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ORIGINAL PAPER

Extension of vector-valued functions and weak–strong principles for differentiable functions of finite order

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Abstract

In this paper we study the problem of extending functions with values in a locally convex Hausdorff space E over a field $\mathbb{K}$, which has weak extensions in a weighted Banach space $\mathcal{F}v(\Omega, \mathbb{K})$ of scalar-valued functions on a set Ω, to functions in a vector-valued counterpart $\mathcal{F}v(\Omega, E)$ of $\mathcal{F}v(\Omega, \mathbb{K})$. Our findings rely on a description of vector-valued functions as continuous linear operators and extend results of Frerick, Jordá and Wengenroth. As an application we derive weak-strong principles for continuously partially differentiable functions of finite order and vector-valued versions of Blaschke's convergence theorem for several spaces.

Keywords extension · Vector-valued · ε-product · Weight · weak-strong principle

Mathematics Subject Classification 46E40 · 46A03 · 46E10

1 Introduction

This paper centres on the problem of extending a vector-valued function $f : \Lambda \to E$ from a subset $\Lambda \subset \Omega$ to a locally convex Hausdorff space E if the scalar-valued functions $e' \circ f$ are extendable for each $e' \in G \subset E'$ under the constraint of preserving the properties, like holomorphy, of the scalar-valued extensions. This problem was considered, among others, by Grothendieck [27, 28], Bierstedt [6], Gramsch [23], Grosse-Erdmann [25, 26], Arendt and Nikolski [2–4], Bonet, Frerick, Jordá and Wengenroth [7, 18, 19, 31, 32] and us [38].

Often, the underlying idea to prove such an extension theorem is to use a representation of an E-valued function by a continuous linear operator. Namely, if

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$\mathcal{F}(\Omega):=\mathcal{F}(\Omega,\mathbb{K})$ is a locally convex Hausdorff space of scalar-valued functions on a set Ω such that the point evaluations δ_x at x belong to the dual $\mathcal{F}(\Omega)'$ for each $x\in\Omega$, then the function $S(u)\colon\Omega\to E$ given by $x\mapsto u(\delta_x)$ is well-defined for every element u of Schwartz' ε-product $\mathcal{F}(\Omega)\varepsilon E:=L_e(\mathcal{F}(\Omega)_\kappa',E)$ where the dual $\mathcal{F}(\Omega)'$ is equipped with the topology of uniform convergence on absolutely convex compacts subsets of $\mathcal{F}(\Omega)$, the space of continuous linear operators $L(\mathcal{F}(\Omega)_\kappa',E)$ is equipped with the topology of uniform convergence on the equicontinuous subsets of $\mathcal{F}(\Omega)_\kappa'$ and E is a locally convex Hausdorff space over the field $\mathbb{K}$. In many cases the function $S(u)$ inherits properties of the functions in $\mathcal{F}(\Omega)$, e.g. if $\mathcal{F}(\Omega)=(\mathcal{O}(\Omega),\tau_{co})$ is the space of holomorphic functions on an open set $\Omega\subset\mathbb{C}$ equipped with the compact-open topology τ_{co}, then the space of functions of the form $S(u)$ with $u\in(\mathcal{O}(\Omega),\tau_{co})\varepsilon E$ coincides with the space $\mathcal{O}(\Omega,E)$ of E-valued holomorphic functions if E is locally complete. Even more is true, namely, that the map $S\colon(\mathcal{O}(\Omega),\tau_{co})\varepsilon E\to(\mathcal{O}(\Omega,E),\tau_{co})$ is a (topological) isomorphism (see [7, p. 232]). So suppose that there is a locally convex Hausdorff space $\mathcal{F}(\Omega,E)$ of E-valued functions on Ω such that the map $S\colon\mathcal{F}(\Omega)\varepsilon E\to\mathcal{F}(\Omega,E)$ is well-defined and at least a (topological) isomorphism into, i.e. to its range. The precise formulation of the extension problem from the beginning is the following question.

Question 1.1 Let Λ be a subset of Ω and G a linear subspace of E'. Let $f\colon\Lambda\to E$ be such that for every $e'\in G$, the function $e'\circ f\colon\Lambda\to\mathbb{K}$ has an extension in $\mathcal{F}(\Omega)$. When is there an extension $F\in\mathcal{F}(\Omega,E)$ of f, i.e. $F_{|\Lambda}=f$?

Even the case $\Lambda=\Omega$ is interesting because then the question is about properties of vector-valued functions and a positive answer is usually called a weak-strong principle. From the connection of $\mathcal{F}(\Omega)\varepsilon E$ and $\mathcal{F}(\Omega,E)$ it is evident to seek for extension theorems for vector-valued functions by extension theorems for continuous linear operators. In this way many of the extension theorems of the aforementioned references are derived but in most of the cases the space $\mathcal{F}(\Omega)$ has to be a semi-Montel (see [23, 28, 38]) or even a Fréchet–Schwartz space (see [7, 18, 23, 25–27, 31, 38]) or E is restricted to be a semi-Montel space (see [6, 38]). The restriction to semi-Montel spaces $\mathcal{F}(\Omega)$ resp. E, i.e. to locally convex spaces in which every bounded set is relatively compact, is quite natural due to the topology of the dual $\mathcal{F}(\Omega)_\kappa'$ in the ε-product $\mathcal{F}(\Omega)\varepsilon E$ and its symmetry $\mathcal{F}(\Omega)\varepsilon E\cong E\varepsilon\mathcal{F}(\Omega)$.

In the present paper we treat the case of a Banach space which we denote by $\mathcal{F}v(\Omega)$ because its topology is induced by a weight v. We use the methods developed in [19] and [32] where, in particular, the special case that $\mathcal{F}v(\Omega)$ is the space of bounded smooth functions on an open set $\Omega\subset\mathbb{R}^d$ in the kernel of a hypoelliptic linear partial differential operator resp. a weighted space of holomorphic functions on an open subset Ω of a Banach space is treated. The lack of compact subsets of the infinite dimensional Banach space $\mathcal{F}v(\Omega)$ is compensated in [19] and [32] by using an auxiliary locally convex Hausdorff space $\mathcal{F}(\Omega)$ of scalar-valued functions on Ω such that the closed unit ball of $\mathcal{F}v(\Omega)$ is compact in $\mathcal{F}(\Omega)$. They share the property that $S\colon\mathcal{F}v(\Omega)\varepsilon E\to\mathcal{F}v(\Omega,E)$ and $S\colon\mathcal{F}(\Omega)\varepsilon E\to\mathcal{F}(\Omega,E)$

are topological isomorphisms into but usually it is only known in the latter case that S is subjective as well under some mild completeness assumption on E. For instance, if $\mathcal{F}v(\Omega, E):=H^{\infty}(\Omega, E)$ is the space of bounded holomorphic functions on an open set $\Omega \subset \mathbb{C}$ with values in a locally complete space E, then the space $\mathcal{F}(\Omega, E):=(\mathcal{O}(\Omega, E), \tau_{co})$ is used in [19].

Let us outline the content of our paper. We give a general approach to the extension problem for Banach function spaces $\mathcal{F}v(\Omega)$. It combines the methods of [19] and [32] with the ones of [35] like in [38] which require that the spaces $\mathcal{F}v(\Omega)$ and $\mathcal{F}v(\Omega, E)$ have a certain structure (see Definition 2.3). To answer Question 1.1 we have to balance the sets $\Lambda \subset \Omega$ and the spaces $G \subset E'$. If we choose Λ to be 'thin', then G has to be 'thick' (see Section 3) and vice versa (see Section 4). In Section 5 we use the results of Section 3 to derive and improve weak-strong principles for differentiable functions of finite order. Section 6 is devoted to vector-valued Blaschke theorems.

2 Notation and Preliminaries

We use essentially the same notation and preliminaries as in [38, Section 2]. We equip the spaces $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\mathbb{C}$ with the usual Euclidean norm $|\cdot|$. By E we always denote a non-trivial locally convex Hausdorff space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$ and, in short, we write E is an lcHs. If $E = \mathbb{K}$, then we set $(p_\alpha)_{\alpha\in\mathfrak{A}}:=\{|\cdot|\}$. For more details on the theory of locally convex spaces see [17, 30] or [40].

By X^Ω we denote the set of maps from a non-empty set Ω to a non-empty set X and by $L(F, E)$ the space of continuous linear operators from F to E where F and E are locally convex Hausdorff spaces. If $E = \mathbb{K}$, we just write $F':=L(F,\mathbb{K})$ for the dual space and G° for the polar set of $G \subset F$. We write $F \cong E$ if F and E are (linearly topologically) isomorphic. We denote by $L_t(F,E)$ the space $L(F, E)$ equipped with the locally convex topology t of uniform convergence on the finite subsets of F if $t = \sigma$, on the absolutely convex, compact subsets of F if $t = \kappa$ and on the bounded subsets of F if $t = b$. We use the symbol $t(F',F)$ for the corresponding topology on F'. A linear subspace G of F' is called separating if $f'(x) = 0$ for every $f' \in G$ implies $x = 0$. This is equivalent to G being $\sigma(F',F)$-dense (and $\kappa(F',F)$-dense) in F' by the bipolar theorem. Further, for a disk $D \subset F$, i.e. a bounded, absolutely convex set, the vector space $F_D:=\bigcup_{n\in\mathbb{N}} nD$ becomes a normed space if it is equipped with gauge functional of D as a norm (see [30, p. 151]). The space F is called locally complete if F_D is a Banach space for every closed disk $D \subset F$ (see [30, 10.2.1 Proposition, p. 197]).

Furthermore, we recall the definition of continuous partial differentiability of a vector-valued function that we need in many examples, especially, for the weak-strong principle for differentiable functions of finite order in Section 5. A function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ to an lcHs E is called continuously partially differentiable (f is $\mathcal{C}^1$) if for the n-th unit vector $e_n \in \mathbb{R}^d$ the limit

$$(\partial^{e_n})^E f(x) := \lim_{\substack{h \to 0 \\ h \in \mathbb{R}, h \neq 0}} \frac{f(x+he_n) - f(x)}{h}$$

exists in E for every $x \in \Omega$ and $(\partial^{e_n})^E f$ is continuous on Ω ($(\partial^{e_n})^E f$ is $\mathcal{C}^0$) for every $1 \leq n \leq d$. For $k \in \mathbb{N}$ a function f is said to be k-times continuously partially differentiable (f is $\mathcal{C}^k$) if f is $\mathcal{C}^1$ and all its first partial derivatives are $\mathcal{C}^{k-1}$. A function f is called infinitely continuously partially differentiable (f is $\mathcal{C}^\infty$) if f is $\mathcal{C}^k$ for every $k \in \mathbb{N}$. For $k \in \mathbb{N}_0 \cup \{\infty\}$ the linear space of all functions $f : \Omega \to E$ which are $\mathcal{C}^k$ is denoted by $\mathcal{C}^k(\Omega, E)$. Let $f \in \mathcal{C}^k(\Omega, E)$. For $\beta \in \mathbb{N}_0^d$ with $|\beta| := \sum_{n=1}^d \beta_n \leq k$ we set $(\partial^{\beta_n})^E f := f$ if $\beta_n = 0$, and

$$(\partial^{\beta_n})^E f := \underbrace{(\partial^{e_n})^E \cdots (\partial^{e_n})^E}_{\beta_n\text{-times}} f$$

if $\beta_n \neq 0$ as well as

$$(\partial^{\beta})^E f := (\partial^{\beta_1})^E \cdots (\partial^{\beta_d})^E f.$$

If $E = \mathbb{K}$, we usually write $\partial^\beta f := (\partial^\beta)^{\mathbb{K}} f$.

In addition, we use the following notion for the relation between the ε-product $\mathcal{F}(\Omega)\varepsilon E$ and the space $\mathcal{F}(\Omega, E)$ of vector-valued functions that has already been described in the introduction.

Definition 2.1 (ε-into-compatible, [38, 2.1 Definition, p. 4]) Let Ω be a non-empty set and E an lcHs. Let $\mathcal{F}(\Omega) \subset \mathbb{K}^\Omega$ and $\mathcal{F}(\Omega, E) \subset E^\Omega$ be lcHs such that $\delta_x \in \mathcal{F}(\Omega)'$ for all $x \in \Omega$. We call the spaces $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ *ε-into-compatible* if the map

$$S : \mathcal{F}(\Omega)\varepsilon E \to \mathcal{F}(\Omega, E), \; u \longmapsto [x \mapsto u(\delta_x)],$$

is a well-defined isomorphism into. We call $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ *ε-compatible* if S is an isomorphism. If we want to emphasise the dependency on $\mathcal{F}(\Omega)$, we write $S_{\mathcal{F}(\Omega)}$ instead of S.

Definition 2.2 (strong, consistent) Let Ω and ω be non-empty sets and E, $\mathcal{F}(\Omega) \subset \mathbb{K}^\Omega$ and $\mathcal{F}(\Omega, E) \subset E^\Omega$ be lcHs. Let $\delta_x \in \mathcal{F}(\Omega)'$ for all $x \in \Omega$ and $T^{\mathbb{K}} : \mathcal{F}(\Omega) \to \mathbb{K}^\omega$ and $T^E : \mathcal{F}(\Omega, E) \to E^\omega$ be linear maps.

