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Practical Model Predictive Control for a Class of Nonlinear Systems Using Linear Parameter-Varying Representations

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ABSTRACT In this paper, a practical *model predictive control* (MPC) for tracking desired reference trajectories is demonstrated for controlling a class of nonlinear systems subject to constraints, which comprises diverse mechanical applications. Owing to the *linear parameter-varying* (LPV) formulation of the associated nonlinear dynamics, the online MPC optimization problem is solvable as a single *quadratic programming* (QP) problem of complexity similar to that of LTI systems. For offset-free tracking, based on the notion of *admissible reference*, the controller ensures convergence to any admissible reference while its deviation from the desired reference is penalized in the stage cost of the optimization problem. This mechanism provides a safety feature under the physical limitations of the system. To guarantee stability and recursive feasibility, a terminal cost as a tracking error penalty term and a terminal constraint associated with both the terminal state and the admissible reference are included. We use tube-based concept to deal with the uncertainty of the scheduling parameter over the prediction horizon. Therefore, the online optimization problem is solved for only the nominal system corresponding to the current value of the scheduling parameter and subject to tightened constraint sets. The proposed approach has been implemented successfully in real-time onto a robotic manipulator, the experimental results illustrates its efficiency and practicality.

INDEX TERMS Constrained systems, linear parameter-varying systems, model predictive control, robust stability, robotic manipulators.

I. INTRODUCTION

The ultimate goal of a control system is to achieve stability and a desired level of performance for plants which often have *nonlinear* (NL) dynamics, constrained levels of operation and are subjected to disturbances and measurement noise. *Model predictive control* (MPC) [1] is a paradigm that can systematically handle such complications. It is a control approach which can optimize system performance online based on predicting its future behavior over a so-called *prediction horizon* (N). However, unless the MPC optimization problem is formulated appropriately, computational complexity or inherent conservatism could affect performance or even result in infeasibility or instability.

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Linear parameter-varying (LPV) approach [2] is a promising framework for controlling nonlinear and *time-varying* (TV) systems using linear control techniques. Numerous successful industrial applications have proven its efficacy, see, e.g., [3], [4]. A *discrete-time* LPV system is represented in state-space form as

$$x(k+1) = A(p(k))x(k) + B(p(k))u(k), \tag{1a}$$

$$y(k) = C(p(k))x(k) + D(p(k))u(k), \tag{1b}$$

where $u(k) \in \mathbb{R}^{n_u}$, $x(k) \in \mathbb{R}^{n_x}$, $y(k) \in \mathbb{R}^{n_y}$ and $p(k) \in \mathbb{R}^{n_p}$, are vectors of the input, state, output and scheduling variable (parameter) of the system at a time index $k \in \mathbb{N}$ and A, B, C, D are parameter-dependent matrices with appropriate dimensions. The representation in (1) provides a modeling framework that can efficiently describe NL/TV systems in a linear setting, in which the relation between the input and output signals is linear, but dependent on p. It is assumed that the scheduling variable is measurable and taking values from a so-called *scheduling range* $\mathbb{P}$. In LPV models of NL applications, p is often associated with the input, state or output (endogenous signals) of the system, thus, LPV representations of NL systems are often referred to as *quasi*-LPV (qLPV) models. The LPV system representation (1) is usually employed for controller design based on linear optimal and robust control methods [5]. In all LPV control strategies, closed-loop stability and performance guarantees during implementation are established under the assumption that p will stay all the time inside $\mathbb{P}$. However, in case of qLPV representations, where p is not a free variable, i.e., endogenous, such assumption cannot be ensured, unless the control design methodology can restrict p from deviation outside $\mathbb{P}$ via input/state constraints. In fact, most of the LPV control strategies based on $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control, e.g., [6], [7], cannot handle signal constraints, which renders achieved stability/performance guarantees based on these strategies void if p violates its region. On the contrary, MPC can optimally tackle such a difficulty by virtue of constraints handling.

Although generalized formulations of MPC for NL systems exist, in the context of nonlinear MPC (NMPC), MPC schemes based on LPV systems (LPVMPC) for controlling NL plants have become popular in the control community since the late 1990s, see [8]–[11] and the recent survey in [12]. LPVMPC provides an intermediate step between conventional linear MPC (LMPC) and NMPC in terms of the achievable control performance with a moderate computational complexity. Usually the achievable control performance of an MPC design approach reflects its degree of conservatism. The strength of the LPVMPC is its ability to be solved using LMPC tools while it can achieve *asymptotic stability* and *recursive feasibility*[1] guarantees for NL systems, yet with a computational cost that is lower than NMPC [11].

However, the major difficulty of an LPVMPC setting is that the scheduling variable is accessible only at the current time instant k, but its future values, which are required over the prediction horizon of the MPC for state prediction, are unknown. Therefore, robust MPC is used to handle such uncertainty of p; however, it is usually conservative if p is assumed to vary arbitrarily fast over its full range, see, e.g., [13] and [14], which might affect achievable performance. Recently, tube-based MPC [15] has been investigated for LPV systems in [16], [17] and [18] to handle the uncertainty of the future values of p by exploiting known bounds on its rate of variation to obtain an admissible range of p over N instead of considering its full range, which can lead to low conservative techniques. However, the computational complexity of the associated optimization problem can be very costly for practical applications. Note that one of the main challenges of any MPC algorithm is to compromise the degree of conservatism with the computational complexity.

[1]In MPC, recursive feasibility is defined as follows: If the MPC optimization problem is initially feasible, then it will remain feasible.

In this paper, to tackle these difficulties, a tube-based setting is carried out, where the bounds on the rate of variation of p can be exploited to construct scheduling tubes containing the N possible future values of p, which can considerably reduce the conservatism of considering its full range. Such tubes are employed to construct state tubes as rigid tubes [15] to which the future trajectories of the state are confined and used for constraint tightening. In contrast to many LPVMPC approaches in the literature, the computation of these tubes in our strategy is performed offline, which significantly reduces the online computations.

To put any LPVMPC algorithm for practical use, it should be suited for reference tracking. Such a control problem has been rarely investigated in the context of LPVMPC, where most of the developed methods have focused on regulating the state of the controlled system to the origin or to a set point. Robust tracking MPC approaches for LPV systems have been developed recently in [19], which have achieved offset-free tracking for piecewise constant references. In [20]–[22] LPVMPC algorithms have been proposed for tracking time-varying reference trajectories, e.g., command trajectories in robotics applications. However, all these approaches cannot guarantee recursive feasibility of the associated MPC optimization problem.

For practical implementation, we extend our proposed MPC strategy to include reference tracking. Inspired by the approach of [23] and [24], for a given desired reference trajectory, the corresponding admissible steady sate and input are parameterized by a parameter vector referred to as the *admissible output*, which is among the decision variables of the optimization problem, and its deviation from the desired reference is penalized in the MPC cost function. This can lead to offset-free tracking if the desired reference is admissible, then, the system is steered toward the closest admissible reference. Such mechanism allows a safety constraint, which protects the system from tracking references beyond its physical limits without external interruption of its operation. Moreover, a larger domain of attraction can be achieved in comparison with standard MPC for tracking. Moreover, at the expense of increasing the number of decision variables associated with the admissible output, the proposed approach is not restricted for tracking piecewise constant references as that of [23]. For guaranteeing recursive feasibility, we utilize the concept of *invariant set for tracking* which is associated with the maximum set of admissible references. That invariant set is employed as a terminal set in the LMPVMPC problem for reference tracking, thus, asymptotic stability of the closed-loop system can be ensured.

To validate the practicality of the proposed technique, it is implemented experimentally on a two *degree-of-freedom* (DOF) robotic manipulator for reference tracking. The unique property of the proposed techniques is that it solves the online LPVMPC optimization problem for the nominal system corresponding to the current value of p. At the expense of losing some performance, such formulation can result in computational burden comparable to that of conventional LMPC.

To summarize, the contributions of this paper are as follows:

1) A general LPV modeling which can accommodate a wide class of mechanical systems that can be represented by rigid body nonlinear dynamics. This allows a straight forward formulation for the MPC reference tracking problem.
2) A computationally convenient LPVMPC algorithm based on *quadratic programming* (QP), which can ensure offset-free tracking and is appropriate for controlling several nonlinear applications.
3) Guarantees for recursive feasibility of the LPVMPC optimization problem and asymptotic stability of the closed-loop system.
4) A successful experimental implementation to control a 2DOF robotic manipulator for reference tracking, which shows comparable results to a recently developed more computationally demanding MPC scheme.

The paper is organized as follows: To introduce the proposed approach, some preliminaries from [17] are given in Section II. The modeling aspects related to the applications considered in this work are illustrated in Section III. The proposed LPVMPC approach for reference tracking is developed in Section IV together with other related computation issues. The application to the 2DOF robotic manipulator and the experimental results are demonstrated in Section V. Finally, in Section VI the conclusion is given.

A. NOTATION AND DEFINITION

Let $\mathbb{N}$ denote the set of non-negative integers including zero. We denote the predicted values of a variable $x(k)$ at time $k+i$ based on the available information at time k as $x_{i|k}$ such that $x_{0|k} = x(k)$. Co$\{\cdot\}$ denotes the convex hull of a set. For any vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the 2-norm, and the weighted norm is defined by $\|x\|_P^2 = x^\top Px$, where $P = P^\top$, $P \in \mathbb{R}^{n\times n}$.

A polytope is a compact polyhedron, which is the intersection of a finite number of half spaces. A (hyper)box is a convex polytope where all the defining hyperplanes are axis parallel. A PC-set is a set that is convex, compact and has a nonempty interior containing the origin. Given two sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$, the Minkowski set addition is defined by $\mathcal{A} \oplus \mathcal{B} := \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ and the Pontryagin set difference is defined by $\mathcal{A} \ominus \mathcal{B} := \{a \mid a \oplus \mathcal{B} \subseteq \mathcal{A}\}$. Let $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$ and a set $\Upsilon \subset \mathbb{R}^{n_a+n_b}$, then the projection of Υ onto a is defined as $Proj_a(\Upsilon) = \{a \in \mathbb{R}^{n_a} \mid \exists\, b \in \mathbb{R}^{n_b}, (a, b) \in \Upsilon\}$.

A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, $f(0) = 0$ and $\lim_{\xi\to\infty} f(\xi) = \infty$.

Definition 1 (Quadratic Stabilizability [25]): The system (1) is quadratically stabilizable if there exists a positive definite function $V : x \to x^\top Px$, where $P = P^\top \succ 0$, $P \in \mathbb{R}^{n_x\times n_x}$ and a control law $u = Kx$, $K \in \mathbb{R}^{n_u\times n_x}$ such that

$$V\left(A^c(p)\,x\right) - V(x) \le -x^\top(Q + K^\top RK)x \qquad (2)$$

$\forall x \in \mathbb{R}^{n_x}$, $p \in \mathbb{P}$, where $Q = Q^\top \succ 0$, $R = R^\top \succ 0$ and

$$A^c(p) = A(p) + B(p)\,K. \qquad (3)$$

Then the origin is globally exponentially stable for $x(k+1) = A^c(p(k))x(k)$, $\forall p \in \mathbb{P}$.

