

# A variant of the inverted Lanczos method

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## Abstract

In this note we study a variant of the inverted Lanczos method which computes eigenvalue approximates of a symmetric matrix  $A$  from the projection to a Krylov space of  $A^{-1}$ . The method turns out to be slightly faster than the Lanczos method at least as long as reorthogonalization is not required. The method is applied to the problem of determining the smallest eigenvalue of a symmetric Toeplitz matrix. It is accelerated taking advantage of symmetry properties of the corresponding eigenvector.

**Keywords:** eigenvalue problem, Lanczos method, Toeplitz matrix, symmetry properties

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## 1 Introduction

In this paper we consider the problem to determine the smallest eigenvalue (or the lower part of the spectrum) of a symmetric and positive definite matrix  $A \in \mathbb{R}^{(n,n)}$  by a variant of the Lanczos method.

It is well known that the accuracy of an approximation to an eigenvalue  $\lambda$  obtained by the Lanczos method is influenced mainly by the relative separation of  $\lambda$  from the other eigenvalues of  $A$ . The bigger the separation is the better is the approximation. Therefore to compute the lower part of the spectrum it is often advantageous to apply the Lanczos method to the inverse  $A^{-1}$ , i.e. to determine the approximations from the projection of the matrix eigenvalue problem  $A^{-1}x = \lambda^{-1}x$  to the Krylov space  $\mathcal{K}_k(A^{-1}, u) := \text{span}\{u, A^{-1}u, \dots, A^{-k+1}u\}$  (cf. [2], [5], [9]).

Motivated by a paper of Melman [6] on bounds of the extreme eigenvalues of real symmetric Toeplitz matrices and by the fact that these bounds can be interpreted as the extreme eigenvalues of the projection of the eigenvalue problem to a Krylov space of  $A^{-1}$  (cf. [11]) we study in this paper a variant of the Lanczos method which produces Ritz values of  $A$  from  $\mathcal{K}_k(A^{-1}, u)$ . A similar combination was considered by Paige, Parlett and van der Vorst [7] who introduced harmonic Ritz values as the reciprocals of the Ritz values of  $A^{-1}$  from  $A\mathcal{K}_k(A, u)$ .

Our method can be executed in a similar way as the inverted Lanczos method. An orthonormal basis  $q_1, \dots, q_k$  of  $\mathcal{K}_k(A^{-1}, u)$  with respect to the scalar product  $\langle x, y \rangle_A := x^T A y$  can be determined by a three term recurrence relation, and with  $Q := (q_1, \dots, q_k) \in \mathbb{R}^{(n,k)}$  the projected eigenproblem  $Q^T A Q y = \lambda Q^T Q y$  is tridiagonal.

Increasing the dimension  $k$  of the Krylov space by 1 the cost of the new variant is identical to that of the inverted Lanczos process. However, for  $k = 1$  the inverted Lanczos method needs one solution of a linear system which is not the case for the modified method. Hence, the cost of the  $k$ -dimensional Lanczos method roughly compares to that of its  $(k-1)$ -dimensional variant. Since the approximate eigenvalues  $\sigma_1^{(k)} \leq \sigma_2^{(k)} \leq \dots \leq \sigma_k^{(k)}$  are shown to satisfy  $\sigma_j^{(k)} \leq \rho_j^{(k-1)} \leq \sigma_j^{((k-1))}$  where  $\rho_j^{(k)}$  denotes the approximation of the  $j$ -th eigenvalue from the  $k$ -dimensional inverted Lanczos method one should prefer the modification given here.

The method suffers the same disadvantage as the Lanczos process, orthogonality is lost in the course of the iteration, and (full or selected) reorthogonalization may be necessary in order to maintain accuracy. Since the Gram-Schmidt process with respect to the inner product  $\langle \cdot, \cdot \rangle_A$  is more expensive than for the Euclidean scalar product the advantage mentioned in the last paragraph disappears if reorthogonalization is required. However, since the convergence to the extreme eigenvalues appears first loss of orthogonality is not a problem if one is only interested in the smallest eigenvalue of  $A$ .

The problem of finding the smallest eigenvalue and corresponding eigenvector of a real symmetric and positive definite Toeplitz matrix is of considerable interest in signal processing (cf. [8]). Section 3 contains the comparison of the variant considered here and the inverted Lanczos method for a large number of examples of this type. Moreover we sketch a modification that takes advantage of symmetry properties of the eigenvectors of symmetric Toeplitz matrices.