(a) We call $(T^E, T^{\mathbb{K}})$ a *consistent* family for $(\mathcal{F}, E)$ if we have for every $u \in \mathcal{F}(\Omega)\varepsilon E$ that $S(u) \in \mathcal{F}(\Omega, E)$ and

$$\forall\, x \in \omega : \; T_x^{\mathbb{K}} := \delta_x \circ T^{\mathbb{K}} \in \mathcal{F}(\Omega)' \quad \text{and} \quad T^E S(u)(x) = u(T_x^{\mathbb{K}}).$$

(b) We call $(T^E, T^{\mathbb{K}})$ a *strong* family for $(\mathcal{F}, E)$ if we have for every $e' \in E'$, $f \in \mathcal{F}(\Omega, E)$ that $e' \circ f \in \mathcal{F}(\Omega)$ and

$$\forall\, x \in \omega : \; T^{\mathbb{K}}(e' \circ f)(x) = (e' \circ T^E(f))(x).$$

This is a special case of [38, 2.2 Definition, p. 4] where the considered family $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ only consists of one pair, i.e. the set M is a singleton. In the introduction we have already hinted that the spaces $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ for which we want to prove extension theorems need to have a certain structure, namely, the following one.

Definition 2.3 (generator) Let Ω and ω be non-empty sets, $\nu : \omega \to (0, \infty)$, $\mathcal{F}(\Omega, E)$ a linear subspace of E^Ω and $T^E : \mathcal{F}(\Omega, E) \to E^\omega$ a linear map. We define the space

$$\mathcal{F}\nu(\Omega, E) := \left\{ f \in \mathcal{F}(\Omega, E) \mid \forall\, \alpha \in \mathfrak{A} : \ |f|_\alpha < \infty \right\}$$

where

$$|f|_\alpha := \sup_{x\in\omega} p_\alpha\left(T^E(f)(x)\right)\nu(x).$$

Further, we call $(T^E, T^{\mathbb{K}})$ the *generator* for $(\mathcal{F}\nu(\Omega), E)$, in short $(\mathcal{F}\nu, E)$. We write $\mathcal{F}\nu(\Omega) := \mathcal{F}\nu(\Omega, \mathbb{K})$ and omit the index α if E is a normed space. If we want to emphasise dependencies, we write $|f|_{\mathcal{F}\nu(\Omega),\alpha}$ instead of $|f|_\alpha$.

This is a special case of [35, Definition 3, p. 1515] where the family of weights only consists of one weight function. For instance, if $\Omega := \omega$, $T^E := id_{E^\Omega}$ and $\nu := 1$ on Ω, then $\mathcal{F}\nu(\Omega, E)$ is the linear subspace of $\mathcal{F}(\Omega, E)$ consisting of bounded functions, in particular, if $\Omega \subset \mathbb{C}$ is open and $\mathcal{F}(\Omega, E) := \mathcal{O}(\Omega, E)$, then $\mathcal{F}\nu(\Omega, E) = H^\infty(\Omega, E)$ is the space of E-valued bounded holomorphic functions on Ω. Due to $(E, (p_\alpha)_{\alpha\in\mathfrak{A}})$ being an lcHs with directed system of seminorms the topology of $\mathcal{F}\nu(\Omega, E)$ generated by $(|\cdot|_\alpha)_{\alpha\in\mathfrak{A}}$ is locally convex and the system $(|\cdot|_\alpha)_{\alpha\in\mathfrak{A}}$ is directed but need not be Hausdorff.

Proposition 2.4 *Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ be ε-into-compatible, $(T^E, T^{\mathbb{K}})$ a consistent family for $(\mathcal{F}, E)$ and a generator for $(\mathcal{F}\nu, E)$ and the map $i : \mathcal{F}\nu(\Omega) \to \mathcal{F}(\Omega)$, $f \mapsto f$, continuous. We set*

$$\mathcal{F}_\varepsilon\nu(\Omega, E) := S(\{u \in \mathcal{F}(\Omega)\varepsilon E \mid u(B^{\circ\mathcal{F}(\Omega)'}_{\mathcal{F}\nu(\Omega)}) \text{ is bounded in } E\})$$

where $B^{\circ\mathcal{F}(\Omega)'}_{\mathcal{F}\nu(\Omega)} := \{y' \in \mathcal{F}(\Omega)' \mid \forall\, f \in B_{\mathcal{F}\nu(\Omega)} : \ |y'(f)| \le 1\}$ and $B_{\mathcal{F}\nu(\Omega)}$ is the closed unit ball of $\mathcal{F}\nu(\Omega)$. Then the following holds.

(a) *$\mathcal{F}\nu(\Omega)$ is Hausdorff and $\delta_x \in \mathcal{F}\nu(\Omega)'$ for all $x \in \Omega$.*
(b) *Let $u \in \mathcal{F}(\Omega)\varepsilon E$. Then*

$$\sup_{y'\in B^{\circ\mathcal{F}(\Omega)'}_{\mathcal{F}\nu(\Omega)}} p_\alpha(u(y')) = |S(u)|_{\mathcal{F}\nu(\Omega),\alpha}, \quad \alpha \in \mathfrak{A}.$$

In particular,

$$\mathcal{F}_{\varepsilon}\nu(\Omega,E)=S(\{u\in\mathcal{F}(\Omega)\varepsilon E\mid\forall\,\alpha\in\mathfrak{A}:\ |S(u)|_{\mathcal{F}\nu(\Omega),\alpha}<\infty\}).$$

(c) *$S(\mathcal{F}\nu(\Omega)\varepsilon E)\subset\mathcal{F}_{\varepsilon}\nu(\Omega,E)\subset\mathcal{F}\nu(\Omega,E)$ as linear spaces. If $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ are even ε-compatible, then $\mathcal{F}_{\varepsilon}\nu(\Omega,E)=\mathcal{F}\nu(\Omega,E)$.*

Proof Part (a) follows from the continuity of i and the ε-into-compatibility of $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$. Let us turn to part (b). Like in [35, Lemma 7, p. 1517] it follows from the bipolar theorem that

$$B_{\mathcal{F}\nu(\Omega)}^{\circ\mathcal{F}(\Omega)'}=\overline{\operatorname{acx}}\{T_x^{\mathbb{K}}(\cdot)\nu(x)\mid x\in\omega\},$$

where $\overline{\operatorname{acx}}$ denotes the closure w.r.t. $\kappa(\mathcal{F}(\Omega)',\mathcal{F}\nu(\Omega))$ of the absolutely convex hull acx of the set $D:=\{T_x^{\mathbb{K}}(\cdot)\nu(x)\mid x\in\omega\}$ on the right-hand side, and that

$$\begin{aligned}\sup_{y'\in B_{\mathcal{F}\nu(\Omega)}^{\circ\mathcal{F}(\Omega)'}}p_\alpha(u(y'))&=\sup_{y'\in\operatorname{acx}(D)}p_\alpha(u(y'))=\sup_{y'\in D}p_\alpha(u(y'))\\&=\sup_{x\in\omega}p_\alpha\big(u(T_x^{\mathbb{K}})\big)\nu(x)=\sup_{x\in\omega}p_\alpha\big(T^E(S(u))(x)\big)\nu(x)\\&=|S(u)|_{\mathcal{F}\nu(\Omega),\alpha}\end{aligned}$$

by consistency, which proves part (b).

Let us address part (c). The continuity of i implies the continuity of the inclusion $\mathcal{F}\nu(\Omega)\varepsilon E\hookrightarrow\mathcal{F}(\Omega)\varepsilon E$ and thus we obtain $u_{|\mathcal{F}(\Omega)'}\in\mathcal{F}(\Omega)\varepsilon E$ for every $u\in\mathcal{F}\nu(\Omega)\varepsilon E$. If $u\in\mathcal{F}\nu(\Omega)\varepsilon E$ and $\alpha\in\mathfrak{A}$, then there are $C_0,C_1>0$ and an absolutely convex compact set $K\subset\mathcal{F}\nu(\Omega)$ such that $K\subset C_1B_{\mathcal{F}\nu(\Omega)}$ and

$$\sup_{y'\in B_{\mathcal{F}\nu(\Omega)}^{\circ\mathcal{F}(\Omega)'}}p_\alpha(u(y'))\leq C_0\sup_{y'\in B_{\mathcal{F}\nu(\Omega)}^{\circ\mathcal{F}(\Omega)'}}\sup_{f\in K}|y'(f)|\leq C_0C_1,$$

which implies $S(\mathcal{F}\nu(\Omega)\varepsilon E)\subset\mathcal{F}_{\varepsilon}\nu(\Omega,E)$. If $f:=S(u)\in\mathcal{F}_{\varepsilon}\nu(\Omega,E)$ and $\alpha\in\mathfrak{A}$, then $S(u)\in\mathcal{F}(\Omega,E)$ and

$$|f|_{\mathcal{F}\nu(\Omega),\alpha}=\sup_{x\in\omega}p_\alpha(u(T_x^{\mathbb{K}})\nu(x))<\infty$$

by consistency, yielding $\mathcal{F}_{\varepsilon}\nu(\Omega,E)\subset\mathcal{F}\nu(\Omega,E)$. If $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ are even ε-compatible, then $S(\mathcal{F}(\Omega)\varepsilon E)=\mathcal{F}(\Omega,E)$, which yields $\mathcal{F}_{\varepsilon}\nu(\Omega,E)=\mathcal{F}\nu(\Omega,E)$ by part (b). □

The canonical situation in part (c) is that $\mathcal{F}_{\varepsilon}\nu(\Omega,E)$ and $\mathcal{F}\nu(\Omega,E)$ coincide as linear spaces for locally complete E as we will encounter in the forthcoming examples, e.g. if $\mathcal{F}\nu(\Omega,E):=H^\infty(\Omega,E)$ and $\mathcal{F}(\Omega,E):=(\mathcal{O}(\Omega,E),\tau_{co})$ for an open set $\Omega\subset\mathbb{C}$. That all three spaces in part (c) coincide is usually only guaranteed by [35, Theorem 14 (ii), p.1524] if E is a semi-Montel space. Therefore the 'mingle-mangle' space $\mathcal{F}_{\varepsilon}\nu(\Omega,E)$ is a good replacement for $S(\mathcal{F}\nu(\Omega)\varepsilon E)$ for our purpose.

3 Extension of vector-valued functions

In this section the sets from which we want to extend our functions are 'thin'. They are so-called sets of uniqueness.

Definition 3.1 (set of uniqueness) Let $\mathcal{F}v(\Omega)$ be a Hausdorff space. A set $U \subset \omega$ is called a *set of uniqenes* for $(T^{\mathbb{K}}, \mathcal{F}v)$ if

$$\forall f \in \mathcal{F}v(\Omega) : \ T^{\mathbb{K}}(f)(x) = 0 \ \ \forall x \in U \quad \Rightarrow \quad f = 0.$$

This definition is a special case of [38, 3.1 Definition, p.8] because $T_x^{\mathbb{K}} \in \mathcal{F}v(\Omega)'$ for all $x \in \omega$ by [35, Remark 5, p. 1516]. The span of $\{T_x^{\mathbb{K}} \mid x \in U\}$ is weak*-dense in $\mathcal{F}v(\Omega)'$ by the bipolar theorem if U is a set of uniqueness for $(T^{\mathbb{K}}, \mathcal{F}v)$. The set $U := \omega$ is always a set of uniqueness for $(T^{\mathbb{K}}, \mathcal{F}v)$ as $\mathcal{F}v(\Omega)$ is an lcHs by assumption. Next, we introduce the notion of a restriction space which is a special case of [38, 3.3 Definition, p.8].

Definition 3.2 (restriction space) Let $G \subset E'$ be a separating subspace, $\mathcal{F}v(\Omega)$ a Hausdorff space and U a set of uniqueness for $(T^{\mathbb{K}}, \mathcal{F}v)$. We denote by $\mathcal{F}v_G(U, E)$ the space of functions $f : U \to E$ such that for every $e' \in G$ there is $f_{e'} \in \mathcal{F}v(\Omega)$ with $T^{\mathbb{K}}(f_{e'})(x) = (e' \circ f)(x)$ for all $x \in U$.

The time has come to use our auxiliary spaces $\mathcal{F}(\Omega)$, $\mathcal{F}(\Omega, E)$ and $\mathcal{F}_\varepsilon v(\Omega, E)$ from Proposition 2.4.

Remark 3.3 Let $(T^E, T^{\mathbb{K}})$ be a strong, consistent family for $(\mathcal{F}, E)$ and a generator for $(\mathcal{F}v, E)$. Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ be ε-into-compatible and the inclusion $\mathcal{F}v(\Omega) \hookrightarrow \mathcal{F}(\Omega)$ continuous. Consider a set of uniqueness U for $(T^{\mathbb{K}}, \mathcal{F}v)$ and a separating subspace $G \subset E'$. For $u \in \mathcal{F}(\Omega)\varepsilon E$ such that $u(B^{\circ \mathcal{F}(\Omega)'}_{\mathcal{F}v(\Omega)})$ is bounded in E, i.e. $S(u) \in \mathcal{F}_\varepsilon v(\Omega, E)$, we set $f := S(u)$. Then $f \in \mathcal{F}(\Omega, E)$ by the ε-into-compatibility and we define $\widetilde{f} : U \to E$, $\widetilde{f}(x) := T^E(f)(x)$. This yields

$$(e' \circ \widetilde{f})(x) = (e' \circ T^E(f))(x) = T^{\mathbb{K}}(e' \circ f)(x) \tag{3.1}$$

for all $x \in U$ and $f_{e'} := e' \circ f \in \mathcal{F}(\Omega)$ for each $e' \in E'$ by the strength of the family. Moreover, $T_x^{\mathbb{K}}(\cdot)v(x) \in B^{\circ \mathcal{F}(\Omega)'}_{\mathcal{F}v(\Omega)}$ for every $x \in \omega$, which implies that for every $e' \in E'$ there are $\alpha \in \mathfrak{A}$ and $C > 0$ such that

$$|f_{e'}|_{\mathcal{F}v(\Omega)} = \sup_{x \in \omega} \left| e'\left(u(T_x^{\mathbb{K}}(\cdot)v(x))\right) \right| \leq C \sup_{y' \in B^{\circ \mathcal{F}(\Omega)'}_{\mathcal{F}v(\Omega)}} p_\alpha(u(y')) < \infty$$

by strength and consistency. Hence $f_{e'} \in \mathcal{F}v(\Omega)$ for every $e' \in E'$ and $\widetilde{f} \in \mathcal{F}v_G(U, E)$.