Definition 2 (Robust Positive Invariant Set [26]): For the system (1), with state and input constraint sets $\mathbb{X}$ and $\mathbb{U}$, respectively, and the control law $u(k) = Kx(k) \in \mathbb{U}$, the set $\mathbb{X}_f \subset \mathbb{X}$ is robustly positively invariant (RPI) if for all $x(k) \in \mathbb{X}_f$ and $p(k) \in \mathbb{P}$, $x(k+1) \in \mathbb{X}_f$.

Definition 3 [Robust Invariant Set for Tracking (RIST)]: It is the set of all initial states and steady states and inputs of the system (1) that can be stabilized by the control law

$$u(k) = K(x(k) - \bar{x}) + \bar{u}, \qquad (4)$$

where $\bar{x}$ and $\bar{u}$ denote the steady state and input, respectively, and K yields the closed-loop system matrix $A^c(p)$ as shown in (3) Schur for all values of $p \in \mathbb{P}$. Moreover, the control law (4) fulfills the input constraints and renders the system state constraints satisfied throughout its evolution.

Definition 3 is extended from [24].

II. PRELIMINARIES

In this section, we review some material from [17]. Then, we introduce a low complexity MPC scheme for regulating the state of an LPV controlled system into the origin, which will be extend in the sequel of the paper into an MPC for tracking a given reference trajectory. The latter will be implemented for controlling the robotic manipulator application in this work, which can be applicable also for a broad class of mechanical systems.

Consider discrete-time LPV systems represented by (1) and let the following assumptions be satisfied.

Assumption 4: (i) The system (1) is quadratically stabilizable.

(ii) The values of $x(k)$ and $p(k)$ are available at every time $k \in \mathbb{N}$.

(iii) The sets $\mathbb{X}$ and $\mathbb{U}$ are polytopic PC-sets.

(iv) The system matrices depend affinely on p, i.e.,

$$A(p) = A^0 + \sum_{j=1}^{n_p} p^j A^j,\ \ B(p) = B^0 + \sum_{j=1}^{n_p} p^j B^j, \qquad (5)$$

where p^j denotes the j^{th}-element of the vector p and A^j, B^j, are constant matrices with appropriate dimensions.

(v) The parameter set $\mathbb{P}$ is a compact hyper-box defined as

$$\mathbb{P} := \{p \in \mathbb{R}^{n_p} \mid p^{j,\min} \le p^j \le p^{j,\max},\ j = 1, \cdots, n_p\}.$$

(vi) The rate of variation of p is denoted as

$$\mathrm{d}p(k) = p(k) - p(k-1),$$

which is bounded such that $\mathrm{d}p \in \mathrm{d}\mathbb{P}$, where

$$\mathrm{d}\mathbb{P} := \{\mathrm{d}p \in \mathbb{R}^{n_p} \mid |\mathrm{d}p^j| \le \mathrm{d}p^{j,\max},\ j = 1, \cdots, n_p\}$$

is a compact hyper-box.

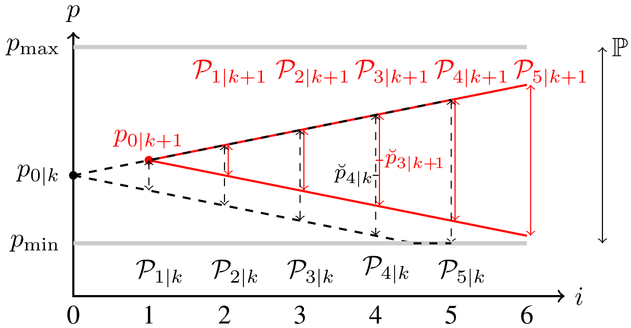

FIGURE 1. An example of two sequences of subsets $\mathcal{P}_{i+1|k}$ (black) and $\mathcal{P}_{i|k+1}$ (red) with one scheduling parameter. Note that the nominal values $\breve{p}_{4|k}$ and $\breve{p}_{3|k+1}$ are depicted at the center of $\mathcal{P}_{4|k}$ and $\mathcal{P}_{3|k+1}$, respectively.

The above assumptions are imposed for the mathematical derivations; however, they are standard in LPV literature and encompass many practical situations [6].

In order to quantify the uncertainty of p over the prediction horizon, let $p_{i|k}$ denote the uncertain scheduling parameter at any step i over N such that $p_{i|k} \in \mathcal{P}_{i|k}$, see Fig. 1, where $\mathcal{P}_{i|k} \subseteq \mathbb{P}$ is a compact hyper-box defined as

$$\mathcal{P}_{i|k} := \Big\{ p_{i|k} \in \mathbb{R}^{n_p} \;\mid\; |p^j_{i|k} - \breve{p}^j_{i|k}| \le \sum_{l=1}^{i} \mathrm{d}p^{j,\max}_{l|k}, \quad j = 1, \cdots, n_\mathrm{p} \Big\}.$$

where $\mathrm{d}p^{j,\max}_{l|k} \le \mathrm{d}p^{j,\max}$ and $\breve{p}_{i|k}$ is a known (nominal) value of the scheduling parameter, which can be computed at the center of $\mathcal{P}_{i|k}$, see Fig. 1 for illustration. Therefore $p_{i|k}$ can be parameterized as

$$p_{i|k} = \breve{p}_{i|k} + \sum_{l=1}^{i} \mathrm{d}p_{l|k}, \tag{6}$$

where $\mathrm{d}p_{l|k} \in \mathrm{d}\mathbb{P}$.

Define the uncertain system over the prediction horizon as

$$x_{i+1|k} = A(p_{i|k})x_{i|k} + B(p_{i|k})u_{i|k}, \quad p_{i|k} \in \mathcal{P}_{i|k}, \tag{7}$$

for all $i = 1, \cdots, N-1$, where $x_{i|k}$ represents the corresponding uncertain state at step i. We can rewrite the system (7) as

$$x_{i+1|k} = A_{i|k}x_{i|k} + B_{i|k}u_{i|k} + w_{i|k}, \tag{8}$$

where $A_{i|k} = A(\breve{p}_{i|k})$, $B_{i|k} = B(\breve{p}_{i|k})$ are known matrices and

$$w_{i|k} := (A(p_{i|k}) - A_{i|k})x_{i|k} + (B(p_{i|k}) - B_{i|k})u_{i|k},$$

where $w_{i|k}$ represents an additive disturbance describing the uncertainty of $p_{i|k}$ such that $w_{i|k} \in \mathcal{W}_{i|k}$,

$$\mathcal{W}_{i|k} := \mathrm{Co}\Big\{ \big(A(p_{i|k}) - A_{i|k}\big)x_{i|k} + \big(B(p_{i|k}) - B_{i|k}\big)u_{i|k}, \;\mid\; p_{i|k} \in \mathcal{P}_{i|k},\ x_{i|k} \in \mathbb{X},\ u_{i|k} \in \mathbb{U} \Big\}, \tag{9}$$

for all $i = 1, \cdots, N-1$. Any set $\mathcal{W}_{i|k}$ is polytopic and contains the origin, see [17]. Based on this setting the value of $x_{1|k}$ in (8) at any time $k \ge 0$ is known as $x_{0|k} = x(k)$ is assumed to be known, thus, $w_{0|k} = 0$.

Introduce the LPV nominal system for (8) as

$$z_{i+1|k} = A_{i|k}z_{i|k} + B_{i|k}v_{i|k}. \tag{10}$$

where $z_{i|k}$ represents the nominal state at step i, $v_{i|k}$ is the nominal input, which is related to $u_{i|k}$ in (8) by

$$u_{i|k} = v_{i|k} + K(x_{i|k} - z_{i|k}), \tag{11}$$

where $K \in \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{x}}$ is referred to as the *disturbance controller* [15], which is used to penalize the error between $x_{i|k}$ and $z_{i|k}$. For simplicity, we consider a robust controller K according to Definition 1. Let $z_{0|k} = x_{0|k}$, thus, $z_{1|k} = x_{1|k}$ as $w_{0|k} = 0$, compare (8) and (10) when $i = 0$. Furthermore, introduce $\varepsilon_{i|k} = x_{i|k} - z_{i|k}$ to denote the difference between the uncertain and the nominal states with the dynamics

$$\varepsilon_{i+1|k} = A^\mathrm{c}_{i|k}\varepsilon_{i|k} + w_{i|k}, \tag{12}$$

where $A^\mathrm{c}_{i|k} = A_{i|k} + B_{i|k}K$ is a Schur matrix for any i, k and $w_{i|k} \in \mathcal{W}_{i|k}$. Moreover, let $\varepsilon_{i|k} \in \mathcal{S}_{i|k}$, where

$$\mathcal{S}_{i|k} := \mathcal{W}_{i-1|k} \oplus A^c_{i-1|k}\mathcal{S}_{i-1|k}, \quad i = 2, \cdots, N, \tag{13}$$

which is a PC-set [17]; note that $\varepsilon_{0|k} = \varepsilon_{1|k} = 0$ and hence $\mathcal{S}_{0|k} = \mathcal{S}_{1|k} = \{0\}$.

The above formulation implies that the state trajectories over N are confined in a state tube with varying cross section and shape according to $\mathcal{S}_{i|k}$, which sometimes is referred to as *heterogeneous* tube [18]. Similarly, every control trajectory over N lies in a control tube with cross sections $K\mathcal{S}_{i|k}$. Constructing heterogeneous tubes yields less conservative tubes in comparison with homothetic or rigid tubes [18]; however, that is at the expense of considerable increase the MPC computational burden.