## 2 An inverted Lanczos method

Let  $A \in \mathbb{R}^{(n,n)}$  be a symmetric and positive definite matrix, let  $u \in \mathbb{R}^n$ ,  $u \neq 0$ , be a given vector, and let

$$\mathcal{K}_k(A^{-1}, u) := \text{span}\{u, A^{-1}u, \dots, A^{-k+1}u\}$$

be the Krylov space corresponding to  $A^{-1}$  and the initial vector  $u$ .

We consider the projection of the eigenvalue problem

$$Ax = \lambda x \quad (1)$$

to  $\mathcal{K}_k(A^{-1}, u)$ , i.e. if the columns of the matrix  $Q \in \mathbb{R}^{(n,k)}$  form a basis of  $\mathcal{K}_k(A^{-1}, u)$ , then we consider the eigenvalue problem

$$Q^T A Q y = \lambda Q^T Q y, \quad y \in \mathbb{R}^k \setminus \{0\}. \quad (2)$$

This method has similar properties as the inverted Lanczos method. An orthonormal basis of  $\mathcal{K}_k(A^{-1}, u)$  with respect to the scalar product  $\langle x, y \rangle_A := x^T A y$  can be determined by a three term recurrence relation such that the projected eigenvalue problem (2) is tridiagonal. To this end consider the following algorithm:

$q_1 = u / \sqrt{u^T A u}$ ;  $\alpha_1 = \|q_1\|_2^2$ ;  $q_0 = 0$ ;  $\beta_0 = 0$ ;  
for  $k = 1, 2, \dots$  until convergence do  
 $\tilde{q}_{k+1} = A^{-1} q_k - \alpha_k q_k - \beta_{k-1} q_{k-1}$ ;  
 $\beta_k = \sqrt{\tilde{q}_{k+1}^T q_k}$ ;  
 $q_{k+1} = \tilde{q}_{k+1} / \beta_k$ ;  
 $\alpha_{k+1} = \|q_{k+1}\|_2^2$ ;  
Solve the eigenvalue problem  
 $y = \lambda \text{tridiag}(\beta_{j-1}, \alpha_j, \beta_j)_{j=1, \dots, k+1} y$ ;  
end

For the sequence  $q_1, q_2, \dots$  it holds that  $q_1^T A q_1 = 1$ , and if  $\{q_1, \dots, q_k\}$  is already shown to be an orthonormal basis of  $\mathcal{K}_k(A^{-1}, u)$  with respect to  $\langle x, y \rangle_A$  for some  $k \in \{1, \dots, n-1\}$  then it follows from  $A^{-1} q_i \in \text{span}\{q_1, \dots, q_{i+1}\}$  that

$$q_k^T q_i = q_k^T A (A^{-1} q_i) = 0 \quad \text{for } i = 1, \dots, k-2 \quad (3)$$

and

$$q_k^T q_{k-1} = q_k^T A (\tilde{q}_k + \alpha_{k-1} q_{k-1} + \beta_{k-2} q_{k-2}) = q_k^T A \tilde{q}_k = \beta_{k-1} q_k^T A q_k = \beta_{k-1}. \quad (4)$$

Hence

$$q_{k+1}^T A q_j = \frac{1}{\beta_k} (A^{-1} q_k - \alpha_k q_k - \beta_{k-1} q_{k-1})^T A q_j = 0 \quad \text{for } j = 1, \dots, k-2,$$

$$q_{k+1}^T A q_{k-1} = \frac{1}{\beta_k} (q_k^T q_{k-1} - \beta_{k-1}) = 0$$

and

$$q_{k+1}^T A q_k = \frac{1}{\beta_k} (\|q_k\|_2^2 - \alpha_k) = 0.$$

Finally it follows from the definition of  $\beta_k$

$$\begin{aligned} q_{k+1}^T A q_{k+1} &= \frac{1}{\beta_k} (A^{-1} q_k - \alpha_k q_k - \beta_{k-1} q_{k-1})^T A q_{k+1} \\ &= \frac{1}{\beta_k} q_k^T q_{k+1} = \frac{1}{\beta_k^2} q_k^T \tilde{q}_{k+1} = 1. \end{aligned}$$

Thus,  $\{q_1, \dots, q_k, q_{k+1}\}$  is an orthonormal basis of  $\mathcal{K}_{k+1}(A^{-1}, u)$ . Moreover, (3), (4) and the definition of  $\alpha_k$  imply that for this basis the projected eigenvalue problem (2) reads

$$y = \lambda \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{k-1} \\ & & & \beta_{k-1} & \alpha_k \end{pmatrix} y =: \theta T_k y. \quad (5)$$

With  $Q_k := (q_1, \dots, q_k) \in \mathbb{R}^{(n,k)}$  the modified Lanczos algorithm can be rewritten as

$$A^{-1} Q_k = Q_k T_k + \beta_k q_{k+1} e_k^T \quad (6)$$

where  $e_k \in \mathbb{R}^k$  denotes the unit vector containing a 1 in its last component and zeros elsewhere. From (6) we obtain an error bound in a similar way as for the Lanczos method.

