- Semidecidable set  $K = \{x \mid x \in \text{dom } \phi_x^{(1)}\}$  is the domain of the partial recursive function given in (274),

$$h(x) = \psi_{\text{univ}}^{(1)}(x, x) = \begin{cases} \phi_x^{(1)}(x) & \text{if } x \in \text{dom } \phi_x^{(1)}, \\ \uparrow & \text{otherwise.} \end{cases}$$

So there exists an index  $e \in \mathbb{N}_0$  such that  $h = \phi_e^{(1)}$ .

- By (312), an enumerator of  $K$  is given by the recursive function

$$f : x \mapsto \begin{cases} \frac{\kappa_2(x)}{a} & \text{if } (e, \kappa_2(x), \tau_2(x)) \in T_1, \\ a & \text{otherwise,} \end{cases}$$

where  $a \in K$ , i.e., program  $P_e$  computes for inputs  $x = 0, 1, 2, \dots$  the outputs  $f(0), f(1), f(2), \dots$

## Recursive Enumeration – Closure Properties

If  $A, B \subseteq \mathbb{N}_0$  are r.e., then  $A \cap B, A \cup B$  are r.e.

## Proof

- Let  $f$  and  $g$  be monadic partial recursive functions with  $A = \text{dom}(f)$  and  $B = \text{dom}(g)$ . Then  $f \cdot g$  with  $(f \cdot g)(x) = f(x) \cdot g(x)$  is partial recursive with  $\text{dom}(f \cdot g) = A \cap B$ . By (289),  $A \cap B$  is r.e.
- Let  $A$  and  $B$  be non-empty sets with enumerators  $f$  and  $g$ , resp.; i.e.,  $A = \{f(0), f(1), f(2), \dots\}$  and  $B = \{g(0), g(1), g(2), \dots\}$ . Then  $A \cup B = \{h(0), h(1), h(2), \dots\}$  with enumerator

$$h : x \mapsto \begin{cases} f(\lfloor x/2 \rfloor) & \text{if } x \text{ is even,} \\ g(\lfloor x/2 \rfloor) & \text{otherwise.} \end{cases}$$

Here  $h(0) = f(0)$ ,  $h(1) = g(0)$ ,  $h(2) = f(1)$ ,  $h(3) = g(1)$  and so on. Function  $h$  defined by cases is recursive and satisfies  $\text{ran}(h) = A \cup B$ . □

## Theorems of Rice and of Rice-Shapiro

- Theorem of Rice states that for any non-trivial property of partial recursive functions, there is no general and effective method to decide whether a partial recursive function has this property or not.
- A property of partial recursive functions is called *trivial* if it holds for all partial recursive functions or for none of them.
- Theorem of Rice-Shapiro posed by Henry Gordon Rice and proved by Norman Shapiro extends of the Theorem of Rice.

## Theorem of Rice (1945)

Let  $\mathcal{C}$  be a class of non-trivial monadic partial recursive functions, i.e.,

$$\emptyset \neq \mathcal{C} \subset \mathcal{R}^{(1)}. \quad (313)$$

The corresponding index set

$$I(\mathcal{C}) = \{x \in \mathbb{N}_0 \mid \phi_x^{(1)} \in \mathcal{C}\} \quad (314)$$

is undecidable.

Special case of Rice-Shapiro (proof below).

## Theorem of Rice – Example

- Let  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto 2x$  be the doubling function and

$$\mathcal{C} = \{g \mid g \text{ is a doubling function}\}.$$

Theorem of Rice says that  $I(\mathcal{C})$  is undecidable; it cannot be decided whether a given number is an index of a doubling function.

- Let  $P$  denote the set of primes and let

$$\mathcal{C} = \{f \mid \text{dom}(f) = P\}.$$

Theorem of Rice implies that  $I(\mathcal{C})$  is undecidable; it cannot be decided whether a given number is an index of a partial recursive function whose domain is  $P$ .

## Extension

- Let  $f, g : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  be functions.  
 $f$  is an *extension* of  $g$ , written  $g \subseteq f$ , if
 
$$\text{dom}(g) \subseteq \text{dom}(f) \wedge g(x) = f(x) \text{ for all } x \in \text{dom}(g). \quad (315)$$
- Relation  $\subseteq$  is an ordering relation on the set of all monadic functions.
- Smallest element is the empty function  $f_{\uparrow}$  with  $\text{dom}(f_{\uparrow}) = \emptyset$ .  
 Maximal elements are the total functions (domain  $\mathbb{N}_0$ ).
- E.g.,  $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 2 & 3 \end{pmatrix}$  is an extension of
 
$$g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \end{pmatrix}.$$

## Finiteness

- Function  $g : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  is *finite* if its domain is finite.
- Each finite function  $g$  is partial recursive, since

$$g(x) = \sum_{a \in \text{dom}(g)} \chi_{=(x, a)} \cdot g(a), \quad x \in \mathbb{N}_0. \quad (316)$$

$\chi_{=}$  is primitive recursive with

$$\chi_{=}(x, y) = \text{csg}[(x \dot{\div} y) + (y \dot{\div} x)]. \quad (317)$$

$x_1 \dot{\div} x_2$  is realized by the URM program  $(S1; S2)S2$ .

- E.g., if  $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 2 & 3 \end{pmatrix}$ , then

$$g(x) = \chi_{=}(x, 1) \cdot 2 + \chi_{=}(x, 2) \cdot 5 + \chi_{=}(x, 3) \cdot 2 + \chi_{=}(x, 4) \cdot 3.$$

## Theorem of Rice-Shapiro (1956)

Let  $\mathcal{C}$  be a class of monadic partial recursive functions whose corresponding index set

$$I(\mathcal{C}) = \{x \in \mathbb{N}_0 \mid \phi_x^{(1)} \in \mathcal{C}\} \quad (318)$$

is r.e.