Based on the above formulation, a tube-based MPC for LPV systems has been introduced in [17]. Given the initial conditions $z_{0|k} = x_{0|k} = x(k)$ and $p_{0|k} = p(k)$, the optimal values of the nominal state $z_{i|k}$, for all $i = 1, \cdots, N-1$, and the nominal control input $v_{i|k}$, for all $i = 0, \cdots, N-1$, can be computed at any $k \in \mathbb{N}$, such that the state and input constraints of the LPV system (8) are satisfied by solving the following optimization problem

$$\min_{v_{0|k}, \cdots, v_{N-1|k}} \sum_{i=0}^{N-1} \|z_{i|k}\|^2_Q + \|v_{i|k}\|^2_R + V_\mathrm{f}(z_{N|k}) \tag{14a}$$

$$\text{subject to} \quad z_{i|k} \subset \mathbb{X} \ominus \mathcal{S}_{i|k}, \quad i = 1, \cdots, N-1, \tag{14b}$$

$$v_{i|k} \subset \mathbb{U} \ominus K\mathcal{S}_{i|k}, \quad i = 0, \cdots, N-1, \tag{14c}$$

$$z_{N|k} \subset \mathbb{X}_\mathrm{f} \ominus \mathcal{S}_{N|k} \tag{14d}$$

and the nominal system dynamics (10) with $z_{0|k} = x(k)$, where $Q = Q^\top \succ 0$ and $R = R^\top \succ 0$ are used as tuning parameters to meet some desired performance in the stage cost of (14a) and $V_\mathrm{f}(\cdot)$ is the terminal cost, which is chosen offline as well as the terminal set $\mathbb{X}_\mathrm{f} \subset \mathbb{X}$ in (14) to guarantee asymptotic stability of the closed-loop system and recursive feasibility of the above optimization problem. The computations of $V_\mathrm{f}(\cdot)$ together with a terminal controller K and $\mathbb{X}_\mathrm{f}$ as an RPI set under K are carried out according to Definitions 1 and 2, respectively. Therefore, according to the basic idea of tube-based MPC [15], the satisfaction of the constraints (14b-d) ensures that $x_{i|k} \in \mathbb{X}$ for all $i = 1, \cdots, N-1$, $u_{i|k} \in \mathbb{U}$ for all $i = 0, 1, \cdots, N-1$ and $x_{N|k} \in \mathbb{X}_\mathrm{f}$, see [17] for more details.

The online implementation of (14) at each time k includes also computing $\mathcal{W}_{i|k}$ for all $i = 1, \cdots, N-1$ using (9) and $\mathcal{S}_{i|k}$ for all $i = 2, \cdots, N$ using (13) as well as performing the constraints tightening in (14b-d). Then, the optimization problem (14) can solved as a QP problem. However, the online computation burden of computing $\mathcal{W}_{i|k}$ and $\mathcal{S}_{i|k}$ might be relatively higher than solving the MPC optimization problem due to the involved sets addition/subtraction.

For practical implementation, we modify in the following the above MPC approach of [17] —at the expense of a probable increase of conservatism— to avoid the online computations of the sets $\mathcal{W}_{i|k}, \mathcal{S}_{i|k}$ and the associated inline tightening of the constraint sets. Consider the following assumption:

Assumption 5: There exists a PC-set $\mathbb{S} \subset \mathbb{X}_\mathrm{f}$ such that

$$\mathcal{S}_{N|k} \subseteq \mathbb{S} \tag{15}$$

holds for all $k \in \mathbb{N}$.

Note that the condition $\mathbb{S} \subset \mathbb{X}_\mathrm{f}$ in Assumption 5 is necessary for $\mathbb{Z}_\mathrm{f}$ to have an interior. We will discuss in Section IV-D the computation of $\mathbb{S}$ (offline), which can satisfy (15). Based on Assumption 5, the tightened constraint sets in (14b-d) can be replaced by the following sets

$$\begin{aligned} z_{1|k} &\subset \mathbb{X}, && (16a)\\ z_{i|k} &\subset \mathbb{Z}, \quad i = 2, \cdots, N-1, && (16b)\\ v_{0|k}, v_{1|k} &\subset \mathbb{U}, && (16c)\\ v_{i|k} &\subset \mathbb{V}, \quad i = 2, \cdots, N-1, && (16d)\\ z_{N|k} &\subset \mathbb{Z}_\mathrm{f}, && (16e)\end{aligned}$$

where

$$\begin{aligned} \mathbb{Z} &\subset \mathbb{X} \ominus \mathbb{S}, && (17a)\\ \mathbb{V} &\subset \mathbb{U} \ominus K\mathbb{S}, && (17b)\\ \mathbb{Z}_\mathrm{f} &\subset \mathbb{X}_\mathrm{f} \ominus \mathbb{S}, \quad \mathbb{Z}_\mathrm{f} \subset \mathbb{Z}. && (17c)\end{aligned}$$

Now, given the set $\mathbb{S}$, all tightened constraint sets $\mathbb{Z}$, $\mathbb{V}$ and $\mathbb{Z}_\mathrm{f}$ can be computed offline, thus, we avoid the online computation of the sets $\mathcal{W}_{i|k}$ and $\mathcal{S}_{i|k}$.

Remark 6: Such formulation implies that the state trajectories over the prediction horizon are confined in a state tube with a center $z_{i|k}$ and a fixed cross section $\mathbb{S}$, i.e., $x_{i|k} \in z_{i|k} \oplus \mathbb{S}, \forall i = 0, 1, \cdots, N$, which sometimes is referred to as rigid tube [27]. Similarly, every control trajectory over N lies in a control tube as $u_{i|k} \in v_{i|k} \oplus K\mathbb{S}$. Note that the feedback policy in (11) can affect the size of $\mathbb{S}$.

Therefore, the online computation involves just updating the system matrices based on the current value of the scheduling variable $p(k)$ and solving the optimization problem

$$\min_{v_{0|k}, \cdots, v_{N-1|k}} \sum_{i=0}^{N-1} \|z_{i|k}\|_Q^2 + \|v_{i|k}\|_R^2 + V_\mathrm{f}(z_{N|k}) \tag{18a}$$

$$\text{subject to} \quad (16a\text{-}e). \tag{18b}$$

To construct the stage cost in (18a) for $i = 0, 1, \cdots, N-1$, one can consider the LPV nominal model (10) or just a simplified LTI model given by

$$z_{i+1|k} = A_{0|k} z_{i|k} + B_{0|k} v_{i|k}, \tag{19}$$

for all $i = 0, 1, \cdots, N-1$, with $z_{0|k} = x(k)$, in this case, the uncertainty of $p_{i|k}$ is parameterized as (6) where $\check{p}_{i|k}$ is replaced by $p_{0|k}$. Considering (10) might give better information about the evolution of p over the prediction horizon; however, it requires the computation of the nominal scheduling variable $\check{p}_{i|k}$ over N, which takes into account the upper bound on the rate of change of p. The proposed simplified version of the MPC compared with that of [17] can result in a lower computational complexity, which is comparable to that of conventional MPC for LTI systems; however, depending on the size of the set $\mathbb{S}$, this might be more conservative than that in [17].

Moreover, at the end of the prediction horizon, i.e., at step N, the state should satisfy the terminal constraint $x_{N|k} \in \mathbb{X}_\mathrm{f}$. This condition is necessary for guaranteeing stability and recursive feasibility. Satisfaction of $x_{N|k} \in \mathbb{X}_\mathrm{f}$ is ensured if the nominal system at step N satisfies the tightened terminal constraint, i.e., $z_{N|k} \in \mathbb{Z}_\mathrm{f}$, this holds true due to the condition (17c), where $x_{N|k} \in z_{N|k} \oplus \mathbb{S}$. Since $x_{i|k} \in \mathbb{X}_\mathrm{f}$ for all $i \geq N$ if $x_{N|k} \in \mathbb{X}_\mathrm{f}$ —as $\mathbb{X}_\mathrm{f}$ is RPI for (1) under the controller K— we can conclude that $z_{i|k} \in \mathbb{Z}_\mathrm{f}$ for all $i \geq N$ if $z_{N|k} \in \mathbb{Z}_\mathrm{f}$. Moreover, since $K\mathbb{X}_\mathrm{f} \subset \mathbb{U}$ holds for all $p \in \mathbb{P}$ as $\mathbb{X}_\mathrm{f}$ is RPI set under K, then, $K\mathbb{Z}_\mathrm{f} \subset \mathbb{V}$, see (17b), i.e.,

$$\forall z_{i|k} \in \mathbb{Z}_\mathrm{f}, \quad K z_{i|k} \in \mathbb{V}, \quad i \geq N. \tag{20}$$

This indicates that $\mathbb{Z}_\mathrm{f}$ is RPI for any nominal system at any $k \geq 0$ under the controller K, i.e.,

$$\forall z_{i|k} \in \mathbb{Z}_\mathrm{f}, \quad z_{i+1|k} \in \mathbb{Z}_\mathrm{f}, \quad i \geq N. \tag{21}$$

Finally, we present a formal statement for the stability of the proposed LPVMPC and the recursive feasibility of its optimization problem (18).

Theorem 7 (Recursive Feasibility and Stability of the LPVMPC for Regulation): Consider the LPV system (1), suppose that Assumptions 4 and 5 are satisfied and let V_f in (18a) be a terminal cost satisfying the condition in Definition 1 and the set $\mathbb{X}_\mathrm{f}$ be a terminal set according to Definition 2, then

(i) the optimization problem (18) is recursively feasible and
(ii) the LPVMPC solution by (18) is asymptotically stabilizing.

The proof of Theorem 7 follows the same lines as of the original approach in [17] taking into account the related simplifications considered here.

III. LPV MODELING OF A CLASS OF NL DYNAMICS

In this section we present a systematic LPV modeling technique of a class of NL dynamics, which describes a wide range of mechanical systems. This is essential for the proposed MPC tracking problem in the next section.

Then, we apply the technique on the NL dynamics of a 2-DOF robotic manipulator as shown in Section V-A.

Consider the Lagrangian formulation of the NL dynamics of an n_q-DOFs mechanical system given by

$$M(q)\ddot{q}(t) + c(q, \dot{q}) + g(q) + \tau_\mathrm{v}(\dot{q}) = \tau, \tag{22}$$

where $q \in \mathbb{R}^{n_\mathrm{q}}$ is the vector of generalized coordinates, M is the inertia matrix, which is assumed to be invertible, the vector c includes Coriolis and centrifugal terms, g contains the terms derived from the potential energy, such as gravitational forces, τ_v denotes friction vector and τ is the vector of control inputs. The states of the system are commonly the vector $\begin{bmatrix} q^\top & \dot{q}^\top \end{bmatrix}^\top$. To simplify the derivation of a qLPV representation of NL systems given by (22), consider the the transformed states given as follows

$$x = \begin{bmatrix} I & 0 \\ 0 & M(q) \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} q \\ M(q)\dot{q} \end{bmatrix},$$

thus,

$$\dot{x} = \begin{bmatrix} \dot{q} \\ M(q)\ddot{q} + \dot{M}(q)\dot{q} \end{bmatrix}. \tag{23}$$

Next, substituting the term $M(q)\ddot{q}$ from (22) into (23), exploiting the relation between the terms $\dot{M}(q)\dot{q}$ and $c(q, \dot{q})$, see, e.g., [28], reformulating the vector $g(q)$ in matrix form as shown below, and considering just viscous friction for the term $\tau_\mathrm{v}(t)$, lead to the following continuous-time general qLPV state-space representation, which is equivalent to the NL model (22),

$$\dot{x} = \tilde{A}(q, \dot{q})x + \tilde{B}u, \tag{24a}$$
$$y = \tilde{C}x \tag{24b}$$

where $u = \tau$,

$$\tilde{A}(q, \dot{q}) = \begin{bmatrix} 0 & \tilde{A}_{12}(q) \\ \tilde{A}_{21}(q) & \tilde{A}_{22}(q, \dot{q}) \end{bmatrix},$$
$$\tilde{B} = \begin{bmatrix} 0 \\ I_{n_\mathrm{q}} \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} I_{n_\mathrm{q}} & 0 \end{bmatrix} \tag{25}$$

with $\tilde{A}_{12}(q) = M^{-1}(q)$, $\tilde{A}_{21}(q)$ is related to the term $g(q)$ and $\tilde{A}_{22}(q, \dot{q})$ is related to the terms c, τ_v. The scheduling parameter can be defined as a function of q and $\dot{q}$, see e.g., [29]. Moreover, the output of the system according to (24b) and (25) is q, which is often the controlled variable. Note that only the A matrix is parameter dependent, which is quite desired in several LPV control design approaches [6]. The structure of the system matrices in (25) will be very useful for developing the proposed MPC tracking problem in the next section.