**Theorem 1** *Let  $y \in \mathbb{R}^k$  such that*

$$y = \theta T_k y, \quad \|y\|_2 = 1.$$

*Then there exists an eigenvalue  $\lambda$  of  $A$  such that*

$$\frac{|\lambda - \theta|}{\lambda} \leq \theta \cdot |\beta_k| \cdot |e_k^T y|. \quad (7)$$

**Proof:** For  $z := Q_k y$  it follows from (6)

$$A^{-1} z = A^{-1} Q_k y = Q_k T_k y + \beta_k q_{k+1} e_k^T y = \frac{1}{\theta} Q_k y + \beta_k q_{k+1} e_k^T y = \frac{1}{\theta} z + \beta_k q_{k+1} e_k^T y.$$

Hence,  $\|q_{k+1}\|_A = 1$  and  $\|z\|_A^2 = y^T Q_k^T A Q_k y = \|y\|_2^2 = 1$  yields

$$\frac{\|A^{-1} z - \frac{1}{\theta} z\|_A}{\|z\|_A} = |\beta_k| \cdot |e_k^T y|,$$

from which we obtain that there exists an eigenvalue  $\lambda$  of  $A$  such that

$$\left| \frac{1}{\lambda} - \frac{1}{\theta} \right| \leq |\beta_k| \cdot |e_k^T y|,$$

i.e. the error bound (7).

q.e.d.

Increasing the dimension  $k$  by 1 the cost is identical for both methods, the inverted Lanczos method and its variant considered here: namely one has to solve 1 linear system and to execute 5 level one operations. Notice however, that for  $k = 1$  the inverted Lanczos methods needs the normalization of the initial vector with respect to the Euclidean inner product and one solution of a linear system whereas in the modified method one only has to normalize the initial vector with respect to  $\langle \cdot, \cdot \rangle_A$ . Hence, the cost of the  $k$ -dimensional inverted Lanczos method roughly compares to that of the  $(k - 1)$ -dimensional modified version.

The following Theorem demonstrates that for  $j = 1, \dots, k$  the approximation  $\rho_j^{(k)}$  to the  $j$ -th eigenvalue  $\lambda_j$  of  $A$  from the inverted Lanczos method using  $\mathcal{K}_k(A^{-1}, u)$  is less accurate than the Ritz value  $\sigma_j^{(k+1)}$  from  $\mathcal{K}_{k+1}(A^{-1}, u)$ .

To prove this comparison result we consider the projected eigenvalue problem (2) corresponding to the basis

$$Q = (u, A^{-1}u, \dots, A^{1-k}u) \quad (8)$$

of  $\mathcal{K}_k(A^{-1}, u)$ . Then equation (2) reads

$$K_M^{(k)} y := \begin{pmatrix} \mu_{-1} & \mu_0 & \dots & \mu_{k-2} \\ \mu_0 & \mu_1 & \dots & \mu_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k-2} & \mu_{k-1} & \dots & \mu_{2k-3} \end{pmatrix} y = \lambda \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{k-1} \\ \mu_1 & \mu_2 & \dots & \mu_k \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k-1} & \mu_k & \dots & \mu_{2k-2} \end{pmatrix} y =: \lambda M_M^{(k)} y. \quad (9)$$

where

$$\mu_j := u^T A^{-j} u. \quad (10)$$

The inverted Lanczos method is the projection of the eigenvalue problem  $A^{-1}x = \lambda^{-1}x$  onto the Krylovspace  $\mathcal{K}_k(A^{-1}, u)$ , and using the basis in (8) it is equivalent to

$$K_L^{(k)} y := \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{k-1} \\ \mu_1 & \mu_2 & \dots & \mu_k \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k-1} & \mu_k & \dots & \mu_{2k-2} \end{pmatrix} y = \lambda \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_k \\ \mu_2 & \mu_3 & \dots & \mu_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_k & \mu_{k+1} & \dots & \mu_{2k-1} \end{pmatrix} y =: \lambda M_L^{(k)} y. \quad (11)$$

These two representations are the basis of the following comparison theorem.

