A Jacobi–Davidson-type projection method for nonlinear eigenvalue problems

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Abstract

This article discusses a projection method for nonlinear eigenvalue problems. The ansatz space is constructed by a Jacobi–Davidson type approach, and the arising eigenproblems of small dimension are solved by safeguarded inverse iteration. The method is applied to a rational eigenvalue problem governing the vibrations of tube bundle immersed in an inviscid compressible fluid.

Key words: nonlinear eigenvalue problem, Jacobi–Davidson method, projection method, Rayleigh functional, minmax characterization
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1 Introduction

In this paper we consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \tag{1}$$

where $T(\lambda) \in \mathbb{R}^{n\times n}$, $\lambda \in J$, is a family of real symmetric matrices, and $J \subset \mathbb{R}$ is an open interval. As in the linear case, $\lambda \in J$ is called an eigenvalue if equation (1) has a nontrivial solution $x \neq 0$, and x is called a corresponding eigenvector. Typically, we assume that n is large, and $T(\lambda)$ is sparse.

For linear sparse eigenproblems $T(\lambda) = \lambda B - A$ the most efficient methods (Lanczos method, Arnoldi method, Jacobi–Davidson method, e.g.) are iterative projection methods, where approximations to the wanted eigenvalues and eigenvectors are obtained from projections of the eigenproblem to subspaces of small dimension which are expanded in the course of the algorithm.

Ruhe [9] and Hager and Wiberg [4], [5] suggested a generalization of this approach to nonlinear eigenvalue problems. Similarly as in the rational Krylov process they construct a sequence V_k of subspaces of $\mathbb{R}^n$ and corresponding Hessenberg matrices H_k which approximate the projection of $T(\sigma)^{-1}T(\lambda_k)$ to V_k. Here σ denotes a shift and λ_k an approximation to the wanted eigenvalue of (1). Then a Ritz vector of H_k corresponding to an eigenvalue of small modulus approximates an eigenvector of the nonlinear problem from which a (hopefully) improved eigenvalue approximation of problem (1) is obtained.

In this paper we propose a further projection method for problem (1). We do not construct a sequence of linear approximations of the nonlinear problem but we update projections of the nonlinear problem to a sequence of nested subspaces of small dimension which are expanded in a similar way as in the Jacobi–Davidson method. Differently from Ruhe's approach the projected problems inherit the symmetry from the original problem (1). Moreover, if the eigenvalues of problem (1) can be characterized as minmax values of a Rayleigh functional then the same holds for the eigenvalues of the projected problem, and it can be solved efficiently by a safeguarded inverse iteration method.

Our paper is organized as follows. Section 2 summarizes minmax properties for nonlinear eigenvalue problems, and Section 3 briefly reviews numerical methods for finite dimensional nonlinear problems. Section 4 introduces the modification of the Jacobi–Davidson method for nonlinear eigenvalue problems, it discusses the solution of the correction equation and a restart procedure to keep the storage requirements at a reasonable size and which can be used to purge unwanted directions as well. In Section 5 we report on our numerical experience for a rational eigenvalue problem governing the the vibrations of a tube bundle immersed in an inviscid compressible fluid.

2 Minmax characterization of eigenvalues

We consider the nonlinear eigenvalue problem $T(\lambda)x = 0$ where $T(\lambda) \in \mathbb{R}^{n\times n}$ is a family of real symmetric matrices for every λ in an open real interval J which may be unbounded.

For a linear symmetric problem $Ax = \lambda x$ all eigenvalues are real, and if they are ordered by magnitude $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ then it is well known that they can be characterized by the minmax principle of Poincaré or by the maxmin principle of Courant and Fischer.

Similar results hold for certain nonlinear eigenvalue problems, too. We assume that the function $f(\lambda, x) := x^T T(\lambda)x$ is continuously differentiable on $J \times \mathbb{R}^n$,

and that for every fixed $x \in \mathbb{R}^n \setminus \{0\}$ the real equation

$$f(\lambda, x) = 0 \tag{2}$$

has at most one solution in J. Then equation (2) implicitly defines a functional p on some subset D of $\mathbb{R}^n \setminus \{0\}$ which replaces the Rayleigh quotient in the variational characterization of eigenvalues of problem (1), and which we call the Rayleigh functional.