Remark 8: In case of underactuated systems [30], which have fewer control inputs than DOFs, some elements of the vector τ in (22) are dependent on each others or zeros, that might limit the controller authority on the system in comparison with that on fully actuated systems. Interestingly, the presented LPV modeling as in (24) can be used for underactuated systems; however, the elements dependency in τ should be taken into account in the MPC optimization problem. That can be easily handled provided that such dependency is linear; otherwise, nonlinear transformations can be performed on τ as commonly used in the context of nonlinear control [31].

Furthermore, since MPC is considered here to control the system, a discrete-time model should be obtained. For simplicity we use Euler's forward (rectangular) discretization, which results in a discrete-time qLPV model as shown in (1) with

$$A(p) = I + \tilde{A}(q, \dot{q})T_s, \; B = \tilde{B}T_s, \; C = \tilde{C}, \; D = 0 \tag{26}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are given in (25) and T_s denotes the sampling time. The rectangular method is an approximative method of discretization; however, it has the important feature that it can preserve the linear dependence over the scheduling variables without introducing any extra complexity [2]. Furthermore, it preserves stability and yields a small discretization error provided that a suitable value of T_s is chosen, see [2] for more details.

IV. PRACTICAL LPVMPC FOR REFERENCE TRACKING

Based on the LPV modeling presented in the previous section, we propose in this section a novel MPC formulation for LPV systems to track a given desired reference trajectory using the notion of admissible reference. For ensuring stability and recursive feasibility, a terminal cost as a tracking error penalty term is added to the stage cost of the related MPC optimization problem and a terminal constraint based on the concept of invariant set for tracking [23] is included; both are computed offline. The MPC optimization problem is formulated based on the nominal LTI model corresponding to the current value of the scheduling variable. Therefore, the control law is obtained online by solving a single QP problem of a complexity similar to that of LMPC. To deal with the uncertainty of the scheduling variable affecting the evolution of the state over the prediction horizon, we utilize the notion of rigid tubes presented in Section II using a set $\mathbb{S}$ satisfying Assumption 5. It can be computed offline taking into account the rate of variation of the scheduling parameter, which can reduce the size of $\mathbb{S}$ leading to a low conservative design.

A. ADMISSIBLE REFERENCE

For the LPV system (1), any admissible steady state and input should satisfy the following equation

$$\begin{bmatrix} A(\bar{p}) - I & B(\bar{p}) \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = 0, \quad \bar{p} \in \mathbb{P} \tag{27}$$

where $\bar{x}$ and $\bar{u}$ are the steady state and input, respectively, and $\bar{p}$ is the steady-state value of the scheduling parameter, which represents —in case of qLPV models— the frozen scheduling parameter associated with $(\bar{x}, \bar{u})$. Note that $(\bar{x}, \bar{u})$ belongs to the null space of the left matrix in (27); moreover, provided that the system is controllable for all $\bar{p} \in \mathbb{P}$, the dimension of the null space is n_u.

Now, consider qLPV models of the nonlinear dynamics (22) as illustrated in Section III and make use of the special structure in (26) with (25), we can rewrite (27) as

$$\left[\begin{array}{cc|c} 0 & \tilde{A}_{12}(\bar{p}) & 0 \\ \tilde{A}_{21}(\bar{p}) & \tilde{A}_{22}(\bar{p}) & I_{n_\mathrm{q}} \end{array}\right]\left[\begin{array}{c} \bar{x} \\ \hdashline \bar{u} \end{array}\right] = 0, \quad \bar{p} \in \mathbb{P}, \tag{28}$$

note here that $n_\mathrm{q} = n_\mathrm{u}$, see Remark 8. It holds that

$$\left[\begin{array}{c} \bar{x} \\ \hdashline \bar{u} \end{array}\right] = L(\bar{p})\bar{y}_s, \quad \bar{p} \in \mathbb{P}, \tag{29}$$

where $\bar{y}_s \in \mathbb{R}^{n_\mathrm{q}}$ characterizes the solution of (28) and $L \in \mathbb{R}^{(n_\mathrm{x}+n_\mathrm{q})\times n_\mathrm{q}}$ can be given as

$$L(\bar{p}) = \left[\begin{array}{c} I_{n_\mathrm{q}} \\ 0 \\ \hdashline -\tilde{A}_{21}(\bar{p}) \end{array}\right], \tag{30}$$

which spans the nullspace of the left matrix in (28). The parameter $\bar{y}_s$ represents the associated admissible output, which is related to $\bar{x}$ by $\bar{y}_s = C\bar{x}$, see (26) and (25). In other words, $\bar{y}_s$ represents admissible references which can be tracked without steady-state tracking error. Moreover, $\bar{p}$ is related to $\bar{y}_s$, this means that for any $\bar{y}_s$ there exists $\bar{p}$ function of $\bar{x}$ for the qLPV models considered in Section III.

B. ADMISSIBLE RIST

Consider system (1) with the constraint sets $\mathbb{X}$ and $\mathbb{U}$ and the special matrices structure in (26) with (25), let also Assumption 4 be satisfied. Moreover, suppose that the system is controllable for all $p \in \mathbb{P}$ by a control law as in (4), which meets the conditions in Definition 3. Therefore, the system matrix $A(p) + BK$ is Schur for all $p \in \mathbb{P}$ and the closed-loop system converges to $(\bar{x}, \bar{u})$ without violating the state and input constraints provided that the initial state and $(\bar{x}, \bar{u})$ belong to the corresponding robust invariant set for tracking. This set can be considered as an RPI set for the augmented system

$$\begin{bmatrix} x(k+1) \\ \bar{y}_s \end{bmatrix} = \underbrace{\begin{bmatrix} A(p(k))+BK & -B(K_1+\tilde{A}_{21}(\bar{p})) \\ 0 & I_{n_\mathrm{q}} \end{bmatrix}}_{\bar{A}^c(p(k),\bar{p})} \begin{bmatrix} x(k) \\ \bar{y}_s \end{bmatrix} \tag{31}$$

subject to the system state and input constraints. The system (31) is derived by substituting (4) into (1a) taking into account (26), (25), using (29) and partitioning $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ such that $K_1 \in \mathbb{R}^{n_\mathrm{u}\times(n_\mathrm{x}-n_\mathrm{u})}$. Note that for a given admissible $\bar{y}_s$ and the corresponding $\bar{p}$, the state of the closed-loop system (31) will converge to the related $\bar{x}$ according to (29) starting from any initial state and the associated $p \in \mathbb{P}$. Since the system (31) subjects to constraints, we want to construct an RPI set in which any initial state will converge to a steady state according to $\bar{y}_s$ for any $p \in \mathbb{P}$. Then, that RPI set can be used as a terminal set for the proposed LPVMPC.

The set of constraints on (31) can be posed as follows

$$\begin{aligned} \bar{\mathbb{X}}_\lambda = \{(x, \bar{y}_s) \mid \; & x \in \mathbb{X}, \;\; \bar{y}_s \in \lambda\bar{\mathbb{Y}}, \;\; \bar{p} \in \mathbb{P}, \\ & (Kx - (K_1+\tilde{A}_{21}(\bar{p}))\bar{y}_s) \in \mathbb{U}\}, \end{aligned} \tag{32}$$

where $\bar{\mathbb{Y}} = Proj_y(\mathbb{X})$ denotes the constraint set of the admissible output and λ is a scalar, for the moment, let $\lambda = 1$. Now, define a set $\Omega \subseteq \bar{\mathbb{X}}_1$ as an admissible robust invariant set for tracking for the system (1) with (26) or as a robust positive invariant set for the augmented system (31), under the constraint set $\bar{\mathbb{X}}_1$, if

$$\forall \begin{bmatrix} x \\ \bar{y}_s \end{bmatrix} \in \Omega : \; \bar{A}^c(p, \bar{p}) \begin{bmatrix} x \\ \bar{y}_s \end{bmatrix} \in \Omega, \;\; \Omega \subseteq \bar{\mathbb{X}}_1, \;\; \forall p \in \mathbb{P}. \tag{33}$$

In order to attain the largest possible domain of attraction of the proposed MPC scheme, we should consider the maximum admissible RIST, which can be determined —using linear programming— similarly as computing a maximum RPI set for the system (31) constrained in $\bar{\mathbb{X}}_1$. However, due to the unity eigenvalues of the augmented matrix $\bar{A}^c$ as shown in (31), the computation of maximum RPI set may not be finitely determined. Apparently, this can be resolved by considering the set $\bar{\mathbb{X}}_\lambda$ in (32) with λ arbitrarily close to 1, note that $\lambda \in (0, 1)$, see [32] for more details. This results in an admissible RIST set smaller but close to the maximum one. Denote such a set as $\bar{\mathbb{X}}_\mathrm{f} \subseteq \bar{\mathbb{X}}_\lambda$, which will be employed for constructing the terminal set in the proposed MPC setting as shown below, therefore, $\bar{\mathbb{X}}_\mathrm{f}$ is the maximum $\Omega \subseteq \bar{\mathbb{X}}_\lambda$.