**Theorem 2** *Let  $\rho_1^{(k)} \leq \rho_2^{(k)} \leq \dots \leq \rho_k^{(k)}$  denote the eigenvalue approximations obtained by the inverted Lanczos method using  $\mathcal{K}_k(A^{-1}, u)$ , and let  $\sigma_1^{(k)} \leq \sigma_2^{(k)} \leq \dots \leq \sigma_k^{(k)}$  denote the eigenvalue approximations of the modified Lanczos method in (9). Then for every  $k < n$  it holds that*

$$\sigma_j^{(k+1)} \leq \rho_j^{(k)} \leq \sigma_j^{(k)}, \quad 1 \leq j \leq k. \quad (12)$$

**Proof:** For  $y := (y_0, y_1, \dots, y_{k-1})^T \in \mathbb{R}^k$  let

$$g := \sum_{j=0}^{k-1} y_j A^{-j-1} u.$$

Then it is easily seen that

$$R_M^{(k)}(y) := \frac{y^T K_M^{(k)} y}{y^T M_M^{(k)} y} = \frac{g^T A^3 g}{g^T A^2 g}$$

and

$$R_L^{(k)}(y) := \frac{y^T K_L^{(k)} y}{y^T M_L^{(k)} y} = \frac{g^T A^2 g}{g^T A g},$$

and for  $\tilde{y} := (0, y_0, y_1, \dots, y_{k-1})^T$  it holds that

$$R_M^{(k+1)}(\tilde{y}) = \frac{g^T A g}{g^T g}.$$

From the Cauchy–Schwarz inequality we obtain

$$(g^T A g)^2 \leq g^T g \cdot g^T A^2 g,$$

i.e.

$$R_M^{(k+1)}(\tilde{y}) \leq R_L^{(k)}(y), \tag{13}$$

and

$$(g^T A^2 g)^2 = (A^{1/2} g)^T (A^{3/2} g) \leq g^T A g \cdot g^T A^3 g,$$

i.e.

$$R_L^{(k)}(y) \leq R_M^{(k)}(y). \tag{14}$$

Let  $W \subset \mathbb{R}^k$  such that

$$\sigma_j^{(k)} = \min_{\dim V=j} \max_{y \in V} R_M^{(k)}(y) = \max_{y \in W} R_M^{(k)}(y).$$

Then inequality (14) implies

$$\max_{y \in W} R_L^{(k)}(y) \leq \max_{y \in W} R_M^{(k)}(y) = \sigma_j^{(k)},$$

and therefore

$$\rho_j^{(k)} = \min_{\dim V=j} \max_{y \in V} R_L^{(k)}(y) \leq \max_{y \in W} R_L^{(k)}(y) \leq \sigma_j^{(k)}.$$

Similarly, let  $Z \subset \mathbb{R}^k$  such that

$$\rho_j^{(k)} = \max_{z \in Z} R_L^{(k)}(z)$$

and let

$$\tilde{Z} := \{\tilde{z} := \begin{pmatrix} 0 \\ z \end{pmatrix} : z \in Z\} \subset \mathbb{R}^{k+1}.$$

Then it follows from inequality (13)

$$\sigma_j^{(k+1)} = \min_{\dim V=j} \max_{y \in V} R_M^{(k+1)}(y) \leq \max_{\tilde{z} \in \tilde{Z}} R_M^{(k+1)}(\tilde{z}) \leq \max_{z \in Z} R_L^{(k)}(z) = \rho_j^{(k)}$$

and this completes the proof.

q.e.d.

Theorem 2 describes the behaviour of the ideal modified Lanczos method in exact arithmetic without roundoff. In floating point arithmetic it behaves similarly as the Lanczos method: roundoff destroys orthogonality properties upon which our analysis depended so far, and (full or selected) reorthogonalization is necessary to maintain its accuracy. Since the Gram–Schmidt process with respect to the scalar product  $\langle \cdot, \cdot \rangle_A$  is more expensive than for the Euclidean inner product the advantage of the modified method proved in Theorem 2 disappears. However, since the extreme eigenvalues usually converge first we can waive reorthogonalization if we are only looking for the smallest eigenvalue of  $A$ . According to Theorem 2 in this case the modified version considered here should be faster than the inverted method.

### 3 Computing the minimum eigenvalue of a symmetric Toeplitz matrix

The problem of finding the smallest eigenvalue and corresponding eigenvector of a real symmetric and positiv definite Toeplitz matrix  $T$  is of considerable interest in signal processing (cf. Pisarenko [8]). In this section we compare the inverted Lanczos method and its modification for this problem.

