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ORIGINAL PAPER

The abstract Cauchy problem in locally convex spaces

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Abstract
We derive necessary and sufficient criteria for the uniqueness and existence of solutions of the abstract Cauchy problem in locally convex Hausdorff spaces. Our approach is based on a suitable notion of an asymptotic Laplace transform and extends results of Langenbruch beyond the class of Fréchet spaces.

Keywords Abstract Cauchy problem · Asymptotic Laplace transform · Asymptotic resolvent · Hyperfunction

Mathematics Subject Classification 34A12 · 47A10 · 46F15 · 44A10

1 Introduction

We study the abstract Cauchy problem in locally convex Hausdorff spaces in the present paper. This is an initial value problem of the form

$$\begin{aligned} x'(t) &= Ax(t), \quad t > 0, \\ x(0) &= x_0 \in E, \end{aligned}$$

where $A\colon D(A) \subset E \to E$ is a sequentially closed linear operator and E a sequentially complete locally convex Hausdorff space over $\mathbb{C}$.

One of the approaches to tackle the abstract Cauchy problem is the theory of C_0-semigroups. The classical theory of C_0-semigroups on Banach spaces (see e.g. [14] and the references therein) has already been extended in several ways. Beyond the realm of Banach spaces it was extended to equicontinuous C_0-semigroups on locally convex Hausdorff spaces in [1, 26, 65, Chap. IX], quasi-equicontinuous C_0-semigroups in [1, 3, 8, 20, 52–54], locally equicontinuous C_0-semigroups in [9, 32, 57], sequentially (locally) equicontinuous C_0-semigroups in [16] and smooth semigroups on convenient algebras in [61].

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Besides the extension of the theory of C_0-semigroups to locally convex Hausdorff spaces the continuity assumptions were weakened as well. Bi-continuous semigroups were introduced in [40, 41], (locally equi-)tight bi-continuous semigroups in [15], integrable semigroups considered in [43], integrated semigroups in [63, 64], distribution semigroups on Banach spaces in [7, 25, 42, 45, 51] and even on locally convex Hausdorff spaces in [58, 59], and (Fourier) hyperfunction semigroups for Banach spaces in [55, 56] (and [18, 19]). We note that most of the classical bi-continuous semigroups are actually quasi-equicontinuous C_0-semigroups with respect to the mixed topology and even quasi-equitight by [33, Theorem 7.4, p. 180] and [39, 3.17 Theorem, p. 13, Section 4].

Apart from the theory of semigroups some of the classical methods for initial value problems were transferred to the setting in locally convex Hausdorff spaces in [46] and the references therein.

A common problem in the mentioned approaches to the abstract Cauchy problem is the development of a suitable notion of a Laplace transform for vector-valued generalised functions. In [4, 5, 47] an appropriate (asymptotic) Laplace transform was developed for Banach-valued locally integrable functions and applied to abstract Cauchy problems, and in [27–31] under minimal regularity assumptions for Banach-valued hyperfunctions as well as in [44] for Fréchet-valued hyperfunctions. This was extended in [38] beyond the class of Fréchet spaces to a large variety of locally convex Hausdorff spaces containing common spaces of distributions.

We use the asymptotic Laplace transform from [38] to study the abstract Cauchy problem for vector-valued hyperfunctions. After recalling the necessary notions and results from [38] in Sect. 2, we characterise the uniqueness of the solutions of the abstract Cauchy problem in Sect. 3, in particular, we derive necessary and sufficient conditions in Theorem 3.2. We use these conditions to phrase sufficient conditions for the uniqueness of solutions in terms of asymptotic left resolvents in Theorems 3.3 and 3.4, generalising corresponding results from [44] (for general notions of resolvents in locally convex Hausdorff spaces see [2] and [12]). In Sect. 4, we turn to the solvability of the abstract Cauchy problem for vector-valued hyperfunctions and present necessary and sufficient conditions for the solvability in Theorem 4.3. Here we use the Laplace transform for vector-valued Laplace hyperfunctions from [11] in combination with the asymptotic Laplace transform for vector-valued hyperfunctions from [38]. In Theorem 4.4, we give a sufficient condition for the solvability of the abstract Cauchy problem in terms of asymptotic right resolvents. Our results on the solvability extend the ones from [44].

2 Notation and preliminaries

We use essentially the same notation and preliminaries as in [38, Section 2]. In the following E is always a locally convex Hausdorff space over $\mathbb{C}$ equipped with a directed system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$, in short, E is a $\mathbb{C}$-lcHs. If E is a normed space, we often write $\|\cdot\|_E$ for the norm on E. We denote by $L(F, E)$ the space of continuous linear maps from a $\mathbb{C}$-lcHs F to E, write $L(F) := L(F, F)$, sometimes use the notion $\langle T, f\rangle := T(f)$, $f \in F$, for $T \in L(F, E)$ and the symbol T^{t} for the dual map of T. If $E = \mathbb{C}$, we write $F' := L(F, \mathbb{C})$ for the dual space of F. We denote by $L_b(F, E)$ the space $L(F, E)$ equipped with the locally convex topology of uniform convergence on the bounded subsets of F.

We denote by $\mathcal{O}(\Omega, E)$ the space of E-valued holomorphic functions on an open set $\Omega \subset \mathbb{C}$ and by $\mathcal{C}^\infty(\Omega, E)$ the space of E-valued infinitely continuously partially differentiable functions on an open set $\Omega \subset \mathbb{R}^2 = \mathbb{C}$. We denote by $\partial^\beta f$ the partial derivative of $f \in \mathcal{C}^\infty(\Omega, E)$ for a multiindex $\beta \in \mathbb{N}_0^2$. We denote by $\mathbb{C}_{\mathrm{Re}>0} := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ the right halfplane, by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ the two-point compactification of $\mathbb{R}$ and set $\overline{\mathbb{C}} := \overline{\mathbb{R}} + i\mathbb{R}$. We define the distance of $z \in \mathbb{C}$ to a set $M \subset \mathbb{C}$ w.r.t. the Euclidean norm $|\cdot|$ via $d(z, M) := \inf_{w\in M} |z - w|$ if $M \neq \emptyset$, and $d(z, M) := \infty$ if $M = \emptyset$. For a compact set $K \subset \overline{\mathbb{R}}$ and $c > 0$ we define the sets

$$U_{\frac{1}{c}}(K) := \{z \in \mathbb{C} \mid d(z, K \cap \mathbb{C}) < c\}$$
$$\cup \begin{cases} \emptyset, & K \subset \mathbb{R}, \\ ]1/c, \infty[+i] - c, c[, & \infty \in K,\ -\infty \notin K, \\ ] - \infty, -1/c[+i] - c, c[, & \infty \notin K,\ -\infty \in K, \\ (] - \infty, -1/c[\cup]1/c, \infty[) + i] - c, c[, & \infty \in K,\ -\infty \in K, \end{cases}$$

and for $n \in \mathbb{N}$

$$S_n(K) := \left(\mathbb{C} \setminus \overline{U_n(K)}\right) \cap \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < n\}.$$

Definition 2.1 [34, 3.2 Definition, p. 12–13] Let E be a $\mathbb{C}$-lcHs and $K \subset \overline{\mathbb{R}}$ compact.

(a) The space of vector-valued slowly increasing infinitely continuously partially differentiable functions outside K is defined as

$$\mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K, E) := \left\{ f \in \mathcal{C}^\infty(\mathbb{C} \setminus K, E) | \forall n \in \mathbb{N}, m \in \mathbb{N}_0, \alpha \in \mathfrak{A} : |||f|||_{n,m,\alpha,K} < \infty \right\}$$

where

$$|||f|||_{n,m,\alpha,K} := \sup_{\substack{z\in S_n(K) \\ \beta\in\mathbb{N}_0^2, |\beta|\leq m}} p_\alpha(\partial^\beta f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}.$$

(b) The space of vector-valued slowly increasing holomorphic functions outside K is defined as

$$\mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K, E) := \left\{ f \in \mathcal{O}(\mathbb{C} \setminus K, E) \mid \forall\, n \in \mathbb{N},\ \alpha \in \mathfrak{A} :\ |||f|||_{n,\alpha,K} < \infty \right\}$$

where

$$|||f|||_{n,\alpha,K} := \sup_{z\in S_n(K)} p_\alpha(f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}.$$

Furthermore, we set

$$bv_K(E) := \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E).$$

We note that $S_1(\overline{\mathbb{R}}) = \emptyset$ and $|||f|||_{1,m,\alpha,\overline{\mathbb{R}}} = -\infty = |||f|||_{1,\alpha,\overline{\mathbb{R}}}$ for any $f : \mathbb{C}\setminus\mathbb{R} \to E$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$. Other common symbols for the spaces $\mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K, E)$ resp. $\mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K, E)$ are $\widetilde{\mathcal{E}}(\overline{\mathbb{C}} \setminus K, E)$ resp. $\widetilde{\mathcal{O}}(\overline{\mathbb{C}} \setminus K, E)$ (see [22, 1.2 Definition, p. 5]).

Definition 2.2 [34, p. 55] A $\mathbb{C}$-lcHs E is called *admissible*, if the Cauchy–Riemann operator

$$\overline{\partial} : \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K, E) \to \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K, E)$$

is surjective for any compact set $K \subset \overline{\mathbb{R}}$. E is called *strictly admissible* if E is admissible and if, in addition,

$$\overline{\partial}\colon \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\Omega, E)$$

is surjective for any open set $\Omega \subset \mathbb{C}$.

If E is strictly admissible and sequentially complete, then the sheaf of E-valued Fourier hyperfunctions is flabby and can be represented by boundary values of exponentially slowly increasing holomorphic functions (see [35, Theorem 5.9, p. 33]). In particular, its subsheaf of E-valued hyperfunctions is flabby under this condition as well. Moreover, we may regard $bv_K(E)$ as the space of E-valued Fourier hyperfunctions with support in $K \subset \overline{\mathbb{R}}$ under this condition by [35, 5.11 Lemma, p. 44].

Theorem 2.3 [34, 5.25 Theorem, p. 98] *If*

(a) *E is a $\mathbb{C}$-Fréchet space, or if*
(b) *$E := F_b'$ where F is a $\mathbb{C}$-Fréchet space satisfying (DN), or if*
(c) *E is a complex ultrabornological PLS-space satisfying (PA),*

then E is strictly admissible.

The definitions of the topological invariants (DN) and (PA) are given in [50, Chap. 29, Definition, p. 359] and [6, Section 4, Eq. (24), p. 577], respectively. Besides every $\mathbb{C}$-Fréchet space, the theorem above covers the space $E = \mathcal{S}(\mathbb{R}^d)_b'$ of tempered distributions, the space $\mathcal{D}(V)_b'$ of distributions and the space $\mathcal{D}_{(\omega)}(V)_b'$ of ultradistributions of Beurling type and many more spaces given in [6], [10, Corollary 4.8, p. 1116], [35, Example 4.4, p. 14–15] and [62].

Definition 2.4 [38, 7.1 Definition, p. 106] Let E be a $\mathbb{C}$-lcHs and $a \in \{0, \infty\}$. We define the space

$$\mathcal{LO}_{[a,\infty]}(E) := \left\{ f \in \mathcal{O}(\mathbb{C}_{\mathrm{Re}>0}, E) \mid \forall\, k \in \mathbb{N},\ \alpha \in \mathfrak{A}:\ \|f\|_{k,\alpha,[a,\infty]} < \infty \right\}$$

where

$$\|f\|_{k,\alpha,[0,\infty]} := \sup_{\mathrm{Re}(z) \geq \frac{1}{k}} p_\alpha(f(z)) e^{-\frac{1}{k}|z|}$$

resp.

$$\|f\|_{k,\alpha,\{\infty\}} := \sup_{\mathrm{Re}(z) \geq \frac{1}{k}} p_\alpha(f(z)) e^{-\frac{1}{k}|z| + k\,\mathrm{Re}(z)}.$$

We omit the index α of the seminorms if E is a normed space, and write $\mathcal{LO}_{[a,\infty]} := \mathcal{LO}_{[a,\infty]}(\mathbb{C})$.