Then for each monadic partial recursive function  $f$ ,

$$f \in \mathcal{C} \iff \exists g[g \in \mathcal{C} \text{ finite} \wedge g \subseteq f], \quad (319)$$

i.e., each function  $f \in \mathcal{C}$  is an extension of a finite function  $g \in \mathcal{C}$ .

### Proof Idea

Use contraposition: If the condition (319) is not met, then  $I(\mathcal{C})$  will not be r.e.

## \*Proof

First, let  $f \in \mathcal{C}$  and assume that no finite function  $g$  which is extended by  $f$  lies in  $\mathcal{C}$ . Take the r.e. set  $K = \{x \mid x \in \text{dom } \phi_x^{(1)}\}$ , let  $e$  be an index for  $K$ , and let  $P_e$  be a GOTO program that computes  $\phi_e^{(1)}$ . Define the function

$$g : (z, t) \mapsto \begin{cases} \uparrow & \text{if } P_e \text{ computes } \phi_e^{(1)}(z) \text{ in } \leq t \text{ steps,} \\ f(t) & \text{otherwise.} \end{cases} \quad (320)$$

The function  $g$  is partial recursive. Thus by the smn theorem, there is a monadic recursive function  $s$  such that

$$g(z, t) = \phi_{s(z)}^{(1)}(t), \quad t, z \in \mathbb{N}_0. \quad (321)$$

Hence,  $\phi_{s(z)}^{(1)} \subseteq f$  for each  $z \in \mathbb{N}_0$ . Consider two cases:

- If  $z \in K$ , the program  $P_e(z)$  halts after, say  $t_0$  steps. Then

$$\phi_{s(z)}^{(1)}(t) = \begin{cases} \uparrow & \text{if } t_0 \leq t, \\ f(t) & \text{otherwise.} \end{cases} \quad (322)$$

Thus  $\phi_{s(z)}^{(1)}$  is finite and hence, by hypothesis,  $\phi_{s(z)}^{(1)}$  does not belong to  $\mathcal{C}$ .

- If  $z \notin K$ , the program  $P_e(z)$  does not halt and so  $\phi_{s(z)}^{(1)} = f$  which implies that  $\phi_{s(z)}^{(1)} \in \mathcal{C}$ .

It follows that the function  $s$  reduces the non-recursively enumerable set  $\bar{K}$  to the set  $I(\mathcal{C})$  and hence the set  $I(\mathcal{C})$  is not r.e.

## Proof (Cont'd)

Conversely, let  $f$  be a monadic partial recursive function that does not belong to  $\mathcal{C}$  and let  $g$  be a finite function in  $\mathcal{C}$  with  $g \subseteq f$ . Define the function

$$h : (z, t) \mapsto \begin{cases} f(t) & \text{if } t \in \text{dom}(g) \text{ or } z \in K, \\ \uparrow & \text{otherwise.} \end{cases} \quad (323)$$

The function  $h$  is partial recursive. Thus by the smn theorem, there is a monadic recursive function  $s$  such that

$$h(z, t) = \phi_{s(z)}^{(1)}(t), \quad t, z \in \mathbb{N}_0. \quad (324)$$

Consider two cases:

- If  $z \in K$ ,  $\phi_{s(z)}^{(1)} = f$  and so  $\phi_{s(z)}^{(1)} \notin \mathcal{C}$ .
- If  $z \notin K$ ,  $\phi_{s(z)}^{(1)}(t) = g(t)$  for all  $t \in \text{dom}(g)$  and  $\phi_{s(z)}^{(1)}$  is undefined elsewhere. Hence,  $\phi_{s(z)}^{(1)} \in \mathcal{C}$ .

It follows that the function  $s$  provides a reduction of the non-recursively enumerable set  $\bar{K}$  to the set  $I(\mathcal{C})$  and hence the set  $I(\mathcal{C})$  is not r.e. □

## Theorem of Rice-Shapiro – Example

Consider the set of total computable (recursive) functions

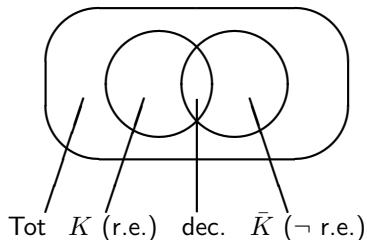
$$\mathcal{T}^{(1)} = \left\{ \phi_x^{(1)} \mid \phi_x^{(1)} \text{ total}, x \in \mathbb{N}_0 \right\}. \quad (325)$$

The index set

$$\text{Tot} = \text{I}(\mathcal{T}^{(1)}) = \left\{ x \mid \phi_x^{(1)} \text{ total} \right\} \quad (326)$$

is not r.e., since each function  $f \in \mathcal{T}^{(1)}$  is total and so cannot be an extension of a finite function  $g \in \mathcal{T}^{(1)}$ ; finite functions are not total.

## Decidable and Undecidable Sets



The decidable sets are at the intersection of r.e. and non-r.e. sets (see (295)).

Tot is not r.e. and its complement  $\overline{\text{Tot}}$  is also not r.e. (see next).  
 → Arithmetical hierarchy.

## Total Computable Functions

$\overline{\text{Tot}}$  is not r.e.

### \*Proof

Consider the dyadic function

$$h(x, y) = \begin{cases} x & \text{if } y \in K, \\ \uparrow & \text{otherwise.} \end{cases}$$

$h$  is partial recursive, since

$$h(x, y) = x \cdot \text{sgn}(\phi_y^{(1)}(y) + 1).$$

Thus there is an index  $e \in \mathbb{N}_0$  such that  $h = \phi_e^{(2)}$ .

