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Research Article

An Extended Analytical Approach to Evaluating Monotonic Functions of Fuzzy Numbers

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This paper presents an extended analytical approach to evaluating continuous, monotonic functions of independent fuzzy numbers. The approach is based on a parametric α-cut representation of fuzzy numbers and allows for the inclusion of parameter uncertainties into mathematical models.

1. Introduction

There is an increasing effort in the scientific community to provide suitable methods for the inclusion of uncertainties into mathematical models. One way to do so is to introduce parametric uncertainty by representing the uncertain model parameters as fuzzy numbers [1] and evaluating the model equations by means of Zadeh's extension principle [2]. The evaluation of this *classical formulation* of the extension principle, however, turns out to be a highly complex task [3]. Fortunately, Buckley and Qu [4] provide an *alternative formulation* that operates on α-cuts and is applicable to continuous functions of independent fuzzy numbers. Powerful numerical techniques have been developed to implement this alternative formulation [5]. These techniques are particularly suitable for very complex simulation models [6]. In engineering design [7], however, the mathematical equations are usually less complex, and hence analytical methods might be more suitable for the inclusion of parameter uncertainties into the computations. For this purpose, a practical analytical approach to evaluating continuous, monotonic functions of independent fuzzy numbers was introduced by the authors [8], which is based on the alternative formulation of the extension principle. In this paper, we extend this approach in terms of computational efficiency depending on certain monotonicity conditions.

An outline of this paper is as follows. In Section 2, we give a definition of fuzzy numbers and present two important types. In Section 3, we introduce the notion of a linguistic variable. In Section 4, we briefly recall Zadeh's extension principle and introduce the alternative formulation based on α-cuts. In Section 5, we describe our extended analytical approach and give four illustrative examples. In Section 6, a practical engineering application is presented. Finally, in Section 7, some conclusions are drawn.

2. Fuzzy Numbers

Fuzzy numbers are a special class of fuzzy sets [9], which can be defined as follows [1].

A normal, convex fuzzy set $\tilde{x}$ over the real line $\mathbb{R}$ is called *fuzzy number* if there is exactly one $\overline{x} \in \mathbb{R}$ with $\mu_{\tilde{x}}(\overline{x}) = 1$ and the membership function is at least piecewise continuous. The value $\overline{x}$ is called the *modal* or *peak value* of $\tilde{x}$.

It is important to note that some authors consider normal, convex fuzzy sets with a core interval also as fuzzy numbers [10]. In [3, 6], these types of fuzzy numbers are denoted as fuzzy intervals. Furthermore, some authors define a fuzzy number having a compact support [11]. Although all concepts presented in this paper can be extended to these definitions of fuzzy numbers, we stick to the definition from [1].

Theoretically, an infinite number of possible types of fuzzy numbers can be defined. However, only few of them are important for engineering applications [6]. These typical fuzzy numbers shall be described in the following.

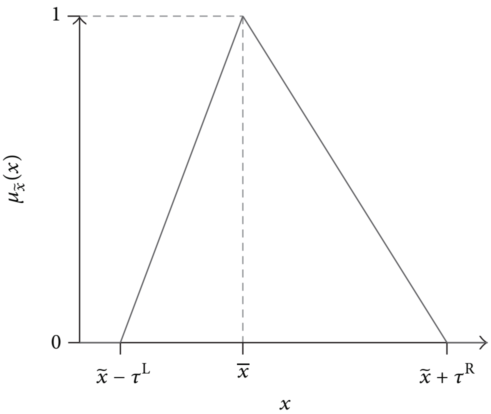

FIGURE 1: Triangular fuzzy number.

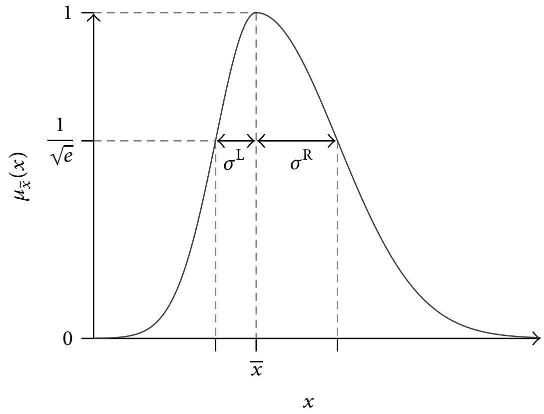

FIGURE 2: Gaussian fuzzy number.

2.1. Triangular Fuzzy Numbers. Due to its very simple, linear membership function, the *triangular fuzzy number* (TFN) is the most frequently used fuzzy number in engineering. In order to define a TFN with the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} 1 + \dfrac{x-\overline{x}}{\tau^{L}}, & \overline{x}-\tau^{L} \le x \le \overline{x}, \\ 1 - \dfrac{x-\overline{x}}{\tau^{R}}, & \overline{x} < x \le \overline{x}+\tau^{R}, \end{cases} \tag{1}$$

we use the parametric notation [6]

$$\tilde{x} = \operatorname{tfn}\left(\overline{x}, \tau^{L}, \tau^{R}\right), \tag{2}$$

where $\overline{x}$ denotes the *modal value*, τ^{L} denotes the *left-hand*, and τ^{R} denotes the *right-hand spread* of $\tilde{x}$ (cf. Figure 1). If $\tau^{L} = \tau^{R}$, the TFN is called *symmetric*. Its α-cuts $x(\alpha) = [x^{L}(\alpha), x^{R}(\alpha)]$ result from the inverse functions of (1) with respect to x:

$$\begin{aligned} x^{L}(\alpha) &= \overline{x} - \tau^{L}(1-\alpha), \quad 0 < \alpha \le 1, \\ x^{R}(\alpha) &= \overline{x} + \tau^{R}(1-\alpha), \quad 0 < \alpha \le 1. \end{aligned} \tag{3}$$

2.2. Gaussian Fuzzy Numbers. Another widely used fuzzy number in engineering is the *Gaussian fuzzy number* (GFN), which is based on the normal distribution from probability theory. In order to define such a GFN with the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} \exp\left[-\dfrac{1}{2}\left(\dfrac{x-\overline{x}}{\sigma^{L}}\right)^{2}\right], & x \le \overline{x}, \\ \exp\left[-\dfrac{1}{2}\left(\dfrac{x-\overline{x}}{\sigma^{R}}\right)^{2}\right], & x > \overline{x}, \end{cases} \tag{4}$$

we use the parametric notation [6]

$$\tilde{x} = \operatorname{gfn}\left(\overline{x}, \sigma^{L}, \sigma^{R}\right), \tag{5}$$

where $\overline{x}$ denotes the *modal value*, σ^{L} denotes the *left-hand*, and σ^{R} denotes the *right-hand standard deviation* of $\tilde{x}$ (cf. Figure 2). If $\sigma^{L} = \sigma^{R}$, the GFN is called *symmetric*. Its α-cuts $x(\alpha) = [x^{L}(\alpha), x^{R}(\alpha)]$ result in

$$\begin{aligned} x^{L}(\alpha) &= \overline{x} - \sigma^{L}\sqrt{-2\ln(\alpha)}, \quad 0 < \alpha \le 1, \\ x^{R}(\alpha) &= \overline{x} + \sigma^{R}\sqrt{-2\ln(\alpha)}, \quad 0 < \alpha \le 1. \end{aligned} \tag{6}$$

3. Linguistic Variables

In decision analysis, linguistic variables are of particular importance [12].

