Linear Algebra and its Applications 732 (2026) 33–73

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Gershgorin-type spectral inclusions for matrices

Simon N. Chandler-Wilde [a,*], Marko Lindner [b]

[a] *Department of Mathematics and Statistics, University of Reading, Reading, RG6 6AX, UK*
[b] *Institut Mathematik, TU Hamburg (TUHH), Hamburg, D-21073, Germany*

ARTICLE INFO

Article history:
Received 4 September 2024
Received in revised form 4 August 2025
Accepted 22 November 2025
Available online 26 November 2025
Submitted by V. Mehrmann

MSC:
15A17
47A10
15B05
47B36

Keywords:
Spectrum
Pseudospectrum
Banded matrix
Toeplitz matrix

ABSTRACT

In this paper we derive sequences of Gershgorin-type inclusion sets for the spectra and pseudospectra of finite matrices. In common with previous generalisations of the classical Gershgorin bound for the spectrum, our inclusion sets are based on a block decomposition. In contrast to previous generalisations that treat the matrix as a perturbation of a block-diagonal submatrix, our arguments treat the matrix as a perturbation of a block-tridiagonal matrix, which can lead to sharp spectral bounds, as we show for the example of large Toeplitz matrices. Our inclusion sets, which take the form of unions of pseudospectra of square or rectangular submatrices, build on our own recent work on inclusion sets for bi-infinite matrices in Chandler-Wilde et al. (2024) [3].

1. Introduction, our main results, and the related literature

Let $A \in \mathbb{C}^{M\times M}$ be an $M \times M$ matrix with complex entries, which we will write in block form as

* Corresponding author.
E-mail addresses: s.n.chandler-wilde@reading.ac.uk (S.N. Chandler-Wilde), lindner@tuhh.de (M. Lindner).

https://doi.org/10.1016/j.laa.2025.11.017

$$A = [a_{ij}]_{i,j=1}^{N} \tag{1.1}$$

where $a_{ij} \in \mathbb{C}^{m_i \times m_j}$ is a subblock of A, with $m_i \in \{1, \dots, M-1\}$, for $i = 1, \dots, N$, so that $M \geq N > 1$ and

$$\sum_{i=1}^{N} m_i = M.$$

The aim of this paper is to derive new families of inclusion sets for the spectrum and pseudospectrum of A. Recall that $\mathrm{Spec}\, A$, the spectrum of A, is its set of eigenvalues, and, for $\varepsilon > 0$,

$$\mathrm{Spec}_\varepsilon A := \{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\|^{-1} \leq \varepsilon\}, \tag{1.2}$$

is the (closed) ε-pseudospectrum of A. Here I is the identity matrix, we adopt the convention that $\|(A - \lambda I)^{-1}\|^{-1} := 0$ when $A - \lambda I$ is singular, so that $\mathrm{Spec}\, A \subset \mathrm{Spec}_\varepsilon A$, for $\varepsilon > 0$, and, except where explicitly indicated otherwise, our norms are 2-norms, so that

$$\|(A - \lambda I)^{-1}\|^{-1} = s_{\min}(A - \lambda I), \qquad \lambda \in \mathbb{C},$$

where $s_{\min}(E)$ denotes the smallest singular value of a matrix E. With our convention the definition (1.2) also makes sense when $\varepsilon = 0$, coinciding with the set of eigenvalues of A, i.e., $\mathrm{Spec}_0 A = \mathrm{Spec}\, A$.

Using our own recent results in [3], where we derived analogous inclusion sets for the spectra and pseudospectra of bi-infinite matrices, we will derive two sequences of inclusion sets for the finite matrix case that we will term, following [3], the τ and τ_1 methods. The starting point for the derivation of each of these methods is to establish inclusion sets for $B = [b_{ij}]_{i,j=1}^{N}$, defined by

$$b_{ij} := \begin{cases} a_{ij}, & |i-j| \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

which we term the *tridiagonal part* of A. We term $C := A - B$ the *remaining part* of A, and set

$$\begin{aligned} r_L(A) &:= \max\{\|a_{2,1}\|, \|a_{3,2}\|, \dots, \|a_{N,N-1}\|\}, \\ r_U(A) &:= \max\{\|a_{1,2}\|, \|a_{2,3}\|, \dots, \|a_{N-1,N}\|\}, \\ r(A) &:= r_L(A) + r_U(A), \end{aligned} \tag{1.3}$$

so $r_L(A)$ and $r_U(A)$ are the maxima of the norms of the submatrices on the first subdiagonal and first superdiagonal of A, respectively, and $r(A)$ is an upper bound for both the maximum row sum and the maximum column sum of the norms of the block off-diagonal entries of B.