Moreover, we assume that

$$\frac{\partial}{\partial\lambda} f(\lambda, x)|_{\lambda=p(x)} > 0 \quad \text{for every } x \in D.$$

For nonlinear eigenvalue problems variational properties using the Rayleigh functional were proved by Duffin, Rogers, Hadeler, and Werner for overdamped problems, i.e. if the Rayleigh functional p is defined in the entire space $\mathbb{R}^n \setminus \{0\}$. Nonoverdamped problems were studied by Barston for quadratic problems, and by Werner and the author for general problems (c.f. [15] and the literature given therein).

In the general case the natural enumeration for which the smallest eigenvalue is the first one, the second smallest is the second one, etc. is not appropriate, but the number of an eigenvalue λ of the nonlinear problem (1) is inherited from the location of the eigenvalue 0 in the spectrum of the matrix $T(\lambda)$.

If $\lambda \in J$ is an eigenvalue of problem (1) then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $k \in \mathbb{N}$ such that

$$0 = \max_{V \in S_k} \min_{v \in V_1} v^T T(\lambda) v$$

where S_k denotes the set of all k–dimensional subspaces of $\mathbb{R}^n$ and $V^1 := \{v \in V : \|v\| = 1\}$ is the unit sphere in V. In this case we call λ a k-th eigenvalue of (1).

With this enumeration the following minmax characterization of the eigenvalues of the nonlinear eigenproblem (1) was proved in [15] (under an additional compactness condition even for the infinite dimensional case):

Theorem 1: *Under the conditions given above the following assertions hold:*

(i) For every $k \in \mathbb{N}$ there is at most one k-th eigenvalue of problem (1) which can be characterized by

$$\lambda_k = \min_{\substack{V \in S_k \\ V \cap D \neq \emptyset}} \max_{v \in V \cap D} p(v). \tag{3}$$

The set of eigenvalues of (1) in J is at most countable.

(ii) *If*

$$\lambda_k = \inf_{\substack{V \in S_k \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v) \in J$$

for some $k \in \mathbb{N}$ then λ_k is the k-th eigenvalue of (1) and the inf *and* sup *are attained by some $V \in S_k$ and some $v \in V \cap D$, i.e. (3) holds.*

(iii) *If for $k < \ell$ there exist a k-th and an ℓ-th eigenvalue λ_k and λ_ℓ in J, then J contains an m-th eigenvalue λ_m for $m = k, \ldots, \ell$, and $\lambda_k \leq \lambda_{k+1} \leq \ldots \lambda_\ell$.*

3 Solving nonlinear eigenvalue problems

In this section we briefly review numerical methods for finite dimensional nonlinear eigenvalue problems $T(\lambda)x = 0$, where $T(\lambda)$ is a family of real symmetric $n \times n$-matrices. We only consider methods for the general problem (1), and do not take into account the rich literature on quadratic or polynomial λ-matrices.

For dense problems algorithms are investigated in [6], [8] and [13] which are all variants of inverse iteration

$$x^{k+1} = \alpha_k T(\lambda_k)^{-1} T'(\lambda_k) x^k. \tag{4}$$

Here α_k is a suitable normalization factor, and λ_k is updated in some way. Similarly as in the linear case inverse iteration converges locally. The convergence is quadratical for simple eigenvalues, and it is even cubic if λ_k is updated by the Rayleigh functional.

Moreover, under the conditions of Section 2 inverse iteration can be safeguarded in a similar way as for linear eigenproblems. Assume that problem (1) has an m-th eigenvalue $\lambda_m \in J$, and let $T(\lambda) = L(\lambda)D(\lambda)L(\lambda)^T$ be an LDL^T-factorization of $T(\lambda)$ for some $\lambda \in J$. If the number $d^+(\lambda)$ of positive diagonal elements of $D(\lambda)$ is less than m, then it can be shown that $\lambda < \lambda_m$, and if $d^+(\lambda) \geq m$ then $\lambda \geq \lambda_m$ (cf. [15]).

An essential disadvantage of inverse iteration is the fact that each eigenvalue has to be determined individually by an iterative process, and that each step of this iteration requires the solution of a linear system. Moreover, the coefficient matrix $T(\lambda_k)$ of system (1) changes in each step, and in contrast to the linear case replacing (4) by $x^{k+1} = \alpha_k T(\sigma)^{-1} T'(\lambda_k) x^k$ with a fixed shift σ results in convergence to an eigenpair of the linear system $T(\sigma)x = \gamma T'(\tilde{\lambda})x$ ($\gamma \neq 0$ depending on the normalization condition) from which we can not recover an eigenpair of the nonlinear problem (1).