Let E be a sequentially complete $\mathbb{C}$-lcHs, $K := [0, \infty]$ or $K := \{\infty\}$ and equip $bv_K(E)$ with its usual quotient topology, which is Hausdorff locally convex by [37, Remark 14, p 22]. By [38, Theorem 7.2 (ii), p. 106] the Laplace transform

$$\mathcal{L} : bv_K(E) \to \mathcal{LO}_K(E), \mathcal{L}([F])(\zeta) := \int_{\gamma_K} F(z) e^{-z\zeta}\, dz,$$

where γ_K is the path along the boundary of $U_{1/c}(K)$ with clockwise orientation (see Fig. 1), does not depend on the choice of $c > 0$ and is a topological isomorphism.

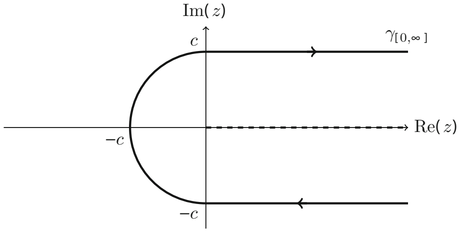


Fig. 1 Path $\gamma_{[0,\infty]}$ with $c > 0$ (cf. [38, Figure 1.2, p. 62])

Let E be an admissible $\mathbb{C}$-lcHs. Then the canonical (restriction) map

$$\mathcal{R}_{[0,\infty[}\colon \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus [0,\infty], E)/\mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus \{\infty\}, E) \to \mathcal{B}([0,\infty[, E),\ [F] \mapsto [F],$$

is a linear isomorphism by [38, Theorem 5.1, p. 96] where

$$\mathcal{B}([0,\infty[, E) := \mathcal{O}(\mathbb{C} \setminus [0,\infty[, E)/\mathcal{O}(\mathbb{C}, E)$$

is the space of hyperfunctions with values in E and support in $[0,\infty[$. The combination of both results leads to the following theorem.

Theorem 2.5 [38, 7.4 Theorem, p. 106] *Let E be an admissible sequentially complete $\mathbb{C}$-lcHs. Then the asymptotic Laplace transform*

$$\mathcal{L}^{\mathcal{B}}\colon \mathcal{B}([0,\infty[, E) \to \mathcal{LO}_{[0,\infty]}(E)/\mathcal{LO}_{\{\infty\}}(E),\ \mathcal{L}^{\mathcal{B}}(f) := [(\mathcal{L} \circ \mathcal{R}^{-1}_{[0,\infty[})(f)],$$

is a linear isomorphism.

3 Uniqueness of solutions of the ACP

In this section, we apply our results on the asymptotic Laplace transform of hyperfunctions with support in $[0,\infty[$ to the abstract Cauchy problem for hyperfunctions with values in an admissible (sequentially) complete $\mathbb{C}$-lcHs. We start with a generalisation of an abstract Cauchy problem for hyperfunctions given in [44, p. 60–61]. Let $(E, (p_\alpha)_{\alpha\in\mathfrak{A}})$ be a sequentially complete $\mathbb{C}$-lcHs. We call

$$x'(t) = Ax(t), \quad t > 0, \quad x(0) = x_0 \in E, \tag{1}$$

an *abstract Cauchy problem (ACP)* where

$$A\colon F := D(A) \subset E \to E$$

is a sequentially closed linear operator with domain $F := D(A)$. Then F is a sequentially complete $\mathbb{C}$-lcHs when equipped with the graph topology τ_A given by the seminorms $(p_{\alpha,A} := p_\alpha + p_\alpha(A\cdot))_{\alpha\in\mathfrak{A}}$, and $A\colon F_A := (F, \tau_A) \to E$ is continuous.

Remark 3.1 (a) If E is a $\mathbb{C}$-Fréchet space and A and F are as above, then $F_A = (F, \tau_A)$ is also a Fréchet space and thus (strictly) admissibile by Theorem 2.3.
(b) If E is a (strictly) admissible space, $F = E$ and $A\colon F \to E$ continuous, then $F_A = E$ as locally convex spaces and so F_A is (strictly) admissible.

An F-valued hyperfunction $[u] \in \mathcal{B}([0,\infty[, F_A)$ is called a *solution of the ACP* (1) (in the sense of hyperfunctions) if

$$\frac{d}{dt}[u] - A[u] = x_0 \otimes \delta_0 \tag{2}$$

where $\delta_0 := [z \mapsto -\frac{1}{2\pi i z}]$ is the Dirac hyperfunction (see [38, 4.11 Example, p. 96]), $x_0 \otimes \delta_0 := x_0\delta_0$, $\frac{d}{dt}[u] := [\frac{d}{dz}u]$ and $A[u] := [z \mapsto Au(z)]$.

We say that the ACP (1) has the *uniqueness property* (in the sense of hyperfunctions) if $[u] = 0$ is the only solution of (2) for $x_0 = 0$. Our next theorem generalises [44, Theorem 7.1, p. 61] and we note that its proof essentially remains the same.

Theorem 3.2 *Let E be an admissible sequentially complete $\mathbb{C}$-lcHs and $A\colon F := D(A) \subset E \to E$ a sequentially closed linear operator with admissible $F_A = (F, \tau_A)$. Then the following statements are equivalent:*

(a) *The ACP (1) has the uniqueness property (in the sense of hyperfunctions).*
(b) *If $h \in \mathcal{LO}_{[0,\infty]}(F_A)$ and $(z - A)h \in \mathcal{LO}_{\{\infty\}}(E)$, then $h \in \mathcal{LO}_{\{\infty\}}(F_A)$.*
(c) *If $h \in \mathcal{LO}_{[0,\infty]}(F_A)$ and $(z - A)h \in \mathcal{LO}_{\{\infty\}}(E)$, then $\{h(t)e^{nt} \mid t \geq \varepsilon\}$ is weakly bounded in F_A for any $n \in \mathbb{N}$ and any (some) $\varepsilon > 0$.*

Proof (a) $\Rightarrow$ (b): Due to Theorem 2.5 there is $[u] \in \mathcal{B}([0, \infty[, F_A)$ such that $[h] = \mathcal{L}^{\mathcal{B}}([u]) \in \mathcal{LO}_{[0,\infty]}(F_A)/\mathcal{LO}_{\{\infty\}}(F_A)$. By [38, 7.10 Proposition, p. 108] and our assumption we have

$$\mathcal{L}^{\mathcal{B}}\left(\frac{d}{dt}[u] - A[u]\right) = (z - A)\mathcal{L}^{\mathcal{B}}([u]) = (z - A)[h] = 0.$$

From Theorem 2.5 and (a) we deduce that $[u] = 0$, which implies $[h] = 0$ and thus $h \in \mathcal{LO}_{\{\infty\}}(F_A)$.

(b) $\Rightarrow$ (c): This follows from $h \in \mathcal{LO}_{\{\infty\}}(F_A)$ by (b) and the definition of the space $\mathcal{LO}_{\{\infty\}}(F_A)$.

(c) $\Rightarrow$ (a): Let $[u] \in \mathcal{B}([0, \infty[, F_A)$ such that $\frac{d}{dt}[u] - A[u] = 0$. Then $[h] := \mathcal{L}^{\mathcal{B}}([u])$ satisfies

$$0 = \mathcal{L}^{\mathcal{B}}\left(\frac{d}{dt}[u] - A[u]\right) = (z - A)[h]$$

and thus $(z - A)h \in \mathcal{LO}_{\{\infty\}}(E)$. Next, we show that $y \circ h \in \mathcal{LO}_{\{\infty\}}$ for any $y \in F_A'$ by the Phragmén–Lindelöf theorem. Let $k \in \mathbb{N}$ and set $S := \{z \in \mathbb{C} \mid -\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}\}$ and $S_0 := \{z \in \mathbb{C} \mid \operatorname{Re}(z) > \frac{1}{k}, \operatorname{Im}(z) > 0\}$. We define the homeomorphism $\theta\colon \overline{S} \to \overline{S}_0$ by $\theta(z) := e^{i\frac{\pi}{4}}z + \frac{1}{k}$ and the function

$$F_0\colon \overline{S} \to \mathbb{C},\ F_0(z) := (y \circ h)(\theta(z))e^{(k+\frac{i}{k})\theta(z)}.$$

Using $y \circ h \in \mathcal{LO}_{[0,\infty]}$, we have for every $z \in \overline{S}$ that

$$\begin{aligned}|F_0(z)| &= |(y \circ h)(\theta(z))|e^{k\operatorname{Re}(\theta(z)) - \frac{1}{k}\operatorname{Im}(\theta(z))}\\ &\leq |(y \circ h)(\theta(z))|e^{-\frac{1}{k}|\theta(z)| + (k+\frac{1}{k})|\operatorname{Re}(\theta(z))|}\\ &\leq |y \circ h|_{k,[0,\infty]}e^{(k+\frac{1}{k})|\theta(z)|} \leq e^{1+\frac{1}{k^2}}|y \circ h|_{k,[0,\infty]}e^{(k+\frac{1}{k})|z|}.\end{aligned}$$

If $\arg(z) = -\frac{\pi}{4}$, then $\theta(z) = |z| + \frac{1}{k}$ and by part (c) there is $\varepsilon > 0$ such that with $\varepsilon_k := \max(0, \varepsilon - \frac{1}{k})$ we get

$$\begin{aligned}&|F_0(z)|\\ &= |(y \circ h)(\theta(z))|e^{k\theta(z)}\\ &\leq e^{k\varepsilon_k + 1}\max_{\substack{\arg(w) = -\frac{\pi}{4}\\ |w| \leq \varepsilon_k}}|(y \circ h)(\theta(w))| + \sup_{\substack{\arg(w) = -\frac{\pi}{4}\\ |w| \geq \varepsilon_k}}|(y \circ h)(\theta(w))|e^{k\theta(w)} =: C_0 < \infty\end{aligned}$$

where we use the continuity of $y \circ h \circ \theta$ as well. If $\arg(z) = \frac{\pi}{4}$, then $\theta(z) = i|z| + \frac{1}{k}$ and we get

$$|F_0(z)| = |(y \circ h)(\theta(z))|e^{1-\frac{1}{k}|z|} \le e^{1+\frac{1}{k^2}}|(y \circ h)(\theta(z))|e^{-\frac{1}{k}|\theta(z)|}$$
$$\le e^{1+\frac{1}{k^2}}|y \circ h|_{k,[0,\infty]} =: C_1 < \infty.$$

Due to the Phragmén–Lindelöf theorem [60, Theorem 3.4, p. 124] (applied to $F(z) := \frac{1}{\max(C_0,C_1)}F_0(z)$) we obtain

$$|F_0(z)| \le \max(C_0, C_1) =: C_2, \ z \in \overline{S},$$

and hence

$$|(y \circ h)(\theta(z))| \le C_2 e^{-k\operatorname{Re}(\theta(z))+\frac{1}{k}\operatorname{Im}(\theta(z))} \le C_2 e^{-k|\operatorname{Re}(\theta(z))|+\frac{1}{k}|\theta(z)|}, \ z \in \overline{S},$$

which implies

$$\sup_{z\in\overline{S}_0} |(y \circ h)(z)|e^{-\frac{1}{k}|z|+k|\operatorname{Re}(z)|} \le C_2 < \infty.$$

Similarly, we get

$$\sup_{z\in\overline{S}_1} |(y \circ h)(z)|e^{-\frac{1}{k}|z|+k|\operatorname{Re}(z)|} < \infty$$

for $S_1 := \{z \in \mathbb{C} \mid \operatorname{Re}(z) > \frac{1}{k}, \operatorname{Im}(z) < 0\}$ by choosing $\theta_1 : \overline{S} \to \overline{S}_1$, $\theta_1(z) := e^{-i\frac{\pi}{4}}z + \frac{1}{k}$, and

$$F_1 : \overline{S} \to \mathbb{C}, \ F_1(z) := (y \circ h)(\theta_1(z))e^{(k-\frac{i}{k})\theta_1(z)}.$$

We conclude that $y \circ h \in \mathcal{LO}_{\{\infty\}}$. The weak-strong principle [36, 3.20 Corollary c), p. 14] yields $h \in \mathcal{LO}_{\{\infty\}}(F_A)$ since $F_A = (F, \tau_A)$ is sequentially complete and $\mathcal{LO}_{\{\infty\}}$ a nuclear Fréchet space by [38, 7.3 Proposition, p. 106]. Hence $[u] = 0$ by Theorem 2.5. □