Under the assumptions of Remark 3.3 the map

$$R_{U,G} : \mathcal{F}_{\varepsilon}\nu(\Omega,E) \to \mathcal{F}\nu_G(U,E),\ f \mapsto (T^E(f)(x))_{x\in U}, \tag{3.2}$$

is well-defined and linear. In addition, we derive from (3.1) that $R_{U,G}$ is injective since U is a set of uniqeness and $G \subset E'$ separating.

Question 3.4 Let the assumptions of Remark 3.3 be fulfilled. When is the injective restriction map

$$R_{U,G} : \mathcal{F}_{\varepsilon}\nu(\Omega,E) \to \mathcal{F}\nu_G(U,E),\ f \mapsto (T^E(f)(x))_{x\in U},$$

surjective?

Due to Proposition 2.4 (c) the Question 1.1 is a special case of this question if $\Lambda \subset \Omega =: \omega$ and $U := \Lambda$ is a set of uniqueness for $(id_{\mathbb{K}^\Omega}, \mathcal{F}\nu)$. To answer Question 3.4 for general sets of uniqueness we have to restrict to a certain class of 'thick' separating subspaces of E'.

Definition 3.5 (determine boundedness [7, p. 230]) A linear subspace $G \subset E'$ *determines boundedness* if every $\sigma(E,G)$-bounded set $B \subset E$ is already bounded in E.

E' itself always determines boundedness by Mackey's theorem. Further examples can be found in [38, 3.10 Remark, p.10] and the references therein. We recall the following extension result for continuous linear operators.

Proposition 3.6 ([19, Proposition 2.1, p. 691]) *Let E be locally complete, $G \subset E'$ determine boundedness, Z a Banach space whose closed unit ball B_Z is a compact subset of an lcHs Y and $X \subset Y'$ be a $\sigma(Y',Z)$-dense subspace. If $\mathsf{A} : X \to E$ is a $\sigma(X,Z)$-$\sigma(E,G)$ continuous linear map, then there exists a (unique) extension $\widehat{\mathsf{A}} \in Y\varepsilon E$ of A such that $\widehat{\mathsf{A}}(B_Z^{\circ Y'})$ is bounded in E where $B_Z^{\circ Y'} := \{y' \in Y' \mid \forall\, z \in B_Z :\ |y'(z)| \leq 1\}$.*

Now, we are able to generalise [19, Theorem 2.2, p. 691] and [32, Theorem 10, p. 5].

Theorem 3.7 *Let E be locally complete, $G \subset E'$ determine boundedness and $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-into-compatible. Let $(T^E, T^{\mathbb{K}})$ be a generator for $(\mathcal{F}\nu, E)$ and a strong, consistent family for $(\mathcal{F}, E)$, $\mathcal{F}\nu(\Omega)$ a Banach space whose closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is a compact subset of $\mathcal{F}(\Omega)$ and U a set of uniqueness for $(T^{\mathbb{K}}, \mathcal{F}\nu)$. Then the restriction map*

$$R_{U,G} : \mathcal{F}_{\varepsilon}\nu(\Omega,E) \to \mathcal{F}\nu_G(U,E)$$

is surjective.

Proof Let $f \in \mathcal{F}\nu_G(U,E)$. We set $X := \operatorname{span}\{T_x^{\mathbb{K}} \mid x \in U\}$, $Y := \mathcal{F}(\Omega)$ and $Z := \mathcal{F}\nu(\Omega)$. The consistency of $(T^E, T^{\mathbb{K}})$ for $(\mathcal{F}, E)$ yields that $X \subset Y'$. From U being a set of uniqueness of Z follows that X is $\sigma(Z',Z)$-dense. Since B_Z is a compact subset of Y, it follows that Z is a linear subspace of Y and the inclusion $Z \hookrightarrow Y$ is continuous, which yields $y'_{|Z} \in Z'$ for every $y' \in Y'$. Thus X is $\sigma(Y',Z)$-dense. Let $\mathsf{A} : X \to E$ be the linear map determined by $\mathsf{A}(T^{\mathbb{K}}_x) := f(x)$. The map A is well-defined since G is $\sigma(E',E)$-dense. Due to

$$e'(\mathsf{A}(T^{\mathbb{K}}_x)) = (e' \circ f)(x) = T^{\mathbb{K}}_x(f_{e'})$$

for every $e' \in G$ and $x \in U$ we have that A is $\sigma(X,Z)$-$\sigma(E,G)$-continuous. We apply Proposition 3.6 and gain an extension $\widehat{\mathsf{A}} \in Y\varepsilon E$ of A such that $\widehat{\mathsf{A}}(B_Z^{\circ Y'})$ is bounded in E. We set $F := S(\widehat{\mathsf{A}}) \in \mathcal{F}_\varepsilon \nu(\Omega,E)$ and get for all $x \in U$ that

$$T^E(F)(x) = T^E S(\widehat{\mathsf{A}})(x) = \widehat{\mathsf{A}}(T^{\mathbb{K}}_x) = f(x)$$

by consistency for $(\mathcal{F},E)$, implying $R_{U,G}(F) = f$. □

For a continuous function $\nu : \mathbb{D} \to (0,\infty)$ and a complex lcHs E we define the Bloch type spaces

$$\mathcal{B}\nu(\mathbb{D},E) := \{f \in \mathcal{O}(\mathbb{D},E) \mid \forall\, \alpha \in \mathfrak{A} :\ |f|_{\nu,\alpha} < \infty\}$$

with

$$|f|_{\nu,\alpha} := \max\left(p_\alpha(f(0)), \sup_{z\in\mathbb{D}} p_\alpha((\partial^1_{\mathbb{C}})^E f(z))\nu(z)\right)$$

and the complex derivative

$$(\partial^1_{\mathbb{C}})^E f(z) := \lim_{\substack{h \to 0\\ h \in \mathbb{C}, h \neq 0}} \frac{f(z+h) - f(z)}{h}, \quad z \in \mathbb{D},\ f \in \mathcal{O}(\mathbb{D},E).$$

If $E = \mathbb{C}$, we write $f'(z) := (\partial^1_{\mathbb{C}})^{\mathbb{C}} f(z)$ for $z \in \mathbb{D}$ and $f \in \mathcal{O}(\mathbb{D})$. We set $\omega := \{0\} \cup \{(1,z) \mid z \in \mathbb{D}\}$, define the operator $T^E : \mathcal{O}(\mathbb{D},E) \to E^\omega$ by

$$T^E(f)(0) := f(0) \quad \text{and} \quad T^E(f)(1,z) := (\partial^1_{\mathbb{C}})^E f(z),$$

and the weight $\nu_* : \omega \to (0,\infty)$ by

$$\nu_*(0) := 1 \quad \text{and} \quad \nu_*(1,z) := \nu(z),\ z \in \mathbb{D}.$$

Then we have for every $\alpha \in \mathfrak{A}$ that

$$|f|_{\nu,\alpha} = \sup_{x\in\omega} p_\alpha\left(T^E(f)(x)\right)\nu_*(x), \quad f \in \mathcal{B}\nu(\mathbb{D},E),$$

and with $\mathcal{F}(\mathbb{D},E) := \mathcal{O}(\mathbb{D},E)$ we observe that $\mathcal{F}\nu_*(\mathbb{D},E) = \mathcal{B}\nu(\mathbb{D},E)$ with generator $(T^E, T^{\mathbb{C}})$.

Corollary 3.8 *Let E be a locally complete lcHs, $G \subset E'$ determine boundedness, $\nu : \mathbb{D} \to (0,\infty)$ continuous and $U_* \subset \mathbb{D}$ have an accumulation point in $\mathbb{D}$.*

If $f:\{0\}\cup(\{1\}\times U_*)\to E$ is a function such that there is $f_{e'}\in\mathcal{B}v(\mathbb{D})$ for each $e'\in G$ with $f_{e'}(0)=e'(f(0))$ and $f_{e'}'(z)=e'(f(1,z))$ for all $z\in U_*$, then there exists a unique $F\in\mathcal{B}v(\mathbb{D},E)$ with $F(0)=f(0)$ and $(\partial_{\mathbb{C}}^1)^E F(z)=f(1,z)$ for all $z\in U_*$.

Proof We take $\mathcal{F}(\mathbb{D}):=(\mathcal{O}(\mathbb{D}),\tau_{co})$ and $\mathcal{F}(\mathbb{D},E):=(\mathcal{O}(\mathbb{D},E),\tau_{co})$. Then we have $\mathcal{F}v_*(\mathbb{D})=\mathcal{B}v(\mathbb{D},E)$ and $\mathcal{F}v_*(\Omega,E)=\mathcal{B}v(\mathbb{D},E)$ with the weight v_* and generator $(T^E,T^{\mathbb{C}})$ for $(\mathcal{F}v_*,E)$ described above. The spaces $\mathcal{F}(\mathbb{D})$ and $\mathcal{F}(\mathbb{D},E)$ are ε-compatible by [7, p. 232] in combination with [36, Remark 5.4, p. 264] and the generator is a strong, consistent family for $(\mathcal{F},E)$ (see e.g. [37, Theorem 4.5, p. 368]). $\mathcal{F}v_*(\mathbb{D})=\mathcal{B}v(\mathbb{D})$ is a Banach space and

$$
\begin{aligned}
|f(z)| &\le |f(0)| + \Big|\int_0^z f'(\zeta)\mathrm{d}\zeta\Big| \le |f(0)| + \frac{|z|}{\min_{\xi\in[0,z]} v(\xi)} \sup_{\zeta\in[0,z]} |f'(\zeta)|v(\zeta)\\
&\le 2\max\Big(1,\frac{|z|}{\min_{\xi\in[0,z]} v(\xi)}\Big)|f|_v
\end{aligned}
$$

for every $z\in\mathbb{D}$, yielding

$$
\max_{|z|\le r}|f(z)| \le 2\max\Big(1,\frac{r}{\min_{|z|\le r} v(z)}\Big)|f|_v
$$

for all $0<r<1$ and $f\in\mathcal{B}v(\mathbb{D})$. We deduce from the inequality above that $B_{\mathcal{F}v_*(\mathbb{D})}$ is compact in the Montel space $(\mathcal{O}(\mathbb{D}),\tau_{co})$. We note that the ε-compatibility of $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ in combination with the consistency of $(T^E,T^{\mathbb{C}})$ for $(\mathcal{F},E)$ gives $\mathcal{F}_\varepsilon v_*(\mathbb{D},E)=\mathcal{F}v_*(\mathbb{D},E)$ as linear spaces by Proposition 2.4 (c). In addition, $U:=\{0\}\cup\{(1,z)\mid z\in U_*\}$ is a set of uniqueness for $(T^{\mathbb{C}},\mathcal{F}v_*)$ by the identity theorem, proving our statement by Theorem 3.7. □

4 Extension of locally bounded functions

To obtain an affirmative answer to Question 3.4 for general separating subspaces of E' we have to restrict to a certain class of 'thick' sets of uniqueness.

Definition 4.1 (fix the topology) Let $\mathcal{F}v(\Omega)$ be a Hausdorff space. $U\subset\omega$ *fixes the topology* in $\mathcal{F}v(\Omega)$ if there is $C>0$ such that

$$
|f|_{\mathcal{F}v(\Omega)} \le C\sup_{x\in U}|T^{\mathbb{K}}(f)(x)|v(x),\quad f\in\mathcal{F}v(\Omega).
$$

In particular, U is a set of uniqueness if it fixes the topology. The present definition of fixing the topology is a special case of [38, 4.1 Definition, p. 18]. Sets that fix the topology appear under many different names, e.g. dominating, (weakly) sufficient or sampling sets (see [38, p.18-19] and the references therein), and they are related to $\ell v(U)$-frames used by Bonet et. al in [8]. For a set U, a function $v:U\to(0,\infty)$ and an lcHs E we set

$$\ell v(U,E) := \{f : U \to E \mid \forall\, \alpha \in \mathfrak{A} :\ \|f\|_\alpha := \sup_{x\in U} p_\alpha(f(x))v(x) < \infty\}. \tag{4.1}$$

If U is countable and fixes the topology in $\mathcal{F}v(\Omega)$, the inclusion $\ell v(U) \hookrightarrow (\mathbb{K}^U, \tau_{co})$ is continuous and $\ell v(U)$ contains the space of sequences (on U) with compact support as a linear subspace, then $(T_x^{\mathbb{K}})_{x\in U}$ is an $\ell v(U)$-frame in the sense of [8, Definition 2.1, p.3]. The next definition is a special case of [38, 4.2 Definition, p. 19].