A remedy against this wrong convergence was proposed by Neumaier [6] who introduced the so called residual inverse iteration which converges linearly with a fixed shift, and quadratically or cubically if the coefficient matrix is altered in every iteration step according to reasonable updates of λ_k.

Closely related to safeguarded inverse iteration and of similar cost for small dimensions is the following method introduced in [13]: For given $\sigma_k \in J$ determine an eigenvector x_k corresponding to the m-largest eigenvalue of $T(\sigma_k)$, and evaluate $\sigma_{k+1} := p(x_k)$. This method converges locally to the m-th eigenvalue $\lambda_m \in J$, and the convergence is quadratic. For positive definite $T'(\lambda)$ for $\lambda \in J$ it is even cubic, if x_k is an eigenvector of the general eigenvalue problem $T(\sigma_k)x_k = \mu_k T'(\sigma_k)x_k$ corresponding to the m-largest eigenvalue μ_k.

For dense problems inverse iteration is a very capable method, however, for large and sparse nonlinear eigenvalue problems inverse iteration is much too expensive.

For sparse linear eigenvalue problems the most efficient methods are iterative projection methods, where approximations of the wanted eigenvalues and corresponding eigenvectors are obtained from projections of the eigenproblem to subspaces which are expanded in the course of the algorithm. Methods of this type for symmetric problems are the Lanczos method, rational Krylov subspace methods, and the Jacobi–Davidson method, e.g. (cf. [1]).

In some sense, Ruhe [9] and Hager and Wiberg [5], [4] generalized this approach to sparse nonlinear eigenvalue problems by nesting the linearization of problem (1) by Regula falsi and the solution of the resulting linear eigenproblem by Arnoldi's method, where the Regula falsi iteration and the Arnoldi recursion are knit together. Similarly as in the rational Krylov process they construct a sequence V_k of subspaces of $\mathbb{R}^n$ and corresponding Hessenberg matrices H_k which approximate the projection of $T(\sigma)^{-1}T(\lambda_k)$ to V_k. Here σ denotes a shift and λ_k an approximation to the wanted eigenvalue of (4). Then a Ritz vector of H_k corresponding to an eigenvalue of small modulus approximates an eigenvector of the nonlinear problem from which a (hopefully) improved eigenvalue approximation of problem (1) is obtained.

In [4] Hager points out that the eigenvalues and eigenvectors are determined one after another. After a Ritz value has converged only the approximate eigenvectors from previous Arnoldi runs, the just converged Ritz vector, and an approximation to a further eigenvector to be computed in the next Arnoldi run are kept, and the rest of the current Krylov space is purged. Hence, each eigenvalue of (1) is determined by an individual approximate Arnoldi process essentially from scratch, and therefore the cost of the rational Krylov method for nonlinear problems is similar to the cost of inverse iteration.

A further severe drawback of the approach of Ruhe, Hager and Wiberg is the

fact that one does not take advantage of symmetry properties of the nonlinear problem (1).

4 A Jacobi–Davidson type projection method

In this section we describe an algorithm which combines the fast convergence of safeguarded inverse iteration with the efficiency of iterative projection methods. It is strongly related to the Jacobi–Davidson method for linear eigenvalue problems.

4.1 The Jacobi–Davidson Method for Linear Eigenvalue Problems

The Jacobi–Davidson method introduced by Sleijpen and van der Vorst (cf. [10], [11]) for the linear eigenvalue problem

$$Ax = \lambda x \tag{5}$$

is an iterative projection method. If V is a matrix with orthonormal columns, and (σ_k, v_k) is an eigenpair of the projected problem $V^T A V v = \sigma v$, then the corresponding Ritz pair (σ_k, u_k), $u_k = V v_k$, approximating an eigenpair of (5) is improved as follows. The matrix V is expanded by an orthogonal correction t of u_k, and V is replaced by $[V, t]$.

The most desirable orthogonal correction t solves the equation

$$A(u_k + t) = \lambda(u_k + t), \quad t \perp u_k. \tag{6}$$

As $t \perp u_k$ the operator A can be restricted to the subspace orthogonal to u_k yielding $(I - u_k u_k^T)A(I - u_k u_k^T)$, and (6) can be rewritten as

$$(I - u_k u_k^T)(A - \lambda I)(I - u_k u_k^T)t = -(A - \sigma_k I)u_k.$$

Here we assumed that u_k is normalized by $\|u_k\| = 1$.