We will shortly, in Sections 1.1-1.2, define our sequences of spectral and pseudospectral inclusion sets for the τ and τ_1 methods, respectively. Our main results are Theorems 1.1 and 1.6, statements that the sets we define are, indeed, inclusion sets. We prove these theorems in §4. Preceding §4 we introduce in §2 concepts and ideas that we need for our arguments. Our main tool is corresponding inclusion sets for bi-infinite matrices recently established in [3]. We recall the results we need from [3] in §3, where we make, necessary for our later arguments, some generalisation of our earlier results as Theorems 3.1 and 3.3.

In §5 we apply our spectral inclusion sets to particular matrices. The examples we choose are mainly Toeplitz, this an attractive class because much is known about the asymptotics of spectra and pseudospectra for large matrix order (e.g., [18,1,2,22]). In §5.1 we treat two canonical tridiagonal examples, the non-normal case where A is a Jordan block, and the real symmetric case where A is a discrete Laplacian. A message from these two simple examples is that for large Toeplitz matrices our sequences of inclusion sets provide accurate approximations to the pseudospectrum, indeed also to the spectrum when the matrix is Hermitian. This message is confirmed by analysis of the general Toeplitz case in §5.2 where we show in Theorem 5.5 that our τ-method sequence of inclusion sets accurately approximates the pseudospectra of large banded Toeplitz matrices, indeed (Theorem 5.6) also the spectrum if the matrix is Hermitian. In Theorem 5.9 we show the same result for the τ_1-method inclusion set sequence for Toeplitz matrices that have symbol in the so-called Wiener algebra class.

Our τ-method sequence of spectral inclusion sets can be seen as an extension to a family, indexed by a parameter $n \in \mathbb{N}$, of previous block-matrix versions of the Gershgorin theorem (see [17,8,9,23]). These are close to the τ method in the simplest case $n = 1$, as we explain in Remark 1.2. In §5.1.2, and in a final §5.3 studying a family of 2-Toeplitz tridiagonal Hermitian matrices, we demonstrate that our new sequences of inclusion sets can give much sharper estimates for the spectrum than these previous block-matrix extensions (see Remark 5.4 and §5.3.2), indeed that these block-matrix extensions can lead to larger inclusion sets than classical Gershgorin. In §5.1.2, §5.2.1, and §5.3.2 we also make comparison in the banded Hermitian case between our τ-method inclusion sets for the spectrum and $\mathbb{R} \cap G$, where G is the classical Gershgorin inclusion set. We show, for the families of matrices we study including the large Hermitian Toeplitz class, that, while $\mathbb{R} \cap G$ is a sharp inclusion set in particular cases, our τ-method sequence provides sharp inclusions in every case; see Remark 5.8, §5.3.2, and Fig. 5.3.

In §6 we make concluding remarks and indicate directions for further work.

In supplementary materials we provide the Matlab codes used to produce Figs. 5.1-5.3. These materials also contain Matlab code for computing the τ-method eigenvalue inclusion set $\mathbb{R} \cap \Sigma_0^n(A)$ whenever A is Hermitian, and application of this code to check the accuracy of formulae (5.40) and (5.47) below.

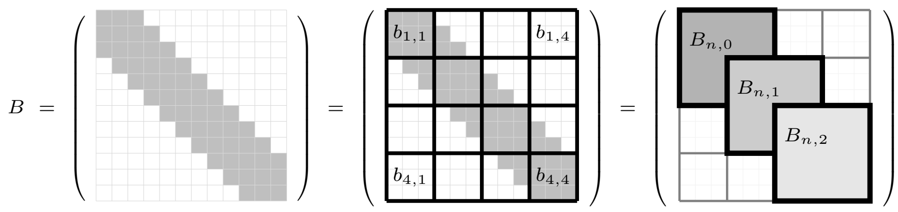

Fig. 1.1. Illustration, for the case that $M = 12$, $N = 4$, $m_i = 3$, $i = 1, \ldots, N$, and $n = 2$, of the part B of the matrix A, of its block tridiagonal representation, and of the sub-matrices $B_{n,k}$, for $k = 0, \ldots, N-n$, that arise in the τ method.