A *linguistic variable* V_{L} is a collection of subsets containing the following elements:

(i) G : set of syntactic rules (e.g., in terms of a grammar) for the linguistic quantification of V_{L};

(ii) T: set of terms t_i, $i \in \mathbb{N}$, resulting from G;

(iii) S: set of semantic rules that assign every term t_i to its (physical) meaning in terms of a fuzzy number $\tilde{t}_i$;

(iv) X: (physically relevant) universal set with the (crisp) elements x.

Figure 3 illustrates a possible description of the linguistic variable *color*. It is based on the continuous spectrum of the wave length λ of visible light: $X = \{\lambda \in \mathbb{R} \mid 380 \le \lambda \le 780\}$ nm. By subjective color perception, the colors t_i are chosen from the set $T = \{\text{violet}, \text{blue}, \text{cyan}, \text{green}, \text{yellow}, \text{red}\}$ of possible colors. Each term $t_i \in T$ is represented as a fuzzy number $\tilde{t}_i$ over the universal set X.

For an easier handling with linguistic variables, they can be transformed into the unit interval $[0, 1]$. These types of linguistic variables are referred to as *normalized linguistic variables* [12].

4. Extension Principle

Zadeh's extension principle [2] allows for extending any real-valued function to a function of fuzzy numbers. More specifically, let $\tilde{x}_1, \dots, \tilde{x}_n$ be *n independent* or *noninteractive* fuzzy numbers, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with

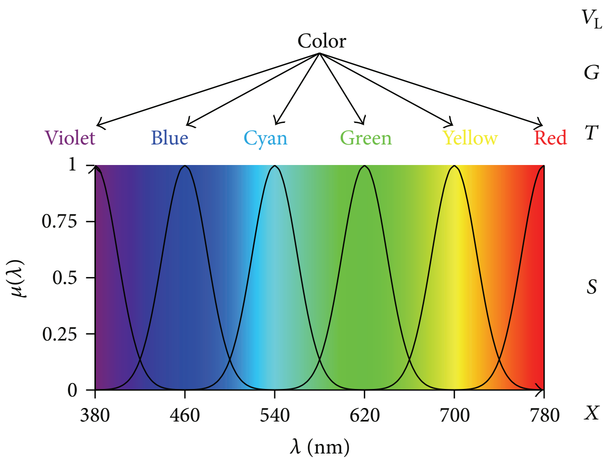

FIGURE 3: Possible description of the linguistic variable *color* according to [6].

$y = f(x_1, \ldots, x_n)$. The fuzzy extension $\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)$ is then defined by

$$\mu_{\tilde{y}}(y) = \sup_{y=f(x_1,\ldots,x_n)} \min\left\{\mu_{\tilde{x}_1}(x_1), \ldots, \mu_{\tilde{x}_n}(x_n)\right\}. \tag{7}$$

In case of *interdependency* between $\tilde{x}_1, \ldots, \tilde{x}_n$, the minimum operator should be replaced by a suitable *triangular norm* [13]. In this paper, however, we restrict ourselves to independent fuzzy numbers.

The evaluation of this *classical formulation* of the extension principle turns out to be a highly complex task [3]. Fortunately, Buckley and Qu [4] provide an *alternative formulation* that operates on α-cuts.

Let $x_1(\alpha), \ldots, x_n(\alpha)$ denote the α-cuts of the n independent fuzzy numbers $\tilde{x}_1, \ldots, \tilde{x}_n$, and let f be continuous. Then, the α-cuts $y(\alpha) = [y^{\mathrm{L}}(\alpha), y^{\mathrm{R}}(\alpha)]$ of $\tilde{y}$ can be computed from

$$\begin{aligned} y^{\mathrm{L}}(\alpha) &= \min\left\{f(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in \Omega(\alpha)\right\}, \\ y^{\mathrm{R}}(\alpha) &= \max\left\{f(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in \Omega(\alpha)\right\}, \end{aligned} \tag{8}$$

where $\Omega(\alpha) = x_1(\alpha) \times \cdots \times x_n(\alpha)$ represent the n-dimensional interval boxes that are spanned by the α-cuts $x_1(\alpha), \ldots, x_n(\alpha)$.

The extended analytical approach, which is presented in the next section, is based on this alternative formulation of the extension principle.

5. Extended Analytical Approach

Basically, our extended analytical approach can be classified into three parts depending on the monotonicity of f: a reduced [8], a general [8], and an extended part.

5.1. Reduced Part. Let the continuous function f be (strictly) monotonic increasing in x_i, $i = 1, \ldots, k$, and (strictly) monotonic decreasing in x_j, $j = 1, \ldots, \ell$, in the domain of interest, and let $k + \ell = n$. Then, the minimum values of f inside of every subdomain $\Omega(\alpha)$ are always found at the left boundaries of $x_i(\alpha)$ and the right boundaries of $x_j(\alpha)$ and its maximum values at the right boundaries of $x_i(\alpha)$ and the left boundaries of $x_j(\alpha)$, respectively. In such case, the α-cuts $y(\alpha) = [y^{\mathrm{L}}(\alpha), y^{\mathrm{R}}(\alpha)]$ of $\tilde{y}$ become

$$\begin{aligned} y^{\mathrm{L}}(\alpha) &= f\left(x_i^{\mathrm{L}}(\alpha), x_j^{\mathrm{R}}(\alpha)\right), \quad 0 < \alpha \le 1, \\ y^{\mathrm{R}}(\alpha) &= f\left(x_i^{\mathrm{R}}(\alpha), x_j^{\mathrm{L}}(\alpha)\right), \quad 0 < \alpha \le 1, \end{aligned} \tag{9}$$

with $x_m(\alpha) = [x_m^{\mathrm{L}}(\alpha), x_m^{\mathrm{R}}(\alpha)]$, $m = 1, \ldots, n$. If (9) is invertible with respect to α, then the membership function of $\tilde{y}$ yields

$$\mu_{\tilde{y}}(y) = \begin{cases} y^{\mathrm{L}}(\alpha)^{-1}, & y^{\mathrm{L}}(0) < y \le y^{\mathrm{L}}(1), \\ y^{\mathrm{R}}(\alpha)^{-1}, & y^{\mathrm{R}}(1) < y < y^{\mathrm{R}}(0). \end{cases} \tag{10}$$

This *reduced* part of our approach can be viewed as an analytical version of the *short transformation method* [14]. Basically, it is equivalent to Lemma 3 from [15] or Corollary 2 from [16].

