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A Note on the Harmonic Extension Approach to Fractional Powers of non-densely defined Operators

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We show to what extend fractional powers of non-densely defined sectorial operators on Banach spaces can still be described by the harmonic extension approach.

1 Introduction

Fractional powers of operators in Banach spaces are a well studied topic in operator theory. Given a closed linear operator A and $\alpha \in \mathbb{C}$, one tries to define A^α, the fractional power of A.

An appropriate point of view is that one plugs in the operator A into the function $z \mapsto z^\alpha$ leading into the realm of the functional calculus of sectorial operators, see e.g. [1,2]. A very detailed study of the entire topic is available in [3].

A recent approach on how to describe fractional powers was rediscovered by Cafarelli and Silvestre in [4] where the authors described fractional powers of the Laplacian by means of taking traces of functions solving the (incomplete) initial value problem for the PDE

$$\partial_t^2 u(t,x) + \frac{1-2\alpha}{t}\partial_t u(t,x) = -\Delta_x u(t,x), \quad (t,x) \in (0,\infty)\times\mathbb{R}^n, \quad u(0,x) = f(x), \quad x \in \mathbb{R}^n, \tag{1}$$

with $\alpha \in (0,1)$ being the fractional power. Because of its origin the technique is sometimes also referred to as harmonic extension approach.

Using a solution u to (1), one can then calculate $(-\Delta)^\alpha$ as

$$c_\alpha\big((-\Delta)^\alpha f\big)(x) = -\lim_{t\to 0+} t^{1-2\alpha}\partial_t u(t,x), \qquad x \in \mathbb{R}^n, \tag{2}$$

with a constant c_α and a solution u of (1). This method may be generalised to the (incomplete) Cauchy problem for the ODE

$$u''(t) + \frac{1-2\alpha}{t}u'(t) = Au(t), \quad t \in (0,\infty), \quad u(0) = x, \tag{3}$$

in a Banach space X with sectorial operator A and initial datum $x \in X$. By now several authors studied the problem beginning with [5] for the case $\alpha = 1/2$ and more recently [6–9] and it was shown that in all studied cases an accordingly generalised version of (2) can be obtained.

In [9] the authors answered the question whether (3) admits a unique solution and to what extend this solution can be used to describe fractional powers for densely defined sectorial operators. The purpose of the work at hand is to comment on the situation if the assumption of A being densely defined is dropped.

2 Main part

Let X be a Banach space. Let $A : \mathcal{D}(A) \to X$ be a sectorial linear operator X (not necessarily densely defined), i.e. $(-\infty, 0) \subseteq \rho(A)$ and $M := \sup_{\lambda>0} \|\lambda(\lambda + A)^{-1}\| < \infty$. Let $\omega_A \in [0,\pi)$ be the angle of sectoriality of A, i.e. the infimum of all $\theta \in [0,\pi)$ such that $\sigma(A)$ is a subset of the closure of the sector $S_\theta := \{z \in \mathbb{C} \setminus (-\infty, 0] \mid |\arg z| < \theta\}$. Moreover, we set $D := \overline{\mathcal{D}(A)}$. It is well-known that the part $A_D \subseteq A$ of A in D, given by $\mathcal{D}(A_D) := \{x \in \mathcal{D}(A) \mid Ax \in D\}$, is again a sectorial operator which is also densely defined in D. Let $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha \in (0,1)$.

Definition 2.1 A solution of (3) is a function $u \in C_\mathrm{b}\big([0,\infty);X\big)$ such that $u(0) = x$, $u(t) \in \mathcal{D}(A)$ for all $t > 0$, $Au \in L_{1,\mathrm{loc}}\big((0,\infty);X\big)$ the ordinary differential equation in (3) is satisfied in the sense of distributions.

In this situation we are going to prove the following theorem.

Theorem 2.2 *The Cauchy problem* (3) *admits a bounded solution* u *if and only if* $x \in D$. *Moreover, the bounded solution* u *is unique, extends holomorphically to the sector* $S_{(\pi-\omega_A)/2}$ *and*

$$c_\alpha\,(A_D)^\alpha\, x = c_\alpha\,(A^\alpha)_D\, x = -\lim_{z\to 0} z^{1-2\alpha}u'(z) \quad in \quad S_\delta.$$

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with $\delta \in \big[0, (\pi - \omega_A)/2\big)$.