By the smn theorem (252), there is a primitive recursive function  $s$  such that

$$h(x, y) = \phi_e^{(2)}(x, y) = \phi_{s(e, y)}^{(1)}(x).$$

## Proof (Cont'd)

We have

$$h(x, y) = \phi_e^{(2)}(x, y) = \phi_{s(e, y)}^{(1)}(x).$$

Since  $e$  is fixed by parametrization (70), write

$$g_y(x) = \phi_{s(e, y)}^{(1)}(x).$$

If  $y \in K$ , then  $g_y = \text{id}_{\mathbb{N}_0}$  and if  $y \notin K$ , then  $g_y = f_{\uparrow}$ .  
 Thus if  $y \in K$ , then  $g_y \in \text{Tot}$  and if  $y \notin K$ , then  $g_y \notin \text{Tot}$ .  
 Therefore,  $K$  is reduced to  $\text{Tot}$  and  $\bar{K}$  is reduced to  $\overline{\text{Tot}}$ .  
 Hence,  $\overline{\text{Tot}}$  is not r.e., since  $\bar{K}$  is not. □

## Theorem of Rice – Revisited (Contraposition)

Let  $\mathcal{C}$  be a class of monadic partial recursive functions.  
If  $I(\mathcal{C})$  is decidable, then  $\mathcal{C}$  is trivial, i.e.,  $\mathcal{C} = \emptyset$  or  $\mathcal{C} = \mathcal{R}^{(1)}$ .

### Proof.

Let  $I(\mathcal{C})$  be decidable.

- Then by (295),  $I(\mathcal{C})$  and  $\overline{I(\mathcal{C})}$  are r.e.
- Thus we may assume that  $\mathcal{C}$  contains the empty function  $f_{\uparrow}$ .
- Since each monadic partial recursive function  $f$  is an extension of  $f_{\uparrow}$ , it follows from Rice-Shapiro that  $\mathcal{C} = \mathcal{R}^{(1)}$ .



Theorem of Rice is a consequence of the Theorem of Rice-Shapiro.

## Recursion Theory

- Operators on the list of partial recursive functions.
- Recursion theorems (Rogers, Kleene). For any recursive function manipulating GOTO programs there is a GOTO program which is semantically invariant under this manipulation.
- Quines as programming gimmick.
- Halting problem revisited.

## Operators

- For each partial recursive function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ , define the *operator*

$$\Phi_f : \left( \phi_e^{(1)} \right)_{e \in \mathbb{N}_0} \rightarrow \left( \phi_{f(e)}^{(1)} \right)_{e \in \mathbb{N}_0}, \quad (327)$$

which maps the sequence of monadic partial recursive functions  $\left( \phi_e^{(1)} \right)_{e \in \mathbb{N}_0}$  to the image sequence  $\left( \phi_{f(e)}^{(1)} \right)_{e \in \mathbb{N}_0}$ .

- If  $f(e)$  is undefined, set  $\phi_{f(e)}^{(1)} = f_{\uparrow}$  (empty function).

## Recursion Theorem (Rodgers, 1967)

For each monadic recursive function  $f$ , there is an index  $e \in \mathbb{N}_0$  such that

$$\phi_e^{(1)} = \phi_{f(e)}^{(1)}. \quad (328)$$

Index  $e$  in (328) is called a *fixed point* of  $f$ .

### Corollary

For each monadic recursive function  $f$ , there is an index  $e \in \mathbb{N}_0$  such that

$$D_e = D_{f(e)}, \quad (329)$$

where  $D_k = \text{dom } \phi_k^{(1)}$ ,  $k \geq 0$ .

## Proof

- Consider the function  $g : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  defined by

$$g(x, y) = \begin{cases} \phi_{f(\phi_x^{(1)}(x))}^{(1)}(y) & \text{if } x \in K, \\ \uparrow & \text{otherwise.} \end{cases}$$

This function is partial recursive, since  $\phi_{f(\phi_x^{(1)}(x))}^{(1)}$  is defined for each  $x \in K = \{x \mid x \in \text{dom } \phi_x^{(1)}\}$  and

$$g(x, y) = \phi_{f(\phi_x^{(1)}(x))}^{(1)}(y) \cdot \text{sgn}(\psi_{\text{univ}}^{(1)}(x, x) + 1).$$

- We can write for all  $x \in \mathbb{N}_0$ ,

$$g(x, \cdot) = \phi_{f(\phi_x^{(1)}(x))}^{(1)}.$$

## Proof (Cont'd)

- By the smn theorem (252), there is a primitive recursive function  $s : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that for all  $x \in \mathbb{N}_0$ ,

$$g(x, \cdot) = \phi_{s(x)}^{(1)}.$$

- Thus

$$\phi_{f(\phi_x^{(1)}(x))}^{(1)} = g(x, \cdot) = \phi_{s(x)}^{(1)}.$$

If  $f(\phi_x^{(1)}(x))$  is undefined,  $s(x)$  will be the index of the empty function.

- Since  $s$  is computable, there is an index  $m$  for  $s$ ; i.e.,  $s = \phi_m^{(1)}$ . Putting  $x = m$  and  $e = \phi_m^{(1)}(m) = s(m)$ , we obtain

$$\phi_{f(e)}^{(1)} \stackrel{\text{def. } e}{=} \phi_{f(\phi_m^{(1)}(m))}^{(1)} = \phi_{s(m)}^{(1)} = \phi_e^{(1)}.$$



## Recursion Theorem – Example

- In view of the increment function  $\nu : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto x + 1$ , there is a number  $e \geq 0$  such that

$$\phi_e^{(1)} = \phi_{e+1}^{(1)}.$$

- In view of the squaring function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto x^2$ , there is a number  $e \geq 0$  such that

$$\phi_e^{(1)} = \phi_{e^2}^{(1)}.$$

- In view of tetration  $A(4, \cdot) : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : x \mapsto A(4, x)$ , there is a number  $e \geq 0$  such that

$$\phi_e^{(1)} = \phi_{A(4,e)}^{(1)}.$$

## Recursion Theorem – Example

There exists an index  $e \in \mathbb{N}_0$  such that

$$D_e = \{e\}. \quad (330)$$

- For each index  $e \in \mathbb{N}_0$  construct an GOTO program  $P_e$  which halts with input  $e$  and otherwise runs forever.
- Define the function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  with

$$f(e) = \rho(P_e),$$

where  $\rho(P)$  is the Gödel number of GOTO program  $P$ . Since  $\rho$  is primitive recursive,  $f$  is primitive recursive.