C. OPTIMIZATION PROBLEM

Now, the proposed MPC optimization problem for reference tracking can be formulated as follows:

$$\begin{aligned} \min_{\substack{v_{0|k}, \cdots, v_{N-1|k} \\ \bar{y}_{0|k}, \cdots, \bar{y}_{N|k}}} \; & \sum_{i=0}^{N-1} \|z_{i|k} - \bar{z}_{i|k}\|_Q^2 + \|v_{i|k} - \bar{v}_{i|k}\|_R^2 \\ & + \|\bar{y}_{i|k} - r_{i|k}\|_T^2 + \|\bar{y}_{N|k} - r_{N|k}\|_T^2 \\ & + \bar{V}_\mathrm{f}(z_{N|k}, \bar{z}_{N|k}) \qquad (34a) \\ \text{subject to} \; & \text{(16a-d)}, \qquad (34b) \\ & \begin{bmatrix} z_{N|k} \\ \bar{y}_{N|k} \end{bmatrix} \in \bar{\mathbb{Z}}_\mathrm{f} \qquad (34c) \end{aligned}$$

and the nominal system dynamics at $p(k)$ with $z_{0|k} = x(k)$, where $T = T^\top \succeq 0$, Q, R are tuning matrices to achieve a desired tracking performance, for all $i = 0, 1, \cdots, N$, $\bar{z}_{i|k}$, $\bar{v}_{i|k}$ are nominal steady state and input, respectively, according to the desired reference trajectory $r_{i|k}$, which are parameterized as in (29) by the corresponding nominal admissible output $\bar{y}_{i|k}$ with $\bar{p} = p(k)$, $\bar{V}_\mathrm{f}(z_{N|k}, \bar{z}_{N|k})$ is the terminal cost for tracking which is given by

$$\bar{V}_\mathrm{f}(z_{N|k}, \bar{z}_{N|k}) = \|z_{N|k} - \bar{z}_{N|k}\|_P^2, \tag{35}$$

and satisfies the condition in Definition 1 and $\bar{\mathbb{Z}}_\mathrm{f}$ is a tightened terminal set given by

$$\bar{\mathbb{Z}}_\mathrm{f} = \bar{\mathbb{X}}_\mathrm{f} \ominus \left(\mathbb{S} \times Proj_y(\mathbb{S})\right), \tag{36}$$

with $\bar{\mathbb{X}}_\mathrm{f}$ is an admissible RIST as discussed in Section IV-B. The nominal states $z_{i|k}$ are computed according to the nominal model in (19) given $p_{0|k} = p(k)$ (for computing the corresponding system matrix), where $z_{0|k} = x(k)$.

The tightened state and input constraint sets $\mathbb{Z}$ and $\mathbb{V}$ are computed offline from (17a) and (17b), respectively. The reference steps $r_{0|k}, \cdots, v_{N|k}$ are given in advance. The nominal control input moves $v_{0|k}, \cdots, v_{N-1|k}$, and the admissible output moves $\bar{y}_{0|k}, \cdots, \bar{y}_{N|k}$ are the decision variables of the optimization problem. Finally, the values of $\bar{z}_{i|k}$ and $\bar{v}_{i|k}$ are substituted in terms of $\bar{y}_{i|k}$ using (29) with $\bar{p} = p(k)$ as these nominal steady state and input are related to the nominal model at $p(k)$, i.e., (19), where its matrices computed at $p(k)$ are frozen over the prediction horizon. Another way, one can use the nominal LPV model in (10) instead of (19); however, the system matrices as well as the matrix L in (30) should be updated at each step of the prediction horizon, which demands a slight increase in computations.

The main result of the paper is presented in the following theorem.

Theorem 9 (Recursive Feasibility and Stability of the LPVMPC for Tracking): Consider the LPV system (1) with the system matrices (26) and (25) and suppose that Assumptions 4 and 5 are satisfied. Let r be a given admissible reference trajectory, $\bar{V}_f$ in (35) be a terminal cost for the optimization problem (34) satisfying the condition in Definition 1 and $\bar{\mathbb{Z}}_f$ in (36) be its terminal set, where $\bar{\mathbb{X}}_f$ satisfies the invariance condition in (33) with Ω and $\bar{\mathbb{X}}_1$ are replaced by $\bar{\mathbb{X}}_f$ and $\bar{\mathbb{X}}_\lambda$, respectively, for a given $\lambda \in (0, 1)$, then

(i) the optimization problem (34) is recursively feasible and
(ii) the LPVMPC solution by (34) is asymptotically stabilizing.

The proof of Theorem 9 is detailed in Appendix A.

Remark 10: The approach in [23] and [24] has considered fixed values for $\bar{z}_{i|k}$ and $\bar{v}_{i|k}$ for all i over N, thus, it handles piecewise constant references, moreover, that can lead to tracking with a large rise time. In contrast, the proposed approach is not restricted to that as the variables $\bar{z}_{i|k}$ and $\bar{v}_{i|k}$ are allowed to be changed over N via the admissible output moves $\bar{y}_{0|k}, \cdots, \bar{y}_{N|k}$, which does increase the number of decision variables, according to the given desired reference. In this perspective, one might realize $\bar{z}_{i|k}$ and $\bar{v}_{i|k}$ as points on state and input trajectories related to the trajectory of the admissible reference rather than steady state and input values discussed in the previous section.

Remark 11: The terms $\|\bar{y}_{i|k} - r_{i|k}\|_T^2$, $i = 0, 1, \cdots, N$ in the cost function (34a) penalize the deviation between the nominal admissible output and the desired reference trajectory. This guarantees offset-free tracking provided that there exists an admissible output trajectory equal to the desired reference; otherwise, it steers the system to the closest admissible output.

Algorithm 1 summarizes the online implementation of the proposed approach. It can be executed provided that Assumptions 4(ii) is satisfied and given the reference trajectory r, the matrices P, Q, R, T and the sets $\mathbb{X}, \mathbb{Z}, \mathbb{U}, \mathbb{V}$ and $\bar{\mathbb{Z}}_f$ which can be computed offline as shown in the next section. At every sample k, the value of $p(k)$ is measured and used to update the matrix A using (26), (25), and hence the block matrix $\tilde{A}_{21}$, and the matrix L as in (30). Then, given $x(k)$ the optimization problem (34) is solved and the receding horizon concept is used, where the control sample $u(k) = v_{0|k}$ is applied to the system.

Algorithm 1 The Proposed LPVMPC for Tracking

Require: $r, P, Q, R, T, \mathbb{X}, \mathbb{Z}, \mathbb{U}, \mathbb{V}$ and $\bar{\mathbb{Z}}_f$ ▷ Offline
Initialization $x(0)$, $p(0)$ and $k = 0$
Repeat ▷ Online
1: Measure $p(k)$ and update $A(p_{0|k})$ and $L(p_{0|k})$.
2: Measure $x(k)$ and solve the optimization (34).
3: Implement the control sample $u(k) = v_{0|k}$.
4: $k \leftarrow k + 1$

D. OFFLINE COMPUTATIONS

The proposed LPVMPC approach involves offline computations including the cost function $\bar{V}_f$ and the sets $\mathbb{Z}, \mathbb{V}, \bar{\mathbb{Z}}_f$. Computing $\bar{V}_f$ together with the controller gain K according to Definition 1 is a standard LMI problem, see [33] for more details. Regarding the computations of the tightened constraint sets $\mathbb{Z}, \mathbb{V}, \bar{\mathbb{Z}}_f$, it is based on the sets $\mathbb{X}, \mathbb{U}, \bar{\mathbb{X}}_f$, respectively, as well as the set $\mathbb{S}$ as shown in (17a,b) and (36). Determining the set $\bar{\mathbb{X}}_f$ is also a standard problem using linear programming which can be computed as a maximum RPI set, as shown in [26], for the system (32) subjected to the constraint set $\bar{\mathbb{X}}_\lambda$ with λ chosen close to 1 so that $\bar{\mathbb{X}}_f$ is finitely determined. Concerning the set $\mathbb{S}$ satisfying Assumption 5, we propose two ways as follows:

1) One way is to consider $\mathbb{S}$ a minimal robust positive invariant (mRPI) set [34] based on a disturbance set given as $Co\{(A(p) - A_0)x + (B(p) - B_0)u \mid p \in \mathbb{P}, x \in \mathbb{X}, u \in \mathbb{U}\}$ corresponding to the whole scheduling range $\mathbb{P}$, where (A_0, B_0) are the related nominal system matrices evaluated at the center of $\mathbb{P}$. In this way, we ignore the fact that the rate of variation of p is bounded. The approach proposed in [34] can be employed to compute an outer approximation of the mRPI set.
2) Another way to compute $\mathbb{S}$, let N and the bound on the rate of variation of p, i.e., dp^{max}, be given, calculate the set $\mathcal{S}_{N|k}$ using (13) on a grid points of $\mathbb{P}$, which we denote as $\mathcal{S}_{N|k}^{i_g}$ at every grid point $i_g = 1, \cdots, n_g$, where n_g is the number of the grid points. Next, use these sets to compute
$$\mathbb{S} = \alpha \cdot Co\left\{\bigcup_{i_g=1}^{n_g} \mathcal{S}_{N|k}^{i_g}\right\}, \tag{37}$$
where $\alpha \geq 1$. Note that, any of the sets $\mathcal{S}_{N|k}^{i_g}$ is a PC-set according to the formulation in Section II. Finally, the obtained $\mathbb{S}$ can be verified on a denser grid to check the validity of Assumption 5; otherwise, it should be enlarged using α and verified again.

The first way is guaranteed to verify condition (15); however, it might be overly conservative, especially, when dp^{max}

is relatively small or N is short, resulting in a significant reduction in the size of $\mathbb{Z}_f$ together with the domain of attraction, then the second method could provide a better solution provided that condition (15) is verified on very dense grid points. It is recommend to compare the obtained $\mathbb{S}$ from both approaches.

V. APPLICATION TO THE CRS A465 ROBOT

In the following, we present the results of implementing the proposed LPVMPC for tracking on the CRS A465 Robotic manipulator shown in Fig 2. It has six rotational joints actuated by DC motors. The angular displacements of the motor shafts are measured by incremental encoders. In this work, we consider the shoulder and the elbow corresponding to q_1 and q_2, respectively, they are the most challenging links to control, since they are affected by gravity, inertial, centripetal, Coriolis and friction torques; the other links are fixed during the experiments. To demonstrate the quality of the proposed control approach we consider two practical trajectories [35] to be tracked and we compare with other MPC scheme developed recently in [22].

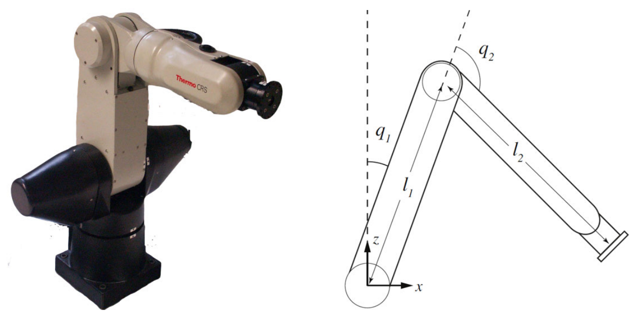

FIGURE 2. The CRS A465 robotic manipulator and a side view of the 2-DOF model.