Example 1. The function $f_1 : \mathbb{R}_+^2 \to \mathbb{R}_+$ with

$$y_1 = f_1(x_1, x_2) = \frac{x_1}{x_1 + x_2} \tag{11}$$

shall be evaluated for the two fuzzy numbers $\tilde{x}_1 = \mathrm{tfn}(2, 2, 3)$ and $\tilde{x}_2 = \mathrm{tfn}(2, 2, 2)$. Since

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= \frac{x_2}{(x_1 + x_2)^2} > 0, \\ \frac{\partial f_1}{\partial x_2} &= \frac{-x_1}{(x_1 + x_2)^2} < 0, \end{aligned} \tag{12}$$

the function f_1 is (strictly) monotonic increasing in x_1 and (strictly) monotonic decreasing in x_2 in the domain $\mathrm{supp}(\tilde{x}_1) \times \mathrm{supp}(\tilde{x}_2) = (0, 5) \times (0, 4)$. Hence, the α-cuts $y_1(\alpha) = [y_1^{\mathrm{L}}(\alpha), y_1^{\mathrm{R}}(\alpha)]$ of $\tilde{y}_1$ are

$$\begin{aligned} y_1^{\mathrm{L}}(\alpha) &= f_1\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right) = \frac{1}{2}\alpha, \\ y_1^{\mathrm{R}}(\alpha) &= f_1\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right) = \frac{3\alpha - 5}{\alpha - 5}. \end{aligned} \tag{13}$$

With $y_1^{\mathrm{L}}(0) = 0$, $y_1^{\mathrm{L}}(1) = 0.5 = y_1^{\mathrm{R}}(1)$, and $y_1^{\mathrm{R}}(0) = 1$, the membership function of $\tilde{y}_1$ yields

$$\mu_{\tilde{y}_1}(y) = \begin{cases} 2y, & 0 < y \le 0.5, \\ \dfrac{5(y-1)}{y-3}, & 0.5 < y < 1. \end{cases} \tag{14}$$

5.2. General Part. Unfortunately, the reduced part of our approach is only valid if the function f does not change its monotonicity within the domain of interest. However, we know from [17, 18] that the global extrema of any monotonic function f are always found at the corner points of $\Omega(\alpha)$. Hence, in order to obtain the analytical solution, we can always proceed as follows.

(1) Evaluate the function f for all the 2^n permutations of the interval boundaries of $x_m(\alpha) = [x_m^{\mathrm{L}}(\alpha), x_m^{\mathrm{R}}(\alpha)]$, $m = 1, \ldots, n$. For example, if $n = 2$, then compute

$$\begin{aligned} y^{\mathrm{LL}}(\alpha) &= f\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right), \\ y^{\mathrm{LR}}(\alpha) &= f\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right), \\ y^{\mathrm{RL}}(\alpha) &= f\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right), \\ y^{\mathrm{RR}}(\alpha) &= f\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right). \end{aligned} \tag{15}$$

(2) Plot these solution candidates in the same diagram.

(3) The analytical solution then corresponds to the maximum envelope formed by the possible solution candidates.

This *general* part of our approach can be viewed as an analytical version of the *reduced transformation method* [19]. Basically, it is equivalent to Lemma 2 from [15] or Corollary 1 from [16].

Example 2. Next, the function $f_2 : \mathbb{R}_+^2 \to \mathbb{R}$ with

$$y_2 = f_2(x_1, x_2) = \frac{(x_1 - 0.2)(x_2 - 1)}{x_1 + x_2} \tag{16}$$

shall be evaluated for the two fuzzy numbers from Example 1. Since

$$\begin{aligned} \frac{\partial f_2}{\partial x_1} &= \frac{(x_2 - 1)(x_2 + 0.2)}{(x_1 + x_2)^2} \begin{cases} \le 0, & 0 < x_2 \le 1, \\ > 0, & 1 < x_2 < 4, \end{cases} \\ \frac{\partial f_2}{\partial x_2} &= \frac{(x_1 - 0.2)(x_1 + 1)}{(x_1 + x_2)^2} \begin{cases} \le 0, & 0 < x_1 \le 0.2, \\ > 0, & 0.2 < x_1 < 5, \end{cases} \end{aligned} \tag{17}$$

the function f_2 changes its monotonicity within the domain $\mathrm{supp}(\tilde{x}_1) \times \mathrm{supp}(\tilde{x}_2) = (0, 5) \times (0, 4)$. Hence, the general part of our approach should be applied. The solution candidates for $y_2(\alpha)$ are

$$\begin{aligned} y_2^{\mathrm{LL}}(\alpha) &= f_2\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right) = \frac{(10\alpha - 1)(2\alpha - 1)}{20\alpha}, \\ y_2^{\mathrm{LR}}(\alpha) &= f_2\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right) = \frac{(10\alpha - 1)(3 - 2\alpha)}{20}, \\ y_2^{\mathrm{RL}}(\alpha) &= f_2\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right) = \frac{3}{5}\frac{(5\alpha - 8)(2\alpha - 1)}{\alpha - 5}, \\ y_2^{\mathrm{RR}}(\alpha) &= f_2\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right) = \frac{3}{5}\frac{(5\alpha - 8)(3 - 2\alpha)}{5\alpha - 9}. \end{aligned} \tag{18}$$

We can see from their plots in Figure 4 that the left branch of the maximum envelope, illustrated by the gray area, is formed by y_2^{RL} for $0 < \alpha \le 0.5$ and by y_2^{LL} for $0.5 < \alpha \le 1$, where the value 0.5 corresponds to the intersection point between y_2^{RL} and y_2^{LL}. Its right branch, on the other hand, is formed by y_2^{LL} for $0 < \alpha \le 0.02$ and by y_2^{RR} for $0.02 < \alpha \le 1$, where the value 0.02 corresponds to the intersection point between y_2^{LL} and y_2^{RR}. Note that the value 0.02 is only approximate. Hence, the α-cuts $y_2(\alpha) = [y_2^{\mathrm{L}}(\alpha), y_2^{\mathrm{R}}(\alpha)]$ of $\tilde{y}_2$ are

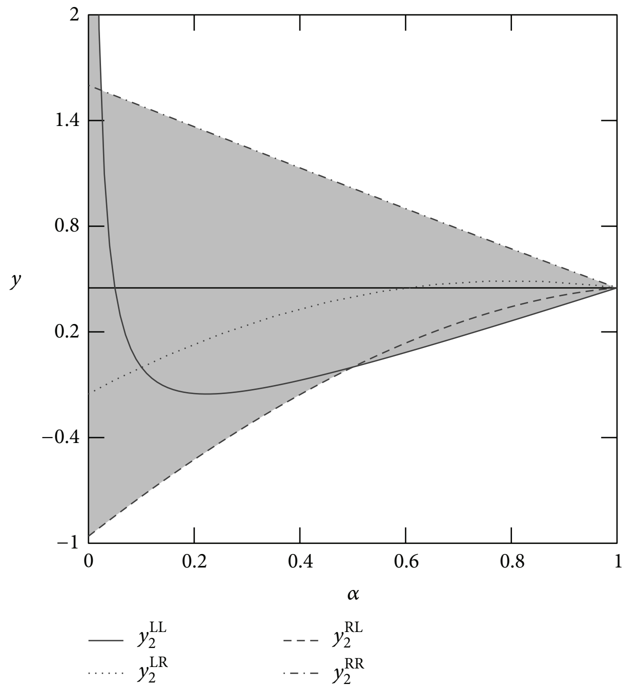

Figure 4: Solution candidates from Example 2.