Assume for convenience that the diagonal of  $T \in \mathbb{R}^{(n,n)}$  is normalized to 1, and that the first row of  $T$  is given by  $(1, t^T)$ , and denote the leading principal submatrix of dimension  $n - 1$  by  $S$ . The most costly step in the algorithms of Section 2 is the solution of the linear system

$$Tv = w. \tag{15}$$

(15) can be solved efficiently in one of the following two ways. Durbin’s algorithm (cf. [4], p. 195) for the corresponding Yule-Walker system  $Sy = -t$  supplies a decomposition  $LT L^T = D$  where  $L$  is a lower triangular matrix and  $D$  is a diagonal matrix. Hence, in every iteration step the solution of equation (15) requires  $2n^2$  flops. This method for solving system (15) is called Levinson-Durbin algorithm.

For large dimensions  $n$  equation (15) can be solved using the Gohberg-Semencul formula for the inverse  $T^{-1}$  (cf. [3])

$$T^{-1} = \frac{1}{1 - y^T t} (GG^T - HH^T), \tag{16}$$

where

$$G := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ y_1 & 1 & 0 & \dots & 0 \\ y_2 & y_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{n-1} & y_{n-2} & y_{n-3} & \dots & 1 \end{pmatrix} \text{ and } H := \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ y_{n-1} & 0 & 0 & \dots & 0 \\ y_{n-2} & y_{n-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1 & y_2 & y_3 & \dots & 0 \end{pmatrix}$$

are Toeplitz matrices and  $y$  denotes the solution of the Yule-Walker system.

The advantages associated with equation (16) are at hand. Firstly, the representation of the inverse of  $T$  requires only  $n$  storage elements. Secondly, the matrices  $G$ ,  $G^T$ ,  $H$  and  $H^T$  are Toeplitz matrices, and hence the solution  $T^{-1}w$  can be calculated in only  $O(n \log n)$  flops using the fast Fourier transform. Experiments show that for  $n \geq 512$  this approach is actually more efficient than the Levinson-Durbin algorithm.

To test the improvement of the modified Lanczos iteration upon the inverted Lanczos method we considered Toeplitz matrices

$$T = \xi \sum_{j=1}^n \eta_j T_{2\pi\theta_j} \quad (17)$$

where  $\xi$  is chosen such that the diagonal of  $T$  is normalized to 1,  $\eta_j$  and  $\theta_j$  are uniformly distributed random numbers in the interval  $[0, 1]$  and  $T_\theta = (\cos(\theta(i - j)))_{i,j=1,\dots,n}$  (cf. Cybenko and Van Loan [1]).

For each of the dimensions  $n = 32, 64, 128, 256, 512$  and  $1024$  we considered 100 test examples. Table 1 contains the average number of flops and of linear systems that had to be solved to determine an approximation to the smallest eigenvalue with relative error less than  $10^{-6}$ . Here we solved the linear systems using the Levinson-Durbin algorithm. In parenthesis we added the average number of flops using the Gohberg-Semencul formula. In either case we had to solve the Yule-Walker system  $Sy = -t$ , and the solution of

$$Tv = \begin{pmatrix} 1 & t^T \\ t & S \end{pmatrix} \begin{pmatrix} \alpha \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is free. We therefore started the method with the unit vector containing a 1 in its first component and zeros elsewhere.

A symmetric Toeplitz matrix  $T$  additionally is persymmetric, i.e.  $T = JTJ$  where  $J$  denotes the flipmatrix containing ones in its secondary diagonal and zeros elsewhere. Hence, it is easily seen that for a simple eigenvalue of  $T$  and corresponding eigenvector  $x$  it holds that  $x = Jx$  or  $x = -Jx$ . In the first case we call  $x$  a symmetric eigenvector and  $\lambda$  an even eigenvalue, in the latter case  $x$  is a skew-symmetric eigenvector and the corresponding eigenvalue  $\lambda$  is called odd.