A. LPV MODELING

Based on the formulation introduced in Section III, we derive now a qLPV model for a 2-DOF of the CRS A465 robotic manipulator. According to (22), $q, \dot{q}, \ddot{q} \in \mathbb{R}^2$ are the joint angular positions, velocities and acceleration, respectively, and the nonlinear matrices are given by

$$M = \begin{bmatrix} b_7 + b_2 + 2b_3\cos(q_2) & b_2 + b_3\cos(q_2) \\ b_7 - b_8 + b_3\cos(q_2) & b_7 \end{bmatrix},$$
$$c = \begin{bmatrix} -b_3\dot{q}_2^2\sin(q_2) - 2b_3\dot{q}_1\dot{q}_2\sin(q_2) \\ b_3\dot{q}_1^2\sin(q_2) \end{bmatrix},$$
$$g = \begin{bmatrix} -b_4\sin(q_1+q_2) - b_5\sin(q_1) \\ -b_4\sin(q_1+q_2) \end{bmatrix}, \ \tau_v = \begin{bmatrix} b_6\dot{q}_1 \\ b_9\dot{q}_2 \end{bmatrix} \quad (38)$$

where $b_1, \cdots, b_9$ are grouped parameters related to the dynamic and kinematic parameters of the robot. We use the parameters of [4], which were experimentally identified for the same robot considered here, the parameters are shown in Table 1. The physical limits of the robot movements are specified by the manufacturer of the robot as the following bounds on its angular positions and velocities

$$|q_1| \leq 90°, \ |q_2| \leq 110°, \ |\dot{q}_1| \leq 180°, \ |\dot{q}_2| \leq 180°, \quad (39)$$

TABLE 1. Constant parameters in (38).

b_1	b_2	b_3	b_4	b_5	b_6	b_7
0.0715	0.0058	0.0114	0.3264	0.3957	0.6254	0.0749
b_8	b_9					
0.0705	1.1261					

and its input constraints are

$$|\tau_1| \leq 5, \quad |\tau_2| \leq 5, \quad (40)$$

which are based on the actuators limits.

Therefore, the qLPV representation of the system is given by (24) with (25), where

$$\tilde{A}_{12}(q) = M^{-1}(q) \quad (41a)$$
$$\tilde{A}_{21}(q) = \frac{b_4\sin(q_1+q_2)}{q_1+q_2}I_2 + \frac{b_5\sin(q_1)}{q_1}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \quad (41b)$$
$$\tilde{A}_{22}(q,\dot{q}) = -\left(\begin{bmatrix}b_6 & 0\\0 & b_9\end{bmatrix} + b_3\dot{q}_1\sin(q_2)\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\right)M^{-1}(q). \quad (41c)$$

To further simplify the obtained qLPV model, we neglect the second-order term $b_3\dot{q}_1\sin(q_2)$ in (41c), this gives

$$\tilde{A}_{22}(q) = -\begin{bmatrix}b_6 & 0\\0 & b_9\end{bmatrix}M^{-1}(q), \quad (42)$$

therefore, the system matrix $\tilde{A}$ becomes dependent only on q, and hence, measuring $\tilde{A}$ in practice is independent of derivatives of signals. In general, the error due to such simplification is usually small, unless when both q_2 and $\dot{q}_1$ are very close to their limits, which is an irregular scenario. Next, we define the scheduling variables of the qLPV model based on (41a,b) and (42), which indicate 4 variables yielding $\tilde{A}$ affine dependent matrix; they are shown together with their bounds and the bounds on their derivatives according to (39) in Table 2. Finally, the discrete-time model of the system can be determined as in (26) taking into account the blocks of $\tilde{A}$ as shown in (41a,b) and (42).

TABLE 2. Scheduling parameters data.

$p_1 = \frac{b_4\sin(q_1+q_2)}{q_1+q_2}$	$-0.098 \leq p_1 \leq 1$	$\lvert\dot{p}_1\rvert \leq 4.0194$
$p_2 = \frac{b_5\sin(q_1)}{q_1}$	$0.637 \leq p_2 \leq 1$	$\lvert\dot{p}_2\rvert \leq 2.1221$
$p_3 = 1/\Delta$	$138.394 \leq p_3 \leq 192.133$	$\lvert\dot{p}_3\rvert \leq 231.95$
$p_4 = \cos(q_2)/\Delta$	$-65.713 \leq p_4 \leq 138.394$	$\lvert\dot{p}_4\rvert \leq 797.79$

$$\Delta = b_7(b_1+b_2+2b_3\cos(q_2)) - (b_2+b_3\cos(q_2)(b_7-b_8+b_3\cos(q_2))$$

B. SIMULATION RESULTS

We consider the discrete-time qLPV model developed in Section V-A, with sampling time $T_s = 0.01$s, which compromises the execution time of the MPC algorithm as well as the bandwidth of the system. Moreover, we scale the scheduling variables in Table 2 to be within ±1, which is preferred for numerical reasons. Note that scaling p changes the constant matrices associated with the affine dependency on p, i.e., the matrices A^j in (5). In discrete-time, the rate of change of p is defined as $\mathrm{d}p = \dot{p}T_s$ and its bounds are computed accordingly using (39) and the definition of p in Table 2, then, they are redefined based on the scaled p as follows:

$$
\begin{aligned}
\mid \mathrm{d}p_1 \mid &\leq 0.073, \quad \mid \mathrm{d}p_2 \mid \leq 0.117 \\
\mid \mathrm{d}p_3 \mid &\leq 0.086, \quad \mid \mathrm{d}p_4 \mid \leq 0.078.
\end{aligned} \tag{43}
$$

To compute the terminal cost $\bar{V}_\mathrm{f}$ (35), the matrix $P \in \mathbb{R}^{4\times4}$ together with the associated controller $K \in \mathbb{R}^{2\times4}$ are obtained using LMIs; K is required for calculating the sets $\bar{\mathbb{X}}_\mathrm{f}$ and $\mathbb{S}$. The RIST $\bar{\mathbb{X}}_\mathrm{f}$ has been computed as discussed in Section IV with $\lambda = 0.975$, which can lead to reasonably offset free tracking ranges of

$$
\mid q_1 \mid \leq 87.75°, \quad \mid q_2 \mid \leq 107.25°, \tag{44}
$$

c.f., (39); note that larger value than that λ could not allow convergence while computing the set $\bar{\mathbb{X}}_\mathrm{f}$. To calculate the set $\mathbb{S}$ satisfying Assumption 5, a prediction horizon $N = 4$ has been chosen and used afterward in the online implementation, then, the second method in Section IV-D has been carried out to compute $\mathbb{S}$, which is a reasonable choice here due to the small values of $\mathrm{d}p_{\max}$ as shown in (43) and the short prediction horizon. According to (37), we obtained the set $\mathbb{S}$ with $n_\mathrm{g} = 3^4$ and $\alpha = 1.1$ and it has been verified on a very dense grid of 81^4 points on the set $\mathbb{P}$. Next, we performed set tightening to obtain the constraint sets $\mathbb{Z}$, $\mathbb{V}$ and $\bar{Z}_\mathrm{f}$ using (17a), (17b) and (36), respectively, which are the required sets to implement Algorithm 1. It is worth mentioning that all the involved set computations and operations have been performed using the Multi-Parametric Toolbox 3.0 [36]. To solve the optimization problem (34), the weighting matrices for the stage cost have been tuned to $Q = 10 \cdot (1, 1, 0, 0)$, $R = 0.001 \cdot diag(1, 1)$ and $T = 1000 \cdot diag(1, 1)$.

To evaluate Algorithm 1, it has been implemented in simulation to track a sinusoidal reference for both q_1 and q_2 which covers the whole range of both as given in (39) and results in an ellipse like shape reference trajectory in the Cartesian space. A projection of the state-phase plane on the (x_1-x_2)-plane is shown in Fig. 3, note that $x_1 = q_1$ and $x_2 = q_2$ according to (23). The figure demonstrates the convergence of the state trajectories of q_1 and q_2 to the admissible command trajectory based on the smaller range (44), c.f., (39). This leads to a small steady-state tracking error with respect to the desired Cartesian command trajectory. In Fig. 3 it is also shown the projection of the computed set $\mathbb{S}$ onto the (x_1, x_2)-plane, which is quite very small leading to a very small difference between the state constraint set $\mathbb{X}$ and the tightened one $\mathbb{Z}$ as depicted in the figure as well by their projections onto the (x_1, x_2)-plane. In addition, the projection of the terminal set $\bar{\mathbb{Z}}_\mathrm{f}$ onto the (x_1, x_2)-plane is shown in Fig. 3. Moreover, for comparison, we have computed the terminal set $\mathbb{Z}_\mathrm{f}^0$ corresponding to the steady state at the origin, its projection onto the (x_1, x_2)-plane is depicted in Fig. 3. Obviously, it is much smaller than the terminal set based on the proposed RIST, which implies larger domain of attraction. For further illustration, Fig. 4 shows the projections of the sets $\mathbb{X}$, $\mathbb{Z}$, $\bar{\mathbb{Z}}_\mathrm{f}$, $\mathbb{Z}_\mathrm{f}^0$ and $\mathbb{S}$ onto the (x_3, x_4)-plane.

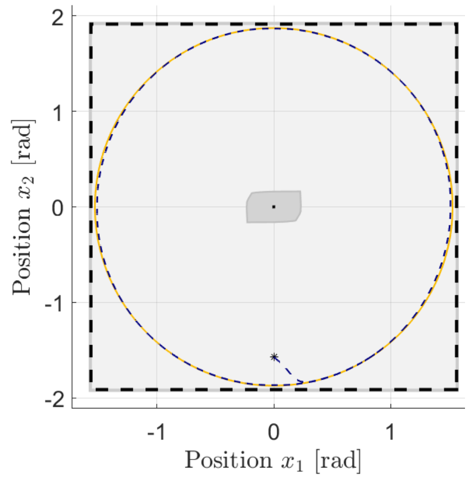

FIGURE 3. Phase plane on the (x_1, x_2)-plane (the transformed state as shown in (23)): the reference (yellow) and the simulated state trajectories of (x_1, x_2) (dashed-blue), and the projection of the sets $\mathbb{X}$ (sold-gray), $\mathbb{Z}$ (dashed-black), $\bar{\mathbb{Z}}_\mathrm{f}$ (filled-light-gray), $\mathbb{Z}_\mathrm{f}^0$ (filled-gray) and $\mathbb{S}$ (the smallest set) onto the (x_1, x_2)-plane. Note that $x_1 = q_1$ and $x_2 = q_2$.

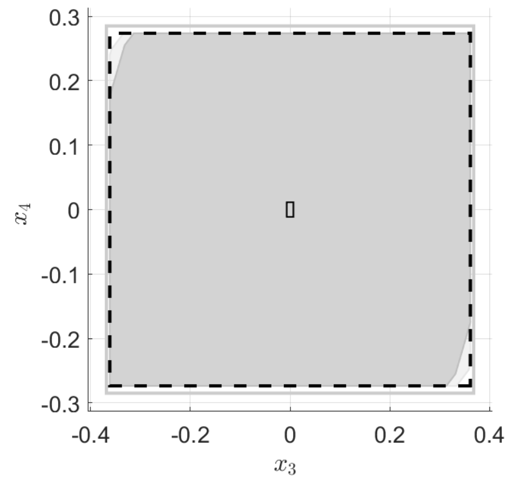

FIGURE 4. Phase plane on the (x_3, x_4)-plane (the transformed state (23)): the projection of the sets $\mathbb{X}$ (sold-gray), $\mathbb{Z}$ (dashed-black), $\bar{\mathbb{Z}}_\mathrm{f}$ (filled-light-gray), $\mathbb{Z}_\mathrm{f}^0$ (filled-gray) and $\mathbb{S}$ (the smallest set) onto the (x_3, x_4)-plane.