Proof. First, let $x \in D$. A calculation similar to the one performed in [9, Proposition 4.8] together with [9, Lemma 4.7] shows that

$$(0,\infty) \ni t \mapsto U(t)x := \frac{t^{2\alpha}}{2 \cdot \Gamma(2\alpha)} \int_0^\infty s^{\alpha-\frac{1}{2}} \mathrm{e}^{-t\sqrt{A+s}} (A+s)^{-\frac{1}{2}} x \, \mathrm{d}s \in X \tag{4}$$

defines a bounded solution to (3). Conversely, let u be a (bounded) solution to (3). Let $c > 0$ and set $\tilde{x} := \mathrm{e}^{-c\sqrt{A}}x$ and $\tilde{u}(t) := \mathrm{e}^{-c\sqrt{A}}u(t)$. From the ODE (3) it follows that for $t > s > 0$ one has the representation

$$t^{1-2\alpha}\tilde{u}'(t) - s^{1-2\alpha}\tilde{u}'(s) = \int_s^t w^{1-2\alpha} A\mathrm{e}^{-c\sqrt{A}} u(w) \mathrm{d}w$$

and therefore $\tilde{y} := \lim_{t\to 0+} t^{1-2\alpha}\tilde{u}'(t)$ exists. By [9, Corollary 5.6], we have $\tilde{u}(t) = v_t(\sqrt{A})\tilde{x} + w_t(\sqrt{A})\tilde{y}$ for small enough $t > 0$ (note that $R(\mathrm{e}^{-c\sqrt{A}}) \subseteq \mathcal{D}\big(v_t(\sqrt{A})\big)$ for small t), where v_t and w_t are certain holomorphic functions, see [9, Definition 3.3 and Lemma 3.4]. Since u is bounded so is $\tilde{u}$. By [9, Theorem 5.8] and the first part of the proof, for $t > 0$ small we have $\tilde{u}(t) = U(t)\tilde{x}$ and therefore $u(t) = U(t)x \in \mathcal{D}(A^\infty)$ by [9, Lemma 4.5], where $\mathcal{D}(A^\infty) := \bigcap_{k\in\mathbb{N}} \mathcal{D}(A^k)$. Since u is continuous at 0 we get $x \in \overline{\mathcal{D}(A^\infty)} \subseteq D$. (Note that $U(\cdot)x$ is strongly continuous at 0 if and only if $x \in D$, see [9, Lemma 4.7].) Moreover, u is unique. By [9, Lemma 5.5], u extends to a holomorphic function on the sector $S_{(\pi-\omega_A)/2}$.

The first equality sign in the last part of the claim was already shown in [10, Proposition 2.9]. In fact, it is generally true that $f(A_D) = f(A)_D$ for all f for which one can define $f(A)$ in the sectorial calculus of A.

Let

$$u_{z,\alpha}(\lambda) := \begin{cases} \frac{1}{2^{\alpha-1}\Gamma(\alpha)}(\lambda z)^\alpha K_\alpha(\lambda z) & \operatorname{Re}\alpha > 0, \\ \frac{1}{2^{-\alpha-1}\Gamma(-\alpha)}(\lambda z)^{-\alpha} K_{-\alpha}(\lambda z) & \operatorname{Re}\alpha < 0, \end{cases}$$

where K_α is the modified Bessel function. Then, we have $u(z) = u_{z,\alpha}(\sqrt{A})x$ by [9, Lemma 4.6]. Inspecting and adapting the proof of [9, Theorem 6.2], we get

$$\begin{aligned} -z^{1-2\alpha}\partial_z u_{z,\alpha}(\sqrt{A})(1+A)^{-\alpha}x &= -z^{1-2\alpha}\big[\lambda \mapsto \partial_z u_{z,\alpha}(\lambda)\big](\sqrt{A_D})(1+A_D)^{-\alpha}x \\ &= -z^{1-2\alpha}\partial_z u_{z,\alpha}(\sqrt{A_D})(1+A_D)^{-\alpha}x \\ &= c_\alpha\big[\lambda \mapsto \lambda^{2\alpha}u_{z,\alpha-1}(\lambda)\big](\sqrt{A_D})(1+A_D)^{-\alpha}x \\ &= c_\alpha u_{z,\alpha-1}(\sqrt{A_D})A_D^\alpha(1+A_D)^{-\alpha}x. \end{aligned}$$

It follows that $\lim_{z\to 0} -z^{1-2\alpha}\partial_z u_{z,\alpha}(\sqrt{A_D})(1+A_D)^{-\alpha}x = c_\alpha A_D^\alpha(1+A_D)^{-\alpha}x$ in S_δ by the same reasoning as in the original proof. Thus, the theorem is proven. □

Remark 2.3 Note that A as well as A_D may be general sectorial operators, so their angle of sectoriality may exceed $\frac{\pi}{2}$. Therefore, they may not be generators of (bounded) C_0-semigroups. However, inspecting the above proof we see that we actually use functional calculus arguments for $\sqrt{A}$ and $\sqrt{A_D}$, whose angle of sectoriality is then below $\frac{\pi}{2}$.

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