$$\begin{aligned} y_2^{\mathrm{L}}(\alpha) &= \begin{cases} \dfrac{3}{5}\dfrac{(5\alpha - 8)(2\alpha - 1)}{\alpha - 5}, & 0 < \alpha \le 0.5, \\ \dfrac{(10\alpha - 1)(2\alpha - 1)}{20\alpha}, & 0.5 < \alpha \le 1, \end{cases} \\ y_2^{\mathrm{R}}(\alpha) &= \begin{cases} \dfrac{(10\alpha - 1)(2\alpha - 1)}{20\alpha}, & 0 < \alpha \le 0.02, \\ \dfrac{3}{5}\dfrac{(5\alpha - 8)(3 - 2\alpha)}{5\alpha - 9}, & 0.02 < \alpha \le 1. \end{cases} \end{aligned} \tag{19}$$

With $y_2^{\mathrm{L}}(0) = -0.96$, $y_2^{\mathrm{L}}(0.5) = 0$, $y_2^{\mathrm{L}}(1) = 0.45 = y_2^{\mathrm{R}}(1)$, $y_2^{\mathrm{R}}(0.02) \approx 1.57$, and $\lim_{\alpha \to 0} y_2^{\mathrm{R}}(\alpha) = \infty$, the membership function of $\tilde{y}_2$ yields

$$\mu_{\tilde{y}_2}(y) = \begin{cases} \dfrac{21}{20} + \dfrac{1}{12}y - \dfrac{1}{60}\sqrt{A}, & -0.96 < y \le 0, \\ \dfrac{3}{10} + \dfrac{1}{2}y + \dfrac{1}{10}\sqrt{B}, & 0 < y \le 0.45, \\ \dfrac{31}{20} - \dfrac{5}{12}y - \dfrac{1}{60}\sqrt{C}, & 0.45 < y \le 1.57, \\ \dfrac{3}{10} + \dfrac{1}{2}y - \dfrac{1}{10}\sqrt{D}, & 1.57 < y < \infty, \end{cases} \tag{20}$$

where

$$A = 25y^2 - 2370y + 1089,$$
$$B = 25y^2 + 30y + 4,$$
$$C = 625y^2 + 750y + 9, \tag{21}$$
$$D = 25y^2 + 30y + 4.$$

5.3. Extended Part. A drawback of the general part of our approach is the fact that a total of 2^n function evaluations have to be carried out to compute the possible solution candidates. However, if some of the variables do not change their monotonicity within the domain of interest, that is, if $k + \ell = q < n$, we can adapt our approach as follows.

(1) Evaluate the function f for $x_i^{\mathrm{L}}(\alpha)$, $i = 1, \ldots, k$, and $x_j^{\mathrm{R}}(\alpha)$, $j = 1, \ldots, \ell$, including all the 2^{n-q} permutations of the interval boundaries of $x_p(\alpha) = [x_p^{\mathrm{L}}(\alpha), x_p^{\mathrm{R}}(\alpha)]$, $p = 1, \ldots, n-q$, to compute the solution candidates for $y^{\mathrm{L}}(\alpha)$.

(2) Evaluate the function f for $x_i^{\mathrm{R}}(\alpha)$, $i = 1, \ldots, k$, and $x_j^{\mathrm{L}}(\alpha)$, $j = 1, \ldots, \ell$, including all the 2^{n-q} permutations of the interval boundaries of $x_p(\alpha) = [x_p^{\mathrm{L}}(\alpha), x_p^{\mathrm{R}}(\alpha)]$, $p = 1, \ldots, n-q$, to compute the solution candidates for $y^{\mathrm{R}}(\alpha)$.

(3) Plot these solution candidates in the same diagram.

(4) The analytical solution then corresponds to the maximum envelope formed by the possible solution candidates.

This *extended* part of our approach requires a total of 2^{n-q+1} function evaluations. Note that, for $q = 1$, the general and the extended part both lead to 2^n function evaluations.

Example 3. Now, the function $f_3 : \mathbb{R}_+^2 \to \mathbb{R}$ with

$$y_3 = f_3(x_1, x_2) = \frac{x_1 - 0.2}{x_1 + x_2} \tag{22}$$

shall be evaluated for the two fuzzy numbers from Example 1. Since

$$\frac{\partial f_3}{\partial x_1} = \frac{x_2 + 0.2}{(x_1 + x_2)^2} > 0,$$
$$\frac{\partial f_3}{\partial x_2} = \frac{0.2 - x_1}{(x_1 + x_2)^2} \begin{cases} \geq 0, & 0 < x_1 \leq 0.2, \\ < 0, & 0.2 < x_1 < 5, \end{cases} \tag{23}$$

the function f_3 is (strictly) monotonic increasing in x_1 but changes its monotonicity in x_2 within the domain $\mathrm{supp}(\tilde{x}_1) \times \mathrm{supp}(\tilde{x}_2) = (0, 5) \times (0, 4)$. Hence, the extended part of our approach should be applied. Note that here, $q = 1$. The solution candidates for $y_3^{\mathrm{L}}(\alpha)$ are

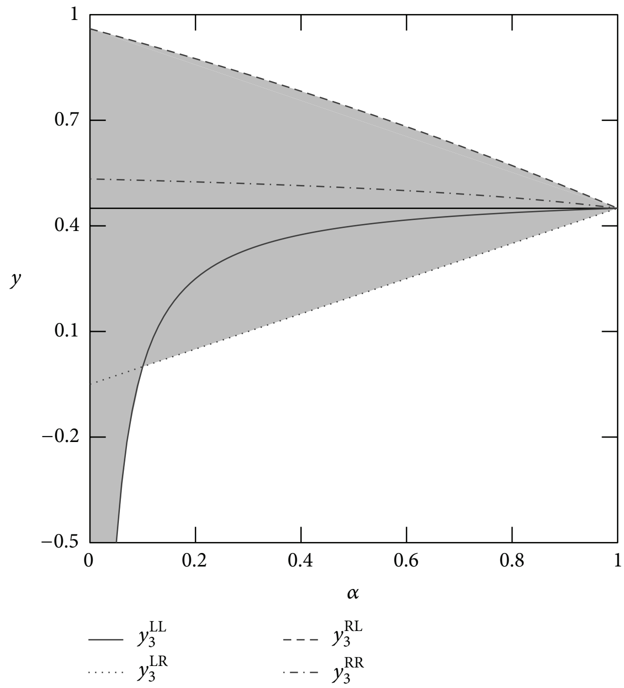

Figure 5: Solution candidates from Example 3.

$$y_3^{\mathrm{LL}}(\alpha) = f_3\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right) = \frac{10\alpha - 1}{20\alpha},$$
$$y_3^{\mathrm{LR}}(\alpha) = f_3\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right) = \frac{1}{2}\alpha - \frac{1}{20}, \tag{24}$$

and for $y_3^{\mathrm{R}}(\alpha)$,

$$y_3^{\mathrm{RL}}(\alpha) = f_3\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{L}}(\alpha)\right) = \frac{3}{5}\frac{5\alpha - 8}{\alpha - 5},$$
$$y_3^{\mathrm{RR}}(\alpha) = f_3\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha)\right) = \frac{3}{5}\frac{5\alpha - 8}{5\alpha - 9}. \tag{25}$$