1.1. *The τ method and earlier generalisations of the Gershgorin theorem*

For $n = 1, \ldots, N$ and $k = 0, \ldots, N-n$, let

$$B_{n,k} := [b_{ij}]_{i,j=k+1}^{k+n},$$

so that $B_{n,k}$ is the $n \times n$ block-tridiagonal matrix consisting of the blocks of B whose rows and columns are in the range $k+1$ through $k+n$. The first sequence of inclusion sets we propose, termed the τ method (τ for truncation), is based on these *truncations* or *finite sections* $B_{n,k}$ of the block-tridiagonal matrix B; see Fig. 1.1. To define these inclusion sets, for $n \in \mathbb{N}$ let

$$\varepsilon_n(A) := 2r(A)\sin(\theta_n(A)/2) + \|C\| \leq 2r(A)\sin(\pi/(2n+4)) + \|C\| \leq \frac{\pi r(A)}{n+2} + \|C\|, \quad (1.4)$$

where $\theta_n(A)$ is the unique solution t in the range $\left[\frac{\pi}{2n+1}, \frac{\pi}{n+2}\right]$ of the equation

$$2\sin\left(\frac{t}{2}\right)\cos\left(\left(n+\frac{1}{2}\right)t\right) \; + \; \frac{r_L(A)r_U(A)}{(r(A))^2}\sin\left((n-1)\,t\right) \; = \; 0. \quad (1.5)$$

Note that, in particular,

$$\varepsilon_1(A) = r(A) + \|C\|.$$

For $\varepsilon \geq 0$ and $n = 1, \ldots, N-1$ (with $N > n \geq 3$ for the second definition), let

$$\sigma_\varepsilon^n(A) := \bigcup_{k=0}^{N-n} \mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} B_{n,k} \cup \bigcup_{m=1}^{n-1} \left(\mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} B_{m,0} \cup \mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} B_{m,N-m}\right),$$

$$\widehat{\sigma}_\varepsilon^n(A) := \bigcup_{k=0}^{N-n} \mathrm{Spec}_{\varepsilon+\varepsilon_{n-2}(A)} B_{n,k},$$

and define the *τ-method inclusion set* $\Sigma_\varepsilon^n(A)$ by

$$\Sigma_\varepsilon^n(A) := \begin{cases} \sigma_\varepsilon^n(A), & \text{if } n \le 2, \\ \sigma_\varepsilon^n(A) \cap \widehat{\sigma}_\varepsilon^n(A), & \text{if } n > 2. \end{cases} \tag{1.6}$$

The following is our main result for the τ method.

Theorem 1.1 *(The τ method inclusion sets). For $\varepsilon \ge 0$ and $n = 1, \dots, N-1$,*

$$\operatorname{Spec}_\varepsilon A \subset \Sigma_\varepsilon^n(A), \qquad \textit{in particular} \qquad \operatorname{Spec} A \subset \Sigma_0^n(A).$$

To aid comprehension and use of the above result we make a number of remarks.

Remark 1.2 *(The case $n = 1$ and previous generalisations of Gershgorin).* In the case that $n = 1$ and $\varepsilon = 0$ the above result is close to previous generalisations of the classical Gershgorin theorem to block matrices. For in that case the bound on $\operatorname{Spec} A$ in Theorem 1.1 reduces to

$$\operatorname{Spec} A \subset \Sigma_0^1(A) = \sigma_0^1(A) = \bigcup_{k=0}^{N-1} \operatorname{Spec}_{r(A)+\|C\|} B_{1,k} = \bigcup_{k=1}^{N} \operatorname{Spec}_{r(A)+\|C\|} a_{k,k}. \tag{1.7}$$

This is reminiscent of block matrix versions of the Gershgorin theorem, discovered independently by Ostrowski [17], Feingold & Varga [8], and Fiedler & Pták [9] (and see [23, Chapter 6]). These read that (e.g., [23, Theorem 6.3])

$$\operatorname{Spec} A \subset \bigcup_{k=1}^{N} G_k, \tag{1.8}$$

where

$$G_k := \{\lambda \in \mathbb{C} : \|(a_{k,k} - \lambda I)^{-1}\|^{-1} \le r_k\}, \quad r_k := \sum_{j=1,\, j\ne k}^{N} \|a_{k,j}\|,$$

and the norms here are any consistent set of matrix norms (see [23, §6.1] for detail). Of course, the set G_k is precisely an ε-pseudospectrum of $a_{k,k}$, defined with respect to the chosen norm, with $\varepsilon = r_k$. If we take each of the matrix norms in the definition of G_k in (1.8) to be the 2-norm, then G_k is our 2-norm pseudospectrum and (1.8) reads

$$\operatorname{Spec} A \subset \bigcup_{k=1}^{N} \operatorname{Spec}_{r_k} a_{k,k}. \tag{1.9}$$