- In view of the function  $f$ , the Recursion theorem (329) ensures the existence of an index  $e \in \mathbb{N}_0$  such that

$$D_e = D_{f(e)} \stackrel{!}{=} \{e\}.$$

## Recursion Theorem – Example

GOTO program  $P_2$  which halts with input  $x_1 = 2$  and otherwise runs forever:

$$\begin{array}{ll}
 (0, \text{if } x_1 = 0, 5, 1) & \text{input } x_1 = 0, \text{ loop forever} \\
 (1, x_1 \leftarrow x_1 - 1, 2) & \\
 (2, \text{if } x_1 = 0, 5, 3) & \text{input } x_1 = 1, \text{ loop forever} \\
 (3, x_1 \leftarrow x_1 - 1, 4) & \\
 (4, \text{if } x_1 = 0, 6, 5) & \text{input } x_1 = 2, \text{ exit} \\
 (5, x_1 \leftarrow x_1 + 1, 5) & \text{input } x_1 > 2, \text{ loop forever}
 \end{array} \tag{331}$$

## Recursion Theorem (Kleene, 1938)

For each dyadic partial recursive function  $f$ , there is an index  $e \in \mathbb{N}_0$  such that

$$\phi_e^{(1)} = f(e, \cdot). \quad (332)$$

### Proof

Let  $f$  be a dyadic partial recursive function.

- By the smn theorem (252), there is a primitive recursive function  $s : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  such that for all  $x \in \mathbb{N}_0$ ,

$$f(x, \cdot) = \phi_{s(x)}^{(1)}.$$

- By the Recursion theorem (328), there is an index  $e \in \mathbb{N}_0$  s.t.

$$\phi_e^{(1)} = \phi_{s(e)}^{(1)}.$$

- Hence, for all  $y \in \mathbb{N}_0$ ,  $f(e, y) = \phi_{s(e)}^{(1)}(y) = \phi_e^{(1)}(y)$ . □

## Recursion Theorem – Quines

Consider the projection function

$$\pi_1^{(2)} : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 : (x, y) \mapsto x.$$

By (332), there is an index  $e \geq 0$  such that for all  $y \in \mathbb{N}_0$ ,

$$\phi_e^{(1)}(y) = \pi_1^{(2)}(e, y) = e.$$

Thus the function  $\phi_e^{(1)}$  outputs its own index when applied to any input value. Such indices are called *quines*.

In programming, a *quine* is a computer program which takes no input and produces a copy of its own source code as its only output (Douglas Hofstadter, 1979).

## Quine in Language C

```
#include <stdio.h>
int main(){
char*s="#include <stdio.h>%cint main(){%cchar*s=%c%c;%cprintf(s,10,10,34,s,34,10);return 0;
printf(s,10,10,34,s,34,10);return 0;}"
```

## Recursion Theorem – Undecidability of $H$

Suppose the complement (296) of the halting problem

$$H = \left\{ (x, y) \in \mathbb{N}_0^2 \mid y \in \text{dom } \phi_x^{(1)} \right\}$$

is semidecidable.

- By (289), there is a dyadic partial recursive function  $f$  such that

$$\bar{H} = \text{dom}(f).$$

- By (332), there is an index  $e \geq 0$  such that

$$\phi_e^{(1)} = f(e, \cdot).$$

- For each  $y \in \mathbb{N}_0$ :  $(e, y) \in \bar{H}$  iff  $(e, y) \in \text{dom}(f)$ , or equivalently,  $f(e, y) = \phi_e^{(1)}(y)$  is defined, which means that  $(e, y) \in H$ . Contradiction!

Thus  $\bar{H}$  is not semidecidable and hence, by (295),  $H$  cannot be decidable. □

## Skills

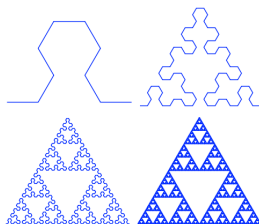
- Computability of a function, e.g. (274).
- Decidability of a set, e.g. (283).
- Existential quantification of a set, e.g.  $Z_0$ .
- Recursive enumeration of a set, e.g. (325).
- Existence of fixed points, e.g. (330).

# Part VIII

## Word Problems

## Word Problems – Motivation

- Investigation of word problems in formal language theory.
- String rewriting systems have an undecidable word problem.
- Notion of string rewriting coincides with the presentation of semigroups.



String rewriting fractals (L-systems)

## Word Problems – Contents

- Semi-Thue Systems
- Thue Systems
- Semigroups
- \*Post's Correspondence Problem
- \*Hilbert's 10th Problem

Reading: Zimmermann, Chapter 8

## Semi-Thue Systems – Inside

- String rewriting systems are historically called semi-Thue systems.
- Introduced by Axel Thue (1914).
- Unsolvable word problem proved independently by Emil Post and A.A. Markov Jr. (1947).

## Semi-Thue Systems

*Semi-Thue system (STS)* is a pair  $(\Sigma, R)$ , where

- $\Sigma$  is an alphabet (i.e., non-empty finite set) and
- $R \subseteq \Sigma^+ \times \Sigma^+$  is a dyadic relation, called *rule set*, on the non-empty strings over  $\Sigma$ ,

$$\Sigma^+ = \Sigma^* \setminus \{\epsilon\}. \quad (333)$$

Each element  $(u, v) \in R$  is called a *rewriting rule* and is written as

$$u \rightarrow_R v. \quad (334)$$

## STS – One-Step Rewriting

Given STS  $(\Sigma, R)$ .