Definition 4.2 (*lb*-restriction space) Let $\mathcal{F}v(\Omega)$ be a Hausdorff space, U fix the topology in $\mathcal{F}v(\Omega)$ and $G \subset E'$ be a separating subspace. We set

$$N_U(f) := \{f(x)v(x) \mid x \in U\}$$

for $f \in \mathcal{FV}_G(U,E)$ and

$$\begin{aligned}\mathcal{FV}_G(U,E)_{lb} &:= \{f \in \mathcal{FV}_G(U,E) \mid N_U(f) \text{ bounded in } E\}\\ &= \mathcal{FV}_G(U,E) \cap \ell v(U,E).\end{aligned}$$

Let us recall the assumptions of Remark 3.3 but now U fixes the topology. Let $(T^E, T^{\mathbb{K}})$ be a strong, consistent family for $(\mathcal{F}, E)$ and a generator for $(\mathcal{F}v, E)$. Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ be ε-into-compatible and the inclusion $\mathcal{F}v(\Omega) \hookrightarrow \mathcal{F}(\Omega)$ continuous. Consider a set U which fixes the topology in $\mathcal{F}v(\Omega)$ and a separating subspace $G \subset E'$. For $u \in \mathcal{F}(\Omega)\varepsilon E$ such that $u(B_{\mathcal{F}v(\Omega)}^{\circ\mathcal{F}(\Omega)'})$ is bounded in E we have $R_{U,G}(f) \in \mathcal{F}v_G(U,E)$ with $f := S(u) \in \mathcal{F}_\varepsilon v(\Omega, E)$ by (3.2). Further, $T_x^{\mathbb{K}}(\cdot)v(x) \in B_{\mathcal{F}v(\Omega)}^{\circ\mathcal{F}(\Omega)'}$ for every $x \in \omega$, which implies that

$$\begin{aligned}\sup_{x\in U} p_\alpha(R_{U,G}(f)(x))v(x) &= \sup_{x\in U} p_\alpha\big(u(T_x^{\mathbb{K}}(\cdot)v(x))\big)\\ &\le \sup_{y'\in B_{\mathcal{F}v(\Omega)}^{\circ\mathcal{F}(\Omega)'}} p_\alpha(u(y')) < \infty\end{aligned}$$

for all $\alpha \in \mathfrak{A}$ by consistency. Hence $R_{U,G}(f) \in \mathcal{F}v_G(U,E)_{lb}$. Therefore the injective linear map

$$R_{U,G} : \mathcal{F}_\varepsilon v(\Omega, E) \to \mathcal{F}v_G(U,E)_{lb},\ f \mapsto (T^E(f)(x))_{x\in U},$$

is well-defined and the question we want to answer is:

Question 4.3 Let the assumptions of Remark 3.3 be fulfilled and U fix the topology in $\mathcal{F}v(\Omega)$. When is the injective restriction map

$$R_{U,G} : \mathcal{F}_\varepsilon v(\Omega, E) \to \mathcal{FV}_G(U,E)_{lb},\ f \mapsto (T^E(f)(x))_{x\in U},$$

surjective?

Proposition 4.4 ([19, *Proposition* 3.1, *p*. 692]) *Let E be locally complete, $G \subset E'$ a separating subspace and Z a Banach space whose closed unit ball B_Z is a compact subset of an lcHs Y. Let $B_1 \subset B_Z^{\circ Y'}$ such that $B_1^{\circ Z} := \{z \in Z \mid \forall\, y' \in B_1 :\ |y'(z)| \le 1\}$ is bounded in Z. If $\mathsf{A} : X := \operatorname{span} B_1 \to E$ is a linear map which is bounded on B_1*

such that there is a $\sigma(E',E)$-dense subspace $G \subset E'$ with $e' \circ \mathsf{A} \in Z$ for all $e' \in G$, then there exists a (unique) extension $\widehat{\mathsf{A}} \in Y\varepsilon E$ of A such that $\widehat{\mathsf{A}}(B_Z^{\circ Y'})$ is bounded in E.

The following theorem is a generalisation of [19, Theorem 3.2, p.693] and [32, Theorem 12, p. 5].

Theorem 4.5 *Let E be locally complete, $G \subset E'$ a separating subspace and $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ be ε-into-compatible. Let $(T^E, T^{\mathbb{K}})$ be a generator for $(\mathcal{F}\nu, E)$ and a strong, consistent family for $(\mathcal{F}, E)$, $\mathcal{F}\nu(\Omega)$ a Banach space whose closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is a compact subset of $\mathcal{F}(\Omega)$ and U fix the topology in $\mathcal{F}\nu(\Omega)$. Then the restriction map*

$$R_{U,G} : \mathcal{F}_{\varepsilon}\nu(\Omega, E) \to \mathcal{F}\nu_G(U, E)_{lb}$$

is surjective.

Proof Let $f \in \mathcal{F}\nu_G(U,E)_{lb}$. We set $B_1 := \{T_x^{\mathbb{K}}(\cdot)\nu(x) \mid x \in U\}$, $X := \operatorname{span} B_1$, $Y := \mathcal{F}(\Omega)$ and $Z := \mathcal{F}\nu(\Omega)$. We have $B_1 \subset Y'$ since $(T^E, T^{\mathbb{K}})$ is a consistent family for $(\mathcal{F}, E)$. If $f \in B_Z$, then

$$|T_x^{\mathbb{K}}(f)\nu(x)| \leq |f|_{\mathcal{F}\nu(\Omega)} \leq 1$$

for all $x \in U$ and thus $B_1 \subset B_Z^{\circ Y'}$. Further on, there is $C > 0$ such that for all $f \in B_1^{\circ Z}$

$$|f|_{\mathcal{F}\nu(\Omega)} \leq C \sup_{x \in U} |T_x^{\mathbb{K}}(f)|\nu(x) \leq C$$

as U fixes the topology in Z, implying the boundedness of $B_1^{\circ Z}$ in Z. Let $\mathsf{A} : X \to E$ be the linear map determined by

$$\mathsf{A}(T_x^{\mathbb{K}}(\cdot)\nu(x)) := f(x)\nu(x).$$

The map A is well-defined since G is $\sigma(E',E)$-dense, and bounded on B_1 because $\mathsf{A}(B_1) = N_U(f)$. Let $e' \in G$ and $f_{e'}$ be the unique element in $\mathcal{F}\nu(\Omega)$ such that $T^{\mathbb{K}}(f_{e'})(x) = (e' \circ f)(x)$ for all $x \in U$, which implies $T^{\mathbb{K}}(f_{e'})(x)\nu(x) = (e' \circ \mathsf{A})(T_x^{\mathbb{K}}(\cdot)\nu(x))$. Again, this allows us to consider $f_{e'}$ as a linear form on X (by setting $f_{e'}(T_x^{\mathbb{K}}(\cdot)\nu(x)) := (e' \circ \mathsf{A})(T_x^{\mathbb{K}}(\cdot)\nu(x))$), which yields $e' \circ \mathsf{A} \in \mathcal{F}\nu(\Omega) = Z$ for all $e' \in G$. Hence we can apply Proposition 4.4 and obtain an extension $\widehat{\mathsf{A}} \in Y\varepsilon E$ of A such that $\widehat{\mathsf{A}}(B_Z^{\circ Y'})$ is bounded in E. We set $F := S(\widehat{\mathsf{A}}) \in \mathcal{F}_{\varepsilon}\nu(\Omega, E)$ and get for all $x \in U$ that

$$T^E(F)(x) = T^E S(\widehat{\mathsf{A}})(x) = \widehat{\mathsf{A}}(T_x^{\mathbb{K}}) = \frac{1}{\nu(x)} \mathsf{A}(T_x^{\mathbb{K}}(\cdot)\nu(x)) = f(x)$$

by consistency for $(\mathcal{F}, E)$, yielding $R_{U,G}(F) = f$. $\square$

Corollary 4.6 *Let E be a locally complete lcHs, $G \subset E'$ a separating subspace, $\nu : \mathbb{D} \to (0, \infty)$ continuous and $U := \{0\} \cup (\{1\} \times U_*)$ fix the topology in $\mathcal{B}\nu(\mathbb{D})$ with $U_* \subset \mathbb{D}$. If $f : U \to E$ is a function in $\ell\nu_*(U, E)$ such that there is $f_{e'} \in \mathcal{B}\nu(\mathbb{D})$ for* *each* $e' \in G$ *with* $f_{e'}(0) = e'(f(0))$ *and* $f'_{e'}(z) = e'(f(1,z))$ *for all* $z \in U_*$, *then there exists a unique* $F \in \mathcal{B}v(\mathbb{D}, E)$ *with* $F(0) = f(0)$ *and* $(\partial^1_{\mathbb{C}})^E F(z) = f(1,z)$ *for all* $z \in U_*$.

Proof Observing that $f \in \mathcal{F}v_{*G}(U,E)_{lb}$ with $\mathcal{F}v_*(\mathbb{D}) = \mathcal{B}v(\mathbb{D})$) our statement follows directly from Theorem 4.5 whose conditions are fulfilled by the proof of Corollary 3.8. □

Sets that fix the topology in $\mathcal{B}v(\mathbb{D})$ play an important role in the characterisation of composition operators on $\mathcal{B}v(\mathbb{D})$ with closed range. Chen and Gauthier give a characterisation in [12] for weights of the form $v(z) = (1-|z|^2)^\alpha$, $z \in \mathbb{D}$, for some $\alpha \geq 1$. We recall the following definitions which are needed to phrase this characterisation. For a continuous function $v : \mathbb{D} \to (0,\infty)$ and a non-constant holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$ we set

$$\tau^v_\phi(z) := \frac{v(z)|\phi'(z)|}{v(\phi(z))}, \; z \in \mathbb{D}, \quad \text{and} \quad \Omega^v_\varepsilon := \{z \in \mathbb{D} \mid \tau^v_\phi(z) \geq \varepsilon\}, \; \varepsilon > 0,$$

and define the *pseudohyperbolic distance*

$$\rho(z,w) := \left| \frac{z-w}{1-\bar{z}w} \right|, \; z, w \in \mathbb{D},$$

(see [12, p. 195-196]). For $0 < r < 1$ a set $E \subset \mathbb{D}$ is called a *pseudo r-net* if for every $w \in \mathbb{D}$ there is $z \in \mathbb{D}$ with $\rho(z,w) \leq r$ (see [12, p.198]).

Theorem 4.7 ([12, Theorem 3.1, p. 199, Theorem 4.3, p. 202]) *Let* $\phi : \mathbb{D} \to \mathbb{D}$ *be a non-constant holomorphic function and* $v(z) = (1-|z|^2)^\alpha$, $z \in \mathbb{D}$, *for some* $\alpha \geq 1$. *Then the following statements are equivalent.*

(i) *The composition operator* $C_\phi : \mathcal{B}v(\mathbb{D}) \to \mathcal{B}v(\mathbb{D})$, $C_\phi(f) := f \circ \phi$, *is bounded below* (*i.e. has closed range*).
(ii) *There is* $\varepsilon > 0$ *such that* $\{0\} \cup (\{1\} \times \phi(\Omega^v_\varepsilon))$ *fixes the topology in* $\mathcal{B}v(\mathbb{D})$.
(iii) *There are* $\varepsilon > 0$ *and* $0 < r < 1$ *such that* $\phi(\Omega^v_\varepsilon)$ *is a pseudo r-net.*

This theorem has some predecessors. The implications $(i) \Rightarrow (iii)$ and (iii), $r < 1/4 \Rightarrow (i)$ for $\alpha = 1$ are due to Ghatage, Yan and Zheng by [20, Proposition 1, p. 2040] and [20, Theorem 2, p.2043]. This was improved by Chen to $(i) \Leftrightarrow (iii)$ for $\alpha = 1$ by removing the restriction $r < 1/4$ in [11, Theorem 1, p. 840]. The proof of the equivalence $(i) \Leftrightarrow (ii)$ given in [21, Theorem 1, p. 1372] for $\alpha = 1$ is due to Ghatage, Zheng and Zorboska. A non-trivial example of a sampling set for $\alpha = 1$ can be found in [21, Example 2, p. 1376] (cf. [12, p. 203]).

Giménez, Malavé and Ramos-Fernández extend Theorem 4.7 by [22, Theorem 3, p.112] and [22, Corollary 1, p. 113] to more general weights of the form $v(z) = \mu(1-|z|^2)$ with some continuous function $\mu : (0,1] \to (0,\infty)$ such that $\mu(r) \to 0$, $r \to 0+$, which can be extended to a holomorphic function μ_0 on $\mathbb{D}(1,1) := \{z \in \mathbb{C} \mid |z-1| < 1\}$ without zeros in $\mathbb{D}(1,1)$ and fulfilling

$\mu(1-|1-z|) \leq C|\mu_0(z)|$ for all $z \in \mathbb{D}(1,1)$ and some $C > 0$ (see [22, p. 109]). Examples of such functions μ are $\mu_1(r) := r^{\alpha}$, $\alpha > 0$, $\mu_2 := r\ln(2/r)$ and $\mu_3(r) := r^{\beta}\ln(1-r)$, $\beta > 1$, for $r \in (0,1]$ (see [22, p. 110]) and with $v(z) = \mu_1(1-|z|^2) = (1-|z|^2)^{\alpha}$, $z \in \mathbb{D}$, one gets Theorem 4.7 back for $\alpha \geq 1$. For $0 < \alpha < 1$ and $v(z) = \mu_1(1-|z|^2)$, $z \in \mathbb{D}$, the equivalence $(i) \Leftrightarrow (ii)$ is given in [47, Proposition 4.4, p. 14] of Yoneda as well and a sufficient condition implying (ii) in [47, Corollary 4.5, p.15]. Ramos-Fernández generalises the results given in [22] to bounded essential weights v on $\mathbb{D}$ by [41, Theorem 4.3, p. 85] and [41, Remark 4.2, p. 84].