Approximating the unknown λ by σ_k we finally arrive at the Jacobi–Davidson correction equation for the update $t \perp u_k$:

$$(I - u_k u_k^T)(A - \sigma_k I)(I - u_k u_k^T)t = -r_k \tag{7}$$

where $r_k := (A - \sigma_k I)u_k$ denotes the residual of (σ_k, u_k).

Implementation details of the Jacobi–Davidson method for various types of eigenvalue problems can be found in [1]. In particular, numerical experiments show (cf. [12]) that the correction equation only needs to be solved approximately. Normally only a small number of steps of a preconditioned Krylov subspace method are sufficient to obtain a good expansion t for the subspace V.

4.2 Modifications for Nonlinear Eigenvalue Problems

To extend the idea of the Jacobi–Davidson method to nonlinear eigenvalue problems of type (1) we use the correction equation

$$(I - \frac{p_k u_k^T}{u_k^T p_k})T(\sigma_k)(I - \frac{u_k u_k^T}{u_k^T u_k})t = -r_k, \quad t \perp u_k, \tag{8}$$

where $p_k := T'(\sigma_k)u_k$ and $r_k := T(\sigma_k)u_k$, and (σ_k, u_k) is the current approximation to an eigenpair of $T(\lambda)x = 0$ obtained from a projected problem

$$V^T T(\lambda) V v = 0, \quad u = Vv.$$

Equation (8) can be understood in the following way. It can be rewritten as

$$T(\sigma_k)t - \alpha p_k = -r_k,$$

where α is chosen such that $t \perp u_k$. Solving for t we obtain

$$t = -u_k + \alpha T(\sigma_k)^{-1} p_k = -u_k + \alpha T(\sigma_k)^{-1} T'(\sigma_k) u_k, \tag{9}$$

and since $u_k = Vx_k$ for some x_k and t is orthogonalized against V to extend the subspace, we can write equation (9) as

$$t = \alpha T(\sigma_k)^{-1} T'(\sigma_k) u_k.$$

This equation demonstrates that the space spanned by the columns of V is expanded by the direction obtained from inverse iteration, which converges cubically if σ_k is chosen as Rayleigh-Functional of u_k. Thus cubic convergence can be expected if equation (8) is solved exactly. Numerical experiments have shown that even a moderate approximate solution of (8) with some steps of preconditioned GMRES also leads to fast convergence which is nearly cubic.

The view of Jacobi–Davidson as accelerated inverse iteration can be found in [10] for the linear case. Equation (8) is already used in [2] for quadratic eigenvalue problems. There the projected problem is solved by linearization of the projected quadratic problem. Our approach is to combine the correction

equation (8) with guarded inverse iteration for the projected problem to solve general nonlinear eigenvalue problems of type (1). The resulting method is given in Algorithm 1.

Algorithm 1 Nonlinear Jacobi-Davidson with guarded inverse iteration

1: Start with $V = v_1/\|v_1\|$
2: $n = 1$, $k = 1$
3: **while** $n \leq$ Number of wanted Eigenvalues **do**
4: Compute the n largest eigenvalue λ_n and the corresponding eigenvector x_n of $V^T T(\lambda) V x = 0$ with guarded inverse iteration.
5: $\sigma_k = \lambda_n$, $u_k = V x_n$
6: **if** $\|T(\sigma_k)u_k\|/\|u_k\| < \epsilon$ **then**
7: PRINT σ_k,u_k
8: $n = n + 1$
9: GOTO 3
10: **end if**
11: Find an approximate solution for the correction equation

$$(I - \frac{p_k u_k^T}{u_k^T p_k}) T(\sigma_k) (I - \frac{u_k u_k^T}{u_k^T u_k}) t = -r_k.$$

12: $t = t - VV^T t$,$\tilde{v} = t/\|t\|$, $V = [V, \tilde{v}]$
13: If necessary perform a purge-operation to reduce the size of the subspace V.
14: $k = k + 1$
15: **end while**

Remarks

1-2: v_1 is an initial approximation to the eigenvector corresponding to the first eigenvalue of (1). If the algorithm shall not start to iterate for the first eigenvalue of (1), but for the k-th eigenvalue, then a suitable k-dimensional start space must be given to the algorithm.
4-5: Here the guarded inverse iteration for nonlinear eigenvalue problems, which is described in Section 3 is used. It can be performed with low cost because the dimension of the projected problem is small.
11: Will be discussed in Subsection 4.3
12: Due to the better numerical stability it is preferable to use modified Gram-Schmidt in implementations of the algorithm.
13: See Subsection 4.4

4.3 Solving the correction equation

The correction equation (8) is solved approximately with some steps of GMRES with an appropriate preconditioner K for $T(\sigma_k)$.

