A Projection Method for a Rational Eigenvalue Problem in Fluid-Structure Interaction

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Abstract. In this paper we consider a rational eigenvalue problem governing the vibrations of a tube bundle immersed in an inviscid compressible fluid. Taking advantage of eigensolutions of appropriate sparse linear eigenproblems the large nonlinear eigenvalue problem is projected to a much smaller one which is solved by inverse iteration.

1 Introduction

Vibrations of a tube bundle immersed in an inviscid compressible fluid are governed under some simplifying assumptions by an elliptic eigenvalue problem with non-local boundary conditions which can be transformed to a rational eigenvalue problem. In [13] we proved that this problem has a countable set of eigenvalues which can be characterized as minmax values of a Rayleigh functional.

To determine eigenvalues and eigenfunctions numerically this approach suggests to apply the Rayleigh-Ritz method yielding a rational matrix eigenvalue problem where the matrices typically are large and sparse. Since for nonlinear eigenvalue problems the eigenvectors do not fulfill an orthogonality condition Lanczos type methods do not apply but each eigenpair has to be determined individually by an iterative process where each iteration step requires the solution of a linear system of large dimension. In this paper we propose a projection method where the ansatz vectors are constructed from the solutions of suitable linear eigenvalue problems. The resulting small dimensional nonlinear eigenproblem then is solved by inverse iteration.

The paper is organized as follows: Section 2 describes the the model for the fluid-structure interaction problem under consideration, and Section 3 summarizes the minmax characterization for nonlinear eigenvalue problems and its application to the fluid-structure interaction problem of section 2. In Section 4 we consider numerical methods for nonlinear eigenvalue problems, and Section 5 contains a numerical example.

2 A Spectral Problem in Fluid-Solid Structures

In this paper we consider a model which governs the free vibrations of a tube bundle immersed in a compressible fluid under the following simplifying assumptions: The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity. Considering small vibrations of the fluid (and the tubes) around the state of rest, it can also be assumed that the fluid is irrotational.

Mathematically this problem can be described in the following way (cf. [2]). Let $\Omega \subset \mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ. We assume that there exists a family $\Omega_j \neq \emptyset$, $j = 1, \dots, k$, (the sections of the tubes) of simply connected open sets such that $\bar{\Omega}_j \subset \Omega$ for every j, $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $j \neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With these notations we set

$$\Omega_0 := \Omega \setminus \bigcup_{j=1}^{k} \Omega_j.$$

Then the boundary of Ω_0 consists of $k+1$ connected components which are Γ and Γ_j, $j = 1, \dots, k$.

We consider the rational eigenvalue problem

Find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega_0)$ such that for every $v \in H^1(\Omega_0)$

$$c^2 \int_{\Omega_0} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega_0} uv \, dx + \sum_{j=1}^{K} \frac{\lambda \rho_0}{k_j - \lambda m_j} \int_{\Gamma_j} un \, ds \cdot \int_{\Gamma_j} vn \, ds. \quad (1)$$

Here

$$H^1(\Omega) = \{u \in L^2(\Omega_0) \, : \, \nabla u \in L^2(\Omega_0)^2\}$$

denotes the standard Sobolev space equipped with the usual scalar product

$$(u, v) := \int_{\Omega_0} (u(x)v(x) + \nabla u(x) \cdot \nabla v(x)) \, dx,$$

and n denotes the outward unit normal on the boundary of Ω_0.

Obviously $\lambda = 0$ is an eigenvalue of (1) with eigenfunction $u = \text{const}$. We reduce the eigenproblem (1) to the space

$$H := \{u \in H^1(\Omega_0) \, : \, \int_{\Omega_0} u(x) \, dx = 0\}$$

and consider the scalar product

$$\langle u, v \rangle := \int_{\Omega_0} \nabla u(x) \cdot \nabla v(x) \, dx.$$

on H which is known to define a norm on H which is equivalent to the norm induced by $(\cdot,\cdot)$.