C. EXPERIMENTAL RESULTS

To evaluate the proposed LPVMPC approach in real-time, we consider here two tracking scenarios: In the first one, the robot tracks typical trajectories from practice, whereas, in the second scenario the robot tracks command trajectories beyond its work space. For comparison we consider the MPC approach in [22]. It is an NMPC scheme based on qLPV models for NL systems with stability guarantees, where the scheduling parameters are predicted in an iterative procedure over the prediction horizon without convergence guarantees and the online nonlinear optimization problem is solved as a sequence of QPs, which might be computationally demanding in comparison with the proposed approach. Moreover,

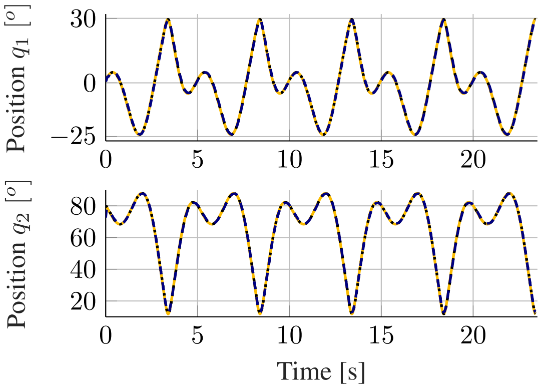

FIGURE 5. Angular positions trajectories of the first scenario: reference (yellow), based on MPC of [22] (dotted-black) and based on the proposed LPVMPC (dashed-blue).

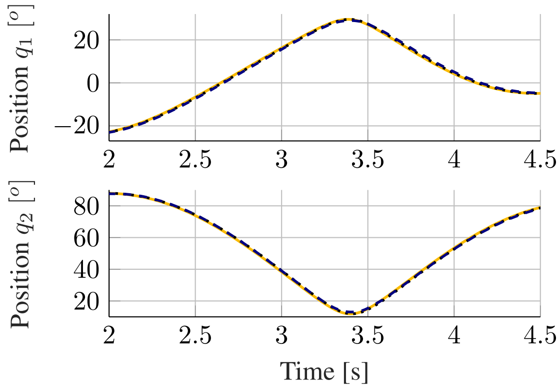

FIGURE 6. Zoom-in angular positions trajectories of the first scenario: reference (yellow), based on MPC of [22] (dotted-black) and based on the proposed LPVMPC (dashed-blue).

the approach in [22] is basically for tracking piece-wise constant references and its stability guarantee was formulated accordingly without recursive feasibility guarantees.

The proposed approach considers the availability of instantaneous information of p and x. Owing to the modeling developed in Section V-A, p depends only on q_1 and q_2, which are directly measured via the incremental encoders attached to the motors shafts. For the state x, it depends also on $\dot{q}_1$ and $\dot{q}_2$, which can be reasonably measured by low-pass filtering and numerically differentiating q_1 and q_2, respectively.

For the first tracking scenario, the trajectory tracking results are shown in Fig. 5 based on the approach of [22] and the proposed approach, both provide perfect tracking which is almost undistinguished from the desired references. To evaluate the tracking capability numerically we use the *best fit rate* (BFR) criterion, which is commonly used for models output validation in system identification [37], it is given by

$$\mathrm{BFR} = 100\% \cdot \max\left(1 - \frac{\|r_i(k) - q_i(k)\|}{\|r_i(k) - r_{m,i}\|}, 0\right),$$

$i = 1, 2$, where $r_{m,i}$ is the mean value of the reference trajectory r_i. The tracking capability of the angular position of the first joint in terms of the BFR is about 98% based on [22] and 96% using the proposed LPVMPC and for the second joint it is 97.91% and 97.14%, respectively. Note that p using the MPC of [22] is predicted over N, which is an advantage in [22] at the expense of its extra online computations, whereas in our approach it is unknown over the prediction horizon. Zoomed-in plots of the trajectories are shown in Fig. 6 to better visualize the deviations from the reference trajectories. Furthermore, the control signals computed by the proposed LPVMPC during this experiment is shown in Fig. 7, which is corresponding to the time interval considered in Fig. 6. The control effort is acceptable and within the specified limits.

Finally, we assess the capability of the LPVMPC to deal with command trajectories beyond the robot workspace. This demonstrates how large the domain of attraction by using the admissible reference concept [23] and it can also be

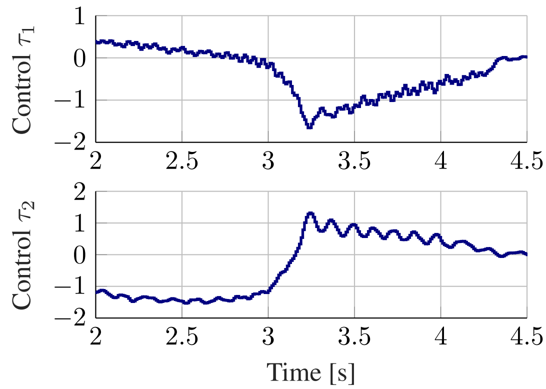

FIGURE 7. Zoom-in control signals of the proposed LPVMPC in the first scenario.

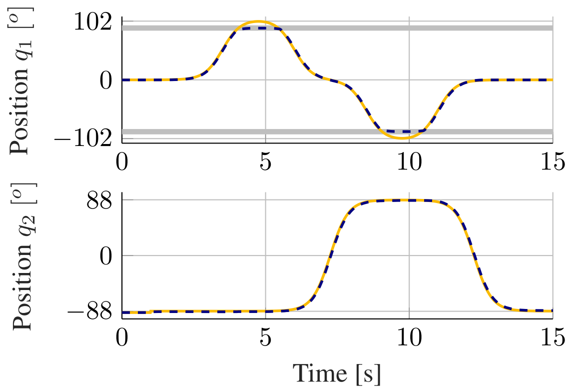

FIGURE 8. Angular positions trajectories of the second scenario: the reference (yellow) and based on the proposed LPVMPC (dashed-blue).

considered as a safety feature of the proposed approach. In this tracking scenario, the desired reference r_1 is assigned to maneuver within $|r_1| \leq 102°$ which is not admissible based on the specified bounds in (39) while r_2 is restricted to $|r_2| \leq 88°$, which is admissible. As shown in Fig.8, the LPVMPC allows only the admissible value for q_1 (44) to be approached with reasonably well tracking capability of both q_1 and q_2 within there admissible values. The corresponding control signals during this experiment are depicted in Fig. 9.

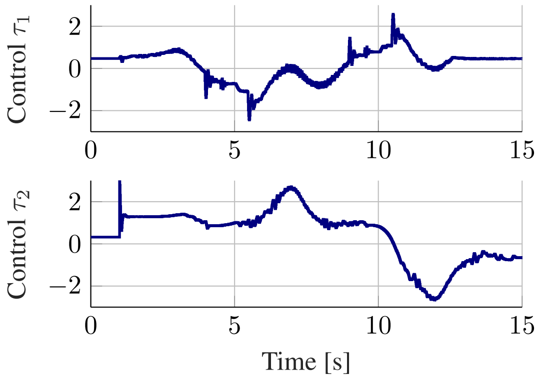

FIGURE 9. Control signals of the proposed LPVMPC in the second scenario.

VI. CONCLUSION

This paper has introduced a novel MPC approach for controlling a variety of nonlinear mechanical systems. The special qLPV representation of the nonlinear dynamics associated with these systems allows practical formulation for the MPC optimization problem, which is solved based on the frozen LPV system at the current scheduling parameter. The approach has several features: (i) the online control law is a solution of single QP problem. (ii) The controller ensures offset free tracking provided that the desired reference trajectory is admissible; otherwise, it steers the system to the closest admissible reference. (iii) Stability of the closed-loop system and feasibility of the optimization problem are guaranteed by including an admissible output trajectory, a related appropriate terminal cost, an admissible robust invariant set for tracking, and tightening the constraint sets. The successful experimental implementation on the 2DOF robot demonstrates its features and performance, which is almost similar to that of a more computationally demanding MPC approach.

APPENDIX A
PROOF OF THEOREM 2

In the following we prove the first part of the theorem, which is the recursive feasibility of the optimization problem (34). Assume that (34) is feasible at time k for an initial conditions $z_{0|k} = x_{0|k} = x(k)$, $p_{0|k} = p(k)$ with the nominal system matrix $A_{0|k}$ and given the reference values $r_{0|k}, \cdots, r_{N|k}$, which results in the optimal control sequence and the optimal sequence of nominal admissible output

$$\{v^*_{0|k}, v^*_{1|k}, \cdots, v^*_{N-1|k}\}, \tag{45a}$$
$$\{\bar{y}^*_{0|k}, \bar{y}^*_{1|k}, \cdots, \bar{y}^*_{N|k}\}, \tag{45b}$$

respectively, these lead to the optimal nominal state sequence $\{z_{0|k}, z^*_{1|k}, \cdots, z^*_{N|k}\}$. When $u(k) = v^*_{0|k}$ is applied to the LPV system, the initial state $x_{0|k}$ will move to the successor state $x(k+1) = x_{1|k} = z^*_{1|k}$. To prove recursive feasibility, we show that at time $k+1$, for the initial conditions $z_{0|k+1} = x_{0|k+1} = x(k+1)$ and $p_{0|k+1} = p(k+1)$, there exists a feasible solution for (34).