Now, we generalise Langenbruch's sufficient criterion [44, Theorem 7.2, p. 62] for the uniqueness property, which itself is a generalisation of Lyubich's uniqueness theorem [49, Theorem 9.2, p. 40]. For this purpose we adapt the notion of an asymptotic left resolvent given by Langenbruch [44, p. 62] (for general notions of resolvents in locally convex spaces see [2] and [12]). Let $A : F := D(A) \subset E \to E$ be a sequentially closed linear operator. We denote by $E_\alpha := (E/\ker p_\alpha)\widehat{}$ and $F_\alpha := (F/\ker p_{\alpha,A})\widehat{}$ the canonical local Banach spaces for p_α resp. $p_{\alpha,A}$ and by $\|x+\ker p_\alpha\|_\alpha := p_\alpha(x)$, $x \in E$, resp. $\|x+\ker p_\alpha\|_{\alpha,A} := p_{\alpha,A}(x)$, $x \in F$, the norms on $E/\ker p_\alpha$ resp. $F/\ker p_{\alpha,A}$, which we extend to norms on the local Banach spaces with the same symbol. Further, we denote by $\kappa_\alpha^F : F_A \to F_\alpha$, $x \mapsto x + \ker p_{\alpha,A}$, the corresponding spectral map of F_A for $\alpha \in \mathfrak{A}$. A set of operators $(R_\alpha(t, A))_{\alpha\in\mathfrak{A}}$ is an *asymptotic left resolvent* if for all $\alpha \in \mathfrak{A}$ there is $t_\alpha > 0$ such that $R_\alpha(t, A) \in L(E, F_\alpha)$ for all $t \ge t_\alpha$ and the continuous linear map $S_\alpha(t) : F_A \to F_\alpha$ given by

$$S_\alpha(t) := R_\alpha(t, A)(t - A) - \kappa_\alpha^F, \quad t \ge t_\alpha, \tag{3}$$

fulfils

$$\forall\, n \in \mathbb{N}\ \exists\, \beta \in \mathfrak{A},\ t_{\alpha,n} \ge t_\alpha,\ C_1, C_2 > 0\ \forall\, t \ge t_{\alpha,n},\ x \in F :$$
$$\|S_\alpha(t)x\|_\alpha \le C_1 p_{\beta,A}(x) \quad \text{and} \quad \|S_\alpha^\beta(t)\|_{L(F_\beta,F_\alpha)} \le C_2 e^{-nt} \tag{4}$$

where $S_\alpha^\beta(t) : F_\beta \to F_\alpha$ is the continuous linear extension of the map $F/\ker p_{\beta,A} \to F_\alpha$, $x + \ker p_{\beta,A} \mapsto S_\alpha(t)x$.

Theorem 3.3 *Let E be an admissible sequentially complete $\mathbb{C}$-lcHs and $A\colon F := D(A) \subset E \to E$ a sequentially closed linear operator with admissible $F_A = (F, \tau_A)$. The ACP (1) has the uniqueness property (in the sense of hyperfunctions) if there is an asymptotic left resolvent $(R_\alpha(t, A))_{\alpha\in\mathfrak{A}}$ such that*

$$\forall\, \alpha \in \mathfrak{A}\ \exists\, \gamma \in \mathfrak{A},\ k \in \mathbb{N},\ C_3, C_4 > 0\ \forall\, t \geq t_\alpha,\ x \in E:$$
$$\|R_\alpha(t, A)x\|_\alpha \leq C_3 p_\gamma(x) \quad \text{and} \quad \|R^\gamma_\alpha(t, C)\|_{L(E_\gamma, F_\alpha)} \leq C_4 e^{kt} \tag{5}$$

where $R^\gamma_\alpha(t, A)\colon E_\gamma \to F_\alpha$ is the continuous linear extension of the map $E/\ker p_\gamma \to F_\alpha$, $x + \ker p_\gamma \mapsto R_\alpha(t, A)x$.

Proof Let $h \in \mathcal{LO}_{[0,\infty]}(F_A)$, $v := (z - A)h \in \mathcal{LO}_{\{\infty\}}(E)$, $\alpha \in \mathfrak{A}$ and $m \in \mathbb{N}$. Then there are $\gamma \in \mathfrak{A}$ and $k \in \mathbb{N}$, and for any $n \in \mathbb{N}$, $n > m$, there is $\beta \in \mathfrak{A}$ such that

$$\begin{aligned}\|\kappa^F_\alpha(h(t))\|_{\alpha,A} &\underset{(3)}{\leq} \|R_\alpha(t, A)v(t)\|_{\alpha,A} + \|S_\alpha(t)h(t)\|_{\alpha,A}\\ &\leq \|R^\gamma_\alpha(t, A)\|_{L(E_\gamma,F_\alpha)} p_\gamma(v(t)) + \|S^\beta_\alpha(t)\|_{L(F_\beta,F_\alpha)} p_{\beta,A}(h(t))\\ &\underset{(4),(5)}{\leq} C_4 e^{kt} p_\gamma(v(t)) + C_2 e^{-nt} p_{\beta,A}(h(t))\end{aligned}$$

for all $t \geq t_{\alpha,n}$. It follows that

$$\begin{aligned}\|\kappa^F_\alpha(h(t))\|_{\alpha,A} e^{mt} &\leq C_4 p_\gamma(v(t)) e^{(k+m)t} + C_2 p_{\beta,A}(h(t)) e^{(m-n)t}\\ &\leq C_4 p_\gamma(v(t)) e^{-\frac{1}{k+m+1}t+(k+m+1)t} + C_2 p_{\beta,A}(h(t)) e^{-\frac{1}{m}t}\\ &\leq C_4 |v|_{k+m+1,\gamma,\{\infty\}} + C_2 |h|_{m,(\beta,A),[0,\infty]}\end{aligned}$$

for all $t \geq t_{\alpha,n}$, which implies for $\varepsilon > 0$

$$\begin{aligned}&\sup_{t\geq\varepsilon} \|\kappa^F_\alpha(h(t))\|_{\alpha,A} e^{mt}\\ &\leq e^{mt_{\alpha,n}} \max_{\min(\varepsilon,t_{\alpha,n})\leq t\leq t_{\alpha,n}} \|\kappa^F_\alpha(h(t))\|_{\alpha,A} + C_4 |v|_{k+m+1,\gamma,\{\infty\}} + C_2 |h|_{m,(\beta,A),[0,\infty]}\\ &< \infty\end{aligned}$$

where we use the continuity of $\|\cdot\|_{\alpha,A} \circ \kappa^F_\alpha \circ h$ as well. Thus $\{h(t)e^{mt} \mid t \geq \varepsilon\}$ is bounded in F_A and we apply Theorem 3.2 (c). □

Langenbruch also formulated a sufficient criterion [44, Theorem 7.3, p. 62] for the uniqueness property by means of an asymptotic existence assumption for the dual operator, which we improve next.

Theorem 3.4 *Let E be an admissible sequentially complete $\mathbb{C}$-lcHs and $A\colon F := D(A) \subset E \to E$ a sequentially closed linear operator with admissible $F_A = (F, \tau_A)$. Then the ACP (1) has the uniqueness property (in the sense of hyperfunctions) if for any $y \in F'_A$ and any $n \in \mathbb{N}$ there are $k \in \mathbb{N}$, $\alpha \in \mathfrak{A}$, $C_1 > 0$ and $t_{y,n} > 0$ such that for any $t \geq t_{y,n}$ there are $\widetilde{y}_{y,n}(t) \in E'$, $s_{y,n}(t) \in F'_A$, such that for all $t \geq t_{y,n}$ and $x \in E$, $z \in F$ it holds that*

$$(t - A^{\mathrm{t}})\widetilde{y}_{y,n}(t) = y + s_{y,n}(t), \quad |\langle \widetilde{y}_{y,n}(t), x\rangle| \leq C_1 p_\alpha(x) e^{kt},$$
$$|\langle s_{y,n}(t), z\rangle| \leq C_1 (p_\alpha(z) + p_\alpha(Az)) e^{-nt}.$$

Proof Let $h \in \mathcal{LO}_{[0,\infty]}(F_A)$ and $v := (z - A)h \in \mathcal{LO}_{\{\infty\}}(E)$. Due to our assumption we have for any $y \in F_A'$ and $n \in \mathbb{N}$

$$\langle y, h(t)\rangle = \langle (t - A^{\mathrm{t}})\widetilde{y}_{y,n}(t), h(t)\rangle - \langle s_{y,n}(t), h(t)\rangle = \langle \widetilde{y}_{y,n}(t), v(t)\rangle - \langle s_{y,n}(t), h(t)\rangle$$

for $t \geq t_{y,n}$, implying

$$|\langle y, h(t)\rangle| \leq C_1 p_\alpha(v(t))e^{kt} + C_1(p_\alpha(h(t)) + p_\alpha(Ah(t)))e^{-nt}.$$

Let $m \in \mathbb{N}$ and choose $n \in \mathbb{N}$ with $n > m$. Then we get

$$\begin{aligned} |\langle y, h(t)\rangle| e^{mt} &\leq C_1 p_\alpha(v(t))e^{(k+m)t} + C_1 p_{\alpha,A}(h(t))e^{(m-n)t} \\ &\leq C_1 p_\alpha(v(t))e^{-\frac{1}{k+m+1}t+(k+m+1)t} + C_1 p_{\alpha,A}(h(t))e^{-\frac{1}{m}t} \\ &\leq C_1 |v|_{k+m+1,\alpha,\{\infty\}} + C_1 |h|_{m,(\alpha,A),[0,\infty]}, \end{aligned}$$

for $t \geq t_{y,n}$, which yields for $\varepsilon > 0$

$$\begin{aligned} &\sup_{t\geq\varepsilon} |\langle y, h(t)\rangle| e^{mt} \\ &\quad \leq e^{mt_{y,n}} \max_{\min(\varepsilon,t_{y,n})\leq t\leq t_{y,n}} |\langle y, h(t)\rangle| + C_1 |v|_{k+m+1,\alpha,\{\infty\}} + C_1 |h|_{m,(\alpha,A),[0,\infty]} < \infty \end{aligned}$$

where we use the continuity of $y \circ h$ as well. Therefore $\{h(t)e^{mt} \mid t \geq \varepsilon\}$ is weakly bounded in F_A for any $m \in \mathbb{N}$ and we apply Theorem 3.2 (c). □

As an application of Theorem 3.4 we consider the uniqueness of the ACP in the setting where $E := F := s(\mathbb{N})_b'$ with the nuclear Fréchet space

$$s(\mathbb{N}) := \{x \in \mathbb{C}^{\mathbb{N}} \mid \forall\, p \in \mathbb{N} : \ |x|_p^{s(\mathbb{N})} := \sup_{i\in\mathbb{N}} |x_i| i^p < \infty\}$$

of rapidly decreasing sequences and $A\colon F \to E$ is a continuous linear operator. Since $s(\mathbb{N})$ is reflexive, we have $(s(\mathbb{N})_b')_b' = s(\mathbb{N})$ and $A^{\mathrm{t}} \in L(s(\mathbb{N}))$ for the dual map by [50, Proposition 23.30 (b), p. 274]. Due to [50, Exercises 4, p. 377] the map A^{t} is given by an infinite matrix $\mathbf{A}^{\mathrm{t}} = (a_{ij})_{i,j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ such that

$$\forall\, \sigma \in \mathbb{R}\ \exists\, s \in \mathbb{R},\ C > 0\ \forall\, i, j \in \mathbb{N} : \ |a_{ij}| \leq C j^s i^{-\sigma}$$

because $s(\mathbb{N})$ coincides with the power series space $\Lambda_\infty((\ln(j))_{j\in\mathbb{N}})$. We also consider the ACP in the classical sense in our next theorem, i.e. the problem

$$x'(t) = Ax(t), \quad t > 0, \quad x(0) = x_0 \in s(\mathbb{N})', \tag{6}$$

where $x \in \mathcal{C}^1([0,\infty[, s(\mathbb{N})_b')$.

Theorem 3.5 *Let $A \in L(s(\mathbb{N})_b')$ and $\mathbf{A}^{\mathrm{t}}$ the infinite matrix that represents $A^{\mathrm{t}} \in L(s(\mathbb{N}))$. Let $(\mathbf{A}^{\mathrm{t}})^l = (a_{ij}^{(l)})_{i,j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ for all $l \in \mathbb{N}$. Consider the following statements:*

(a) $\forall\, p \in \mathbb{N}\ \exists\, q \in \mathbb{N},\ C > 0\ \forall\, l, i, j \in \mathbb{N} : \ |a_{ij}^{(l)}| \leq C^l j^q i^{-p}$
(b) *The ACP* (1) *has the uniqueness property (in the sense of hyperfunctions).*
(c) *The ACP* (6) *has the uniqueness property (in the classical sense).*

We have the chain of implications (a) ⇒ (b) ⇒ (c).