5 Weak-strong principles for differentiable functions of finite order

This section is dedicated to $\mathcal{C}^k$-weak-strong principles for differentiable functions. So the question is:

Question 5.1 Let E be an lcHs, $G \subset E'$ a separating subspace, $\Omega \subset \mathbb{R}^d$ open and $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f : \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^k(\Omega)$ for each $e' \in G$, does $f \in \mathcal{C}^k(\Omega, E)$ hold?

An affirmative answer to the preceding question is called a $\mathcal{C}^k$-weak-strong principle. It is a result of Bierstedt [6, 2.10 Lemma, p. 140] that for $k = 0$ the $\mathcal{C}^0$-weak-strong principle holds if $\Omega \subset \mathbb{R}^d$ is open (or more general a $k_{\mathbb{R}}$-space), $G = E'$ and E is such that every bounded set is already precompact in E. For instance, the last condition is fulfilled if E is a semi-Montel or Schwartz space. The $\mathcal{C}^0$-weak-strong principle does not hold for general E by [33, Beispiel, p. 232].

Grothendieck sketches in a footnote [27, p. 39] (cf. [29, Chap. 3, Sect. 8, Corollary 1, p. 134]) the proof that for $k < \infty$ a weakly-$\mathcal{C}^{k+1}$ function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ with values in a quasi-complete lcHs E is already $\mathcal{C}^k$, i.e. that from $e' \circ f \in \mathcal{C}^{k+1}(\Omega)$ for all $e' \in E'$ it follows $f \in \mathcal{C}^k(\Omega, E)$. A detailed proof of this statement is given by Schwartz in [44], simultaneously weakening the condition from quasi-completeness of E to sequential completeness and from weakly-$\mathcal{C}^{k+1}$ to weakly-$\mathcal{C}^{k,1}_{loc}$.

Theorem 5.2 ([44, Appendice, Lemme II, Remarques 1^0), *p*. 146-147]) *Let E be a sequentially complete lcHs, $\Omega \subset \mathbb{R}^d$ open and $k \in \mathbb{N}_0$.*

(a) *If $f : \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^{k,1}_{loc}(\Omega)$ for all $e' \in E'$, then $f \in \mathcal{C}^k(\Omega, E)$.*
(b) *If $f : \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^{k+1}(\Omega)$ for all $e' \in E'$, then $f \in \mathcal{C}^k(\Omega, E)$.*

Here $\mathcal{C}^{k,1}_{loc}(\Omega)$ denotes the space of functions in $\mathcal{C}^k(\Omega)$ whose partial derivatives of order k are locally Lipschitz continuous. Part (b) clearly implies a $\mathcal{C}^\infty$-weak-strong principle for open $\Omega \subset \mathbb{R}^d$, $G = E'$ and sequentially complete E. This can be generalised to locally complete E. Waelbroeck has shown in [46, Proposition 2, p. 411] and [45, Definition 1, p. 393] that the $\mathcal{C}^\infty$-weak-strong principle holds if Ω is a manifold, $G = E'$ and E is locally complete. It is a result of Bonet, Frerick and Jordá that the $\mathcal{C}^\infty$-weak-strong principle still holds by [7, Theorem 9, p. 232] if $\Omega \subset \mathbb{R}^d$ is open, $G \subset E'$ determines boundedness and E is locally complete. Due to [34, 2.14 Theorem, p. 20] of Kriegl and Michor an lcHs E is locally complete if and only if the $\mathcal{C}^\infty$-weak-strong principle holds for $\Omega = \mathbb{R}$ and $G = E'$.

One of the goals of this section is to improve Theorem 5.2. We start with the following definition. For $k \in \mathbb{N}_0$ we define the space of k-times continuously partially differentiable E-valued functions on an open set $\Omega \subset \mathbb{R}^d$ whose partial derivatives up to order k are continuously extendable to the boundary of Ω by

$$\mathcal{C}^k(\overline{\Omega}, E) := \{f \in \mathcal{C}^k(\Omega, E) \mid (\partial^\beta)^E f \text{ cont. extendable on } \overline{\Omega} \\ \text{for all } \beta \in \mathbb{N}_0^d,\ |\beta| \leq k\}$$

which we equip with the system of seminorms given by

$$|f|_{\mathcal{C}^k(\overline{\Omega}),\alpha} := \sup_{\substack{x \in \Omega \\ \beta \in \mathbb{N}_0^d, |\beta| \leq k}} p_\alpha((\partial^\beta)^E f(x)), \quad f \in \mathcal{C}^k(\overline{\Omega}, E),\ \alpha \in \mathfrak{A}.$$

The space of functions in $\mathcal{C}^k(\overline{\Omega}, E)$ such that all its k-th partial derivatives are γ -Hölder continuous with $0 < \gamma \leq 1$ is given by

$$\mathcal{C}^{k,\gamma}(\overline{\Omega}, E) := \left\{f \in \mathcal{C}^k(\overline{\Omega}, E) \mid \forall\, \alpha \in \mathfrak{A} :\ |f|_{\mathcal{C}^{k,\gamma}(\overline{\Omega}),\alpha} < \infty\right\}$$

where

$$|f|_{\mathcal{C}^{k,\gamma}(\overline{\Omega}),\alpha} := \max\left(|f|_{\mathcal{C}^k(\overline{\Omega}),\alpha}, \sup_{\beta \in \mathbb{N}_0^d, |\beta| = k} |(\partial^\beta)^E f|_{\mathcal{C}^{0,\gamma}(\Omega),\alpha}\right)$$

with

$$|f|_{\mathcal{C}^{0,\gamma}(\Omega),\alpha} := \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{p_\alpha(f(x) - f(y))}{|x - y|^\gamma}.$$

We set

$$\omega_1 := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq k\} \times \Omega,$$
$$\omega_2 := \{\beta \in \mathbb{N}_0^d \mid |\beta| = k\} \times (\Omega^2 \setminus \{(x, x) \mid x \in \Omega\})$$

as well as $\omega := \omega_1 \cup \omega_2$. We define the operator $T^E : \mathcal{C}^k(\Omega, E) \to E^\omega$ by

$$T^E(f)(\beta, x) := (\partial^\beta)^E(f)(x) \quad , (\beta, x) \in \omega_1,$$
$$T^E(f)(\beta, (x, y)) := (\partial^\beta)^E(f)(x) - (\partial^\beta)^E(f)(y) \quad , (\beta, (x, y)) \in \omega_2.$$

and the weight $\nu : \omega \to (0, \infty)$ by

$$v(\beta,x):=1,\ (\beta,x)\in\omega_1,\quad \text{and}\quad v(\beta,(x,y)):=\frac{1}{|x-y|^{\gamma}},\ (\beta,(x,y))\in\omega_2.$$

By setting $\mathcal{F}(\Omega,E):=\mathcal{C}^k(\overline{\Omega},E)$ and observing that

$$|f|_{\mathcal{C}^{k,\gamma}(\overline{\Omega}),\alpha}=\sup_{x\in\omega}p_\alpha(T^E(f)(x))v(x),\quad f\in\mathcal{C}^{k,\gamma}(\overline{\Omega},E),\ \alpha\in\mathfrak{A},$$

we have $\mathcal{F}v(\Omega,E)=\mathcal{C}^{k,\gamma}(\overline{\Omega},E)$ with generator $(T^E,T^{\mathbb{K}})$.

Corollary 5.3 *Let E be a locally complete lcHs, $G\subset E'$ determine boundedness, $\Omega\subset\mathbb{R}^d$ open and bounded, $k\in\mathbb{N}_0$ and $0<\gamma\leq 1$. If $f:\Omega\to E$ is such that $e'\circ f\in\mathcal{C}^{k,\gamma}(\overline{\Omega})$ for all $e'\in G$, then $f\in\mathcal{C}^{k,\gamma}(\overline{\Omega},E)$.*

Proof We take $\mathcal{F}(\Omega):=\mathcal{C}^k(\overline{\Omega})$ and $\mathcal{F}(\Omega,E):=\mathcal{C}^k(\overline{\Omega},E)$ and have $\mathcal{F}v(\Omega)=\mathcal{C}^{k,\gamma}(\overline{\Omega})$ and $\mathcal{F}v(\Omega,E)=\mathcal{C}^{k,\gamma}(\overline{\Omega},E)$ with the weight v and generator $(T^E,T^{\mathbb{K}})$ for $(\mathcal{F}v,E)$ described above. Due to the proof of [35, Example 20, p. 1529] and the first part of the proof of [35, Theorem 14, p. 1524] the spaces $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ are ε-into-compatible for any lcHs E (the condition that E has metric ccp in [35, Example 20, p.1529] is only needed for ε-compatibility). Another consequence of [35, Example 20, p. 1529] is that

$$T^E(S(u))(\beta,x)=(\partial^\beta)^E(S(u))(x)=u(\delta_x\circ(\partial^\beta)^{\mathbb{K}})=u(T^{\mathbb{K}}_{\beta,x}),\quad (\beta,x)\in\omega_1,$$

holds for all $u\in\mathcal{F}(\Omega)\varepsilon E$, implying

$$\begin{aligned}T^E(S(u))(\beta,(x,y))&=T^E(S(u))(\beta,x)-T^E(S(u))(\beta,y)=u(T^{\mathbb{K}}_{\beta,x})-u(T^{\mathbb{K}}_{\beta,y})\\&=u(T^{\mathbb{K}}_{\beta,(x,y)}),\quad (\beta,(x,y))\in\omega_2.\end{aligned}$$

Thus $(T^E,T^{\mathbb{K}})$ is a consistent family for $(\mathcal{F},E)$ and its strength is easily seen. In addition, $\mathcal{F}v(\Omega)=\mathcal{C}^{k,\gamma}(\overline{\Omega})$ is a Banach space by [15, Theorem 9.8, p. 110] (cf. [1, 1.7 Hölderstetige Funktionen, p. 46]) whose closed unit ball is compact in $\mathcal{F}(\Omega)=\mathcal{C}^k(\overline{\Omega})$ by [15, Theorem 14.32, p. 232] (cf. [1, 8.6 Einbettungssatz in Hölder-Räumen, p. 338]). Moreover, the ε-into-compatibility of $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ in combination with the consistency of $(T^E,T^{\mathbb{K}})$ for $(\mathcal{F},E)$ implies $\mathcal{F}_\varepsilon v(\Omega,E)\subset\mathcal{F}v(\Omega,E)$ as linear spaces by Proposition 2.4 (c). Hence our statement follows from Theorem 3.7 with the set of uniqueness $U:=\{0\}\times\Omega$ for $(T^{\mathbb{K}},\mathcal{F}v)$. □

Next, we use the preceding corollary to generalise the Theorem 5.2 of Grothendieck and Schwartz on weakly $\mathcal{C}^{k+1}$-functions. For $k\in\mathbb{N}_0$ and $0<\gamma\leq 1$ we define the space of k-times continuously partially differentiable E-valued functions with locally γ-Hölder continuous partial derivatives of k-th order on an open set $\Omega\subset\mathbb{R}^d$ by

$$\mathcal{C}^{k,\gamma}_{loc}(\Omega,E):=\{f\in\mathcal{C}^k(\Omega,E)\mid \forall\, K\subset\Omega \text{ compact},\ \alpha\in\mathfrak{A}:\ |f|_{K,\alpha}<\infty\}$$

where

$$|f|_{K,\alpha} := \max\Big(|f|_{\mathcal{C}^k(K),\alpha}, \sup_{\beta\in\mathbb{N}_0^d, |\beta|=k} |(\partial^\beta)^E f|_{\mathcal{C}^{0,\gamma}(K),\alpha} \Big)$$

with

$$|f|_{\mathcal{C}^k(K),\alpha} := \sup_{\substack{x\in K\\ \beta\in\mathbb{N}_0^d,\, |\beta|\leq k}} p_\alpha((\partial^\beta)^E f(x))$$

and

$$|f|_{\mathcal{C}^{0,\gamma}(K),\alpha} := \sup_{\substack{x,y\in K\\ x\neq y}} \frac{p_\alpha(f(x)-f(y))}{|x-y|^\gamma}.$$

Using Corollary 5.3, we are able to improve Theorem 5.2 to the following form.