*One-step rewriting relation* is the dyadic relation

$$\Rightarrow_R \subseteq \Sigma^+ \times \Sigma^+$$

such that for all  $s, t$  in  $\Sigma^+$ ,

$$s \Rightarrow_R t \quad :\iff \quad (335)$$

$$s = xuy, t = xvy, u \rightarrow_R v \text{ for some } x, y \in \Sigma^*, u, v \in \Sigma^+,$$

i.e., string  $s$  is rewritten by string  $t$  if substring  $u$  of  $s$  is replaced by substring  $v$  of  $t$  by using the rewriting rule  $u \rightarrow_R v$ :

$$s = xuy$$

$$u \rightarrow_R v$$

$$t = xvy$$

## STS – Multi-Step Rewriting

Given STS  $(\Sigma, R)$ .

*Multi-steps rewriting* or *derivation* is captured by the reflexive transitive closure of  $\Rightarrow_R$  denoted by  $\Rightarrow_R^*$ .

For any strings  $s, t \in \Sigma^+$ ,  $s \Rightarrow_R^* t$  iff

- $s = t$  or
- there is a finite sequence  $s_0, s_1, \dots, s_m$  of elements in  $\Sigma^+$  such that
  - $s = s_0$ ,
  - $s_i \Rightarrow_R s_{i+1}$  for  $0 \leq i \leq m - 1$ , and
  - $s_m = t$ .

Let  $R \subseteq A \times A$  be a relation. Put  $R^0 = \text{id}_A$  and  $R^{n+1} = R \circ R^n$  for all  $n \geq 0$ . Then  $R^* = \bigcup_{n \geq 0} R^n$  is the *reflexive transitive closure* of  $R$ ;  $R^*$  is the smallest reflexive and transitive relation containing  $R$ .

## STS – Example

Consider the semi-Thue system  $(\Sigma, R)$  with alphabet  $\Sigma = \{a, b\}$  and rule set

$$R = \{(ab, a), (ba, b)\}.$$

The derivation

$$\begin{array}{ll} baab \Rightarrow_R bab, & ba \rightarrow_R b, \\ \Rightarrow_R ba, & ab \rightarrow_R a, \\ \Rightarrow_R b, & ba \rightarrow_R b, \end{array}$$

shows that

$$baab \Rightarrow_R^* b.$$

## STS – Word Problem

Given semi-Thue system  $(\Sigma, R)$  and two strings  $s, t \in \Sigma^+$ .

Decide whether or not

$$s \Rightarrow_R^* t. \quad (336)$$

This problem is undecidable.

## Proof

Reduce the halting problem for SGOTO-2 programs to this word problem.

## STS – SGOTO-2 Programs

Given SGOTO-2 program consisting of  $n$  instructions,

$$P = s_0; s_1; \dots; s_{n-1}$$

- By definition, instruction  $s_i$  has the label  $i$ ,  $0 \leq i \leq n - 1$ .
- Assume label  $n$ , called *exit label*, which does not address an instruction, is the only label signifying termination.

Define STS  $(\Sigma, R_P)$  that simulates the behavior of SGOTO-2 program  $P$ .

## STS – Configurations

- *Configuration* of the two-register machine is given by a triple

$$(j, x, y), \quad (337)$$

where  $j \geq 0$  is the actual instruction number of SGOTO program,  $x$  is the content of  $R_1$  and  $y$  is the content of  $R_2$ .

- Register content can be encoded in unary format,

$$\bar{x} = \overbrace{LL \dots L}^x \quad \text{and} \quad \bar{y} = \overbrace{LL \dots L}^y. \quad (338)$$

- Now each configuration of the two-register machine can be written as a string

$$a\bar{x}j\bar{y}b \quad (339)$$

over the alphabet  $\Sigma = \{a, b, 0, 1, 2, \dots, n-1, n, L\}$ .

## STS – Encoding of Instructions

Configuration of two-register machine:

$$aL \dots LjL \dots Lb.$$

SGOTO-2 instructions are assigned appropriate rewriting rules:

GOTO-2 instructions	Rewriting rules	
$(j, x_1 \leftarrow x_1 + 1, k)$	$(j, Lk)$	(340)
$(j, x_2 \leftarrow x_2 + 1, k)$	$(j, kL)$	
$(j, x_1 \leftarrow x_1 - 1, k)$	$(Lj, k), (aj, ak)$	
$(j, x_2 \leftarrow x_2 - 1, k)$	$(jL, k), (jb, kb)$	
$(j, \text{if } x_1 = 0, k, l)$	$(Lj, Ll), (aj, ak)$	
$(j, \text{if } x_2 = 0, k, l)$	$(jL, lL), (jb, kb)$	

Moreover, there are two clean-up rewriting rules:

$$(Ln, n) \quad \text{and} \quad (anL, an). \quad (341)$$

## STS – Example

SGOTO-2 program  $P$  given by

$$(0, \text{if } x_2 = 0, 3, 1)$$

$$(1, x_1 \leftarrow x_1 + 1, 2)$$

$$(2, x_2 \leftarrow x_2 - 1, 0)$$

computes the addition of two numbers,

$$\|P\|_{2,1}(x, y) = x + y, \quad x, y \in \mathbb{N}_0.$$

Associated STS over  $\Sigma = \{a, b, 0, 1, 2, 3, L\}$  has rewriting rules:

Instructions	Rewriting rules
$(0, \text{if } x_2 = 0, 3, 1)$	$(0L, 1L), (0b, 3b),$
$(1, x_1 \leftarrow x_1 + 1, 2)$	$(1, L2),$
$(2, x_2 \leftarrow x_2 - 1, 0)$	$(2L, 0), (2b, 0b),$
	$(L3, 3), (a3L, a3).$

## STS – Example (Cont'd)

Sample computation with initial register values  $x_1 = 3$  and  $x_2 = 2$ :