Proof (a) $\Rightarrow$ (b): We will use Theorem 3.4. The complete space $E := F := s(\mathbb{N})_b'$ is admissible by Theorem 2.3 (c) and [10, Corollary 4.8, p. 1116]. Since $A \in L(s(\mathbb{N})_b')$ and $s(\mathbb{N})$ is reflexive, we have $(s(\mathbb{N})_b')_A = s(\mathbb{N})_b'$ and $s(\mathbb{N}) = (s(\mathbb{N})_b')_b'$. Let $y = (y_j)_{j\in\mathbb{N}} \in (s(\mathbb{N})_b')_b' = s(\mathbb{N})$ and set

$$Y(t)^{(m)} := \sum_{l=0}^{m}(\mathbf{A}^{\mathrm{t}})^l y t^{-l-1} = yt^{-1} + \sum_{l=1}^{m}\left(\sum_{j=1}^{\infty} a_{ij}^{(l)} y_j\right)_{i\in\mathbb{N}} t^{-l-1} \in s(\mathbb{N})$$

for $m \in \mathbb{N}_0$ and $t > 0$. We claim that $(Y(t)^{(m)})_{m\in\mathbb{N}_0}$ converges in $s(\mathbb{N})$ if t is big enough. We note that for any $m, n \in \mathbb{N}$, $m \geq n$, and $i, k \in \mathbb{N}$ it holds

$$\begin{aligned}\left|\sum_{l=n}^{m}\sum_{j=1}^{\infty} a_{ij}^{(l)} y_j t^{-l-1}\right| i^p &\leq \sum_{l=n}^{m}\sum_{j=1}^{\infty}|a_{ij}^{(l)}||y_j|t^{-l-1}i^p \leq \sum_{l=n}^{m}\sum_{j=1}^{\infty} C^l j^q i^{-p}|y_j|t^{-l-1}i^p \\ &= \sum_{j=1}^{\infty} j^q|y_j|\sum_{l=n}^{m} C^l t^{-l-1} \leq \sum_{j=1}^{\infty} j^{-2}\sup_{r\in\mathbb{N}}|y_r|r^{q+2}t^{-1}\sum_{l=n}^{m}\left(\frac{C}{t}\right)^l \\ &= \frac{\pi^2}{6}\|y\|_{q+2}^{s(\mathbb{N})} t^{-1}\sum_{l=n}^{m}\left(\frac{C}{t}\right)^l, \end{aligned} \tag{7}$$

which implies that $(Y(t)^{(m)})_{m\in\mathbb{N}_0}$ is a Cauchy sequence in $s(\mathbb{N})$ if $t > C$. Hence the limit

$$Y(t) := \lim_{m\to\infty} Y(t)^{(m)} = \sum_{l=0}^{\infty}(\mathbf{A}^{\mathrm{t}})^l y t^{-l-1} = \sum_{l=0}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}^{(l)} y_j\right)_{i\in\mathbb{N}} t^{-l-1}$$

exists in the complete space $s(\mathbb{N})$ if $t > C$. Furthermore, we have

$$\begin{aligned}(t - A^{\mathrm{t}})Y(t)^{(m)} &= tY(t)^{(m)} - A^{\mathrm{t}}Y(t)^{(m)} = \sum_{l=0}^{m}(\mathbf{A}^{\mathrm{t}})^l y t^{-l} - \sum_{l=0}^{m}(\mathbf{A}^{\mathrm{t}})^{l+1} y t^{-l-1} \\ &= y - (\mathbf{A}^{\mathrm{t}})^{m+1} y t^{-m-1}\end{aligned}$$

as well as

$$\begin{aligned}|((\mathbf{A}^{\mathrm{t}})^{m+1}y)_i|t^{-m-1}i^p &\leq \sum_{j=1}^{\infty}|a_{ij}^{(m+1)}||y_j|t^{-m-1}i^p \leq \sum_{j=1}^{\infty} C^{m+1} j^q i^{-p}|y_j|t^{-m-1}i^p \\ &\leq \frac{\pi^2}{6}|y|_{q+2}^{s(\mathbb{N})}\left(\frac{C}{t}\right)^{m+1}\end{aligned}$$

for all $m \in \mathbb{N}_0$ and $t > 0$, yielding that

$$(t - A^{\mathrm{t}})Y(t) = \lim_{m\to\infty}(t - A^{\mathrm{t}})Y(t)^{(m)} = y - \lim_{m\to\infty}(\mathbf{A}^{\mathrm{t}})^{m+1} y t^{-m-1} = y$$

in $s(\mathbb{N})$ if $t > C$. The topology of $s(\mathbb{N})_b'$ is induced by the seminorms

$$p_B(x) := \sup_{w\in B}|x(w)|, \quad x \in s(\mathbb{N})',$$

for bounded sets $B \subset s(\mathbb{N})$. We remark that

$$|Y(t)|_p^{s(\mathbb{N})} = \sup_{i\in\mathbb{N}} |Y(t)_i| i^p \underset{(7)}{\leq} \sup_{i\in\mathbb{N}} t^{-1}|y_i| i^p + \frac{\pi^2}{6}|y|_{q+2}^{s(\mathbb{N})} t^{-1} \sum_{l=1}^{\infty}\Big(\frac{C}{t}\Big)^l$$
$$= t^{-1}|y|_p^{s(\mathbb{N})} + \frac{\pi^2}{6}|y|_{q+2}^{s(\mathbb{N})} \frac{\frac{C}{t}}{t(1-\frac{C}{t})} \leq \frac{1}{2C}|y|_p^{s(\mathbb{N})} + \frac{\pi^2}{12C}|y|_{q+2}^{s(\mathbb{N})} =: K_p$$

if $t > 2C$. Thus $Y(t) \in \{w \in s(\mathbb{N}) \mid \forall\, p \in \mathbb{N}: \ |w|_p^{s(\mathbb{N})} \leq K_p\} =: B_0$ if $t > 2C$, and B_0 is a bounded set in $s(\mathbb{N})$. So for $x \in s(\mathbb{N})'$ we have

$$|\langle Y(t), x\rangle| = |x(Y(t))| \leq \sup_{w\in B_0} |x(w)| = p_{B_0}(x)$$

if $t > 2C$. Hence we may apply Theorem 3.4 with $t_{y,n} := 2C$, $\widetilde{y}_{y,n} := Y$, $s_{y,n} := 0$ for $n \in \mathbb{N}$ as well as $k := C_1 := 1$ and $\alpha := B_0$.

(b) $\Rightarrow$ (c): Let $x \in \mathcal{C}^1([0,\infty[, s(\mathbb{N})_b')$ be a solution of the ACP (6) for $x_0 := 0$. Then x defines a hyperfunction $[u]$ in $\mathcal{B}([0,\infty[, s(\mathbb{N})_b')$ (for instance by [10, Theorem 6.9, p. 1125] as in [24, Theorem 1.3.10, p. 25] and [24, Theorem 1.3.13 b), p. 31]) which solves (2) for $x_0 = 0$. Thus $[u] = 0$ by the uniqueness property in the sense of hyperfunctions, implying $x = 0$ on $[0,\infty[$. □

4 Solvability of the ACP

Let us turn to the question of existence of a solution of the ACP (1). Following [44, p. 64], this boils down to solving the equation $(\lambda - A)S(\lambda) = x_0$ only approximately near the half-circle $S_\infty := \{\infty e^{i\varphi} \mid |\varphi| < \frac{\pi}{2}\}$ at ∞, and the approximate solution is needed only in the local Banach spaces of $F_A = (F, \tau_A)$. The precise characterisation of existence of a solution given in Theorem 4.3 below uses the Laplace transform of E-valued Laplace hyperfunctions from [11]. We recall what is needed. Let

$$H := \varinjlim_{K\in\mathbb{N}} \big(\varprojlim_{k\in\mathbb{N}} H_{K,k}\big)$$

be the inductive limit of the projective limit $\varprojlim_{k\in\mathbb{N}} H_{K,k}$ where

$$H_{K,k} := \{f \in \mathcal{O}(\Omega_K) \mid \|f\|_{K,k}^H := \sup_{z\in\Omega_K} |f(z)|e^{k\,\mathrm{Re}(z)} < \infty\}$$

and

$$\Omega_K := \Big\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < \frac{\mathrm{Re}(z)}{K} + \frac{1}{K^2}\Big\}$$

(see Fig. 2). By [11, Definition 2.3, p. 133] an E-valued *Laplace hyperfunction* (in the sense of Domański and Langenbruch) is a continuous linear operator $T: H \to E$ for complete E. Its Laplace transform $\mathscr{L}(T)$ is not a single holomorphic function but a compatible family of holomorphic functions, a so-called spectral-valued holomorphic function, for whose definition we need to direct the index set $\mathfrak{A}$ of the seminorms of E first.

Let E be a complete $\mathbb{C}$-lcHs with a directed system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$, i.e. for $\alpha, \beta \in \mathfrak{A}$ there are $\gamma \in \mathfrak{A}$ and $C_1 > 0$ such that $\max(p_\alpha, p_\beta) \leq C_1 p_\gamma$. We write $\alpha \leq \beta$ for $\alpha, \beta \in \mathfrak{A}$ if there is $C_2 > 0$ such $p_\alpha \leq C_2 p_\beta$. Then $\leq$ is a preorder on $\mathfrak{A}$ and $(\mathfrak{A}, \leq)$ a directed set due

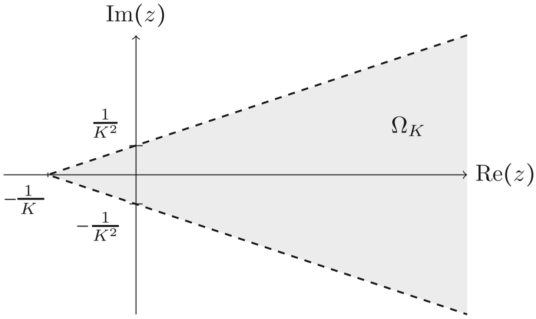


Fig. 2 Ω_K for $K \in \mathbb{N}$

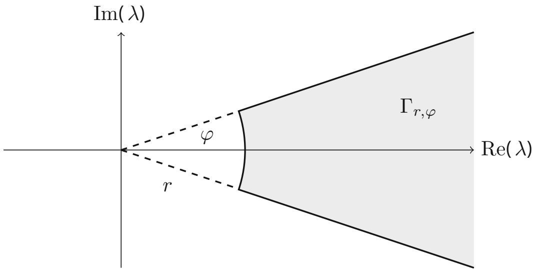


Fig. 3 $\Gamma_{r,\varphi}$ for $0 < \varphi < \frac{\pi}{2}$ and $r \geq 0$ (cf. [38, Figure 1.11, p. 111])

to the system of seminorms being directed. Furthermore, for $\alpha, \beta \in \mathfrak{A}$ with $\alpha \leq \beta$ we denote by $\kappa_\alpha^\beta \colon E_\beta \to E_\alpha$ the linking maps of the local Banach spaces, which are the continuous linear extensions of the maps $E/\ker p_\alpha \to E/\ker p_\beta$, $x + \ker p_\alpha \mapsto x + \ker p_\beta$, and by $\kappa_\alpha^E \colon E \to E_\alpha$, $x \mapsto x + \ker p_\alpha$, the spectral maps. With these definitions E becomes a projective limit of its local Banach spaces E_α, i.e. $E = \lim_{\longleftarrow\, \alpha\in\mathfrak{A}} E_\alpha$ (see [23, p. 151–152]).

Let $\mathcal{E} := (E_\alpha)_{\alpha\in\mathfrak{A}}$, and $\mathcal{G} := (G_\alpha)_{\alpha\in\mathfrak{A}}$ be a directed family of non-empty domains in $\mathbb{C}$, i.e. they are open and connected sets and $G_\beta \subset G_\alpha$ for $\alpha \leq \beta$ (see [11, p. 131]). By [11, Definition 2.1, p. 132] a family $\mathcal{S} := (S_\alpha)_{\alpha\in\mathfrak{A}}$ is called a *spectral-valued* (or $\mathcal{E}$*-valued*) *holomorphic function* (denoted by $\mathcal{S}\colon \mathcal{G} \to \mathcal{E}$) if

(i) $S_\alpha \colon G_\alpha \to E_\alpha$ is holomorphic for all $\alpha \in \mathfrak{A}$, and
(ii) (compatibility) $\forall\, \alpha, \beta \in \mathfrak{A},\ \alpha \leq \beta :\ \kappa_\alpha^\beta \circ S_\beta = S_{\alpha|G_\beta}$.