We can see from their plots in Figure 5 that the left branch of the maximum envelope is formed by y_3^{LL} for $0 < \alpha \leq 0.1$ and by y_3^{LR} for $0.1 < \alpha \leq 1$, where the value 0.1 corresponds to their intersection point. Its right branch, on the other hand, is entirely formed by y_3^{RL}. Hence, the α-cuts $y_3(\alpha) = [y_3^{\mathrm{L}}(\alpha), y_3^{\mathrm{R}}(\alpha)]$ of $\tilde{y}_3$ are

$$y_3^{\mathrm{L}}(\alpha) = \begin{cases} \dfrac{10\alpha - 1}{20\alpha}, & 0 < \alpha \leq 0.1, \\ \dfrac{1}{2}\alpha - \dfrac{1}{20}, & 0.1 < \alpha \leq 1, \end{cases}$$
$$y_3^{\mathrm{R}}(\alpha) = \frac{3}{5}\frac{5\alpha - 8}{\alpha - 5}, \quad 0 < \alpha \leq 1. \tag{26}$$

With $\lim_{\alpha\to 0} y_3^{\mathrm{L}}(\alpha) = -\infty$, $y_3^{\mathrm{L}}(0.1) = 0$, $y_3^{\mathrm{L}}(1) = 0.45 = y_3^{\mathrm{R}}(1)$, and $y_3^{\mathrm{R}}(0) = 0.96$, the membership function of $\tilde{y}_3$ yields

$$\mu_{\tilde{y}_3}(y) = \begin{cases} \dfrac{1}{10}\dfrac{1}{1-2y}, & -\infty < y \le 0, \\ 2y + \dfrac{1}{10}, & 0 < y \le 0.45, \\ \dfrac{1}{5}\dfrac{25y-24}{y-3}, & 0.45 < y < 0.96. \end{cases} \tag{27}$$

Example 4. Finally, the function $f_4 : \mathbb{R}_+^3 \to \mathbb{R}$ with

$$y_4 = f_4(x_1, x_2, x_3) = \frac{x_1 x_2 - 1}{x_1 + x_2 + x_3} \tag{28}$$

shall be evaluated for the two fuzzy numbers from Example 1 and $\tilde{x}_3 = \mathrm{tfn}(3, 3, 2)$. Since

$$\begin{aligned} \frac{\partial f_4}{\partial x_1} &= \frac{x_2^2 + x_2 x_3 + 1}{(x_1 + x_2 + x_3)^2} > 0, \\ \frac{\partial f_4}{\partial x_2} &= \frac{x_1^2 + x_1 x_3 + 1}{(x_1 + x_2 + x_3)^2} > 0, \\ \frac{\partial f_4}{\partial x_3} &= \frac{1 - x_1 x_2}{(x_1 + x_2 + x_3)^2} \begin{cases} \le 0, & x_1 x_2 \ge 1, \\ > 0, & x_1 x_2 < 1, \end{cases} \end{aligned} \tag{29}$$

the function f_4 is (strictly) monotonic increasing in both x_1 and x_2 but changes its monotonicity in x_3 within the domain $\mathrm{supp}(\tilde{x}_1) \times \mathrm{supp}(\tilde{x}_2) \times \mathrm{supp}(\tilde{x}_3) = (0,5) \times (0,4) \times (0,5)$. Hence, the extended part of our approach should be applied. The solution candidates for $y_4^{\mathrm{L}}(\alpha)$ are

$$\begin{aligned} y_4^{\mathrm{LLL}}(\alpha) &= f_4\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha), x_3^{\mathrm{L}}(\alpha)\right) = \frac{4\alpha^2 - 1}{7\alpha}, \\ y_4^{\mathrm{LLR}}(\alpha) &= f_4\left(x_1^{\mathrm{L}}(\alpha), x_2^{\mathrm{L}}(\alpha), x_3^{\mathrm{R}}(\alpha)\right) = \frac{4\alpha^2 - 1}{5 + 2\alpha}, \end{aligned} \tag{30}$$

and for $y_4^{\mathrm{R}}(\alpha)$,

$$\begin{aligned} y_4^{\mathrm{RRL}}(\alpha) &= f_4\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha), x_3^{\mathrm{L}}(\alpha)\right) = \frac{6\alpha^2 - 22\alpha + 19}{9 - 2\alpha}, \\ y_4^{\mathrm{RRR}}(\alpha) &= f_4\left(x_1^{\mathrm{R}}(\alpha), x_2^{\mathrm{R}}(\alpha), x_3^{\mathrm{R}}(\alpha)\right) = \frac{6\alpha^2 - 22\alpha + 19}{14 - 7\alpha}. \end{aligned} \tag{31}$$

We can see from their plots in Figure 6 that the left branch of the maximum envelope is formed by y_4^{LLL} for $0 < \alpha \le 0.5$ and by y_4^{LLR} for $0.5 < \alpha \le 1$, where the value 0.5 corresponds to their intersection point. Its right branch, on the other hand, is entirely formed by y_4^{RRL}. Hence, the α-cuts $y_4(\alpha) = [y_4^{\mathrm{L}}(\alpha), y_4^{\mathrm{R}}(\alpha)]$ of $\tilde{y}_4$ are

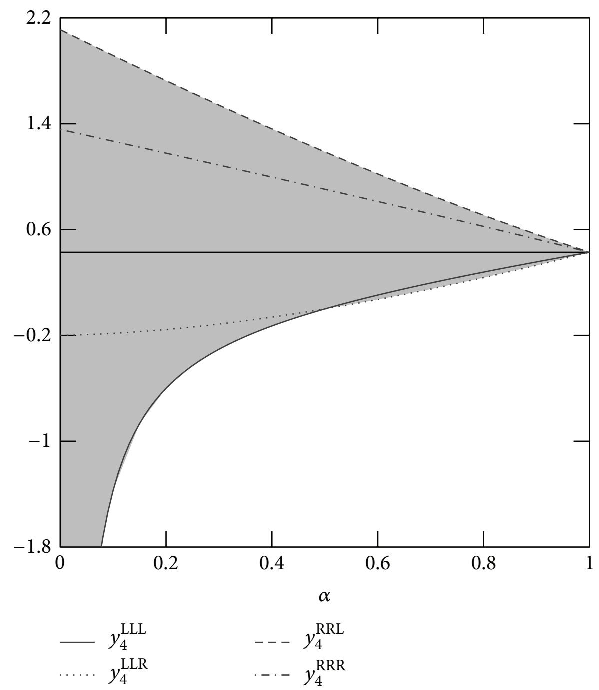

Figure 6: Solution candidates from Example 4.

$$y_4^{\mathrm{L}}(\alpha) = \begin{cases} \dfrac{4\alpha^2 - 1}{7\alpha}, & 0 < \alpha \le 0.5, \\ \dfrac{4\alpha^2 - 1}{5 + 2\alpha}, & 0.5 < \alpha \le 1, \end{cases}$$
$$y_4^{\mathrm{R}}(\alpha) = \frac{6\alpha^2 - 22\alpha + 19}{9 - 2\alpha}, \quad 0 < \alpha \le 1. \tag{32}$$