The operator $T(\sigma_k)$ is restricted to map the subspace $\{u_k\}^\perp$ to $\{T'(\sigma_k)u_k\}^\perp$ in the correction equation (8). Therefore, the preconditioner also has to be modified, and instead of K we use the preconditioner

$$\tilde{K} := (I - \frac{p_k u_k^T}{u_k^T p_k}) K (I - \frac{u_k u_k^T}{u_k^T u_k})$$

for the restricted operator

$$\tilde{T}(\sigma_k) := (I - \frac{p_k u_k^T}{u_k^T p_k}) T(\sigma_k) (I - \frac{u_k u_k^T}{u_k^T u_k}).$$

With left-preconditioning equation (8) becomes

$$\tilde{K}^{-1}\tilde{T}(\sigma_k)t = -\tilde{K}^{-1}r_k, \quad t \perp u_k. \tag{10}$$

We apply a Krylov solver to equation (10) with initial guess $t = 0$. For the linear case this was already discussed in [12], and the transfer to equation (10) is straight forward.

Since the operator $\tilde{K}^{-1}\tilde{T}(\sigma_k)$ maps the space $\{u_k\}^\perp$ into itself all iterates are contained in $\{u_k\}^\perp$, and therefore in each step we have to perform the matrix-vector product

$$y = \tilde{K}^{-1}\tilde{T}(\sigma_k)v \tag{11}$$

for some $v \in \{u_k\}^\perp$.

This can be done in 2 steps. First multiply v by $\tilde{T}(\sigma_k)$ which yields

$$\tilde{y} = (I - \frac{p_k u_k^T}{u_k^T p_k}) T(\sigma_k) v,$$

and then solve

$$\tilde{K}y = \tilde{y}, \quad y \perp u_k.$$

This equation can be rewritten as

$$Ky - \alpha p_k = \tilde{y},$$

where α is determined from the condition that $y \perp u_k$. Thus, we finally obtain

$$y = K^{-1}\tilde{y} - \frac{u_k^T K^{-1}\tilde{y}}{u_k^T K^{-1} p_k} K^{-1} p_k \tag{12}$$

To conclude, the approximate solution of the the linear system (8) by a preconditioned Krylov solver requires one matrix-vector product to get $K^{-1}p_k$, and one more product in each step to obtain $K^{-1}\tilde{y}$. Therefore, taking into account the projections in the correction equation raises the number of matrix-vector multiplications only by one.

4.4 Restarts

As the subspaces expand in the course of the algorithm the increasing storage or the computational overhead may make it necessary to restart and purge some of the basis vectors. An obvious way to restart motivated by the linear theory is to take an orthonormal basis of the space spanned by the eigenvectors of (1) already computed. However, for nonlinear problems this space is not appropriate.

The proof of the minmax characterization

$$\lambda_\ell = \min_{\substack{W \in S_\ell \\ W \cap D \neq \emptyset}} \max_{w \in W \cap D} \; p(w)$$

in [15] shows that the minimum is attained by the invariant subspace W of $T(\lambda_\ell)$ corresponding to the ℓ largest eigenvalues. Hence, if W denotes an orthonormal basis of W as well the ℓ-th eigenvalue of the projected problem

$$W^T T(\lambda) W x = 0$$

is λ_ℓ, and W or any subspace of $\mathbb{R}^n$ containing W yields a perfect restart.

If σ_k is the current approximation to λ_ℓ obtained from the projection of (1) using the matrix V with orthonormal columns we therefore determine the matrix S of eigenvectors corresponding to the ℓ largest eigenvalues of

$$V^T T(\sigma_k) V x = \mu x \tag{13}$$

and continue with the reduced basis $\tilde{V} = VS$. If we restart with this ℓ dimensional space we may cut off valuable information that is contained in the discarted part of the subspace V. Actually in one of our examples we observed that it took unusually many steps to determine the next eigenvalue

after a restart. We therefore continued with $\tilde{\ell}$ eigenvectors of (13) where $\tilde{\ell}$ is slightly larger than ℓ, say $\tilde{\ell} = \ell + 3$.