For $0 < \varphi < \frac{\pi}{2}$ and $r \geq 0$ we set

$$\Gamma_{r,\varphi} := \{\rho e^{i\psi} \mid \rho \geq r,\ |\psi| \leq \varphi\}.$$

An open set $U \subset \mathbb{C}$ is called *postsectorial* (see [47, p. 37], [48, p. 150]) if

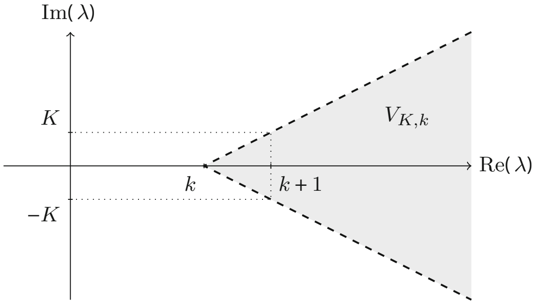
$$\forall\, 0 < \varphi < \frac{\pi}{2}\ \exists\, r > 0 :\ \Gamma_{r,\varphi} \subset U$$

(see Fig. 3). Further, we define the set $H_{\exp}(\mathcal{E})$ of all $\mathcal{E}$-valued holomorphic functions $\mathcal{S}\colon \mathcal{G} \to \mathcal{E}$ where $\mathcal{G}$ consists of postsectorial domains and

$$\forall\, \alpha \in \mathfrak{A},\ K \in \mathbb{N},\ 0 < \varphi < \tfrac{\pi}{2}\ \exists\, r > 0 :\ \Gamma_{r,\varphi} \subset G_\alpha \text{ and } \sup_{\lambda\in\Gamma_{r,\varphi}} \|S_\alpha(\lambda)\|_\alpha e^{-\frac{1}{K}\operatorname{Re}(\lambda)} < \infty$$

(see [11, Definition 2.6, p. 134]). Considering the elements of $H_{\exp}(\mathcal{E})$ as germs near S_∞, we note that $H_{\exp}(\mathcal{E})$ is a vector space canonically.

Fig. 4 $V_{K,k}$ for $K, k \in \mathbb{N}$



Theorem 4.1 [11, Theorem 2.4, p. 134, Corollary 3.5, p. 145] *Let E be a complete $\mathbb{C}$-lcHs which is the projective limit of a spectrum of Banach spaces $\mathcal{E} := (E_\alpha)_{\alpha\in\mathfrak{A}}$. Then the Laplace transform $\mathscr{L}\colon L(H, E) \to H_{\exp}(\mathcal{E})$ is a linear bijection such that $\mathscr{L}(\frac{d}{dt}T) = \lambda\mathscr{L}(T)$.*

Remark 4.2 The definition of $H_{\exp}(\mathcal{E})$ in [11, Definition 2.6, p. 134] is actually phrased with a family $\mathcal{G}$ of conoidal sets. An open set $G \subset \mathbb{C}$ is called *conoidal* if for every $K \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that

$$V_{K,k} := \left\{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > k + \frac{|\operatorname{Im}(\lambda)|}{K}\right\} \subset G$$

(see [11, Definition 2.7, p. 134] and Fig. 4). We note that an open set $G \subset \mathbb{C}$ is conoidal if and only if G is postsectorial.

Proof First, we observe that $V_{K,k} = k+\Gamma_{0,\varphi(K)}$ with $\varphi(K) := \arctan(K)$ for every $K, k \in \mathbb{N}$. Let G be conoidal and $0 < \varphi < \frac{\pi}{2}$. We choose $K \in \mathbb{N}$ such that $\varphi(K) > \varphi$. Then there are $k \in \mathbb{N}$ and $r > 0$ with $\Gamma_{r,\varphi} \subset k + \Gamma_{0,\varphi(K)} = V_{K,k} \subset G$ because G is conoidal.

Let G be postsectorial and $K \in \mathbb{N}$. Then $0 < \varphi(K) < \frac{\pi}{2}$ and there is $r > 0$ such that $\Gamma_{r,\varphi(K)} \subset G$. We choose $k \in \mathbb{N}$ with $k > r$ and get $V_{K,k} = k + \Gamma_{0,\varphi(K)} \subset \Gamma_{r,\varphi(K)} \subset G$. □

Let E be as above and $A\colon F := D(A) \subset E \to E$ a closed linear operator. We equip F with the graph topology τ_A, which makes it a complete space. We denote by $\kappa_\alpha^\beta\colon F_\beta \to F_\alpha$ for $\alpha, \beta \in \mathfrak{A}$ with $\alpha \leq \beta$ the linking maps of its local Banach spaces. Then $F_A = (F, \tau_A)$ is a projective limit of its local Banach spaces F_α. By the definition of the graph topology the map $A\colon F_A \to E$ is continuous and for any $\alpha \in \mathfrak{A}$ there are $\beta \in \mathfrak{A}$ and $C_1 > 0$ such that $p_\alpha(Ax) \leq C_1 p_{\beta,A}(x)$ for all $x \in F$ (e.g. any $\beta \in \mathfrak{A}$ with $\alpha \leq \beta$). This defines a continuous linear operator $A_\alpha^\beta\colon F_\beta \to E_\alpha$ as the extension of the continuous linear map $F/\ker p_{\beta,A} \to E/\ker p_\alpha$, $x + \ker p_{\beta,A} \mapsto Ax + \ker p_\alpha$ (well-defined because $\ker p_{\beta,A} \subset \ker p_\alpha \circ A$). Moreover, we call $I_\alpha^\beta\colon F_\beta \to E_\alpha$ the continuous linear extension of the map $F/\ker p_{\beta,A} \to E/\ker p_\alpha$, $x + \ker p_{\beta,A} \mapsto x + \ker p_\alpha$, for $\alpha \leq \beta$ (well-defined because $\alpha \leq \beta$ implies $\ker p_{\beta,A} \subset \ker p_\alpha$).

Theorem 4.3 *Let E be an admissible complete $\mathbb{C}$-lcHs with local Banach spaces $\mathcal{E} := (E_\alpha)_{\alpha\in\mathfrak{A}}$, let $A\colon F := D(A) \subset E \to E$ be a closed linear operator and $F_A = (F, \tau_A)$ admissible with local Banach spaces $\mathcal{F} := (F_\alpha)_{\alpha\in\mathfrak{A}}$. For $x_0 \in E$ the following are equivalent:*

(a) *The ACP (1) has a solution (in the sense of hyperfunctions).*

(b) *There is a spectral-valued holomorphic function* $\mathcal{S} := (S_\alpha)_{\alpha\in\mathfrak{A}} \in H_{\exp}(\mathcal{F})$ *such that for any* $\alpha \in \mathfrak{A}$ *there is* $\beta \in \mathfrak{A}$, $\alpha \leq \beta$, *such that*

$$s_\alpha^\beta \colon G_\beta \to E_\alpha,\ s_\alpha^\beta(\lambda) := (\lambda I_\alpha^\beta - A_\alpha^\beta)S_\beta(\lambda) - \kappa_\alpha^E(x_0), \tag{8}$$

is well-defined and

$$\forall\, j \in \mathbb{N}, 0 < \varphi < \tfrac{\pi}{2}\, \exists\, r > 0 :\ \Gamma_{r,\varphi} \subset G_\beta \text{ and } \sup_{\lambda\in\Gamma_{r,\varphi}} \|s_\alpha^\beta(\lambda)\|_\alpha e^{j\operatorname{Re}(\lambda) - \frac{1}{j}|\operatorname{Im}(\lambda)|} < \infty. \tag{9}$$

Proof (a) ⇒ (b): Let $[u] \in \mathcal{B}([0,\infty[, F_A)$ be a solution of (2) and $[h] := \mathcal{L}^{\mathcal{B}}([u]) \in \mathcal{LO}_{[0,\infty]}(F_A)/\mathcal{LO}_{\{\infty\}}(F_A)$. It follows that

$$(\lambda - A)[h] = \mathcal{L}^{\mathcal{B}}(x_0 \otimes \delta_0) = [x_0]$$

in $\mathcal{LO}_{[0,\infty]}(E)/\mathcal{LO}_{\{\infty\}}(E)$. We set $G_\alpha := \mathbb{C}_{\operatorname{Re}>0}$ and $S_\alpha := \kappa_\alpha^F \circ h$ for $\alpha \in \mathfrak{A}$. Then there is $f \in \mathcal{LO}_{\{\infty\}}(E)$ such that with $\beta = \alpha$

$$\begin{aligned}
\kappa_\alpha^E(x_0) + s_\alpha^\alpha(\lambda) &= (\lambda I_\alpha^\alpha - A_\alpha^\alpha)S_\alpha(\lambda) = (\lambda I_\alpha^\alpha - A_\alpha^\alpha)\kappa_\alpha^F(h(\lambda)) \\
&= (\lambda I_\alpha^\alpha - A_\alpha^\alpha)(h(\lambda) + \ker p_{\alpha,A}) = \lambda h(\lambda) - Ah(\lambda) + \ker p_\alpha \\
&= (\lambda - A)h(\lambda) + \ker p_\alpha = x_0 + f(\lambda) + \ker p_\alpha = \kappa_\alpha^E(x_0) + f(\lambda)
\end{aligned}$$

and thus $s_\alpha^\alpha(\lambda) = f(\lambda)$ for $\lambda \in G_\alpha$. Let $j \in \mathbb{N}$ and $0 < \varphi < \frac{\pi}{2}$. We note that $\Gamma_{r,\varphi} \subset G_\alpha$ for any $r > 0$ and with $k \in \mathbb{N}$ such that $\frac{1}{k} \leq r$ and $k \geq j$ we obtain

$$\begin{aligned}
\sup_{\lambda\in\Gamma_{r,\varphi}} \|s_\alpha^\alpha(\lambda)\|_\alpha e^{j\operatorname{Re}(\lambda) - \frac{1}{j}|\operatorname{Im}(\lambda)|} &\leq \sup_{\operatorname{Re}(\lambda)\geq\frac{1}{k}} p_\alpha(f(\lambda))e^{j|\operatorname{Re}(\lambda)| - \frac{1}{j}|\lambda| + \frac{1}{j}|\operatorname{Re}(\lambda)|} \\
&\leq \sup_{\operatorname{Re}(\lambda)\geq\frac{1}{k}} p_\alpha(f(\lambda))e^{-\frac{1}{j+1}|\lambda| + (j+1)|\operatorname{Re}(\lambda)|} \\
&\leq |f|_{k+1,\alpha,\{\infty\}}.
\end{aligned}$$

(b) ⇒ (a): First, we observe that for $\alpha, \beta \in \mathfrak{A}$ with $\alpha \leq \beta$ the map $s_\alpha^\beta \colon G_\beta \to E_\alpha$ is well-defined by our considerations above this theorem. In addition, s_α^β is holomorphic on G_β because I_α^β and A_α^β are linear and continuous and S_β holomorphic by (i). We observe that for $\lambda \in G_\beta$ there is $(h_n(\lambda))_{n\in\mathbb{N}}$ in F such that $S_\beta(\lambda) = \lim_{n\to\infty}(h_n(\lambda) + \ker p_{\beta,A})$ in F_β and

$$s_\alpha^\beta(\lambda) + \kappa_\alpha^E(x_0) = (\lambda I_\alpha^\beta - A_\alpha^\beta)S_\beta(\lambda) = \lim_{n\to\infty}((\lambda - A)h_n(\lambda) + \ker p_\alpha). \tag{10}$$

Now, we want to construct an $\mathcal{E}$-valued holomorphic function $\widetilde{s}$ on a suitable family $\widetilde{G}$ of postsectorial domains using our maps s_α^β. For $\alpha \in \mathfrak{A}$ we set

$$M_\alpha := \{\beta \in \mathfrak{A} \mid \alpha \leq \beta \text{ and (9) is satisfied}\} \quad \text{and} \quad \widetilde{G}_\alpha := \bigcup_{\beta\in M_\alpha} G_\beta.$$