With $\lim_{\alpha\to 0} y_4^{\mathrm{L}}(\alpha) = -\infty$, $y_4^{\mathrm{L}}(0.5) = 0$, $y_4^{\mathrm{L}}(1) \approx 0.43 \approx y_4^{\mathrm{R}}(4)$, and $y_4^{\mathrm{R}}(0) \approx 2.11$, the membership function of $\tilde{y}_4$ yields

$$\mu_{\tilde{y}_4}(y) = \begin{cases} \dfrac{7}{8}y + \dfrac{1}{8}\sqrt{E}, & -\infty < y \le 0, \\ \dfrac{1}{4}y + \dfrac{1}{4}\sqrt{F}, & 0 < y \le 0.43, \\ \dfrac{11}{6} - \dfrac{1}{6}y - \dfrac{1}{6}\sqrt{G}, & 0.43 < y < 2.11, \end{cases} \tag{33}$$

where

$$\begin{aligned} E &= 49y^2 + 16, \\ F &= y^2 + 20y + 4, \\ G &= y^2 + 32y + 7. \end{aligned} \tag{34}$$

6. Engineering Application

In order to illustrate the extended analytical approach in a more practical context, we consider a simplified version of

Table 1: Linguistic weights of the criteria and ratings of the alternative materials.

Criterion j			1 Low weight	2 Low cost
Weight w_j			H	MH
Alternative i	1	Polymer composite	H	L
	2	Aluminum alloy	MH	L

the case study from [20], where the material for an automotive bumper beam has to be selected. Here, two alternative materials (*polymer composite* and *aluminum alloy*) have to be evaluated against the criteria *low weight* and *low cost* using the normalized linguistic variable *value scale* from Figure 7. The corresponding linguistic weights and ratings are summarized in Table 1.

For computing the fuzzy overall rating $\tilde{r}_i$ of each alternative i, we use the fuzzy weighted average

$$\tilde{r}_i = \frac{\sum_{j=1}^{n} \tilde{w}_j \tilde{r}_{ij}}{\sum_{j=1}^{n} \tilde{w}_j}, \tag{35}$$

where $\tilde{r}_{ij}$ denotes the fuzzy rating of the alternative i according the criterion j and $\tilde{w}_j$ denotes the fuzzy weight of the criterion j (see Table 1). Since

$$\begin{aligned} \frac{\partial r_i}{\partial r_{ik}} &= \frac{w_k}{\sum_{j=1}^{n} w_j} > 0, \\ \frac{\partial r_i}{\partial w_k} &= \frac{r_{ik}}{\sum_{j=1}^{n} w_j} - \frac{\sum_{j=1}^{n} w_j r_{ij}}{\left(\sum_{j=1}^{n} w_j\right)^2} \\ &= \frac{\sum_{j=1}^{k-1} w_j \left(r_{ik} - r_{ij}\right) + \sum_{j=k+1}^{n} w_j \left(r_{ik} - r_{ij}\right)}{\left(\sum_{j=1}^{n} w_j\right)^2}, \end{aligned} \tag{36}$$

r_i is (strictly) monotonic increasing in r_{ij} but may change its monotonicity in w_j within the domain $(0,1)^{2n}$. Hence, the extended part of our approach should be applied. The solution candidates for $r_1^{\mathrm{L}}(\alpha)$ are

$$\begin{aligned} r_1^{\mathrm{LLLL}}(\alpha) &= r_1\left(w_1^{\mathrm{L}}(\alpha), w_2^{\mathrm{L}}(\alpha), r_{11}^{\mathrm{L}}(\alpha), r_{12}^{\mathrm{L}}(\alpha)\right) \\ &= \frac{1}{6}\frac{2\alpha^2 + 11\alpha + 16}{2\alpha + 7}, \\ r_1^{\mathrm{LRLL}}(\alpha) &= r_1\left(w_1^{\mathrm{L}}(\alpha), w_2^{\mathrm{R}}(\alpha), r_{11}^{\mathrm{L}}(\alpha), r_{12}^{\mathrm{L}}(\alpha)\right) \\ &= \frac{13}{54}\alpha + \frac{8}{27}, \\ r_1^{\mathrm{RLLL}}(\alpha) &= r_1\left(w_1^{\mathrm{R}}(\alpha), w_2^{\mathrm{L}}(\alpha), r_{11}^{\mathrm{L}}(\alpha), r_{12}^{\mathrm{L}}(\alpha)\right) \\ &= \frac{5}{54}\alpha + \frac{4}{9}, \\ r_1^{\mathrm{RRLL}}(\alpha) &= r_1\left(w_1^{\mathrm{R}}(\alpha), w_2^{\mathrm{R}}(\alpha), r_{11}^{\mathrm{L}}(\alpha), r_{12}^{\mathrm{L}}(\alpha)\right) \\ &= \frac{1}{6}\frac{2\alpha^2 - 7\alpha - 24}{2\alpha - 11}, \end{aligned} \tag{37}$$

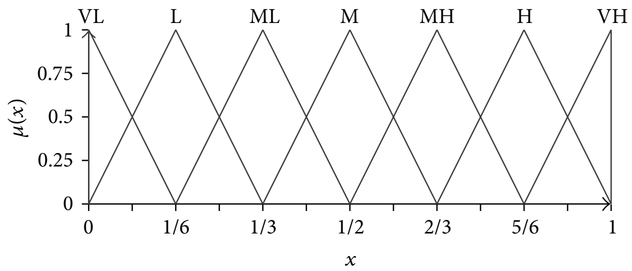

Figure 7: Normalized linguistic variable *value scale*. VL: very low, L: low, ML: medium low, M: medium, MH: medium high, H: high, and VH: very high.

and for $r_1^{\mathrm{R}}(\alpha)$,

$$\begin{aligned} r_1^{\mathrm{LLRR}}(\alpha) &= r_1\left(w_1^{\mathrm{L}}(\alpha), w_2^{\mathrm{L}}(\alpha), r_{11}^{\mathrm{R}}(\alpha), r_{12}^{\mathrm{R}}(\alpha)\right) \\ &= -\frac{1}{6}\frac{2\alpha^2 - \alpha - 30}{2\alpha + 7}, \\ r_1^{\mathrm{LRRR}}(\alpha) &= r_1\left(w_1^{\mathrm{L}}(\alpha), w_2^{\mathrm{R}}(\alpha), r_{11}^{\mathrm{R}}(\alpha), r_{12}^{\mathrm{R}}(\alpha)\right) \\ &= -\frac{5}{54}\alpha + \frac{17}{27}, \\ r_1^{\mathrm{RLRR}}(\alpha) &= r_1\left(w_1^{\mathrm{R}}(\alpha), w_2^{\mathrm{L}}(\alpha), r_{11}^{\mathrm{R}}(\alpha), r_{12}^{\mathrm{R}}(\alpha)\right) \\ &= -\frac{13}{54}\alpha + \frac{7}{9}, \\ r_1^{\mathrm{RRRR}}(\alpha) &= r_1\left(w_1^{\mathrm{R}}(\alpha), w_2^{\mathrm{R}}(\alpha), r_{11}^{\mathrm{R}}(\alpha), r_{12}^{\mathrm{R}}(\alpha)\right) \\ &= -\frac{1}{6}\frac{2\alpha^2 - 19\alpha + 46}{2\alpha - 11}. \end{aligned} \tag{38}$$