If $T'(\lambda)$ is positive definite then we replace S in the last paragraph by the matrix containing ℓ or $\tilde{\ell}$ eigenvectors of generalized matrix eigenvalue problem

$$V^T T(\sigma_k) V x = \mu V^T T'(\sigma_k) V x$$

of the ℓ or $\tilde{\ell}$ largest eigenvalues. This modification of the restart procedure can be motivated by the linear maxmin theory. λ_ℓ is an ℓ-th eigenvalue and x_ℓ a corresponding eigenvector of problem (1) if and only if $\mu_\ell = 0$ is the ℓ-largest eigenvalue of the linear problem

$$T(\lambda_\ell) x = \mu x$$

and x_ℓ is a corresponding eigenvector.

If σ_k is a good approximation to λ_ℓ, then

$$T(\lambda_\ell) \approx T(\sigma_k) - \eta T'(\sigma_k)$$

is a first order approximation, and we can approximate (1) by the generalized eigenproblem

$$(T(\sigma_k) - \eta T'(\sigma_k)) x_\ell = 0. \tag{14}$$

The ℓ largest eigenvalue η_ℓ is near 0, if σ_k is a good approximation to λ_ℓ, and it can be characterized by a maxmin principle

$$0 \approx \eta_\ell = \max_{V \in S_\ell} \min_{v \in V \setminus \{0\}} \frac{v^T T(\sigma_k) v}{v^T T'(\sigma_k) v}.$$

The maximum is attained by the subspace spanned by the eigenvectors corresponding to the ℓ largest eigenvalues of (14) which motivates the choice of S.

5 Numerical example

To test our method we consider a mathematical model which describes the problem governing free vibrations of a tube bundle immersed in a slightly compressible fluid under the following simplifying assumptions: The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to

be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity. Considering small vibrations of the fluid (and the tubes) around the state of rest, it can also be assumed that the fluid is irrotational.

Mathematically this problem can be described in the following way (cf. [7], [3]). Let $\Omega \subset \mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ. We assume that there exists a family $\Omega_j \neq \emptyset$, $j = 1, \ldots, K$, (the sections of the tubes) of simply connected open sets such that $\bar{\Omega}_j \subset \Omega$ for every j, $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $j \neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With these notations we set $\Omega_0 := \Omega \setminus \bigcup_{j=1}^{K} \Omega_j$. Then the boundary of Ω_0 consists of $K+1$ connected components which are Γ and Γ_j, $j = 1, \ldots, K$.

We denote by $H^1(\Omega_0) = \{u \in L^2(\Omega_0) : \nabla u \in L^2(\Omega_0)^2\}$ the standard Sobolev space equipped with the usual scalar product. Then the eigenfrequencies and the eigenmodes of the fluid-solid structure are governed by the following variational eigenvalue problem (cf. [7], [3])

Find $\lambda \in \mathbb{R}$ *and* $u \in H^1(\Omega_0)$ *such that for every* $v \in H^1(\Omega_0)$

$$c^2 \int_{\Omega_0} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega_0} uv \, dx + \sum_{j=1}^{K} \frac{\lambda \rho_0}{k_j - \lambda m_j} \int_{\Gamma_j} un \, ds \cdot \int_{\Gamma_j} vn \, ds. \tag{15}$$

Here u is the potential of the velocity of the fluid, c denotes the speed of sound in the fluid, ρ_0 is the specific density of the fluid, k_j represents the stiffness constant of the spring system supporting tube j, m_j is the mass per unit length of the tube j, and n is the outward unit normal on the boundary of Ω_0.

We consider the rational eigenvalue problem (15) where Ω is the ellipse with center $(0,0)$ and length of semiaxes 8 and 4, and Ω_j, $j = 1, \ldots, 9$ are circles with radius 0.3 and centers $(-4,-2)$, $(0,-2)$, $(4,-2)$, $(-5,0)$, $(0,0)$, $(5,0)$, $(-4,2)$, $(0,2)$ and $(4,2)$. We assume that all constants in problem (15) are equal to 1.