Corollary 5.4 *Let E be a locally complete lcHs, $G \subset E'$ determine boundedness, $\Omega \subset \mathbb{R}^d$ open, $k \in \mathbb{N}_0$ and $0 < \gamma \leq 1$.*

(a) *If $f : \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega)$ for all $e' \in G$, then $f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$.*
(b) *If $f : \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^{k+1}(\Omega)$ for all $e' \in G$, then $f \in \mathcal{C}^{k,1}_{loc}(\Omega, E)$.*

Proof Let us start with (a). Let $f : \Omega \to E$ be such that $e' \circ f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega)$ for all $e' \in G$. Let $(\Omega_n)_{n\in\mathbb{N}}$ be an exhaustion of Ω with open, relatively compact sets $\Omega_n \subset \Omega$. Then the restriction of $e' \circ f$ to Ω_n is an element of $\mathcal{C}^{k,\gamma}(\overline{\Omega_n})$ for every $e' \in G$ and $n \in \mathbb{N}$. Due to Corollary 5.3 we obtain that $f \in \mathcal{C}^{k,\gamma}(\overline{\Omega_n}, E)$ for every $n \in \mathbb{N}$. Thus $f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$ since differentiability is a local property and for each compact $K \subset \Omega$ there is $n \in \mathbb{N}$ such that $K \subset \Omega_n$.

Let us turn to (b), i.e. let $f : \Omega \to E$ be such that $e' \circ f \in \mathcal{C}^{k+1}(\Omega)$ for all $e' \in G$. Since $\Omega \subset \mathbb{R}^d$ is open, for every $x \in \Omega$ there is $\varepsilon_x > 0$ such that $\overline{\mathbb{B}(x, \varepsilon_x)} \subset \Omega$ where $\overline{\mathbb{B}(x, \varepsilon_x)}$ is the closure of $\mathbb{B}(x, \varepsilon_x) := \{y \in \mathbb{R}^d \mid |y - x| < \varepsilon_x\}$. For all $e' \in G$, $\beta \in \mathbb{N}_0^d$ with $|\beta| = k$ and $w, y \in \mathbb{B}(x, \varepsilon_x)$, $w \neq y$, it holds that

$$\frac{|(\partial^\beta)^{\mathbb{K}}(e' \circ f)(w) - (\partial^\beta)^{\mathbb{K}}(e' \circ f)(y)|}{|w-y|} \leq C \max_{1\leq n\leq d} \max_{z\in\overline{\mathbb{B}(x,\varepsilon_x)}} |(\partial^{\beta+e_n})^{\mathbb{K}}(e' \circ f)(z)|$$

by the mean value theorem applied to the real and imaginary part where $C := \sqrt{d}$ if $\mathbb{K} = \mathbb{R}$ and $C := 2\sqrt{d}$ if $\mathbb{K} = \mathbb{C}$. Thus $e' \circ f \in \mathcal{C}^{k,1}_{loc}(\Omega)$ for all $e' \in G$ since for each compact set $K \subset \Omega$ there are $m \in \mathbb{N}$ and $x_i \in \Omega$, $1 \leq i \leq m$, such that $K \subset \bigcup_{i=1}^m \mathbb{B}(x_i, \varepsilon_{x_i})$. It follows from part (a) that $f \in \mathcal{C}^{k,1}_{loc}(\Omega, E)$. $\square$

If $\Omega = \mathbb{R}$, $\gamma = 1$ and $G = E'$, then part (a) of Corollary 5.4 is already known by [34, 2.3 Corollary, p. 15]. A 'full' $\mathcal{C}^k$-weak-strong principle for $k < \infty$, i.e. the conditions of part (b) imply $f \in \mathcal{C}^{k+1}(\Omega, E)$, does not hold in general (see [34, p. 11-12]) but it holds if we restrict the class of admissible lcHs E.

Theorem 5.5 *Let E be a semi-Montel space, $G \subset E'$ determine boundedness, $\Omega \subset \mathbb{R}^d$ open and $k \in \mathbb{N}$. If $f : \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^k(\Omega)$ for all $e' \in G$, then $f \in \mathcal{C}^k(\Omega, E)$.*

Proof Let $f : \Omega \to E$ be such that $e' \circ f \in \mathcal{C}^k(\Omega)$ for all $e' \in G$. Due to Corollary 5.4 (b) we already know that $f \in \mathcal{C}^{k-1,1}_{loc}(\Omega, E)$ since semi-Montel spaces are quasi-complete and thus locally complete. Now, let $x \in \Omega$, $\varepsilon_x > 0$ such that $\overline{\mathbb{B}(x, \varepsilon_x)} \subset \Omega$, $\beta \in \mathbb{N}_0^d$ with $|\beta| = k - 1$ and $n \in \mathbb{N}$ with $1 \leq n \leq d$. The set

$$B := \left\{ \frac{(\partial^\beta f)^E(x + he_n) - (\partial^\beta f)^E f(x)}{h} \mid h \in \mathbb{R},\ 0 < h \leq \varepsilon_x \right\}$$

is bounded in E because $f \in \mathcal{C}^{k-1,1}_{loc}(\Omega, E)$. As E is semi-Montel, the closure $\overline{B}$ is compact in E. Let $(h_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $0 < h_m \leq \varepsilon_x$ for all $m \in \mathbb{N}$. From the compactness of $\overline{B}$ we deduce that there is a subnet $(h_{m_\iota})_{\iota\in I}$, where I is a directed set, of $(h_m)_{m\in\mathbb{N}}$ and $y_x \in \overline{B}$ with

$$y_x = \lim_{\iota\in I} \frac{(\partial^\beta f)^E(x + h_{m_\iota} e_n) - (\partial^\beta f)^E f(x)}{h_{m_\iota}} =: \lim_{\iota\in I} y_\iota.$$

Further, we note that the limit

$$\partial^{\beta+e_n}(e' \circ f)(x) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}, h \neq 0}} \frac{\partial^\beta(e' \circ f)(x + he_n) - \partial^\beta(e' \circ f)(x)}{h} \tag{5.1}$$

exists for all $e' \in G$ and that $(e'(y_\iota))_{\iota\in I}$ is a subnet of the net of difference quotients on the right-hand side of (5.1) as $\partial^\beta(e' \circ f) = e' \circ (\partial^\beta)^E f$. Therefore

$$\begin{aligned}
\partial^{\beta+e_n}(e' \circ f)(x) &= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}, h \neq 0}} e'\left(\frac{(\partial^\beta)^E f(x + he_n) - (\partial^\beta)^E f(x)}{h}\right) \\
&= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}, 0 < h \leq \varepsilon_x}} e'\left(\frac{(\partial^\beta)^E f(x + he_n) - (\partial^\beta)^E f(x)}{h}\right) \\
&= \lim_{\iota\in I} e'(y_\iota) = e'(y_x)
\end{aligned} \tag{5.2}$$

for all $e' \in G$. By [38, 4.10 Proposition (i), p.21] the topology $\sigma(E, G)$ and the initial topology of E coincide on $\overline{B}$. Combining this fact with (5.2), we deduce that

$$(\partial^{\beta+e_n})^E f(x) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}, h \neq 0}} \frac{(\partial^{\beta})^E f(x+he_n) - (\partial^{\beta})^E f(x)}{h} = y_x.$$

Further, $e' \circ (\partial^{\beta+e_n})^E f = \partial^{\beta+e_n}(e' \circ f)$ is continuous on $\overline{\mathbb{B}(x,\varepsilon_x)}$ for all $e' \in G$, meaning that the restriction of $(\partial^{\beta+e_n})^E f$ on $\overline{\mathbb{B}(x,\varepsilon_x)}$ to $(E,\sigma(E,G))$ is continuous, and the range $(\partial^{\beta+e_n})^E f(\overline{\mathbb{B}(x,\varepsilon_x)})$ is bounded in E. As before we use that $\sigma(E,G)$ and the initial topology of E coincide on $(\partial^{\beta+e_n})^E f(\overline{\mathbb{B}(x,\varepsilon_x)})$, which implies that the restriction of $(\partial^{\beta+e_n})^E f$ on $\overline{\mathbb{B}(x,\varepsilon_x)}$ is continuous w.r.t. the initial topology of E. Since continuity is a local property and $x \in \Omega$ is arbitrary, we conclude that $(\partial^{\beta+e_n})^E f$ is continuous on Ω. □

In the special case that $\Omega = \mathbb{R}$, $G = E'$ and E is a Montel space, i.e. a barrelled semi-Montel space, a different proof of the preceding weak-strong principle can be found in the proof of [10, Lemma 4, p. 15]. This proof uses the Banach–Steinhaus theorem and needs the barrelledness of the Montel space E_b'. Our weak-strong principle Theorem 5.5 does not need the barrelledness of E, hence can be applied to non-barrelled semi-Montel spaces like $E = (\mathcal{C}^{\infty}_{\overline{\partial},b}(\mathbb{D}), \beta)$ where β is the strict topology (see the definition above Corollary 6.7, [38, 3.14 Proposition, p. 12] and [38, 3.15 Remark, p. 13]).

Besides the 'full' $\mathcal{C}^k$-weak-strong principle for $k < \infty$ and semi-Montel E, part (b) of Corollary 5.4 also suggests an 'almost' $\mathcal{C}^k$-weak-strong principle in terms of [16, 3.1.6 Rademacher's theorem, p. 216], which we prepare next.

Definition 5.6 (generalised Gelfand space) We say that an lcHs E is a *generalised Gelfand space* if every Lipschitz continuous map $f : [0,1] \to E$ is differentiable almost everywhere w.r.t to the one-dimensional Lebesgue measure.

If E is a real Fréchet space ($\mathbb{K} = \mathbb{R}$), then this definition coincides with the definition of a Fréchet–Gelfand space given in [39, Definition 2.2, p. 17]. In particular, every real nuclear Fréchet lattice (see [24, Theorem 6, Corollary, p.375, 378]) and more general every real Fréchet–Montel space is a generalised Gelfand space by [39, Theorem 2.9, p. 18]. If E is a Banach space, then this definition coincides with the definition of a Gelfand space given in [14, Definition 4.3.1, p.106-107] by [5, Proposition 1.2.4, p.18]. A Banach space is a Gelfand space if and only if it has the Radon-Nikodým property by [14, Theorem 4.3.2, p. 107]. Thus separable duals of Banach spaces, reflexive Banach spaces and $\ell^1(\Gamma)$ for any set Γ are generalised Gelfand spaces by [14, Theorem 3.3.1 (Dunford-Pettis), p. 79], [14, Corollary 3.3.4 (Phillips), p. 82] and [14, Corollary 3.3.8, p.83]. The Banach spaces c_0, ℓ^{∞}, $\mathcal{C}([0,1])$, $L^1([0,1])$ and $L^{\infty}([0,1])$ do not have the Radon–Nikodým property and hence are not generalised Gelfand spaces by [5, Proposition 1.2.9, p. 20], [5, Example 1.2.8, p. 20] and [5, Proposition 1.2.10, p. 21].

Corollary 5.7 *Let E be a locally complete generalised Gelfand space, $G \subset E'$ determine boundedness, $\Omega \subset \mathbb{R}$ open and $k \in \mathbb{N}$. If $f : \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^k(\Omega)$*

for all $e' \in G$, *then* $f \in \mathcal{C}^{k-1,1}_{loc}(\Omega, E)$ *and the derivative* $(\partial^k)^E f(x)$ *exists for Lebesgue almost all* $x \in \Omega$.

Proof The first part follows from Corollary 5.4 (b). Now, let $[a, b] \subset \Omega$ be a bounded interval. We set $F : [0, 1] \to E$, $F(x) := (\partial^{k-1})^E f(a + x(b - a))$. Then F is Lipschitz continuous as $f \in \mathcal{C}^{k-1,1}_{loc}(\Omega, E)$. This yields that F is differentiable on [0, 1] almost everywhere because E is a generalised Gelfand space, implying that $(\partial^{k-1})^E f$ is differentiable on $[a, b]$ almost everywhere. Since the open set $\Omega \subset \mathbb{R}$ can be written as a countable union of disjoint open intervals I_n, $n \in \mathbb{N}$, and each I_n is a countable union of closed bounded intervals $[a_m, b_m]$, $m \in \mathbb{N}$, our statement follows from the fact that the countable union of null sets is a null set. □

To the best of our knowledge there are still some open problems for continuously partially differentiable functions of finite order.

Question 5.8

(i) Are there other spaces than semi-Montel spaces E for which the 'full' $\mathcal{C}^k$-weak-strong principle Theorem 5.5 with $k < \infty$ is true? For instance, if $k = 0$, then it is still true if E is an lcHs such that every bounded set is already precompact in E by [6, 2.10 Lemma, p. 140]. Does this hold for $0 < k < \infty$ as well?
(ii) Does the 'almost' $\mathcal{C}^k$-weak-strong principle Corollary 5.7 also hold for $d > 1$?
(iii) For every $\varepsilon > 0$ does there exist a function $g \in \mathcal{C}^k(\mathbb{R}, E)$ such that $\lambda(\{x \in \Omega \mid f(x) \neq g(x)\}) < \varepsilon$ in Corollary 5.7 where λ is the one-dimensional Lebesgue measure. In the case that $E = \mathbb{R}^n$ this is true by [16, Theorem 3.1.15, p. 227].
(iv) Is there a 'Radon–Nikodým type' characterisation of generalised Gelfand spaces like in the Banach case?

6 Vector-valued Blaschke theorems

In this section we prove several convergence theorems for Banach-valued functions in the spirit of Blaschke's convergence theorem [9, Theorem 7.4, p. 219] as it is done in [3, Theorem 2.4, p. 786] and [3, Corollary 2.5, p. 786-787] for bounded holomorphic functions and more general in [19, Corollary 4.2, p. 695] for bounded functions in the kernel of a hypoelliptic linear partial differential operator. Blaschke's convergence theorem says that if $(z_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ is a sequence of distinct elements with $\sum_{n\in\mathbb{N}}(1 - |z_n|) = \infty$ and if $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $H^\infty(\mathbb{D})$ such that $(f_k(z_n))_k$ converges in $\mathbb{C}$ for each $n \in \mathbb{N}$, then there is $f \in H^\infty(\mathbb{D})$ such that $(f_k)_k$ converges uniformly to f on the compact subsets of $\mathbb{D}$, i.e. w.r.t. to τ_{co}.