By the Lax–Milgram lemma the variational eigenvalue problem (1) is equivalent to the nonlinear eigenvalue problem

Determine λ and $u \in H$ such that

$$T(\lambda)u := (-I + \lambda A + \sum_{j=1}^{K} \frac{\rho_0 \lambda}{k_j - \lambda m_j} B_j)u = 0 \tag{2}$$

where the linear symmetric operators A and B_j are defined by

$$\langle Au, v\rangle := \int_{\Omega_0} uv\,dx \quad \textit{for every } u, v \in H \textit{ and}$$

$$\langle B_j u, v\rangle := \int_{\Gamma_j} un\,ds \cdot \int_{\Gamma_j} vn\,ds \quad \textit{for every } u, v \in H.$$

A is completely continuous by Rellich's embedding theorem and $w := B_j u$, $j = 1, \dots, K$, is the weak solution in H of the elliptic problem

$$\Delta w = 0 \text{ in } \Omega_0, \quad \frac{\partial}{\partial n} w = 0 \text{ on } \partial\Omega_0 \setminus \Gamma_j, \ \frac{\partial}{\partial n} w = n \cdot \int_{\Gamma_j} un\,ds \text{ on } \Gamma_j.$$

By the continuity of the trace operator B_j is continuous, and since the range of B_j is twodimensional spanned by the solutions $w_i \in H$ of

$$\Delta w_i = 0 \text{ in } \Omega_0, \quad \frac{\partial}{\partial n} w = 0 \text{ on } \partial\Omega_0 \setminus \Gamma_j, \ \frac{\partial}{\partial n} w = n_i \text{ on } \Gamma_j, \ i = 1, 2,$$

it is even completely continuous.

In Conca et al. [2] it is shown that the eigenvalues are the characteristic values of a linear compact operator acting on a Hilbert space. The operator associated with this eigenvalue problem is not selfadjoint, but it can be symmetrized in the sense that one can prove the existence of a selfadjoint operator which has the same spectrum as the original operator. The following section describes a framework to prove the existence of countably many eigenvalues taking advantage of a minmax characterization for nonlinear eigenvalue problems.

3 Characterization of eigenvalues

We consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \tag{3}$$

where $T(\lambda)$ is a selfadjoint and bounded operator on a real Hilbert space H for every λ in an open real interval J. As in the linear case $\lambda \in J$ is called an eigenvalue of problem (3) if equation (3) has a nontrivial solution $x \neq 0$.

We assume that

$$f \,:\, \begin{cases} J \times H \to \mathbb{R} \\ (\lambda, x) \;\mapsto \langle T(\lambda)x, x\rangle \end{cases}$$

is continuously differentiable, and that for every fixed $x \in H^0$, $H^0 := H \setminus \{0\}$, the real equation

$$f(\lambda, x) = 0 \tag{4}$$

has at most one solution in J. Then equation (4) implicitly defines a functional p on some subset D of H^0 which we call the Rayleigh functional.

We assume that

$$\frac{\partial}{\partial\lambda} f(\lambda, x)\big|_{\lambda=p(x)} > 0 \quad \text{for every } x \in D.$$

Then it follows from the implicit function theorem that D is an open set and that p is continuously differentiable on D.

For the linear eigenvalue value problem $T(\lambda) := \lambda I - A$ where $A \,:\, H \to H$ is selfadjoint and continuous the assumptions above are fulfilled, p is the Rayleigh quotient and $D = H^0$. If A additionally is completely continuous then A has a countable set of eigenvalues which can be characterized as minmax and maxmin values of the Rayleigh quotient.

For nonlinear eigenvalue problems variational properties using the Rayleigh functional were proved by Duffin [3] and Rogers [9] for the finite dimensional case and by Hadeler [4], [5], Rogers [10], and Werner [15] for the infinite dimensional case if the problem is overdamped, i.e. if the Rayleigh functional p is defined in the whole space H^0. Nonoverdamped problems were studied by Werner and the author [14]. In this case the natural numbering for which the smallest eigenvalue is the first one, the second smallest is the second one, etc. is not appropriate, but the number of an eigenvalue λ of the nonlinear problem (3) is obtained from the location of the eigenvalue 0 in the spectrum of the linear operator $T(\lambda)$.

We assume that for every fixed $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that the linear operator $T(\lambda) + \nu(\lambda) I$ is completely continuous. If $\lambda \in J$ is an eigenvalue of $T(\cdot)$ then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $n \in \mathbb{N}$ such that

$$0 = \max_{V \in H_n} \min_{v \in V_1} \langle T(\lambda)v, v\rangle$$

where H_n denotes the set of all n–dimensional subspaces of H and $V^1 := \{v \in V \,:\, \|v\| = 1\}$ is the unit ball in V. In this case we call λ an n-th eigenvalue of (3).

With this numbering the following minmax characterization of the eigenvalues of the nonlinear eigenproblem (3) was proved in [14]:

Theorem 1. *Under the conditions given above the following assertions hold:*

(i) For every $n \in \mathbb{N}$ there is at most one n-th eigenvalue of problem (3) which can be characterized by

$$\lambda_n = \min_{\substack{V \in H_n \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v). \tag{5}$$

The set of eigenvalues of (3) in J is at most countable.