Discretizing problem (15) by finite elements one gets a rational matrix eigenvalue problem

$$T(\lambda)x := -Ax + \lambda Bx + \frac{\lambda}{1-\lambda} Cx = 0 \tag{16}$$

where C collects the contributions of all tubes. A, B, and C are symmetric

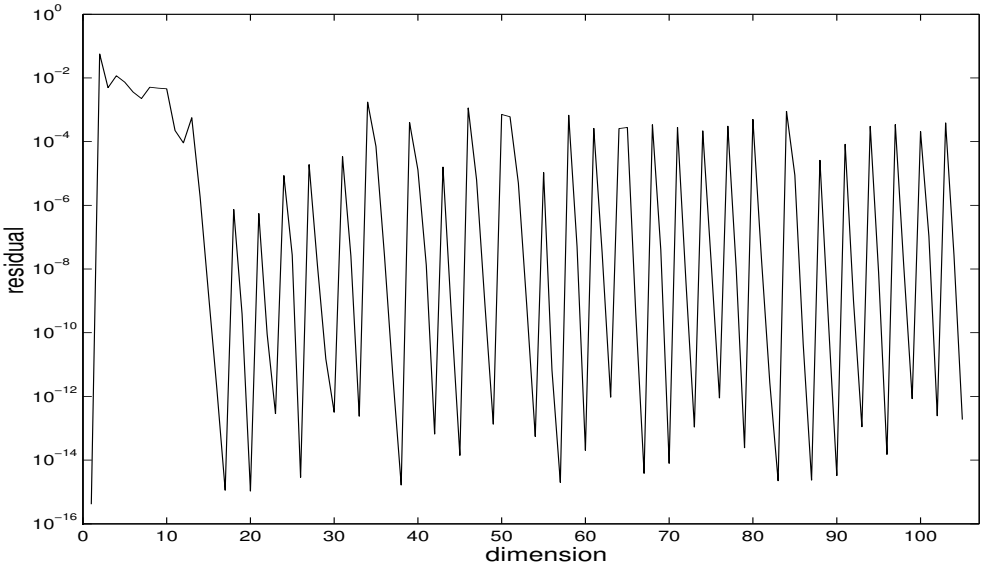

Fig. 1: Eigenvalues in (0,1); no restarts

matrices, A and C are positive semidefinite, and B is positive definite. In our example the dimension is $n = 36040$.

Problem (16) has 28 eigenvalues $\lambda_1 \leq \ldots \leq \lambda_{28}$ in the interval $J_1 = (0, 1)$ and an infinite number of eigenvalues $\tilde{\lambda}_{11} \leq \tilde{\lambda}_{12} \leq \ldots$ in $(1, \infty)$, 20 of which are contained in $J_2 := (1, 3)$.

We determined the approximations to the eigenvalues $\lambda_1, \ldots, \lambda_{28} \in J_1$ without restarts. We terminated the iteration for an eigenvalue if the relative residual $\|T(\sigma_k)u_k\|/\|u_k\|$ was less than 10^{-12}.

We solved the correction equations by preconditioned GMRES, and we terminated the GMRES iteration after at most 10 steps or if the initial residual was reduced by a factor 10^{-3}. As preconditioner we chose the LU factorization of $T(\sigma)$ since this could be obtained quite inexpensively in our example. We brushed the preconditioner up and replaced it by the LU factorization of $T(\sigma_k)$ with the current eigenvalue approximation σ_k if GMRES was not able to obtain the required residual reduction by 10^{-3} within 5 steps. Since some iterations are needed to gather information we applied this rule beginning with the computation of λ_3.

With these parameters the algorithm needed 16 LU decompositions and 438 GMRES steps in total to determine all 28 eigenvalues in J_1. The dimension of the subspace V grew only to 106. The convergence history is given in Figure 1.