SGOTO-2 instruction	Configuration	Derivation	Rule
(0, if $x_2 = 0, 3, 1$ )	(0, 3, 2)	<i>aLLL0LLb</i>	(0L, 1L)
(1, $x_1 \leftarrow x_1 + 1, 2$ )	(1, 3, 2)	<i>aLLL1LLb</i>	(1, L2)
(2, $x_2 \leftarrow x_2 - 1, 0$ )	(2, 4, 2)	<i>aLLLL2LLb</i>	(2L, 0)
(0, if $x_2 = 0, 3, 1$ )	(0, 4, 1)	<i>aLLLL0Lb</i>	(0L, 1L)
(1, $x_1 \leftarrow x_1 + 1, 2$ )	(1, 4, 1)	<i>aLLLL1Lb</i>	(1, L2)
(2, $x_2 \leftarrow x_2 - 1, 0$ )	(2, 5, 1)	<i>aLLLLL2Lb</i>	(2L, 0)
(0, if $x_2 = 0, 3, 1$ )	(0, 5, 0)	<i>aLLLLL0b</i>	(0b, 3b)
	(3, 5, 0)	<i>aLLLLL3b</i>	(L3, 3)
		<i>aLLLL3b</i>	(L3, 3)
		<i>aLLL3b</i>	(L3, 3)
		<i>aLL3b</i>	(L3, 3)
		<i>aL3b</i>	(L3, 3)
		<i>a3b</i>	

## STS – One-Step Reduction

Configuration  $(j, x, y)$  of SGOTO-2 program  $P$  with  $j < n$  leads to successor configuration  $(k, u, v)$  in one step, i.e. by (128)

$$(j, x, y) \vdash_P (k, u, v), \quad (342)$$

iff STS  $(\Sigma, R_P)$  provides the one-step derivation

$$a\bar{x}j\bar{y}b \Rightarrow_{R_P} a\bar{u}k\bar{v}b. \quad (343)$$

There is no other one-step rewriting rule in the STS applicable to the string  $a\bar{x}j\bar{y}b$  by the definition of the rule set.

## STS – Multi-Step Reduction

Configuration  $(j, x, y)$  of SGOTO-2 program  $P$  with  $j < n$  leads to configuration  $(k, u, v)$  in multiple steps, i.e. by (128)

$$(j, x, y) \vdash_P^* (k, u, v), \quad (344)$$

iff STS  $(\Sigma, R_P)$  yields the derivation

$$a\bar{x}j\bar{y}b \xRightarrow{*}_{R_P} a\bar{u}k\bar{v}b. \quad (345)$$

Proof by induction.

## STS – Termination

- When the SGOTO-2 program terminates, its final configuration is of the form

$$(n, x, y) \quad (346)$$

where  $n$  is the unique exit label.

- This corresponds in the STS to the word

$$a\bar{x}n\bar{y}b \quad (347)$$

which can be further rewritten according to the clean-up rules (341):

$$a\bar{x}n\bar{y}b \xrightarrow{*}_{RP} anb. \quad (348)$$

## STS – Main Theorem

The word problem for semi-Thue systems is undecidable.

### Proof.

- SGOTO-2 program  $P$  started in configuration  $(0, x, y)$  halts iff in the corresponding STS there is a derivation

$$a\bar{x}0\bar{y}b \xRightarrow{*}_{R_P} anb. \quad (349)$$

- This provides an effective reduction of the halting problem for SGOTO-2 programs to the word problem for semi-Thue systems (336).
- Halting problem for SGOTO-2 programs is undecidable implies word problem for STS is undecidable.



## Thue Systems

- Thue systems form a subclass of semi-Thue systems.
- Word problem for Thue systems is still undecidable.
- Thue systems correspond to presentations of semigroups.

## Thue Systems

*Thue system* (TS) is an STS  $(\Sigma, R)$  whose relation  $R$  is symmetric, i.e.,

$$u \rightarrow_R v \implies v \rightarrow_R u. \quad (350)$$

In the TS  $(\Sigma, R)$ , the reflexive transitive closure  $\Rightarrow_R^*$  of the one-step rewriting relation  $\Rightarrow_R$  is also symmetric and hence an *equivalence relation* on  $\Sigma^+$ .

In the TS  $(\Sigma, R)$ , we have for all strings  $s, t \in \Sigma^*$  by symmetry,

$$s \Rightarrow_R^* t \iff t \Rightarrow_R^* s. \quad (351)$$

## Thue Systems – Word Problem

Given Thue system  $(\Sigma, R)$  and two strings  $s, t \in \Sigma^+$ .

Decide whether or not

$$s \Rightarrow_R^* t. \quad (352)$$

This problem is undecidable.

### Proof

Using Post's Lemma.

## Thue Systems – Symmetric Closure

Given STS  $(\Sigma, R)$ .

- *Symmetric closure* of rule set  $R$  is the symmetric relation

$$R^{(s)} = R \cup R^{-1}, \quad (353)$$

where

$$R^{-1} = \{(v, u) \mid (u, v) \in R\} \quad (354)$$

is the *inverse relation* of  $R$ .

- $R^{(s)}$  is the smallest symmetric relation containing  $R$  and  $(\Sigma, R^{(s)})$  is a Thue system.
- Relation  $\Rightarrow_{R^{(s)}}^*$  is the reflexive transitive and symmetric closure of  $\Rightarrow_R$  and hence an equivalence relation on  $\Sigma^+$ .

## Thue Systems – Multi-Step Reduction

For any strings  $s, t \in \Sigma^+$ ,  $s \xRightarrow{*}_{R(s)} t$  iff

- $s = t$  or
- there is a finite sequence  $s_0, s_1, \dots, s_m$  of elements in  $\Sigma^+$  such that
  - $s = s_0$ ,
  - $s_i \Rightarrow_R s_{i+1}$  or  $s_{i+1} \Rightarrow_R s_i$  for  $0 \leq i \leq m - 1$ , and
  - $s_m = t$ .

## Thue Systems – Multi-Step Reduction

Derivations in STS  $(\Sigma, R)$  and associated TS  $(\Sigma, R^{(s)})$  are related as follows:

$$s \Rightarrow_R^* t \implies s \Rightarrow_{R^{(s)}}^* t, \quad s, t \in \Sigma^+, \quad (355)$$

since  $R \subseteq R^{(s)}$ .