Proposition 6.1 ([19, Proposition 4.1, p. 695]) *Let* $(E, \|\cdot\|)$ *be a Banach space*, Z *a Banach space whose closed unit ball* B_Z *is a compact subset of an lcHs* Y *and let* $(\mathsf{A}_\iota)_{\iota\in I}$ *be a net in* $Y\varepsilon E$ *such that*

$$\sup_{\iota\in I}\{\|\mathsf{A}_\iota(y)\| \mid y\in B_Z^{\circ Y'}\}<\infty.$$

Assume further that there exists a $\sigma(Y',Z)$-dense subspace $X\subset Y'$ such that $\lim_\iota \mathsf{A}_\iota(x)$ exists for each $x\in X$. Then there is $\mathsf{A}\in Y\varepsilon E$ with $\mathsf{A}(B_Z^{\circ Y'})$ bounded and $\lim_\iota \mathsf{A}_\iota=\mathsf{A}$ uniformly on the equicontinuous subsets of Y', i.e. for all equicontinuous $B\subset Y'$ and $\varepsilon>0$ there exists $\varsigma\in I$ such that

$$\sup_{y\in B}\|\mathsf{A}_\iota(y)-\mathsf{A}(y)\|<\varepsilon$$

for each $\iota\geq\varsigma$.

Next, we generalise [19, Corollary 4.2, p. 695].

Corollary 6.2 *Let $(E,\|\cdot\|)$ be a Banach space and $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-into-compatible. Let $(T^E,T^{\mathbb{K}})$ be a generator for $(\mathcal{F}\nu,E)$ and a strong, consistent family for $(\mathcal{F},E)$, $\mathcal{F}\nu(\Omega)$ a Banach space whose closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is a compact subset of $\mathcal{F}(\Omega)$ and U a set of uniqueness for $(T^{\mathbb{K}},\mathcal{F}\nu)$.*

If $(f_\iota)_{\iota\in I}\subset\mathcal{F}_\varepsilon\nu(\Omega,E)$ is a bounded net in the space $\mathcal{F}\nu(\Omega,E)$ such that $\lim_\iota T^E(f_\iota)(x)$ exists for all $x\in U$, then there is $f\in\mathcal{F}_\varepsilon\nu(\Omega,E)$ such that $(f_\iota)_{\iota\in I}$ converges to f in $\mathcal{F}(\Omega,E)$.

Proof We set $X:=\operatorname{span}\{T_x^{\mathbb{K}}\mid x\in U\}$, $Y:=\mathcal{F}(\Omega)$ and $Z:=\mathcal{F}\nu(\Omega)$. As in the proof of Theorem 3.7 we observe that X is $\sigma(Y',Z)$-dense in Y'. From $(f_\iota)_{\iota\in I}\subset\mathcal{F}_\varepsilon\nu(\Omega,E)$ follows that there are $\mathsf{A}_\iota\in\mathcal{F}(\Omega)\varepsilon E$ with $S(\mathsf{A}_\iota)=f_\iota$ for all $\iota\in I$. Since $(f_\iota)_{\iota\in I}$ is a bounded net in $\mathcal{F}\nu(\Omega,E)$, we note that

$$\begin{aligned}\sup_{\iota\in I}\sup_{x\in\omega}\|\mathsf{A}_\iota(T_x^{\mathbb{K}}(\cdot)\nu(x))\| &= \sup_{\iota\in I}\sup_{x\in\omega}\|T^ES(\mathsf{A}_\iota)(x)\|\nu(x)\\ &= \sup_{\iota\in I}\sup_{x\in\omega}\|T^Ef_\iota(x)\|\nu(x)=\sup_{\iota\in I}|f_\iota|_{\mathcal{F}\nu(\Omega,E)}<\infty\end{aligned}$$

by consistency. Further, $\lim_\iota S(\mathsf{A}_\iota)(T_x^{\mathbb{K}})=\lim_\iota T^E(f_\iota)(x)$ exists for each $x\in U$, implying the existence of $\lim_\iota S(\mathsf{A}_\iota)(x)$ for each $x\in X$ by linearity. We apply Proposition 6.1 and obtain $f:=S(\mathsf{A})\in\mathcal{F}_\varepsilon\nu(\Omega,E)$ such that $(\mathsf{A}_\iota)_{\iota\in I}$ converges to A in $\mathcal{F}(\Omega)\varepsilon E$. From $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ being ε-into-compatible it follows that $(f_\iota)_{\iota\in I}$ converges to f in $\mathcal{F}(\Omega,E)$. □

First, we apply the preceding corollary to k-times continuously partially differentiable E-valued functions with locally γ-Hölder continuous partial derivatives of order k.

Corollary 6.3 *Let E be a Banach space, $\Omega\subset\mathbb{R}^d$ open and bounded, $k\in\mathbb{N}_0$ and $0<\gamma\leq 1$. If $(f_\iota)_{\iota\in I}$ is a bounded net in $\mathcal{C}^{k,\gamma}(\overline{\Omega},E)$ such that*

(i) $\lim_\iota f_\iota(x)$ *exists for all* x *in a dense subset* $U \subset \Omega$, *or if*
(ii) $\lim_\iota (\partial^{e_n})^E f_\iota(x)$ *exists for all* $1 \leq n \leq d$ *and* x *in a dense subset* $U \subset \Omega$, Ω *is connected and there is* $x_0 \in \overline{\Omega}$ *such that* $\lim_\iota f_\iota(x_0)$ *exists and* $k \geq 1$,

then there is $f \in \mathcal{C}^{k,\gamma}(\overline{\Omega}, E)$ such that $(f_\iota)_{\iota \in I}$ converges to f in $\mathcal{C}^k(\overline{\Omega}, E)$.

Proof Like in Corollary 5.3 we choose $\mathcal{F}(\Omega) := \mathcal{C}^k(\overline{\Omega})$ and $\mathcal{F}(\Omega, E) := \mathcal{C}^k(\overline{\Omega}, E)$ as well as $\mathcal{F}\nu(\Omega) := \mathcal{C}^{k,\gamma}(\overline{\Omega})$ and $\mathcal{F}\nu(\Omega, E) := \mathcal{C}^{k,\gamma}(\overline{\Omega}, E)$ with the weight ν and generator $(T^E, T^{\mathbb{K}})$ for $(\mathcal{F}\nu, E)$ described above of Corollary 5.3. By the proof of Corollary 5.3 all conditions of Corollary 6.2 are satisfied, which implies our statement. □

Corollary 6.4 *Let* E *be a Banach space,* $\Omega \subset \mathbb{R}^d$ *open,* $k \in \mathbb{N}_0$ *and* $0 < \gamma \leq 1$. *If* $(f_\iota)_{\iota \in I}$ *is a bounded net in* $\mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$ *such that*

(i) $\lim_\iota f_\iota(x)$ *exists for all* x *in a dense subset* $U \subset \Omega$, *or if*
(ii) $\lim_\iota (\partial^{e_n})^E f_\iota(x)$ *exists for all* $1 \leq n \leq d$ *and* x *in a dense subset* $U \subset \Omega$, Ω *is connected and there is* $x_0 \in \Omega$ *such that* $\lim_\iota f_\iota(x_0)$ *exists and* $k \geq 1$,

then there is $f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$ such that $(f_\iota)_{\iota \in I}$ converges to f in $\mathcal{C}^k(\Omega, E)$.

Proof Let $(\Omega_n)_{n \in \mathbb{N}}$ be an exhaustion of Ω with open, relatively compact sets $\Omega_n \subset \Omega$ such that $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$ and, in addition, $x_0 \in \Omega_1$ and Ω_n is connected for each $n \in \mathbb{N}$ in case (ii). The restriction of $(f_\iota)_{\iota \in I}$ to Ω_n is a bounded net in $\mathcal{C}^{k,\gamma}(\overline{\Omega}_n, E)$ for each $n \in \mathbb{N}$. By Corollary 6.3 there is $F_n \in \mathcal{C}^{k,\gamma}(\overline{\Omega}_n, E)$ for each $n \in \mathbb{N}$ such that the restriction of $(f_\iota)_{\iota \in I}$ to Ω_n converges to F_n in $\mathcal{C}^k(\overline{\Omega}_n, E)$ since $U \cap \Omega_n$ is dense in Ω_n due to Ω_n being open and x_0 being an element of the connected set Ω_n in case (ii). The limits F_{n+1} and F_n coincide on Ω_n for each $n \in \mathbb{N}$. Thus the definition $f := F_n$ on Ω_n for each $n \in \mathbb{N}$ gives a well-defined function $f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$, which is a limit of $(f_\iota)_{\iota \in I}$ in $\mathcal{C}^k(\Omega, E)$. □

Corollary 6.5 *Let* E *be a Banach space,* $\Omega \subset \mathbb{R}^d$ *open and* $k \in \mathbb{N}_0$. *If* $(f_\iota)_{\iota \in I}$ *is a bounded net in* $\mathcal{C}^{k+1}(\Omega, E)$ *such that*

(i) $\lim_\iota f_\iota(x)$ *exists for all* x *in a dense subset* $U \subset \Omega$, *or if*
(ii) $\lim_\iota (\partial^{e_n})^E f_\iota(x)$ *exists for all* $1 \leq n \leq d$ *and* x *in a dense subset* $U \subset \Omega$, Ω *is connected and there is* $x_0 \in \Omega$ *such that* $\lim_\iota f_\iota(x_0)$ *exists,*

then there is $f \in \mathcal{C}^{k,1}_{loc}(\Omega, E)$ such that $(f_\iota)_{\iota \in I}$ converges to f in $\mathcal{C}^k(\Omega, E)$.

Proof By Corollary 5.4 (b) $(f_\iota)_{\iota \in I}$ is a bounded net in $\mathcal{C}^{k,1}_{loc}(\Omega, E)$. Hence our statement is a consequence of Corollary 6.4. □

The preceding result directly implies a $\mathcal{C}^\infty$-smooth version.

Corollary 6.6 *Let* E *be a Banach space and* $\Omega \subset \mathbb{R}^d$ *open. If* $(f_\iota)_{\iota \in I}$ *is a bounded net in* $\mathcal{C}^\infty(\Omega, E)$ *such that*

(i) $\lim_\iota f_\iota(x)$ *exists for all* x *in a dense subset* $U \subset \Omega$, *or if*
(ii) $\lim_\iota (\partial^{e_n})^E f_\iota(x)$ *exists for all* $1 \leq n \leq d$ *and* x *in a dense subset* $U \subset \Omega$, Ω *is connected and there is* $x_0 \in \Omega$ *such that* $\lim_\iota f_\iota(x_0)$ *exists*,

then there is $f \in \mathcal{C}^\infty(\Omega, E)$ such that $(f_\iota)_{\iota\in I}$ converges to f in $\mathcal{C}^\infty(\Omega, E)$.

For an open set $\Omega \subset \mathbb{R}^d$, an lcHs E and a linear partial differential operator $P(\partial)^E : \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\Omega, E)$ which is hypoelliptic if $E = \mathbb{K}$ we define the space of bounded zero solutions

$$\mathcal{C}^\infty_{P(\partial),b}(\Omega, E) := \{f \in \mathcal{C}^\infty_{P(\partial)}(\Omega, E) \mid \forall\, \alpha \in \mathfrak{A} : \ \|f\|_{\infty,\alpha} := \sup_{x\in\Omega} p_\alpha(f(x)) < \infty\}$$

where

$$\mathcal{C}^\infty_{P(\partial)}(\Omega, E) := \{f \in \mathcal{C}^\infty(\Omega, E) \mid f \in \ker P(\partial)^E\}.$$

Apart from the topology given by $(\|\cdot\|_{\infty,\alpha})_{\alpha\in\mathfrak{A}}$ there is another weighted locally convex topology on $\mathcal{C}^\infty_{P(\partial),b}(\Omega, E)$ which is of interest, namely, the one induced by the seminorms

$$|f|_{\nu,\alpha} := \sup_{x\in\Omega} p_\alpha(f(x))|\nu(x)|, \quad f \in \mathcal{C}^\infty_{P(\partial),b}(\Omega, E),$$

for $\nu \in \mathcal{C}_0(\Omega)$ and $\alpha \in \mathfrak{A}$. We write $(\mathcal{C}^\infty_{P(\partial),b}(\Omega, E), \beta)$ for $\mathcal{C}^\infty_{P(\partial),b}(\Omega, E)$ equipped with the *strict topology* β which is induced by the seminorms $(|\cdot|_{\nu,\alpha})_{\nu\in\mathcal{C}_0(\Omega),\alpha\in\mathfrak{A}}$. Now, we phrase an improved version of [19, Corollary 4.2, p. 695] .