We can see from their plots in Figure 8 that the left branch of the maximum envelope is formed by r_1^{LRLL} and the right branch by r_1^{RLRR}. Hence, the α-cuts $r_1(\alpha) = [r_1^{\mathrm{L}}(\alpha), r_1^{\mathrm{R}}(\alpha)]$ of $\tilde{r}_1$ are

$$\begin{aligned} r_1^{\mathrm{L}}(\alpha) &= +\frac{13}{54}\alpha + \frac{8}{27}, \\ r_1^{\mathrm{R}}(\alpha) &= -\frac{13}{54}\alpha + \frac{7}{9}. \end{aligned} \tag{39}$$

With $r_1^{\mathrm{L}}(0) \approx 0.30$, $r_1^{\mathrm{L}}(1) \approx 0.54 \approx r_1^{\mathrm{R}}(1)$, and $r_1^{\mathrm{R}}(0) \approx 0.78$, the membership function of $\tilde{r}_1$ yields

$$\mu_{\tilde{r}_1}(r_1) = \begin{cases} +\frac{54}{13}r_1 - \frac{16}{13}, & 0.30 < r_1 \leq 0.54, \\ -\frac{54}{13}r_1 + \frac{42}{13}, & 0.54 < r_1 < 0.78. \end{cases} \tag{40}$$

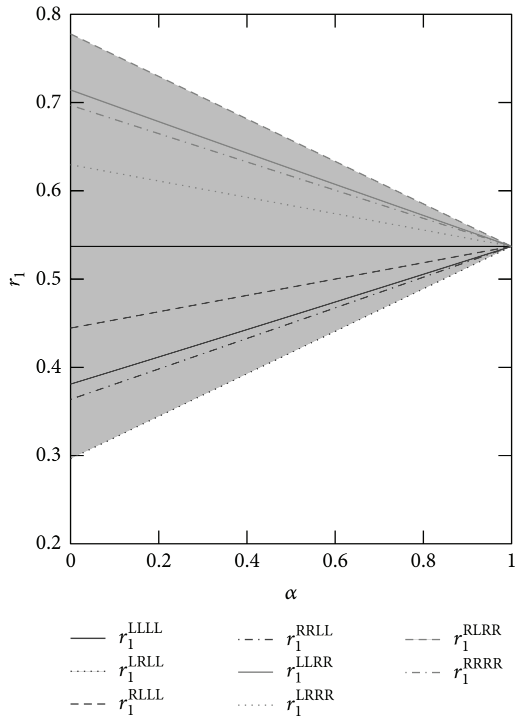

Figure 8: Solution candidates for $r_1(\alpha)$.

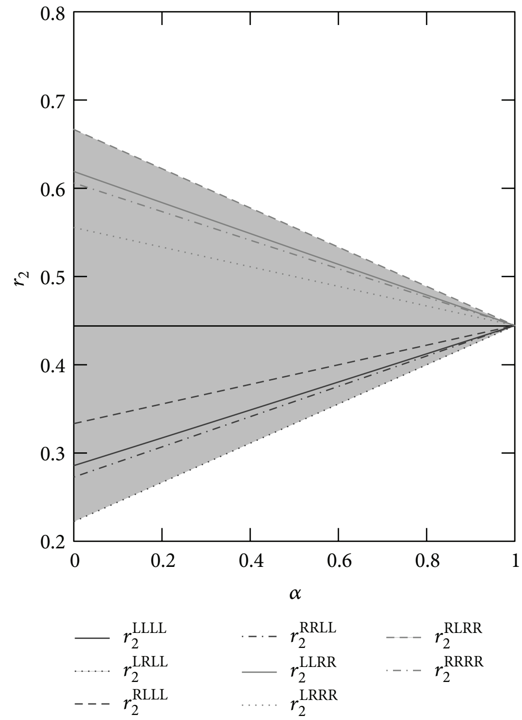

Figure 9: Solution candidates for $r_2(\alpha)$.

Furthermore, the solution candidates for $r_2^{\mathrm{L}}(\alpha)$ are

$$\begin{aligned}
r_2^{\mathrm{LLLL}}(\alpha) &= r_2\left(w_1^{\mathrm{L}}(\alpha), w_2^{\mathrm{L}}(\alpha), r_{21}^{\mathrm{L}}(\alpha), r_{22}^{\mathrm{L}}(\alpha)\right) \\
&= \frac{1}{3}\frac{(\alpha+3)(\alpha+2)}{2\alpha+7}, \\
r_2^{\mathrm{LRLL}}(\alpha) &= r_2\left(w_1^{\mathrm{L}}(\alpha), w_2^{\mathrm{R}}(\alpha), r_{21}^{\mathrm{L}}(\alpha), r_{22}^{\mathrm{L}}(\alpha)\right) \\
&= \frac{2}{9}\alpha + \frac{2}{9}, \\
r_2^{\mathrm{RLLL}}(\alpha) &= r_2\left(w_1^{\mathrm{R}}(\alpha), w_2^{\mathrm{L}}(\alpha), r_{21}^{\mathrm{L}}(\alpha), r_{22}^{\mathrm{L}}(\alpha)\right) \\
&= \frac{1}{9}\alpha + \frac{1}{3}, \\
r_2^{\mathrm{RRLL}}(\alpha) &= r_2\left(w_1^{\mathrm{R}}(\alpha), w_2^{\mathrm{R}}(\alpha), r_{21}^{\mathrm{L}}(\alpha), r_{22}^{\mathrm{L}}(\alpha)\right) \\
&= \frac{1}{3}\frac{\alpha^2-4\alpha-9}{2\alpha-11},
\end{aligned} \tag{41}$$

and for $r_2^{\mathrm{R}}(\alpha)$,

$$\begin{aligned}
r_2^{\mathrm{LLRR}}(\alpha) &= r_2\left(w_1^{\mathrm{L}}(\alpha), w_2^{\mathrm{L}}(\alpha), r_{21}^{\mathrm{R}}(\alpha), r_{22}^{\mathrm{R}}(\alpha)\right) \\
&= -\frac{1}{3}\frac{\alpha^2-13}{2\alpha+7}, \\
r_2^{\mathrm{LRRR}}(\alpha) &= r_2\left(w_1^{\mathrm{L}}(\alpha), w_2^{\mathrm{R}}(\alpha), r_{21}^{\mathrm{R}}(\alpha), r_{22}^{\mathrm{R}}(\alpha)\right) \\
&= -\frac{1}{9}\alpha + \frac{5}{9}, \\
r_2^{\mathrm{RLRR}}(\alpha) &= r_2\left(w_1^{\mathrm{R}}(\alpha), w_2^{\mathrm{L}}(\alpha), r_{21}^{\mathrm{R}}(\alpha), r_{22}^{\mathrm{R}}(\alpha)\right) \\
&= -\frac{2}{9}\alpha + \frac{2}{3}, \\
r_2^{\mathrm{RRRR}}(\alpha) &= r_2\left(w_1^{\mathrm{R}}(\alpha), w_2^{\mathrm{R}}(\alpha), r_{21}^{\mathrm{R}}(\alpha), r_{22}^{\mathrm{R}}(\alpha)\right) \\
&= -\frac{1}{3}\frac{(\alpha-5)(\alpha-4)}{2\alpha-11}.
\end{aligned} \tag{42}$$