But if in TS  $(\Sigma, R^{(s)})$

$$s \xrightarrow{R^{(s)}}^* t,$$

holds, then

$$\text{neither } s \xrightarrow{R}^* t \text{ nor } t \xrightarrow{R}^* s$$

need to be valid in the STS  $(\Sigma, R)$ .

## Thue Systems – Example

Consider STS  $(\Sigma, R)$  with alphabet  $\Sigma = \{a, b\}$  and rule set

$$R = \{(ab, a), (ba, b)\}.$$

- Associated TS  $(\Sigma, R^{(s)})$  has rule set

$$R^{(s)} = R \cup \{(a, ab), (b, ba)\}.$$

- In the STS, rewriting is strictly antitone leading to smaller strings, i.e., if  $u \Rightarrow_R^* v$  and  $u \neq v$ , then  $|u| > |v|$ , e.g.,

$$baab \Rightarrow_R bab \Rightarrow_R ba \Rightarrow_R b.$$

- In the TS,  $baab \Rightarrow_{R^{(s)}} bab \Rightarrow_{R^{(s)}} baba$ , but in the STS

neither  $baab \xRightarrow_R^* baba$  nor  $baba \xRightarrow_R^* baab$ .

## Post's Lemma

Given SGOTO-2 program  $P$  with  $n$  instructions and exit label  $n$ .

Let  $(\Sigma, R_P)$  be the corresponding semi-Thue system, and  $(\Sigma, R_P^{(s)})$  be the associated Thue system.

For each configuration  $(j, x, y)$  of  $P$ ,

$$a\bar{x}j\bar{y}b \xrightarrow{*}_{R_P} anb \iff a\bar{x}j\bar{y}b \xrightarrow{*}_{R_P^{(s)}} anb. \quad (356)$$

Compare (356) with the general case (355).

## Proof

- By (355), the direction from left-to-right is clear:

$$a\bar{x}j\bar{y}b \xRightarrow{*}_{R_P} anb \implies a\bar{x}j\bar{y}b \xRightarrow{*}_{R_P^{(s)}} anb$$

since  $R_P \subseteq R_P^{(s)}$ .

- Conversely, let  $(j, x, y)$  be a configuration of  $P$  and

$$a\bar{x}j\bar{y}b \xRightarrow{*}_{R_P^{(s)}} anb.$$

Then there is a rewriting sequence in the TS:

$$w_0 = a\bar{x}j\bar{y}b \Rightarrow_{R_P^{(s)}} w_1 \Rightarrow_{R_P^{(s)}} \dots \Rightarrow_{R_P^{(s)}} w_q = anb. \quad (357)$$

Assume that the length  $q$  of this derivation is minimal.

## Proof (Cont'd)

Suppose derivation (357) cannot be established by the STS.

- Then the sequence (357) contains a reverse rewriting step

$$w_p \leftarrow_{R_P} w_{p+1}, \quad 0 \leq p \leq q - 1.$$

Choose the index  $p$  maximal with this property.

- Since there is no rewriting rule applicable to  $w_q = anb$ , we have  $p + 1 < q$ . This leads to the following situation:

$$w_p \leftarrow_{R_P} w_{p+1} \Rightarrow_{R_P} w_{p+2}. \quad (358)$$

But by construction, there is at most one rule applicable to the string  $w_{p+1} = a\bar{x}j\bar{y}b$  given by instruction  $s_j$  and configuration  $(j, x, y)$ .

- Thus the words  $w_p$  and  $w_{p+2}$  must be identical and hence the derivation (357) can be shortened by deleting the string  $w_{p+1}$  contradicting the (minimality) assumption.  $\square$

## Semigroups

- Word problems are also encountered in abstract algebra.
- Thue systems give rise to semigroups in a natural way.
- Word problem for semigroups is undecidable (similar for groups).

## Semigroups

A *semigroup* is a pair  $(S, \cdot)$ , where

- $S$  is a non-empty set and
- $\cdot : S \times S \rightarrow S$  is a dyadic operation which is *associative*, i.e., for all  $a, b, c \in S$ ,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c). \quad (359)$$

- A semigroup  $(S, \cdot)$  is *commutative* if for all  $a, b \in S$ ,

$$a \cdot b = b \cdot a. \quad (360)$$

Reading: Zimmermann, Appendix A

### Example

Let  $\Sigma$  be an alphabet. The pair  $(\Sigma^+, \cdot)$  is the semigroup of all non-empty words over  $\Sigma$  and  $\cdot$  is the concatenation of words.

## Equivalence Relation

Let  $(\Sigma, R^{(s)})$  be a Thue system.

- Rewriting relation  $\xRightarrow{*}_{R^{(s)}}$  on  $\Sigma^+$  is an equivalence relation.
- Two strings  $s, t \in \Sigma^+$  are *equivalent* if

$$s \xRightarrow{*}_{R^{(s)}} t. \quad (361)$$

- *Equivalence class* of string  $s \in \Sigma^+$  is the set of all strings in  $\Sigma^+$  that can be derived from  $s$  by a finite number of rewriting steps, i.e.,

$$[s] = \{t \in \Sigma^+ \mid s \xRightarrow{*}_{R^{(s)}} t\}. \quad (362)$$

- Note that for all strings  $s, t \in \Sigma^+$ , we have

$$[s] = [t] \iff s \xRightarrow{*}_{R^{(s)}} t. \quad (363)$$

## Quotient Set

Let  $(\Sigma, R^{(s)})$  be a Thue system.

- *Quotient set* of the equivalence relation is given by the set of all equivalence classes,

$$S = S(\Sigma, R^{(s)}) = \{[s] \mid s \in \Sigma^+\}. \quad (364)$$

## Semigroups

Let  $(\Sigma, R^{(s)})$  be a Thue system.

Quotient set  $S = S(\Sigma, R^{(s)})$  forms a semigroup under the operation (concatenation)

$$[s] \cdot [t] = [st], \quad s, t \in \Sigma^+. \quad (365)$$

### Proof

- Operation is well-defined.