Corollary 6.7 *Let* E *be a Banach space*, $\Omega \subset \mathbb{R}^d$ *open*, $P(\partial)^{\mathbb{K}}$ *a hypoelliptic linear partial differential operator and* $U \subset \Omega$ *a set of uniqueness for* $(id_{\mathbb{K}^\Omega}, \mathcal{C}^\infty_{P(\partial),b})$. *If* $(f_\iota)_{\iota\in I}$ *is a bounded net in* $(\mathcal{C}^\infty_{P(\partial),b}(\Omega, E), \|\cdot\|_\infty)$ *such that* $\lim_\iota f_\iota(x)$ *exists for all* $x \in U$, *then there is* $f \in \mathcal{C}^\infty_{P(\partial),b}(\Omega, E)$ *such that* $(f_\iota)_{\iota\in I}$ *converges to* f *in* $(\mathcal{C}^\infty_{P(\partial),b}(\Omega, E), \beta)$.

Proof We take $\mathcal{F}(\Omega) := (\mathcal{C}^\infty_{P(\partial),b}(\Omega), \beta)$ and $\mathcal{F}(\Omega, E) := (\mathcal{C}^\infty_{P(\partial),b}(\Omega, E), \beta)$ as well as $\mathcal{F}\nu(\Omega) := (\mathcal{C}^\infty_{P(\partial),b}(\Omega), \|\cdot\|_\infty)$ and $\mathcal{F}\nu(\Omega, E) := (\mathcal{C}^\infty_{P(\partial),b}(\Omega, E), \|\cdot\|_\infty)$ with the weight $\nu(x) := 1$, $x \in \Omega$, and generator $(id_{E^\Omega}, id_{\Omega^{\mathbb{K}}})$ for $(\mathcal{F}\nu, E)$. The generator is strong and consistent for $(\mathcal{F}, E)$ and $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ are ε-compatible by [38, 3.14 Proposition, p. 12]. The space $\mathcal{F}\nu(\Omega)$ is a Banach space as a closed subspace of the Banach space $(\mathcal{C}_b(\Omega), \|\cdot\|_\infty)$. Its closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is τ_{co}-compact because $(\mathcal{C}^\infty_{P(\partial)}(\Omega), \tau_{co})$ is a Fréchet–Schwartz space, in particular, a Montel space. Thus $B_{\mathcal{F}\nu(\Omega)}$ is $\|\cdot\|_\infty$ -bounded and τ_{co}-compact, which implies that it is also β-compact by [13, Proposition 1 (viii), p. 586] and [13, Proposition 3, p.590]. In addition, the ε-compatibility of $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ in combination with the consistency of $(id_{E^\Omega}, id_{\mathbb{K}^\Omega})$ for $(\mathcal{F}, E)$ gives $\mathcal{F}_\varepsilon\nu(\Omega, E) = \mathcal{F}\nu(\Omega, E)$ as linear spaces by Proposition 2.4 (c), verifying our statement by Corollary 6.2. □

A direct consequence of Corollary 6.7 is the following remark.

Remark 6.8 Let E be a Banach space, $\Omega \subset \mathbb{R}^d$ open, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator and $(f_\iota)_{\iota\in I}$ a bounded net in $(\mathcal{C}^\infty_{P(\partial),b}(\Omega, E), \|\cdot\|_\infty)$. Then the following statements are equivalent:

(i) (f_ι) converges pointwise.
(ii) (f_ι) converges uniformly on compact subsets of Ω.
(iii) (f_ι) is β-convergent.

In the case of complex-valued bounded holomorphic functions of one variable, i.e. $E = \mathbb{C}$, $\Omega \subset \mathbb{C}$ is open and $P(\partial) = \overline{\partial}$ is the Cauchy–Riemann operator, convergence w.r.t. β is known as bounded convergence (see [42, p. 13-14, 16]) and the preceding remark is included in [43, 3.7 Theorem, p. 246] for connected sets Ω. Let us return to Bloch type spaces.

Corollary 6.9 *Let E be a Banach space, $v : \mathbb{D} \to (0, \infty)$ continuous and $U_* \subset \mathbb{D}$ have an accumulation point in $\mathbb{D}$. If $(f_\iota)_{\iota\in I}$ is a bounded net in $\mathcal{B}v(\mathbb{D}, E)$ such that $\lim_\iota f_\iota(0)$ and $\lim_\iota (\partial^1_{\mathbb{C}})^E f_\iota(z)$ exist for all $z \in U_*$, then there is $f \in \mathcal{B}v(\mathbb{D}, E)$ such that $(f_\iota)_{\iota\in I}$ converges to f in $(\mathcal{O}(\mathbb{D}, E), \tau_{co})$.*

Proof Due to the proof of Corollary 3.8 all conditions needed to apply Corollary 6.2 are fulfilled, which proves our statement. □

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References

1. Alt, H. W.: Lineare Funktionalanalysis. Mem. Am. Math. Soc. 16, 6th edition, Springer, Berlin (2012). https://doi.org/10.1007/978-3-642-22261-0
2. Arendt, W.: Vector-valued holomorphic and harmonic functions. Concr. Oper. **3**(1), 68–76 (2016). https://doi.org/10.1515/conop-2016-0007
3. Arendt, W., Nikolski, N.: Vector-valued holomorphic functions revisited. Math. Z. **234**(4), 777–805 (2000). https://doi.org/10.1007/s002090050008
4. Arendt, W., Nikolski, N.: Addendum: Vector-valued holomorphic functions revisited. Math. Z. **252**(3), 687–689 (2006). https://doi.org/10.1007/s00209-005-0858-x
5. Arendt, W., Batty, C. J. K., Hieber, M., Neubrander, F.: Vector-Valued Laplace Transforms and Cauchy Problems. Monogr. Math. 96, 2nd edition, Birkhäuser, Basel (2011). https://doi.org/10.1007/978-3-0348-0087-7

6. Bierstedt, K.-D.: Gewichtete Räume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt. II. J. Reine Angew. Math. **260**, 133–146 (1973). https://doi.org/10.1515/crll.1973.260.133
7. Bonet, J., Frerick, L., Jordá, E.: Extension of vector-valued holomorphic and harmonic functions. Stud. Math. **183**(3), 225–248 (2007). https://doi.org/10.4064/sm183-3-2
8. Bonet, J., Fernández, C., Galbis, A., Ribera, J.M.: Frames and representing systems in Fréchet spaces and their duals. Banach J. Math. Anal. **11**(1), 1–20 (2017). https://doi.org/10.1215/17358787-3721183
9. Burckel, R. B.: An Introduction to Classical Complex Analysis. Pure Appl. Math. (Amst.) 82, Academic Press, New York (1979). https://doi.org/10.1016/S0079-8169(08)61287-8
10. Carroll, R.W.: Some singular Cauchy problems. Ann. Mat. Pura Appl. (4) **1**, 1–31 (1961). https://doi.org/10.1007/BF02414262
11. Chen, H.: Boundedness from below of composition operators on the Bloch spaces. Sci. China Ser. A **46**(6), 838–846 (2003). https://doi.org/10.1360/02ys0212
12. Chen, H., Gauthier, P.: Boundedness from below of composition operators on α-Bloch spaces. Canad. Math. Bull. **51**(2), 195–204 (2008). https://doi.org/10.4153/CMB-2008-021-2
13. Cooper, J.B.: The strict topology and spaces with mixed topologies. Proc. Am. Math. Soc. **30**(3), 583–592 (1971). https://doi.org/10.2307/2037739
14. Diestel, J., Uhl, J. J.: Vector Measures. Math. Surveys 15, AMS, Providence, RI (1977). https://doi.org/10.1090/surv/015
15. Driver, B. K.: Analysis Tools with Examples. https://doi.org/http://www.math.ucsd.edu/~bdriver/DRIVER/Book/anal.pdf (2004). Accessed 5 Nov 2021
16. Federer, H.: Geometric Measure Theory. Grundlehren Math. Wiss. 153, Springer, Berlin (1969)
17. Floret, K., Wloka, J.: Einführung in die Theorie der lokalkonvexen Räume. Lecture Notes in Math, vol. 56. Springer, Berlin (1968). https://doi.org/10.1007/BFb0098549
18. Frerick, L., Jordá, E.: Extension of vector-valued functions. Bull. Belg. Math. Soc. Simon Stevin **14**(3), 499–507 (2007). https://doi.org/10.36045/bbms/1190994211
19. Frerick, L., Jordá, E., Wengenroth, J.: Extension of bounded vector-valued functions. Math. Nachr. **282**(5), 690–696 (2009). https://doi.org/10.1002/mana.200610764
20. Ghatage, P., Yan, J., Zheng, D.: Composition operators with closed range on the Bloch space. Proc. Am. Math. Soc. **129**(7), 2039–2044 (2001). https://doi.org/10.2307/2669002
21. Ghatage, P., Zheng, D., Zorboska, N.: Sampling sets and closed range composition operators on the Bloch space. Proc. Am. Math. Soc. **133**(5), 1371–1377 (2005). https://doi.org/10.2307/4097789
22. Giménez, J., Malavé, R., Ramos-Fernández, J.C.: Composition operators on μ-Bloch type spaces. Rend. Circ. Mat. Palermo (2) **59**(1), 107–119 (2010). https://doi.org/10.1007/s12215-010-0007-1
23. Gramsch, B.: Ein Schwach-Stark-Prinzip der Dualitätstheorie lokalkonvexer Räume als Fortsetzungsmethode. Math. Z. **156**(3), 217–230 (1977). https://doi.org/10.1007/BF01214410
24. Grosse-Erdmann, K.-G.: Lebesgue's theorem of differentiation in Fréchet lattices. Proc. Am. Math. Soc. **112**(2), 371–379 (1991). https://doi.org/10.2307/2048729
25. Grosse-Erdmann, K.-G.: The Borel-Okada theorem revisited. Fernuniversität Hagen, Hagen, Habilitation (1992)
26. Grosse-Erdmann, K.-G.: A weak criterion for vector-valued holomorphy. Math. Proc. Camb. Phil. Soc. **136**(2), 399–411 (2004)
27. Grothendieck, A.: Sur certains espaces de fonctions holomorphes. I. J. Reine Angew. Math. **192**, 35–64 (1953). https://doi.org/10.1515/crll.1953.192.35
28. Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires. Mem. Am. Math. Soc. 16, 4th edition, AMS, Providence (1966). https://doi.org/10.1090/memo/0016
29. Grothendieck, A.: Topological vector spaces. Notes on mathematics and its applications, Gordon and Breach, New York (1973)
30. Jarchow, H.: Locally Convex Spaces. Math. Leitfäden, Teubner, Stuttgart (1981). https://doi.org/10.1007/978-3-322-90559-8
31. Jordá, E.: Extension of vector-valued holomorphic and meromorphic functions. Bull. Belg. Math. Soc. Simon Stevin **12**(1), 5–21 (2005). https://doi.org/10.36045/bbms/1113318125
32. Jordá, E.: Weighted vector-valued holomorphic functions on Banach spaces. Abstr. Appl. Anal. (2013). https://doi.org/10.1155/2013/501592
33. Kaballo, W.: Aufbaukurs Funktionalanalysis und Operatortheorie. Springer, Berlin (2014). https://doi.org/10.1007/978-3-642-37794-5
34. Kriegl, A., Michor, P. W.: The Convenient Setting of Global Analysis. Math. Surveys Monogr. 53, AMS, Providence (1997). https://doi.org/10.1090/surv/053

35. Kruse, K.: Weighted spaces of vector-valued functions and the ε-product. Banach J. Math. Anal. **14**(4), 1509–1531 (2020). https://doi.org/10.1007/s43037-020-00072-z
36. Kruse, K.: Vector-valued holomorphic functions in several variables. Funct. Approx. Comment. Math. **63**(2), 247–275 (2020). https://doi.org/10.7169/facm/1861
37. Kruse, K.: Series representations in spaces of vector-valued functions via Schauder decompositions. Math. Nachr. **294**(2), 354–376 (2021). https://doi.org/10.1002/mana.201900172
38. Kruse, K.: Extension of vector-valued functions and sequence space representation. Preprint. https://arxiv.org/abs/1808.05182v5 (2021)
39. Mankiewicz, P.: On the differentiability of Lipschitz mappings in Fréchet spaces. Stud. Math. **45**(1), 15–29 (1973). https://doi.org/10.4064/sm-45-1-15-29
40. Meise, R., Vogt, D.: Introduction to Functional Analysis. Oxf. Grad. Texts Math. 2, Clarendon Press, Oxford (1997)
41. Ramos-Fernández, J.C.: Composition operators between μ-Bloch spaces. Extr. Math. **26**(1), 75–88 (2011)
42. Rubel, L.A.: Bounded convergence of analytic functions. Bull. Am. Math. Soc. **77**(1), 13–24 (1971)
43. Rubel, L.A., Shields, A.L.: The space of bounded analytic functions on a region. Ann. Inst. Fourier (Grenoble) **16**(1), 235–277 (1966). https://doi.org/10.5802/aif.231
44. Schwartz, L.: Espaces de fonctions différentiables à valeurs vectorielles. J. Anal. Math. **4**, 88–148 (1954). https://doi.org/10.1007/BF02787718
45. Waelbroeck, L.: Some theorems about bounded structures. J. Funct. Anal. **1**(4), 392–408 (1967). https://doi.org/10.1016/0022-1236(67)90009-2
46. Waelbroeck, L.: Differentiable mappings into b-spaces. J. Funct. Anal. **1**(4), 409–418 (1967). https://doi.org/10.1016/0022-1236(67)90010-9
47. Yoneda, R.: Composition operators on the weighted Bloch space and the weighted Dirichlet spaces, and BMOA with closed range. Complex Var. Elliptic Equ. **63**(5), 730–747 (2018). https://doi.org/10.1080/17476933.2017.1345887