Using the feasible solution at time k, we can construct a feasible solution at time $k+1$ as follows. The reference trajectory is defined in advance, then,

$$r_{i|k+1} = r_{i+1|k}, \quad \forall i = 0, 1, \cdots, N-1, \tag{46}$$

and hence, the solution for the admissible output trajectory at time $k+1$ can be chosen as

$$\bar{y}_{i|k+1} = \bar{y}^*_{i+1|k}, \quad \forall i = 0, 1, \cdots, N-1. \tag{47}$$

Next, at any prediction step $i|k+1$, $i = 0, \cdots, N-1$, the nominal system is

$$z_{i|k+1} = A_{0|k+1} z_{i-1|k+1} + B v_{i-1|k+1}. \tag{48}$$

Note that if $p_{0|k+1} \neq p_{0|k}$, then $A_{0|k+1} \neq A_{0|k}$. Since the variation rate of p is bounded, we have $p_{0|k+1} \in \mathcal{P}_{1|k}$, see Fig. 1. Now, we need to show that, there exists $v_{i-1|k+1} \in \mathbb{U} \ominus K\mathbb{S}$ yielding $z_{i|k+1} \in \mathbb{X} \ominus \mathbb{S}$, c.f., (17). Let

$$v_{i-1|k+1} = v^*_{i|k} + K(z_{i-1|k+1} - z^*_{i|k}), \tag{49}$$

and substitute (49) into (48), then, one can rewrite (48) as

$$z_{i|k+1} = z^*_{i+1|k} + A^c_{0|k}(z_{i-1|k+1} - z^*_{i|k}) + (A_{0|k+1} - A_{0|k}) z_{i-1|k+1}, \tag{50}$$

which implies that $z_{i|k+1} \in z^*_{i+1|k} \oplus \mathcal{S}_{i+1|k}$, see (13). Since $z^*_{i+1|k} \in \mathbb{X} \ominus \mathbb{S}$, by imposing the constraint on $z_{i|k+1}$, it holds that $(z^*_{i+1|k} \oplus \mathcal{S}_{i+1|k}) \cap (\mathbb{X} \ominus \mathbb{S}) \neq \emptyset$, i.e., the solution set of $z_{i|k+1}$ subject to the constraint $\mathbb{X} \ominus \mathbb{S}$ is nonempty, which implies feasibility of $z_{i|k+1} \in \mathbb{X} \ominus \mathbb{S}$. For $v_{i-1|k+1}$, (49) ensures that $v_{i-1|k+1} \in v^*_{i|k} \oplus K\mathcal{S}_{i|k}$, where $v^*_{i|k} \in \mathbb{U} \ominus K\mathbb{S}$, again, the solution set for $v_{i-1|k+1}$ is $(v^*_{i|k} \oplus K\mathcal{S}_{i|k}) \cap (\mathbb{U} \ominus K\mathbb{S}) \neq \emptyset$, which concludes feasibility of $v_{i-1|k+1} \in \mathbb{U} \ominus K\mathbb{S}$.

Now, consider step $N-1|k+1$ and the nominal system (48) with $i = N-1$, using the same reasoning as above, one can show the feasibility of the constraints at this step. Furthermore, (50) at $i = N-1$ indicates that $z_{N-1|k+1} \in z^*_{N|k} \oplus \mathcal{S}_{N|k}$ and according to (47) $\bar{y}_{N-1|k+1} = \bar{y}^*_{N|k}$, we can conclude that $(z_{N-1|k+1}, \bar{y}_{N-1|k+1}) \in \bar{\mathbb{X}}_f$ as $(z^*_{N|k}, \bar{y}^*_{N|k}) \in \bar{\mathbb{X}}_f \ominus (\mathbb{S} \times Proj_y(\mathbb{S}))$ and $\mathcal{S}_{N|k} \in \mathbb{S}$ based on Assumption 5, therefore, $(z_{N-1|k+1}, \bar{y}_{N-1|k+1})$ is already in the RIST, and hence, we guarantee convergence to steady state and input according to the control law, see Definition 3. By choosing

$$v_{N-1|k+1} = \bar{v}^*_{N|k} + K(z_{N-1|k+1} - \bar{z}^*_{N|k}), \tag{51}$$

at step N, the feasibility of $v_{N-1|k+1} \in \mathbb{U} \ominus K\mathbb{S}$ can be ensured using similar argument as above. Substituting (51) into (50), then, it follows that $z_{N|k+1} \in z^*_{N+1|k} \ominus \mathcal{S}_{N+1|k}$. Moreover, using the relation between the nominal systems at k and $k+1$, according to $dp^{\max}$, one can show that $\bar{y}_{N|k+1} \in \bar{y}^*_{N+1|k} \oplus Proj_y(\mathcal{S}_{N+1|k})$. Since $(z^*_{N+1|k}, \bar{y}^*_{N+1|k}) \in \bar{\mathbb{X}}_f \ominus (\mathbb{S} \times Proj_y(\mathbb{S}))$, we can conclude that the solution set of $(z_{N+1|k}, \bar{y}_{N|k+1})$ using (51) is nonempty, i.e., $(z_{N+1|k}, \bar{y}_{N|k+1}) \oplus (\mathcal{S}_{N+1|k} \times \{0\}) \cap \{\bar{\mathbb{X}}_f \ominus (\mathbb{S} \times Proj_y(\mathbb{S})\} \neq \emptyset$, and hence, feasibility of $(z_{N|k+1}, \bar{y}_{N|k+1}) \in \{\bar{\mathbb{X}}_f \ominus (\mathbb{S} \times Proj_y(\mathbb{S})\}$ can be ensured. Therefore, it is possible to construct sequences

$$\{v_{0|k+1}, v_{1|k+1}, \cdots, v_{N-1|k+1}\}, \tag{52a}$$
$$\{\bar{y}_{0|k+1}, \bar{y}_{1|k+1}, \cdots, \bar{y}_{N|k+1}\}, \tag{52b}$$

which can lead to a feasible solution for (14) at time $k+1$. This completes the proof of the first part of the theorem.

The proof of the second part of the theorem, the asymptotic stability, is demonstrated as follows. We show that the cost function (34a) is a Lyapaunov function for the closed-loop system under the proposed LPVMPC controller. To simplify the notation, we denote the stage cost in (34a) by $\ell_{i|k+j} = \|e_{i|k+j}\|_Q^2 + \|v_{i|k+j} - \bar{v}_{i|k+j}\|_R^2 + \|\bar{y}_{i|k+j} - \bar{r}_{i|k+j}\|_T^2$, for all $i = 0, 1, \cdots, N-1, j = 0, 1, \cdots$, where $e_{i|k+j} = z_{i|k+j} - \bar{z}_{i|k+j}$, and we use $\ell_{N|k+j} = \|\bar{y}_{N|k+j} - \bar{r}_{N|k+j}\|_T^2$. Furthermore, we indicate by J_{k+j} the cost function (34a) at time $k+j$, which is a function of $z_{0|k+j}, p_{0|k+j}, r_{0|k+j}, \cdots, r_{N|k+j}$.

As shown above, the non-optimal solution (52) at time $k+1$, which is constructed based on the optimal solution (45) at time k, yields $(z_{N-1|k+1}, \bar{y}_{N-1|k+1}) \in \bar{\mathbb{X}}_{\mathrm{f}}$. Next, using the solution at time $k+1$, we can construct similarly as shown above a feasible solution at time $k+2$, which can yield $(z_{N-2|k+2}, \bar{y}_{N-2|k+2}) \in \bar{\mathbb{X}}_{\mathrm{f}}$. Proceeding in the same way, we can conclude that $(z_{N-i|k+j}, \bar{y}_{N-i|k+j}) \in \bar{\mathbb{X}}_{\mathrm{f}}$ for all $i = j = 1, 2, \cdots, N$. Each constructed feasible solution can provide a non-optimal cost $J_{k+j}, j = 1, 2, \cdots$, which bounds from above the corresponding optimal one.

Now, consider the difference between the costs at times $k+N$ and $k+N-1$ as

$$\begin{aligned}\Delta J_{k+N} &= J_{k+N} - J_{k+N-1}\\ &= \sum_{i=0}^{N} \ell_{i|k+N} - \ell_{i|k+N-1}\\ &\quad + \bar{V}_{\mathrm{f}}(e_{N|k+N}) - \bar{V}_{\mathrm{f}}(e_{N|k+N-1}). \end{aligned} \tag{53}$$

Note that the control policy over $\bar{\mathbb{X}}_{\mathrm{f}}$ is $v_{N-i|k+j} = \bar{v}_{N-i|k+j} + Ke_{N-i|k+j}$, $\forall i = j = 0, 1, \cdots, N$. Then, substituting into $\ell_{i|k+N}$ renders $\ell_{i|k+N} = \|e_{i|k+N}\|_{Q+K^\top RK}^2 + \|\bar{y}_{i|k+N} - r_{i|k+N}\|_T^2$, $\forall i = 0, 1, \cdots, N-1$ and $\ell_{N|k+N} = \|\bar{y}_{N|k+N} - \bar{r}_{N|k+N}\|_T^2$. From (2) it holds that $Q + K^\top RK < P - A^{c\top}(p) PA^c(p)$, $\forall p \in \mathbb{P}$, therefore, we can conclude that

$$\ell_{i|k+N} \leq \|e_{i|k+N}\|_P^2 - \|e_{i+1|k+N}\|_P^2 + \|\bar{y}_{i|k+N} - r_{i|k+N}\|_T^2,$$

where $A^c{}_{0|k+N} = A_{0|k+N} + BK$ and $e_{i+1|k+N} = A^c{}_{0|k+N} e_{i|k+N}$. A condition for $\ell_{i|k+N-1}$ can be similarly obtained. Substitute these two conditions into (53) and consider the following: According to (46) and (47), we have $r_{i|k+N} = r_{i+1|k+N-1}$ and $\bar{y}_{i|k+N} = \bar{y}_{i+1|k+N-1}$, respectively, for all $i = 0, \cdots, N-1$, moreover, $e_{0|k+N} = e_{1|k+N-1}$, $\bar{V}_{\mathrm{f}}(e_{N|k+N-1}) = \|e_{N|k+N-1}\|_P^2$ and $\bar{V}_{\mathrm{f}}(e_{N|k+N}) = \|e_{N|k+N}\|_P^2$. Thus, one can cancel all similar terms in ΔJ_{k+N} (53) related to $\ell_{i|k+N}, \ell_{i|k+N-1}$ for all $i = 1, \cdots, N$, $\bar{V}_{\mathrm{f}}(e_{N|k+N-1})$ and $\bar{V}_{\mathrm{f}}(e_{N|k+N})$. Finally, if r is an admissible reference, then, it follows that $\Delta J_{k+N} \leq -\ell_{0|k+N-1}$. Employing quadratic stage cost here ensures existence of a $\mathcal{K}_\infty$-function $\alpha(\|e_{0|k+N-1}\|)$ such that $\ell_{0|k+N-1} \geq \alpha(\|e_{0|k+N-1}\|)$. Therefore, the cost function J, which is positive definite, is monotonically decreasing [25], consequently, it is a parameter-dependent Lyapunov function for the closed-loop system consisting of any of the nominal systems and the MPC control law. The convergence of J to zero implies the convergence of the nominal state z asymptotically to the steady state $\bar{z}$, which ensures the convergence of $\bar{y}$ to r. On the other hand, if r is not admissible, the convergence of z to $\bar{z}$ ensures the convergence of $\bar{y}$ to $\tilde{y} = \arg\min_{\bar{y} \in \bar{\mathbb{Y}}} \|\bar{y} - r\|_T^2$, see Lemma 1 in [24] and Lemma 3 in [23] for more details. In this case, convergence of J is ensured but not to zero.

Regarding the convergence of the state x, we have $x_{0|k+1} = z_{1|k}$ for all $k \geq 0$, and hence, $x(k+1) = z_{1|k}$. The asymptotic stability of the nominal system and the fact that $x(k+1) = x_{0|k+1} = z_{1|k}$ implies that $x(k+1)$ also will converge asymptotically to the corresponding steady state. ■

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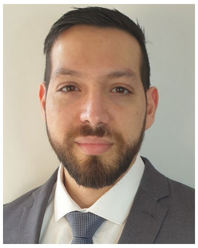

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