We can see from their plots in Figure 9 that the left branch of the maximum envelope is formed by r_2^{LRLL} and the right branch by r_2^{RLRR}. Hence, the α-cuts $r_2(\alpha) = [r_2^{\mathrm{L}}(\alpha), r_2^{\mathrm{R}}(\alpha)]$ of $\tilde{r}_2$ are

$$\begin{aligned}
r_2^{\mathrm{L}}(\alpha) &= +\frac{2}{9}\alpha + \frac{2}{9}, \\
r_2^{\mathrm{R}}(\alpha) &= -\frac{2}{9}\alpha + \frac{2}{3}.
\end{aligned} \tag{43}$$

With $r_2^{\mathrm{L}}(0) \approx 0.22$, $r_2^{\mathrm{L}}(1) \approx 0.44 \approx r_2^{\mathrm{R}}(1)$, and $r_2^{\mathrm{R}}(0) \approx 0.67$, the membership function of $\tilde{r}_2$ yields

$$\mu_{\tilde{r}_2}(r_2) = \begin{cases} +\dfrac{9}{2}r_2 - 1, & 0.22 < r_2 \leq 0.44, \\ -\dfrac{9}{2}r_2 + 3, & 0.44 < r_2 < 0.67. \end{cases} \tag{44}$$

The membership functions of $\tilde{r}_1$ and $\tilde{r}_2$ are illustrated in Figure 10. There, we can see that polymer composite seems to be more appropriate as bumper beam material than aluminum alloy.

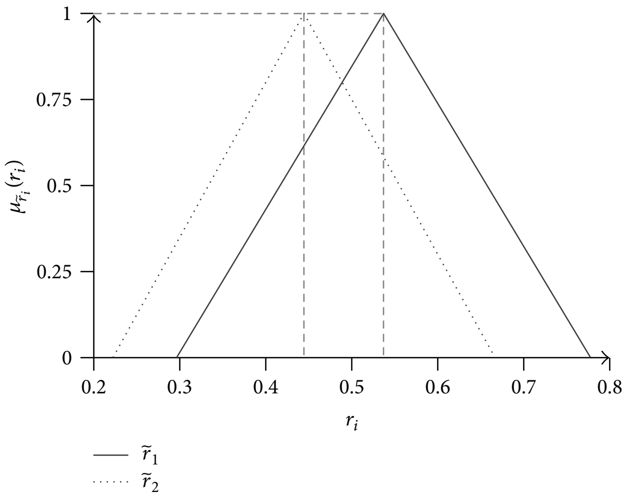

Figure 10: Membership functions of $\tilde{r}_1$ and $\tilde{r}_2$.

7. Conclusions

The proposed extended analytical approach is a very practical tool for the inclusion of parameter uncertainties into mathematical models. It is valid for continuous, monotonic functions of independent fuzzy numbers but can also be applied to fuzzy intervals as defined in [3, 6].

An analytical solution has the advantage that the degrees of membership of the fuzzy output can be computed for any value within the support, whereas a numerical solution only provides a finite number of values. Furthermore, our approach also allows a symbolic processing of uncertainties.

In further research activities, this approach shall be generalized to nonmonotonic functions of independent fuzzy numbers, where the influence of interdependency shall be investigated as well.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, NY, USA, 1980.

[2] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I," *Information Sciences*, vol. 8, no. 3, pp. 199–249, 1975.

[3] A. Klimke, *Uncertainty modeling using fuzzy arithmetic and sparse grids [Ph.D. thesis]*, University of Stuttgart, 2006.

[4] J. J. Buckley and Y. Qu, "On using α-cuts to evaluate fuzzy equations," *Fuzzy Sets and Systems*, vol. 38, no. 3, pp. 309–312, 1990.

[5] D. Moens and M. Hanss, "Non-probabilistic finite element analysis for parametric uncertainty treatment in applied mechanics: recent advances," *Finite Elements in Analysis and Design*, vol. 47, no. 1, pp. 4–16, 2011.

[6] M. Hanss, *Applied Fuzzy Arithmetic: An Introduction with Engineering Applications*, Springer, Berlin, Germany, 2005.

[7] G. Pahl, W. Beitz, J. Feldhusen, and K. H. Grote, *Engineering Design: A Systematic Approach*, Springer, London, UK, 2007.

[8] A. Seibel and J. Schlattmann, "An analytical approach to evaluating monotonic functions of fuzzy numbers," in *Proceedings of the 8th Conference of the European Society for Fuzzy Logic and Technology*, pp. 289–293, Milano, Italy, 2013.

[9] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[10] A. Kaufmann and M. M. Gupta, *Introduction to Fuzzy Arithmetic*, Van Nostrand Reinhold, New York, NY, USA, 1991.

[11] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall PTR, Upper Saddle River, NJ, USA, 1995.

[12] B. Möller and M. Beer, *Fuzzy Randomness: Uncertainty in Civil Engineering and Computational Mechanics*, Springer, Berlin, Germany, 2004.

[13] K. Scheerlinck, *Metaheuristic versus tailor-made approaches to optimization problems in the biosciences [Ph.D. thesis]*, Ghent University, 2011.

[14] S. Donders, D. Vandepitte, J. van de Peer, and W. Desmet, "Assessment of uncertainty on structural dynamic responses with the short transformation method," *Journal of Sound and Vibration*, vol. 288, no. 3, pp. 523–549, 2005.

[15] H. Q. Yang, H. Yao, and J. D. Jones, "Calculating functions of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 55, no. 3, pp. 273–283, 1993.

[16] J. Fortin, D. Dubois, and H. Fargier, "Gradual numbers and their application to fuzzy interval analysis," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 2, pp. 388–402, 2008.

[17] W. M. Dong and F. S. Wong, "Fuzzy weighted averages and implementation of the extension principle," *Fuzzy Sets and Systems*, vol. 21, no. 2, pp. 183–199, 1987.

[18] W. Dong and H. C. Shah, "Vertex method for computing functions of fuzzy variables," *Fuzzy Sets and Systems*, vol. 24, no. 1, pp. 65–78, 1987.

[19] M. Hanss, "The transformation method for the simulation and analysis of systems with uncertain parameters," *Fuzzy Sets and Systems*, vol. 130, no. 3, pp. 277–289, 2002.

[20] L. V. Vanegas and A. W. Labib, "Application of new fuzzy-weighted average (NFWA) method to engineering design evaluation," *International Journal of Production Research*, vol. 39, no. 6, pp. 1147–1162, 2001.