(ii) If

$$\lambda_n = \inf_{\substack{V\in H_n\\ V\cap D\neq\emptyset}} \sup_{v\in V\cap D} p(v) \in J$$

for some $n \in \mathbb{N}$ then λ_n is the n-th eigenvalue of (3) and (5) holds.

(iii) If there exists the m-th and the n-th eigenvalue λ_m and λ_n in J and $m < n$ then J contains a k-th eigenvalue λ_k for $m < k < n$ and

$$\inf J < \lambda_m \leq \lambda_{m+1} \leq \ldots \leq \lambda_n < \sup J.$$

(iv) If $\lambda_1 \in J$ and $\lambda_n \in J$ for some $n \in \mathbb{N}$ then for every $j \in \{1,\ldots,n\}$ the space $V \in H_j$ with $V \cap D \neq \emptyset$ and $\lambda_j = \sup_{u\in V\cap D} p(u)$ is contained in D, and the characterization (5) can be replaced by

$$\lambda_j = \min_{\substack{V\in H_j\\ V_1\subset D}}, \max_{v\in V_1} p(v) \quad j = 1,\ldots,n.$$

For the nonlinear eigenproblem (2) the general conditions obviously are satisfied for every open interval $J \subset \mathbb{R}_+$ which does not contain k_j/m_j for $j = 1,\ldots,K$. Namely, $T(\lambda)$ is selfadjoint and bounded on H and $T(\lambda) + I$ is completely continuous for every $\lambda \in J$. Moreover for fixed $u \in H^0$

$$f(\lambda,u) = -c^2 \int_{\Omega_0} |\nabla u|^2\,dx + \lambda \int_{\Omega_0} u^2\,dx + \sum_{j=1}^{K} \frac{\lambda\rho_0}{k_j - \lambda m_j} \Big| \int_{\Gamma_j} un\,ds \Big|^2 \tag{6}$$

is monotonely increasing with respect to λ. Hence, every open interval J such that $k_j/m_j \notin J$ for $j = 1,\ldots,K$ contains at most countably many eigenvalues which can be characterized as minmax value of the Rayleigh functional p defined by $f(\lambda,u) = 0$ where f is defined in (6).

Comparing p with the Rayleigh quotient of the linear eigenvalue problem

Determine $\lambda \in \mathbb{R}$ and $u \in H^0$ such that for every $v \in H$

$$c^2 \int_{\Omega_0} \nabla u \cdot \nabla v\,dx = \lambda\Big(\int_{\Omega_0} uv\,dx + \sum_{j=1}^{K} \frac{\rho_0}{k_j - \kappa m_j} \int_{\Gamma_j} un\,ds \cdot \int_{\Gamma_j} vn\,ds \Big). \tag{7}$$

we obtained the following inclusion result for the eigenvalues in $J_1 := (0, \min_j \frac{k_j}{m_j})$.

Theorem 2. *Let $\kappa \in J_1$. If the linear eigenvalue problem (7) has m eigenvalues $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_m$ in J_1 then the nonlinear eigenvalue problem (1) has m eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ in J_1, and the following inclusion holds*

$$\min(\mu_j,\kappa) \leq \lambda_j \leq \max(\mu_j,\kappa),\ j = 1,\ldots,m.$$

λ_j is a j-th eigenvalue of (1).

Similarly, for eigenvalues greater than $\max_j \frac{k_j}{m_j}$ we obtained the inclusion in Theorem 3. Here we compared the Rayleigh functional of (1) in the interval $(\max_j \frac{k_j}{m_j}, \infty)$ with the Rayleigh quotient of the linear eigenproblem:

Find $\lambda \in \mathbb{R}$ and $u \in H^0$ such that for every $v \in H$

$$c^2 \int\limits_{\Omega_0} \nabla u \cdot \nabla v \, dx + \sum_{j=1}^{K} \frac{\kappa \rho_0}{\kappa m_j - k_j} \int\limits_{\Gamma_j} un \, ds \cdot \int\limits_{\Gamma_j} vn \, ds = \lambda \int\limits_{\Omega_0} uv \, dx. \quad (8)$$

Theorem 3. *Let $\kappa > \max_j \frac{k_j}{m_j}$. If the m-smallest eigenvalue μ_m of the linear eigenvalue problem (8) satisfies $\mu_m > \max_j \frac{k_j}{m_j}$ then the nonlinear eigenvalue problem (1) has an m-th eigenvalue λ_m, and*

$$\min(\mu_m, \kappa) \le \lambda_m \le \max(\mu_m, \kappa).$$

Theorems 2 and 3 are proved in [13]. Inclusions of eigenvalues λ of problem (1) which lie between $\min_j \frac{k_j}{m_j}$ and $\max_j \frac{k_j}{m_j}$ are under investigation.