The sets $\widetilde{G}_\alpha \subset \mathbb{C}$ are non-empty by assumption as well as open, connected and postsectorial as they are unions of such sets. Next, we show that $M_\alpha \subset M_\gamma$ for $\alpha, \gamma \in \mathfrak{A}$ with $\gamma \leq \alpha$, which then implies $\widetilde{G}_\alpha \subset \widetilde{G}_\gamma$ and means that $\widetilde{\mathcal{G}} := (\widetilde{G}_\alpha)_{\alpha\in\mathfrak{A}}$ is directed. Let $\beta \in M_\alpha$. Then $\gamma \leq \beta$ and it holds by (10) that

$$\kappa_{\gamma}^{\alpha}(s_{\alpha}^{\beta}(\lambda))+\kappa_{\gamma}^{E}(x_0)=\kappa_{\gamma}^{\alpha}(s_{\alpha}^{\beta}(\lambda)+\kappa_{\alpha}^{E}(x_0))=\lim_{n\to\infty}\kappa_{\gamma}^{\alpha}((\lambda-A)h_n(\lambda)+\ker p_{\alpha})$$
$$=\lim_{n\to\infty}((\lambda-A)h_n(\lambda)+\ker p_{\gamma})=s_{\gamma}^{\beta}(\lambda)+\kappa_{\gamma}^{E}(x_0)$$

and thus $\kappa_{\gamma}^{\alpha}(s_{\alpha}^{\beta}(\lambda)) = s_{\gamma}^{\beta}(\lambda)$ for $\lambda \in G_{\beta}$. We deduce that there is $C_1 > 0$ such that $\|s_{\gamma}^{\beta}(\lambda)\|_{\gamma} \leq C_1\|s_{\alpha}^{\beta}(\lambda)\|_{\alpha}$ for any $\lambda \in G_{\beta}$ from the continuity of $\kappa_{\gamma}^{\alpha}\colon E_{\alpha} \to E_{\gamma}$. Therefore s_{γ}^{β} satisfies the estimate (9) with α replaced by γ, which means that $\beta \in M_{\gamma}$.

Now, let $\beta_1, \beta_2 \in M_{\alpha}$. Then $\emptyset \neq (G_{\beta_1}\cap G_{\beta_2}) \subset G_{\alpha}$ as $\alpha \leq \beta_1, \beta_2$ and $\mathcal{G}$ is directed and consists of postsectorial sets. For $\lambda \in G_{\beta_1}\cap G_{\beta_2}$ there are $(h_{i,n}(\lambda))_{n\in\mathbb{N}}$ in F such that $S_{\beta_i}(\lambda) = \lim_{n\to\infty}(h_{i,n}(\lambda) + \ker p_{\beta_i,C})$ in F_{β_i} and

$$s_{\alpha}^{\beta_i}(\lambda)+\kappa_{\alpha}^{E}(x_0)=\lim_{n\to\infty}((\lambda-A)h_{i,n}(\lambda)+\ker p_{\alpha})$$

by (10) for $i = 1, 2$. Due to the compatibility (ii) for $\mathcal{S}$ we get

$$S_{\alpha}(\lambda) \underset{(ii)}{=} (\kappa_{\alpha}^{\beta_i}\circ S_{\beta_i})(\lambda)=\lim_{n\to\infty}(h_{i,n}(\lambda)+\ker p_{\alpha,A})$$

for $i = 1, 2$, which yields $\lim_{n\to\infty}(h_{1,n}(\lambda) - h_{2,n}(\lambda) + \ker p_{\alpha,A}) = 0$ in F_{α} and thus in $(F/\ker p_{\alpha,A})$ as well. It follows that $\lim_{n\to\infty}(h_{1,n}(\lambda) - h_{2,n}(\lambda)) \in \ker p_{\alpha,A}$ and so

$$s_{\alpha}^{\beta_1}(\lambda)-s_{\alpha}^{\beta_2}(\lambda)=(\lambda-A)(\lim_{n\to\infty}(h_{1,n}(\lambda)-h_{2,n}(\lambda)))+\ker p_{\alpha}=\ker p_{\alpha},$$

implying $s_{\alpha}^{\beta_1} = s_{\alpha}^{\beta_2}$ on $G_{\beta_1}\cap G_{\beta_2}$. Therefore the map $\widetilde{s}_{\alpha}\colon \widetilde{G}_{\alpha} \to E_{\alpha}$ given by $\widetilde{s}_{\alpha} := s_{\alpha}^{\beta}$ on G_{β} for $\beta \in M_{\alpha}$ is well-defined and holomorphic on $\widetilde{G}_{\alpha}$. This gives us (i) for $\widetilde{s} := (\widetilde{s}_{\alpha})_{\alpha\in\mathfrak{A}}\colon \widetilde{\mathcal{G}} \to \mathcal{E}$.

Let us turn to the compatibility condition (ii) for $\widetilde{s}$. Let $\alpha, \gamma \in \mathfrak{A}$ with $\alpha \leq \gamma$. Then for any $\beta \in M_{\gamma} \subset M_{\alpha}$ and $\lambda \in G_{\beta}$ we have by (10)

$$(\kappa_{\alpha}^{\gamma}\circ\widetilde{s}_{\gamma})(\lambda)=(\kappa_{\alpha}^{\gamma}\circ s_{\gamma}^{\beta})(\lambda)=\lim_{n\to\infty}\kappa_{\alpha}^{\gamma}((\lambda-A)h_n(\lambda)-x_0+\ker p_{\gamma})$$
$$=\lim_{n\to\infty}((\lambda-A)h_n(\lambda)-x_0+\ker p_{\alpha})=s_{\alpha}^{\beta}(\lambda)=\widetilde{s}_{\alpha}(\lambda)$$

and we conclude that $\widetilde{s}$ fulfils (ii) and is an $\mathcal{E}$-valued holomorphic function.

Let $\alpha \in \mathfrak{A}$, $K \in \mathbb{N}$ and $0 < \varphi < \frac{\pi}{2}$ and choose $\beta \in M_{\alpha}$. Due to (9) for $j = 2$ there is $r > 0$ such that $\Gamma_{r,\varphi} \subset G_{\beta} \subset \widetilde{G}_{\alpha}$. We observe that for $\lambda \in \Gamma_{r,\varphi}$ it holds that $\operatorname{Re}(\lambda) > 0$ and

$$-\frac{1}{K}\operatorname{Re}(\lambda)\leq\frac{1}{2}|\operatorname{Im}(\lambda)|-\frac{1}{2}|\operatorname{Im}(\lambda)|\leq\frac{\arctan(\varphi)}{2}\operatorname{Re}(\lambda)-\frac{1}{2}|\operatorname{Im}(\lambda)|$$
$$\leq 2\operatorname{Re}(\lambda)-\frac{1}{2}|\operatorname{Im}(\lambda)|,$$

which implies

$$\sup_{\lambda\in\Gamma_{r,\varphi}}\|\widetilde{s}_{\alpha}(\lambda)\|_{\alpha}e^{-\frac{1}{K}\operatorname{Re}(\lambda)}\leq\sup_{\lambda\in\Gamma_{r,\varphi}}\|s_{\alpha}^{\beta}(\lambda)\|_{\alpha}e^{2\operatorname{Re}(\lambda)-\frac{1}{2}|\operatorname{Im}(\lambda)|}<\infty.$$

We conclude that $\widetilde{s} \in H_{\exp}(\mathcal{E})$.

By Theorem 4.1 and the definition of $\widetilde{s}$ in connection with (8) there are $T \in L(H, F_A)$ and $\widetilde{T} \in L(H, E)$ such that $\mathscr{L}(T) = \mathcal{S}$ and $\mathscr{L}(\widetilde{T}) = \widetilde{s}$ as well as

$$\Big(\frac{d}{dt}-A\Big)T=x_0\otimes\delta_0+\widetilde{T} \tag{11}$$

where δ_0 is the Dirac distribution, i.e. $\delta_0(f) := f(0)$, and $(x_0 \otimes \delta_0)(f) := x_0 f(0)$ for $f \in H$. As in [44, Theorem 7.6, p. 65] we translate this equation from Laplace hyperfunctions to hyperfunctions using the functions $f_\lambda(t) := \frac{-1}{2\pi i}\frac{e^{(t-\lambda)^2}}{t-\lambda}$ for $\lambda \notin [0,\infty[$. Since $f_\lambda \in H$ (for every $\lambda \notin [0,\infty[$ there is $K \in \mathbb{N}$ such that $\lambda \notin \Omega_K$), the functions

$$u_T\colon \mathbb{C}\setminus[0,\infty[\to F_A,\ u_T(\lambda) := \langle T, f_\lambda\rangle,$$

and analogously $u_{\widetilde{T}}\colon \mathbb{C}\setminus[0,\infty[\to E$ are defined. The difference quotients of f_λ w.r.t. λ converge in H, which yields that u_T is holomorphic and

$$\frac{d}{d\lambda}u_T(\lambda) = \Big\langle T, \frac{d}{d\lambda}f_\lambda\Big\rangle = \Big\langle T, -\frac{d}{dt}f_\lambda\Big\rangle = \Big\langle \frac{d}{dt}T, f_\lambda\Big\rangle, \quad \lambda \in \mathbb{C}\setminus[0,\infty[.$$

Hence we get for $\lambda \in \mathbb{C}\setminus[0,\infty[$

$$\begin{aligned}\Big(\frac{d}{d\lambda} - A\Big)u_T(\lambda) &= \Big\langle\Big(\frac{d}{dt} - A\Big)T, f_\lambda\Big\rangle \underset{(11)}{=} \langle x_0\otimes\delta_{t=0} + \widetilde{T}, f_\lambda\rangle = x_0\frac{-1}{2\pi i}\frac{e^{\lambda^2}}{-\lambda} + u_{\widetilde{T}}(\lambda)\\ &= -x_0 f_0(\lambda) + u_{\widetilde{T}}(\lambda) = (x_0\otimes(-f_0))(\lambda) + u_{\widetilde{T}}(\lambda).\end{aligned}$$

Since $[-f_0] = -\delta_0$ in $\mathcal{B}([0,\infty[)$ by [38, 4.11 Example, p. 96], we only need to show that $u_{\widetilde{T}} \in \mathcal{O}(\mathbb{C}, E)$ because then $-[u_T] \in \mathcal{B}([0,\infty[, F_A)$ is a solution of the ACP (1). Now, we repeat the argument from [44, Theorem 7.6, p. 65]. For $j \in \mathbb{N}$ and $R \in L(H,E)$ we set $\langle \tau_{-j}R, f\rangle := \langle R, f(\cdot + j)\rangle$ for $f \in H$. Then

$$\mathscr{L}(\tau_{-j}R) = e^{-j(\cdot)}\mathscr{L}(R) \tag{12}$$

by the definition of the Laplace transform $\mathscr{L}$ in [11, p. 133–134]. It follows from (9) that $e^{j(\cdot)}\widetilde{s} \in H_{\exp}(\mathcal{E})$ and thus there exists $\widetilde{T}_j \in L(H,E)$ such that $\mathscr{L}(\widetilde{T}_j) = e^{j(\cdot)}\widetilde{s}$ by Theorem 4.1, implying

$$\mathscr{L}(\tau_{-j}\widetilde{T}_j) \underset{(12)}{=} e^{-j(\cdot)}\mathscr{L}(\widetilde{T}_j) = \widetilde{s} = \mathscr{L}(\widetilde{T})$$

and therefore $\tau_{-j}\widetilde{T}_j = \widetilde{T}$ by Theorem 4.1 again. We deduce for any $j \in \mathbb{N}$ that

$$u_{\widetilde{T}}(\lambda) = \langle \tau_{-j}\widetilde{T}_j, f_\lambda\rangle = \langle \widetilde{T}_j, f_\lambda(\cdot + j)\rangle = \langle \widetilde{T}_j, f_{\lambda - j}\rangle$$

is holomorphic for $\lambda \notin [j,\infty[$ because $\widetilde{T}_j \in L(H,E)$, which proves our statement. □

Our next goal is to generalise Langenbruch's sufficient criterion [44, Theorem 7.7, p. 66] for the solvability of the ACP (1), which is done by using a suitable notion of an asymptotic right resolvent. Let E be a complete $\mathbb{C}$-lcHs and $A\colon F := D(A) \subset E \to E$ a closed linear operator. If E is bornological, i.e.