We repeated the computation of the eigenvalues in J_1 now restarting if the dimension exceeded 40 and reducing it to $\min(10, 3 + \#\text{of eigenvalue})$. This time the algorithm needed 13 LU factorizations and 470 GMRES steps. The

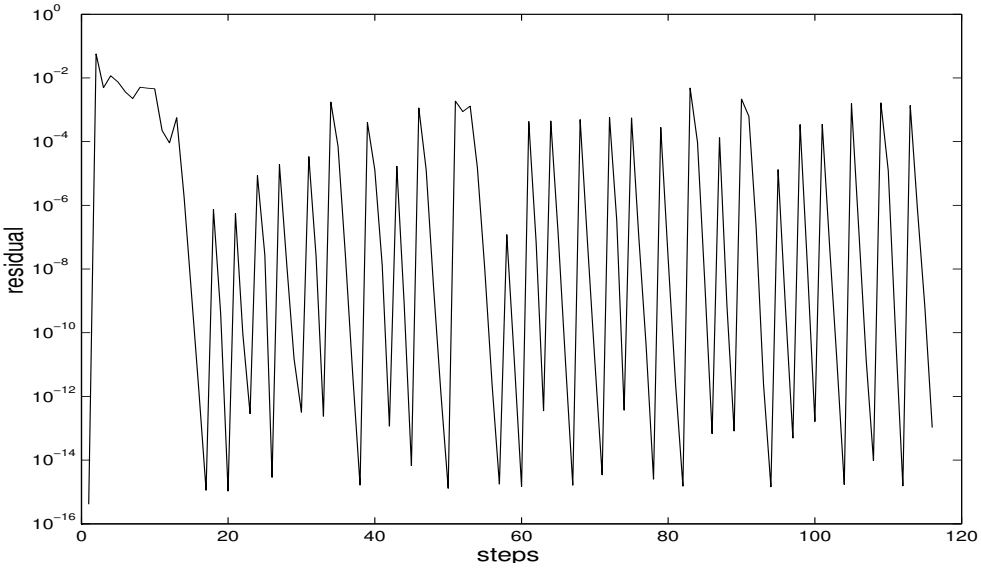

Fig. 2: Eigenvalues in (0,1); restarted

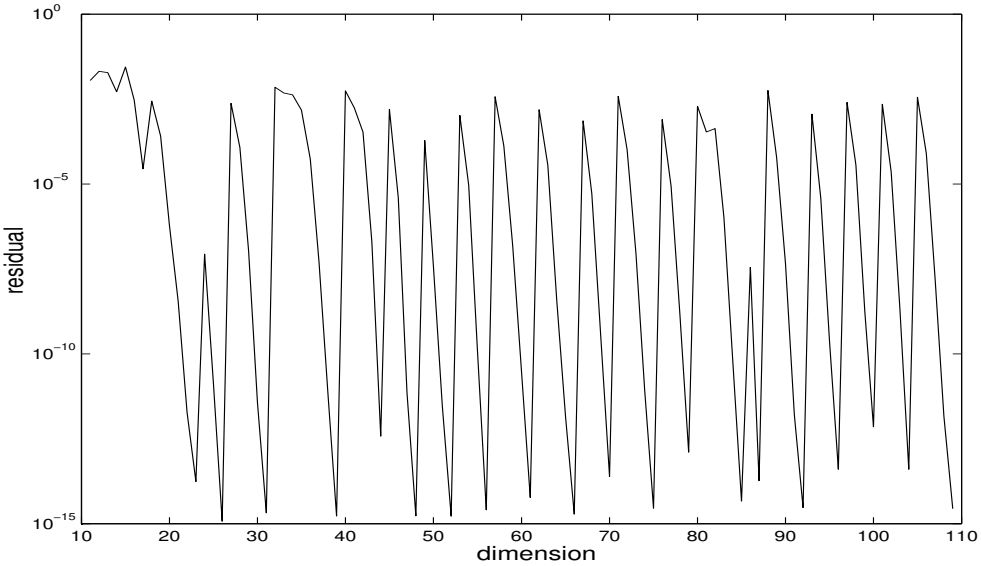

Fig. 3: Eigenvalues in (1,3); no restarts

convergence history in Figure 2 is very similar to that one without restarts.

Finally we determined the eigenvalues in the interval $J_2 := (1, 3)$. Since the smallest eigenvalue in this interval is an eleventh eigenvalue we needed a subspace of dimension 11 that approximates the invariant subspace of $T(\tilde{\lambda}_{11})$ corresponding to the 11 largest eigenvalues to start the algorithm. Not knowing $\tilde{\lambda}_{11}$ we started with the 11 dimensional subspace $Kx = \mu Mx$ corresponding to the 11 smallest eigenvalues. In this case the algorithm needed 15 LU factorizations and 340 GMRES steps. The convergence history is given in Figure 3.

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