4 Algorithms for nonlinear eigenvalue problems

In this section we consider the finite dimensional nonlinear eigenvalue problem

$$T(\lambda)x = 0 \quad (9)$$

(for instance a finite element approximation of problem (1)) where $T(\lambda)$ is a family of real symmetric $n \times n$-matrices satisfying the conditions of the last section, and we assume that the dimension n of problem (9) is large.

For sparse linear eigenvalue problems the most efficient methods are iterative projection methods, where approximations of the wanted eigenvalues and corresponding eigenvectors are obtained from projections of the eigenproblem to subspaces which are expanded in the course of the algorithm. Methods of this type for symmetric problems are the Lanczos method, rational Krylov subspace methods and the Jacobi-Davidson method, e.g. (cf. [1]).

Generalizations of these methods to nonlinear eigenvalue problems do not exist. The numerical methods for nonlinear problems studied in [6], [7], [8], [11], [12] are all variants of inverse iteration

$$x^{k+1} = \alpha_k T(\lambda_k)^{-1} T'(\lambda_k) x^k \quad (10)$$

where α_k is a suitable normalization factor and λ_k is updated in some way. Similarly as in the linear case inverse iteration is quadratically convergent for simple eigenvalues, and the convergence is even cubic if $\lambda_k = p(x_k)$ where p denotes the Rayleigh functional of problem (9).

An essential disadvantage of inverse iteration is the fact that each eigenvalue has to be determined individually by an iterative process, and that each step of this iteration requires the solution of a linear system. Moreover, the coefficient matrix $T(\lambda_k)$ of system (10) changes in each step, and in contrast to the linear case replacing (10) by

$$x^{k+1} = \alpha_k T(\sigma)^{-1} T'(\lambda_k) x^k$$

with a fixed shift σ results in convergence to an eigenpair of the linear system $T(\sigma)x = \gamma T'(\tilde{\lambda})x$ ($\gamma \neq 0$ depending on the normalization condition) from which we can not recover an eigenpair of the nonlinear problem (9).

A remedy against this wrong convergence was proposed by Neumaier [8] who introduced the so called residual inverse iteration which converges linearly with a fixed shift, and quadratically or cubically if the coefficient matrix changes in every iteration step according to reasonable updates of λ_k.

For large and sparse nonlinear eigenvalue problems inverse iteration is much too expensive. For the rational eigenvalue problem (1) the proof of the Inclusion Theorems 2 and 3 demonstrates that eigenvectors of the linear systems (7) and (8), respectively, are good approximations to eigenvectors of the nonlinear problem, at least if the shift κ is close to the corresponding eigenvalue. This suggests the following projection method if we are interested in eigenvalues of the nonlinear problem (1) in an interval $J \subset (0, \min_j \frac{k_j}{m_j})$ or $J \subset (\max_j \frac{k_j}{m_j}, \infty)$.

Projection method

1. Choose a small number of shifts $\kappa_1, \ldots, \kappa_r \in J$.
2. For $j = 1, \ldots, r$ determine the eigenvectors u_{jk}, $k = 1, \ldots, s_j$, of the linear problem (7) with shift κ_j corresponding to eigenvalues in J.
3. Let U be the matrix with columns u_{jk}, $j = 1, \ldots, r$, $k = 1, \ldots, s_j$. Determine the QR factorization with column pivoting which produces the QR factorization of UE where E denotes a permutation matrix such that the absolute values of the diagonal elements of R are monotonely decreasing.
4. For every j with $|r_{jj}| < \tau \cdot |r_{11}|$ drop the j-th column of Q where $\tau \in [0, 1)$ is a given tolerance, and denote by V the space that is spanned by the remaining columns of Q.
5. Project the nonlinear eigenvalue problem (1) to V and solve the projected problem by inverse iteration with variable shifts.

5 A numerical example

Consider the rational eigenvalue problem (1) where Ω is the L-shaped region $\Omega := (-8, 8) \times (-8, 8) \setminus ([0, 8) \times (-8, 0])$, Ω_j, $j = 1, 2, 3$ are circles with radius 1 and centers $(-4, -4)$, $(-4, 4)$ and $(4, 4)$, and $k_j = m_j = 1$, $j = 1, 2, 3$.