$$E = \lim_{\substack{\longrightarrow\\ \mathscr{B}\in\mathfrak{B}^E}} E_{\mathscr{B}}$$

where $\mathfrak{B}^E$ is the system of bounded closed absolutely convex subsets of E and $E_{\mathscr{B}} := \operatorname{span}(\mathscr{B})$ equipped with the gauge norm induced by $\mathscr{B} \in \mathfrak{B}^E$, then the topological identities

$$L_b(E,E) = \lim_{\substack{\longleftarrow\\ (\mathscr{B},\alpha)\in\mathfrak{B}^E\times\mathfrak{A}}} L(E_{\mathscr{B}}, E_\alpha) \quad\text{and}\quad L_b(E,F_A) = \lim_{\substack{\longleftarrow\\ (\mathscr{B},\alpha)\in\mathfrak{B}^E\times\mathfrak{A}}} L(E_{\mathscr{B}}, F_\alpha)$$

hold by [11, p. 136–137]. This means that the local Banach spaces of $L_b(E,E)$ and $L_b(E,F_A)$ are the spaces $L(E_{\mathscr{B}}, E_\alpha)$ and $L(E_{\mathscr{B}}, F_\alpha)$ equipped with the operator norm, respectively. We set $\mathfrak{C} := \mathfrak{B}^E \times \mathfrak{A}$. A spectral-valued holomorphic operator function

$$\mathcal{R} := (R_{\mathscr{A},\alpha})_{(\mathscr{A},\alpha)\in\mathfrak{C}}\colon \mathcal{G} := (G_{\mathscr{A},\alpha})_{(\mathscr{A},\alpha)\in\mathfrak{C}} \to \mathcal{L}(E, F_A) := (L(E_{\mathscr{A}}, F_\alpha))_{(\mathscr{A},\alpha)\in\mathfrak{C}}$$

is called an *asymptotic right resolvent* if $\mathcal{R} \in H_{\exp}(\mathcal{L}(E, F_A))$ and if there is a spectral-valued holomorphic function

$$\mathcal{T} := (T_{\mathscr{A},\alpha})_{(\mathscr{A},\alpha)\in\mathfrak{C}} : \widetilde{\mathcal{G}} := (\widetilde{G}_{\mathscr{A},\alpha})_{(\mathscr{A},\alpha)\in\mathfrak{C}} \to \mathcal{L}(E) := (L(E_{\mathscr{A}}, E_\alpha))_{(\mathscr{A},\alpha)\in\mathfrak{C}}$$

such that for any $(\mathscr{A}, \alpha) \in \mathfrak{C}$ there is $(\mathscr{B}, \beta) \in \mathfrak{C}$, $(\mathscr{A}, \alpha) \leq (\mathscr{B}, \beta)$, such that

$$(\lambda I^\beta_\alpha - A^\beta_\alpha)R_{\mathscr{B},\beta}(\lambda) = \kappa^E_{\alpha\ |E_{\mathscr{B}}} + T_{\mathscr{B},\alpha}(\lambda), \quad \lambda \in G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha}, \tag{13}$$

and for any $j \in \mathbb{N}$ and any $0 < \varphi < \frac{\pi}{2}$ there is $r > 0$ with $\Gamma_{r,\varphi} \subset (G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha})$ and

$$\sup_{\lambda\in\Gamma_{r,\varphi}} \|T_{B,\alpha}(\lambda)\|_{L(E_{\mathscr{B}},E_\alpha)} e^{j\operatorname{Re}(\lambda) - \frac{1}{j}|\operatorname{Im}(\lambda)|} < \infty. \tag{14}$$

Theorem 4.4 *Let E be an admissible complete bornological $\mathbb{C}$-lcHs and $A\colon F := D(A) \subset E \to E$ a closed linear operator and $F_A = (F, \tau_A)$ admissible. The ACP (1) has a solution (in the sense of hyperfunctions) for any $x_0 \in E$ if A admits an asymptotic right resolvent.*

Proof In order to apply Theorem 4.3 we have to construct a suitable spectral-valued holomorphic function $\mathcal{S} := (S_\alpha)_{\alpha\in\mathfrak{A}} \in H_{\exp}(\mathcal{F})$. For $x_0 \in E$ we choose $\mathscr{A} \in \mathfrak{B}^E$ such that $x_0 \in \mathscr{A}$. For $\alpha \in \mathfrak{A}$ we set

$$M_\alpha := \{(\mathscr{B}, \beta) \in \mathfrak{C} \mid (\mathscr{A}, \alpha) \leq (\mathscr{B}, \beta),\ (13) \text{ and } (14) \text{ are satisfied}\}$$

and

$$G_\alpha := \bigcup_{(\mathscr{B},\beta)\in M_\alpha} (G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha}).$$

The sets $G_\alpha \subset \mathbb{C}$ are non-empty by assumption as well as open, connected and postsectorial as they are unions of such sets. Next, we show that $M_\alpha \subset M_\gamma$ for $\alpha, \gamma \in \mathfrak{A}$ with $\gamma \leq \alpha$, which then implies $G_\alpha \subset G_\gamma$ and means that $\mathcal{G}_0 := (G_\alpha)_{\alpha\in\mathfrak{A}}$ is directed. Let $(\mathscr{B}, \beta) \in M_\alpha$. Then $(\mathscr{A}, \gamma) \leq (\mathscr{B}, \beta)$ and we note that

$$\kappa^{\mathscr{B},\alpha}_{\mathscr{B},\gamma}(f) = \kappa^\alpha_\gamma \circ f \tag{15}$$

for all $f \in L(E_{\mathscr{B}}, E_\alpha)$ where $\kappa^{\mathscr{B},\alpha}_{\mathscr{B},\gamma}\colon L(E_{\mathscr{B}}, E_\alpha) \to L(E_{\mathscr{B}}, E_\gamma)$ is the linking map of the local Banach spaces. It holds by (13) and the compatibility condition (ii) for $\mathcal{T}$ that

$$\begin{aligned}(\lambda I^\beta_\gamma - A^\beta_\gamma)R_{\mathscr{B},\beta}(\lambda) &= \kappa^\alpha_\gamma(\lambda I^\beta_\alpha - A^\beta_\alpha)R_{\mathscr{B},\beta}(\lambda) = \kappa^\alpha_\gamma(\kappa^E_{\alpha\ |E_{\mathscr{B}}} + T_{\mathscr{B},\alpha}(\lambda))\\ &= \kappa^E_{\gamma\ |E_{\mathscr{B}}} + \kappa^\alpha_\gamma \circ T_{\mathscr{B},\alpha}(\lambda) \underset{(ii),(15)}{=} \kappa^E_{\gamma\ |E_{\mathscr{B}}} + T_{\mathscr{B},\gamma}(\lambda)\end{aligned} \tag{16}$$

for all $\lambda \in G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha}$. Since $(G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha}) \subset (G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\gamma})$, the identity theorem implies that (16) holds on the connected set $G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\gamma}$ as well. Moreover, from the inclusion $(G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha}) \subset (G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\gamma})$ and $T_{\mathscr{B},\gamma}(\lambda) = (\kappa^\alpha_\gamma \circ T_{\mathscr{B},\alpha})(\lambda)$ for all $\lambda \in \widetilde{G}_{\mathscr{B},\alpha}$ it follows that (14) holds with α replaced by γ too. Hence $(\mathscr{B}, \beta) \in M_\gamma$, implying $M_\alpha \subset M_\gamma$.

Now, let $(\mathscr{B}_1, \beta_1), (\mathscr{B}_2, \beta_2) \in M_\alpha$. Then $\emptyset \neq (G_{\mathscr{B}_1,\beta_1} \cap G_{\mathscr{B}_2,\beta_2}) \subset G_{\mathscr{A},\alpha}$ and the compatibility (ii) for $\mathcal{R}$ yields

$$\langle \kappa^{\beta_i}_\alpha \circ R_{\mathscr{B}_i,\beta_i}(\lambda), x_0\rangle = \langle (\kappa^{\mathscr{B}_i,\beta_i}_{\mathscr{A},\alpha} \circ R_{\mathscr{B}_i,\beta_i})(\lambda), x_0\rangle \underset{(ii)}{=} R_{\mathscr{A},\alpha}(\lambda)(x_0)$$

for all $\lambda \in G_{\mathscr{B}_i,\beta_i}$ and $i = 1, 2$ where $\kappa^{\mathscr{B}_i,\beta_i}_{\mathscr{A},\alpha}\colon L(E_{\mathscr{B}_i}, F_{\beta_i}) \to L(E_{\mathscr{A}}, F_\alpha)$ is the linking map of the local Banach spaces. This implies $\langle \kappa^{\beta_1}_\alpha \circ R_{\mathscr{B}_1,\beta_1}(\lambda), x_0\rangle = \langle \kappa^{\beta_2}_\alpha \circ R_{\mathscr{B}_2,\beta_2}(\lambda), x_0\rangle$

for all $\lambda \in G_{\mathscr{B}_1,\beta_1} \cap G_{\mathscr{B}_2,\beta_2}$. Therefore the map $S_\alpha \colon G_\alpha \to F_\alpha$ given by $S_\alpha(\lambda) := \langle \kappa_\alpha^\beta \circ R_{\mathscr{B},\beta}(\lambda), x_0\rangle$ on $G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha}$ for $(\mathscr{B},\beta) \in M_\alpha$ is well-defined and holomorphic on G_α. This gives us (i) for $S := (S_\alpha)_{\alpha\in\mathfrak{A}} \colon \mathcal{G}_0 \to \mathcal{F}$.

Let us turn to the compatibility condition (ii) for S. Let $\alpha, \gamma \in \mathfrak{A}$ with $\alpha \leq \gamma$. Then for any $(\mathscr{B},\beta) \in M_\gamma \subset M_\alpha$ and $\lambda \in G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha}$ we have

$$(\kappa_\alpha^\gamma \circ S_\gamma)(\lambda) = \langle \kappa_\alpha^\gamma \circ \kappa_\gamma^\beta \circ R_{\mathscr{B},\beta}(\lambda), x_0\rangle = \langle \kappa_\alpha^\beta \circ R_{\mathscr{B},\beta}(\lambda), x_0\rangle = S_\alpha(\lambda).$$

We derive that S fulfils (ii) and is an $\mathcal{F}$-valued holomorphic function.

Since $\mathcal{R} \in H_{\exp}(\mathcal{L}(E, F_A))$, for $(\mathscr{B},\beta) \in M_\alpha$ and any $K \in \mathbb{N}$ and any $0 < \varphi < \frac{\pi}{2}$, there is $r > 0$ such that $\Gamma_{r,\varphi} \subset G_{\mathscr{B},\beta}$ and

$$\sup_{\lambda\in\Gamma_{r,\varphi}} \|R_{\mathscr{B},\beta}(\lambda)\|_{L(E_{\mathscr{B}},F_\beta)} e^{-\frac{1}{K}\operatorname{Re}(\lambda)} < \infty.$$

The set $G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha}$ is postsectorial and so there is $t \geq r$ with $\Gamma_{t,\varphi} \subset \Gamma_{r,\varphi}$ and $\Gamma_{t,\varphi} \subset (G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha})$. We remark that the continuity of κ_α^β implies that there is $C_1 > 0$ such that

$$\begin{aligned}\|S_\alpha(\lambda)\|_\alpha &= \|\langle \kappa_\alpha^\beta \circ R_{\mathscr{B},\beta}(\lambda), x_0\rangle\|_\alpha \leq C_1 \|R_{\mathscr{B},\beta}(\lambda)x_0\|_{\beta,A}\\ &\leq C_1 \|R_{\mathscr{B},\beta}(\lambda)\|_{L(E_{\mathscr{B}},F_\beta)} \|x_0\|_{E_{\mathscr{B}}}\end{aligned}$$

for all $\lambda \in G_{\mathscr{B},\beta} \cap \widetilde{G}_{\mathscr{B},\alpha}$. It follows that

$$\sup_{\lambda\in\Gamma_{t,\varphi}} \|S_\alpha(\lambda)\|_\alpha e^{-\frac{1}{K}\operatorname{Re}(\lambda)} \leq C_1\|x_0\|_{E_{\mathscr{B}}} \sup_{\lambda\in\Gamma_{r,\varphi}} \|R_{\mathscr{B},\beta}(\lambda)\|_{L(E_{\mathscr{B}},F_\beta)} e^{-\frac{1}{K}\operatorname{Re}(\lambda)} < \infty$$

and we conclude that $S \in H_{\exp}(\mathcal{F})$.