Indeed, let  $[s] = [s']$  and  $[t] = [t']$ , where  $s, s', t, t' \in \Sigma^+$ .

Then by (363),

$$s \xrightarrow{*}_{R^{(s)}} s' \text{ and } t \xrightarrow{*}_{R^{(s)}} t'.$$

Thus

$$st \xrightarrow{*}_{R^{(s)}} s't \xrightarrow{*}_{R^{(s)}} s't'$$

and hence by (363),

$$[st] = [s't'].$$

## Proof (Cont'd)

- Operation is associative.

Indeed, let  $r, s, t \in \Sigma^+$ . Then

$$\begin{aligned} [r] \cdot ([s] \cdot [t]) &= [r] \cdot [st], \text{ by (365)} \\ &= [r(st)], \text{ by (365)} \\ &= [(rs)t], (\Sigma^+, \cdot) \text{ is associative,} \\ &= [rs] \cdot [t], \text{ by (365)} \\ &= ([r] \cdot [s]) \cdot [t], \text{ by (365).} \end{aligned}$$



## Semigroups – Presentation

Semigroup  $S = S(\Sigma, R^{(s)})$  has a *presentation* in terms of generators and relations,

$$S = \langle a_1, \dots, a_n \mid u_1 = v_1, \dots, u_m = v_m \rangle, \quad (366)$$

where

$$\Sigma = \{a_1, \dots, a_n\}$$

is the alphabet and

$$R = \{(u_1, v_1), \dots, (u_m, v_m)\}$$

is the rule set.

Reading: Zimmermann, Appendix A

## Presentation – Example

Given Thue system  $(\Sigma, R^{(s)})$  with alphabet  $\Sigma = \{a, b\}$  and rule set  $R^{(s)}$ , where

$$R = \{(ab, ba), (a, aa), (b, bbb)\}$$

and so

$$R^{(s)} = \{(ab, ba), (a, aa), (b, bbb), (ba, ab), (aa, a), (bbb, b)\}.$$

- Semigroup  $S = S(\Sigma, R^{(s)})$  has the presentation

$$S = \langle a, b \mid ab = ba, a = aa, b = bbb \rangle.$$

- Each word of  $\Sigma^+$  has the form

$$a^{i_1} b^{j_1} \dots a^{i_k} b^{j_k},$$

where  $k \geq 1$  and at least one of the numbers  $i_1, \dots, i_k, j_1, \dots, j_k \geq 0$  is non-zero.

## Example (Cont'd)

As seen, each word of  $\Sigma^+$  has the form

$$w = a^{i_1} b^{j_1} \dots a^{i_k} b^{j_k} \neq \epsilon.$$

- In view of the rule  $(ba, ab)$ ,

$$w = a^i b^j$$

where  $i, j \geq 0$  and  $i, j$  are not both zero.

- In view of the rules  $(aa, a)$  and  $(bbb, b)$ ,

$$w = a^i b^j$$

where  $0 \leq i \leq 1$ ,  $0 \leq j \leq 2$  and  $i, j$  are not both zero.

## Example (Cont'd)

As seen, each word of  $\Sigma^+$  can be transformed to

$$w = a^i b^j$$

where  $0 \leq i \leq 1$ ,  $0 \leq j \leq 2$  and  $i, j$  are not both zero.

- This leads to the semigroup

$$S = \{[a], [ab], [abb], [b], [bb]\}.$$

- Multiplication table of  $S$ :

$\cdot$	$[a]$	$[ab]$	$[abb]$	$[b]$	$[bb]$
$[a]$	$[a]$	$[ab]$	$[abb]$	$[ab]$	$[abb]$
$[ab]$	$[ab]$	$[abb]$	$[ab]$	$[abb]$	$[ab]$
$[abb]$	$[abb]$	$[ab]$	$[abb]$	$[ab]$	$[abb]$
$[b]$	$[ab]$	$[abb]$	$[ab]$	$[bb]$	$[b]$
$[bb]$	$[abb]$	$[ab]$	$[abb]$	$[b]$	$[bb]$

## Example (Cont'd)

Multiplication in semigroup  $S$ :

- $[ab] \cdot [ab] = [abb]$ , since
  - $[ab] \cdot [ab] = [ab \cdot ab] = [abab]$  and
  - $[abab] = [abb]$  as

$$abab \Rightarrow_{R(s)} aabb \Rightarrow_{R(s)} abb.$$

- $[ab] \cdot [abb] = [ab]$ , since
  - $[ab] \cdot [abb] = [ab \cdot abb] = [ababb]$  and
  - $[ababb] = [ab]$  as

$$ababb \Rightarrow_{R(s)} aabbb \Rightarrow_{R(s)} abbb \Rightarrow_{R(s)} ab.$$

Multiplication is commutative because of the rules  $ab \rightarrow_R ba$  and  $ba \rightarrow_R ab$ .

## Semigroups – Word Problem

Given semigroup  $(S, \cdot)$  and two elements  $s, t \in S$ .  
Decide whether or not

$$s = t. \quad (367)$$

This problem is undecidable.

Let  $(\Sigma, R^{(s)})$  be a Thue System,  $S = S(\Sigma, R^{(s)})$  be the corresponding semigroup and  $s, t \in \Sigma^+$ .  
Decide whether or not

$$[s] = [t]. \quad (368)$$

### Proof.

By (363), the word problem for Thue systems can be reduced to word problem for semigroups. But the word problem for Thue systems is undecidable and so is the word problem for semigroups. □

## Skills

- Multi-step rewriting in semi-Thue system.
- Representation of computation of SGOTO program by derivation of associated semi-Thue system.
- Multi-step rewriting in Thue system.
- Symmetrizing semi-Thue system to provide Thue system.
- Representation of Thue system as semigroup including multiplication table.

## Final Words

- Introduction to computability theory
- Exam: written, electronic, 2 hours, look into former exams.
- Exam date: August 26, 2021.
- Preparation hour for the exam by Merve Cakir.

All the best and good luck in the exam!