We discretized this eigenvalue problem with linear elements obtaining a matrix eigenvalue problem of dimension $n = 10820$ which has 25 eigenvalues $\lambda_1 \leq \ldots \leq \lambda_{25}$ in the interval $J_1 = (0, 1)$ and 16 eigenvalues $\tilde{\lambda}_{20} \leq \ldots \leq \tilde{\lambda}_{35}$ in $J_2 := (1, 2)$.

With 2 shift parameters $\kappa_1 := 0$ and $\kappa_2 = 0.999$ and the drop tolerances $\tau = 1e-1,\ 1e-2,\ 1e-3$ we obtained eigenvalue approximations the relative errors of which are displayed in Figure 1. The dimensions of the projected eigenvalue problems were 25, 32 and 37 respectively.

With Gaussian knots $\kappa_1 = \frac{1}{2}(1 - \frac{1}{\sqrt{3}})$ and $\kappa_2 = \frac{1}{2}(1 + \frac{1}{\sqrt{3}})$ we obtained smaller relative errors, however in this case the projected problem found only 24 approximate eigenvalues missing $\lambda_{25} = 0.9945$.

Figure 2 shows the relative errors which were obtained with shift parameters $\kappa_1 = 1.001$ and $\kappa_2 = 2$ in problem (8) and dimensions 16, 24 and 29, respectively, of the nonlinear projected problem.

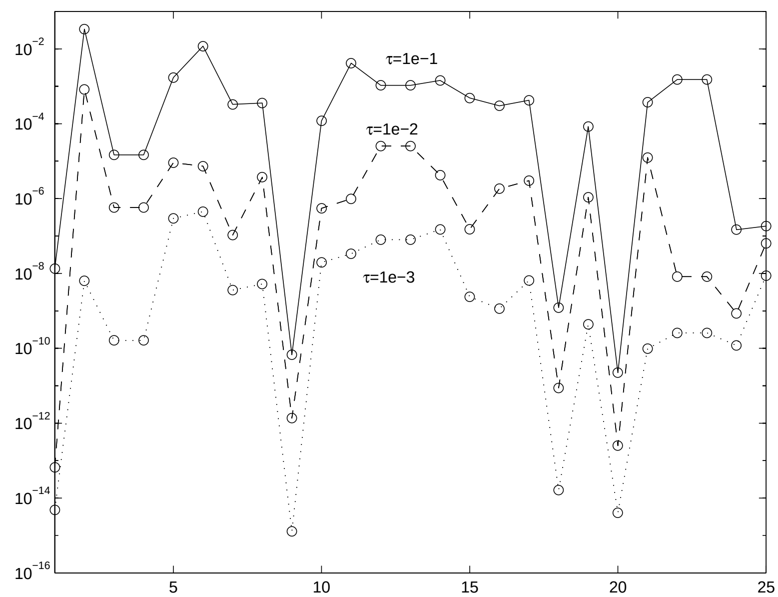

Fig. 1: relative errors; eigenvalue in (0,1)

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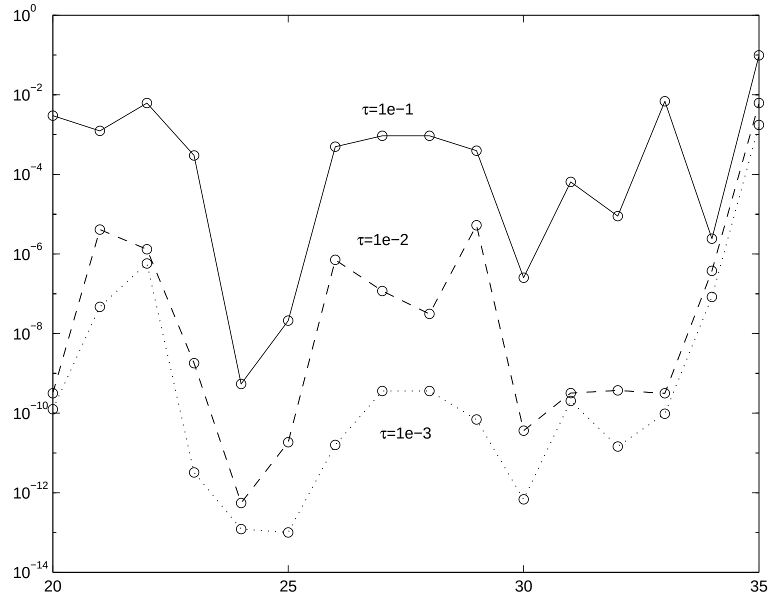

Fig. 2: relative errors; eigenvalues in (1,2)

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