We define $s_\alpha^\beta \colon G_\beta \to E_\alpha$ for β with $(\mathscr{B},\beta) \in M_\alpha$ by (8) as before and note that for $(\mathscr{B}_1,\gamma) \in M_\beta \subset M_\alpha$

$$\begin{aligned}s_\alpha^\beta(\lambda) &= (\lambda I_\alpha^\beta - A_\alpha^\beta)S_\beta(\lambda) - \kappa_\alpha^E(x_0) = (\lambda I_\alpha^\beta - A_\alpha^\beta)\kappa_\beta^\gamma R_{\mathscr{B}_1,\gamma}(\lambda)x_0 - \kappa_\alpha^E(x_0)\\ &= (\lambda I_\alpha^\gamma - A_\alpha^\gamma)R_{\mathscr{B}_1,\gamma}(\lambda)x_0 - \kappa_\alpha^E(x_0) \underset{(13)}{=} \kappa_\alpha^E(x_0) + T_{\mathscr{B}_1,\alpha}(\lambda)x_0 - \kappa_\alpha^E(x_0)\\ &= T_{\mathscr{B}_1,\alpha}(\lambda)x_0\end{aligned}$$

for all $\lambda \in (G_{\mathscr{B}_1,\gamma} \cap \widetilde{G}_{\mathscr{B}_1,\beta}) \subset (G_{\mathscr{B}_1,\gamma} \cap \widetilde{G}_{\mathscr{B}_1,\alpha})$. As $(\mathscr{B}_1,\gamma) \in M_\beta$, for any $j \in \mathbb{N}$ and any $0 < \varphi < \frac{\pi}{2}$ there is $r > 0$ with $\Gamma_{r,\varphi} \subset (G_{\mathscr{B}_1,\gamma} \cap \widetilde{G}_{\mathscr{B}_1,\beta})$ and

$$\sup_{\lambda\in\Gamma_{r,\varphi}} \|T_{\mathscr{B}_1,\beta}(\lambda)\|_{L(E_{\mathscr{B}_1},E_\beta)} e^{j\operatorname{Re}(\lambda) - \frac{1}{j}|\operatorname{Im}(\lambda)|} < \infty. \tag{17}$$

From the compatibility condition (ii) of $\mathcal{T}$ we deduce that

$$s_\alpha^\beta(\lambda) = T_{\mathscr{B}_1,\alpha}(\lambda)x_0 = \langle (\kappa_{\mathscr{B}_1,\alpha}^{\mathscr{B}_1,\beta} \circ T_{\mathscr{B}_1,\beta})(\lambda), x_0\rangle$$

for all $\lambda \in G_{\mathscr{B}_1,\gamma} \cap \widetilde{G}_{\mathscr{B}_1,\beta}$ where $\kappa_{\mathscr{B}_1,\alpha}^{\mathscr{B}_1,\beta} \colon L(E_{\mathscr{B}_1}, E_\beta) \to L(E_{\mathscr{B}_1}, E_\alpha)$ is the linking map of the local Banach spaces. The continuity of the linking map implies that there is $C_2 > 0$ such that

$$\begin{aligned}\|s_\alpha^\beta(\lambda)\|_\alpha &= \|\langle (\kappa_{\mathscr{B}_1,\alpha}^{\mathscr{B}_1,\beta} \circ T_{\mathscr{B}_1,\beta})(\lambda), x_0\rangle\|_\alpha \leq C_2\|T_{\mathscr{B}_1,\beta}(\lambda)x_0\|_\beta\\ &\leq C_2\|T_{\mathscr{B}_1,\beta}(\lambda)\|_{L(E_{\mathscr{B}_1},E_\beta)}\|x_0\|_{E_{\mathscr{B}_1}}\end{aligned}$$

for all $\lambda \in G_{\mathscr{B}_1,\gamma} \cap \widetilde{G}_{\mathscr{B}_1,\beta}$. In combination with (17) we get (9). Applying Theorem 4.3, we obtain our statement. □

We illustrate Theorem 4.3 by an application to the one-dimensional heat equation in the space of tempered distributions. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space, i.e.

$$\mathcal{S}(\mathbb{R}) := \{f \in \mathcal{C}^{\infty}(\mathbb{R}) \mid \forall\, n \in \mathbb{N}_0 :\ |f|_n^{\mathcal{S}(\mathbb{R})} < \infty\}$$

where

$$|f|_n^{\mathcal{S}(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ m\in\mathbb{N}_0, m\leq n}} |f^{(m)}(x)|(1+|x|^2)^{n/2}.$$

Further, we equip the space $\mathcal{C}^{\infty}(\mathbb{R})$ with its usual topology of uniform convergence of partial derivatives up to any order on compact subsets of $\mathbb{R}$.

Theorem 4.5 *Let $x_0 \in \mathcal{C}^{\infty}(\mathbb{R})'$. Then the ACP*

$$x'(t) = \Delta x(t), \quad t > 0, \quad x(0) = x_0,$$

has a solution $x \in \mathcal{B}([0,\infty[, \mathcal{S}(\mathbb{R})_b')$ in the sense of hyperfunctions.

Proof We set $f\colon \mathbb{C}_{\mathrm{Re}>0} \times \mathbb{R} \to \mathbb{C}$, $f(\lambda, s) := \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|s|}$, where $\sqrt{\cdot}$ is the principal square root, i.e. $\sqrt{\lambda} = \sqrt{|\lambda|}(\cos(\frac{\arg(\lambda)}{2}) + i\sin(\frac{\arg(\lambda)}{2}))$ with the principal argument $\arg(\lambda) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$. Then $f(\lambda, \cdot)$ is continuous and thus Borel-measurable for all $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$ and

$$\int_{\mathbb{R}} |f(\lambda, s)| ds = \frac{1}{2\sqrt{|\lambda|}} \int_{\mathbb{R}} e^{-\sqrt{|\lambda|}\cos(\frac{\arg(\lambda)}{2})|s|} ds \leq \frac{1}{2\sqrt{|\lambda|}} \int_{\mathbb{R}} e^{-\sqrt{\frac{|\lambda|}{2}}|s|} ds = \frac{\sqrt{2}}{|\lambda|}, \tag{18}$$

which means that $f(\lambda,\cdot) \in L^1(\mathbb{R}, \mathbb{C})$ for all $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$. Therefore the distributional convolution $f(\lambda,\cdot) * x_0 \in \mathcal{S}(\mathbb{R})'$ for all $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$ by [17, Theorem 7.1.15, p. 166] where

$$\langle f(\lambda,\cdot) * x_0, \psi\rangle := \langle f(\lambda,\cdot), \check{x}_0 * \psi\rangle = \int_{\mathbb{R}} f(\lambda, s)(\check{x}_0 * \psi)(s) ds$$

and

$$(\check{x}_0 * \psi)(s) := \langle x_0, \psi(s+\cdot)\rangle, \ s \in \mathbb{R},$$

for $\psi \in \mathcal{S}(\mathbb{R})$. The map $\lambda \mapsto \langle f(\lambda,\cdot) * x_0, \psi\rangle$ is holomorphic for all $\psi \in \mathcal{S}(\mathbb{R})$ by differentiation under the integral w.r.t. the parameter λ due to (18) and [13, 5.8 Satz, p. 148–149] with the majorant $g_K(s) := \frac{1}{2\sqrt{C_K}} e^{-\sqrt{\frac{C_K}{2}}|s|} |(\check{x}_0 * \psi)(s)|$, $s \in \mathbb{R}$, where $C_K := \min_{\lambda\in K} |\lambda|$ for any compact disc $K \subset \mathbb{C}_{\mathrm{Re}>0}$. As $\mathcal{S}(\mathbb{R})$ is reflexive and $\mathcal{S}(\mathbb{R})_b'$ complete, this means that $\lambda \mapsto f(\lambda,\cdot) * x_0$ is weakly holomorpic and thus holomorphic, i.e. $(\lambda \mapsto f(\lambda,\cdot) * x_0) \in \mathcal{O}(\mathbb{C}_{\mathrm{Re}>0}, \mathcal{S}(\mathbb{R})_b')$, by [21, 16.7.2 Theorem, p. 362–363]. Since $x_0 \in \mathcal{C}^{\infty}(\mathbb{R})'$, there are $C_0 \geq 0$ and $n \in \mathbb{N}_0$ such that for all $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$ and $\psi \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned} |\langle f(\lambda,\cdot) * x_0, \psi\rangle| &= \left|\int_{\mathbb{R}} f(\lambda, s)(\check{x}_0 * \psi)(s) ds\right| \\ &\leq \frac{\sqrt{2}}{|\lambda|} \sup_{s\in\mathbb{R}} |(\check{x}_0 * \psi)(s)| \leq \frac{\sqrt{2}C_0}{|\lambda|} \sup_{s\in\mathbb{R}} \sup_{\substack{y\in[-n,n]\\ m\leq n}} |\psi^{(m)}(s+y)| \\ &\leq \frac{\sqrt{2}C_0}{|\lambda|} \sup_{\substack{s\in\mathbb{R}\\ m\leq n}} |\psi^{(m)}(s)|(1+|s|^2)^{\frac{n}{2}} = \frac{\sqrt{2}C_0}{|\lambda|} |\psi|_n^{\mathcal{S}(\mathbb{R})}, \end{aligned}$$

implying for every bounded set $\beta \subset \mathcal{S}(\mathbb{R})$ that

$$\sup_{\psi\in\beta} |\langle f(\lambda,\cdot) * x_0, \psi\rangle| \leq \frac{\sqrt{2}C_0}{|\lambda|} \sup_{\psi\in\beta} |\psi|_n^{\mathcal{S}(\mathbb{R})} = \frac{\sqrt{2}C_0 C_{n,\beta}}{|\lambda|}$$

with $C_{n,\beta} := \sup_{\psi\in\beta} |\psi|_n^{\mathcal{S}} < \infty$. We deduce that for all bounded sets $\beta \subset \mathcal{S}(\mathbb{R})$, $K \in \mathbb{N}$, $0 < \varphi < \frac{\pi}{2}$ and any $r \geq 1$ it holds that $\Gamma_{r,\varphi} \subset \mathbb{C}_{\mathrm{Re}>0}$ and

$$\begin{aligned} \sup_{\lambda\in\Gamma_{r,\varphi}} \sup_{\psi\in\beta} |\langle f(\lambda,\cdot) * x_0, \psi\rangle| e^{-\frac{1}{K}\mathrm{Re}(\lambda)} &\leq \sup_{\lambda\in\Gamma_{r,\varphi}} \sup_{\psi\in\beta} |\langle f(\lambda,\cdot) * x_0, \psi\rangle||\lambda| \\ &\leq \sqrt{2}C_0 C_{n,\beta} < \infty. \end{aligned} \tag{19}$$

Furthermore, we have

$$(\lambda - \Delta)(f(\lambda,\cdot) * x_0) = ((\lambda - \Delta) f(\lambda,\cdot)) * x_0 = \delta_{s=0} * x_0 = x_0 \tag{20}$$

for all $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$ (cf. [31, p. 249–251]). The complete space $E := F := \mathcal{S}(\mathbb{R})'_b$ is admissible by [35, Example 4.4, p. 14–15], $\Delta\colon E \to F$ continuous, in particular $F = F_\Delta$, and $x_0 \in \mathcal{C}^\infty(\mathbb{R})' \subset \mathcal{S}(\mathbb{R})'$ as linear spaces. Setting $S_\beta(\lambda) := \kappa_\beta^E(f(\lambda,\cdot) * x_0)$ for $\lambda \in G_\beta := \mathbb{C}_{\mathrm{Re}>0}$ and

$$s_\alpha^\beta(\lambda) := (\lambda I_\alpha^\beta - \Delta_\alpha^\beta) S_\beta(\lambda) - \kappa_\alpha^E(x_0) \underset{(20)}{=} 0, \quad \lambda \in G_\beta,$$

for bounded sets $\alpha, \beta \subset \mathcal{S}(\mathbb{R})$ with $\alpha \subset \beta$, we conclude the existence of a solution $x \in \mathcal{B}([0,\infty[, \mathcal{S}(\mathbb{R})'_b)$ in the sense of hyperfunctions from Theorem 4.3 and (19). □

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