Table 1. Average number of flops and linear systems

dim	Lanczos method			Modification		
	flops		steps	flops		steps
32	1.46E4	(6.13E4)	5.59	1.38E4	(5.49E4)	5.05
64	5.51E4	(2.66E5)	6.00	5.11E4	(2.39E5)	5.46
128	2.11E5	(6.04E5)	6.11	1.95E5	(5.50E5)	5.64
256	8.90E5	(1.49E6)	6.62	8.17E5	(1.36E6)	6.06
512	3.76E6	(3.70E6)	7.08	3.51E6	(3.45E6)	6.60
1024	1.56E7	(9.26E6)	7.38	1.46E7	(8.77E6)	6.93

If the initial vector  $u$  of a Krylov space is symmetric and skew-symmetric, respectively, obviously  $\mathcal{K}_k(T^{-1}, u)$  contains only symmetric and skew-symmetric elements, too, and a projection method yields approximations only to even and odd eigenvalues, respectively. If the symmetry class of the principal eigenvector is known then we will choose the initial vector  $u$  in the same class.

However, the symmetry class of the principal eigenvector is known in advance only for a small class of symmetric Toeplitz matrices (cf. Trench [10]). In the general case the inverted Lanczos method can be performed simultaneously for a symmetric and a skew-symmetric initial vector such that in each step only one solution of a linear system has to be determined (cf. [12]). Similarly for the modified method considered here we can exploit symmetry of the eigenvectors.

We assume that the dimension  $n = 2m$  is even (the modifications for odd dimensions are obvious). Let  $p_1, q_1 \in \mathbb{R}^m$  be given. Then the following algorithm simultaneously produces orthonormal bases  $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_k$  of  $\mathcal{K}_k(T^{-1}, \tilde{p}_1)$  and  $\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k$  of  $\mathcal{K}_k(T^{-1}, \tilde{q}_1)$  where

$$\tilde{p}_1 := \begin{pmatrix} p_1 \\ Jp_1 \end{pmatrix} \quad \text{and} \quad \tilde{q}_1 := \begin{pmatrix} q_1 \\ -Jq_1 \end{pmatrix}$$

are symmetric and skew-symmetric initial vectors, respectively. For convenience we assume that  $\tilde{p}_1$  and  $\tilde{q}_1$  are normalized such that  $\tilde{p}_1^T T \tilde{p}_1 = 1$  and  $\tilde{q}_1^T T \tilde{q}_1 = 1$ .

$$\alpha_1 = 2\|p_1\|_2^2; \gamma_1 = 2\|q_1\|_2^2; \beta_0 = 0; \gamma_0 = 0; p_0 = 0; q_0 = 0;$$

for  $k = 1, 2, \dots$  until convergence do

$$w = \begin{pmatrix} p_k + q_k \\ J(p_k - q_k) \end{pmatrix};$$

$$v = T^{-1}w;$$

$$\tilde{p} = (v(1:m) + Jv(m+1:n))/2;$$

$$\tilde{q} = v(1:m) - \tilde{p};$$

$$p_{k+1} = \tilde{p} - \alpha_k p_k - \beta_{k-1} p_{k-1};$$

$$q_{k+1} = \tilde{q} - \gamma_k q_k - \delta_{k-1} q_{k-1};$$

Table 2. Average number of flops and linear systems  
Symmetry exploited

dim	Lanczos method			Modification		
	flops		steps	flops		steps
32	1.14E4	(4.42E4)	4.23	1.04E4	(3.67E4)	3.59
64	4.06E4	(1.82E5)	4.35	3.81E4	(1.62E5)	3.95
128	1.54E5	(4.17E5)	4.42	1.38E5	(3.62E5)	3.91
256	6.21E5	(1.01E6)	4.60	5.68E5	(9.08E5)	4.18
512	2.57E6	(2.53E6)	4.82	2.31E6	(2.28E6)	4.33
1024	1.06E7	(6.65E6)	5.04	9.69E6	(6.14E6)	4.58

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 $\beta_k = \sqrt{2p_{k+1}^T p_k};$ 
 $\delta_k = \sqrt{2q_{k+1}^T q_k};$ 
 $p_{k+1} = p_{k+1} / \beta_k;$ 
 $q_{k+1} = q_{k+1} / \delta_k;$ 
 $\alpha_{k+1} = 2\|p_{k+1}\|_2^2;$ 
 $\gamma_{k+1} = 2\|q_{k+1}\|_2^2;$ 
Solve the linear eigenvalue problems
 $y = \lambda \text{tridiag}(\beta_{j-1}, \alpha_j, \beta_j)_{j=1,\dots,k+1} y;$ 
 $z = \lambda \text{tridiag}(\delta_{j-1}, \gamma_j, \delta_j)_{j=1,\dots,k+1} z;$ 
end

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For the same test problems as before the Table 2 contains the average number of flops and of linear systems that had to be solved for the inverted Lanczos method and its modification.

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