In the case that $N = M$, so that each $a_{k,k} \in \mathbb{C}$ is a scalar, (1.8) and (1.9) reduce to the classical Gershgorin theorem [11,23], that

$$\operatorname{Spec} A \subset G := \bigcup_{k=1}^{N} G_k, \text{ where } G_k = \{\lambda \in \mathbb{C} : |a_{k,k} - \lambda| \le r_k\}, \;\; r_k = \sum_{j=1,\, j\neq k}^{N} |a_{k,j}|, \tag{1.10}$$

with G_k commonly termed the kth *Gershgorin disc.*

To make the connection between (1.9) and (1.7), note that, where $D = \operatorname{diag}(a_{1,1}, \ldots, a_{N,N})$ is the main diagonal of A, each r_k in (1.9) is a lower bound for the mixed infinity and 2-norm of $A - D$, defined by

$$\|A - D\|_{\infty,2} := \max_{k \in \{1,\ldots,N\}} \sum_{j=1,\, j\neq k}^{N} \|a_{k,j}\|,$$

with equality for some k, while $r(A) + \|C\|$ in (1.7) is an upper bound for $\|A - D\|$, the 2-norm of $A - D$, with equality, for example, when B is diagonal so that $r(A) = 0$ and $A - D = C$. □

Remark 1.3 *(The case $n \ge 2$ and an example).* Theorem 1.1 is equivalent to a statement that $\operatorname{Spec}_\varepsilon A \subset \sigma_\varepsilon^n(A)$ for $n \in \mathbb{N}$ and also $\operatorname{Spec} A \subset \widehat{\sigma}_\varepsilon^n(A)$ for $n > 2$. Both $\sigma_\varepsilon^n(A)$ and $\widehat{\sigma}_\varepsilon^n(A)$ are unions of the η-pseudospectra of finitely many of the submatrices $B_{m,k}$, with $m = n$ for $\widehat{\sigma}_\varepsilon^n(A)$ and $1 \le m \le n$ for $\sigma_\varepsilon^n(A)$, and with $\eta \le \varepsilon + \pi r(A)/n + \|C\|$ for each pseudospectrum. For the example in Fig. 1.1 with $N = 4$, and taking $n = 2$ as in the figure caption, the Theorem 1.1 bound on $\operatorname{Spec} A$ is

$$\begin{aligned}
\operatorname{Spec} A \subset \Sigma_0^2(A) &= \sigma_0^2(A) \\
&= \operatorname{Spec}_{\varepsilon_2(A)} B_{2,0} \,\cup\, \operatorname{Spec}_{\varepsilon_2(A)} B_{2,1} \,\cup\, \operatorname{Spec}_{\varepsilon_2(A)} B_{2,2} \\
&\qquad \cup \operatorname{Spec}_{\varepsilon_2(A)} B_{1,0} \,\cup\, \operatorname{Spec}_{\varepsilon_2(A)} B_{1,3} \\
&= \operatorname{Spec}_{\varepsilon_2(A)} B_{2,0} \,\cup\, \operatorname{Spec}_{\varepsilon_2(A)} B_{2,1} \,\cup\, \operatorname{Spec}_{\varepsilon_2(A)} B_{2,2} \\
&\qquad \cup \operatorname{Spec}_{\varepsilon_2(A)} a_{1,1} \,\cup\, \operatorname{Spec}_{\varepsilon_2(A)} a_{4,4},
\end{aligned}$$

since $B_{1,0} = b_{1,1} = a_{1,1}$ and $B_{1,3} = b_{4,4} = a_{4,4}$, and note that $b_{1,1}$, $b_{4,4}$, $B_{2,0}$, $B_{2,1}$, and $B_{2,2}$ are displayed in Fig. 1.1 and that $\varepsilon_2(A) \le 2r(A)\sin(\pi/8) + \|C\| < 0.77r(A) + \|C\|$. □

Remark 1.4 *(The case that A is banded).* In the case that the $M \times M$ matrix A is banded with band-width w, the block-matrix representation (1.1) of A is tridiagonal if the block dimensions are large enough, in particular if $m_i \ge w$, for $i = 1, \ldots, N$, in which case $B = A$ and $C = 0$. (In Fig. 1.1 the matrix B is banded with order $M = 12$ and band-width $w = 2$, so that, taking $N = 4$ and $m_i = 3$ for $i = 1, \ldots N$, as in Fig. 1.1, its block representation is tridiagonal.) We show examples in §5.1 and §5.2.1 that illustrate that, for large matrices A, the inclusion set $\Sigma_\varepsilon^n(A)$ for $\operatorname{Spec}_\varepsilon A$ can be sharp in such cases if n is large enough ($1 \ll n \ll N$), so that each η-pseudospectrum of $B_{m,k}$ in the definition of $\Sigma_0^n(A)$ has a small value of $\eta \le \pi r(A)/n$. □

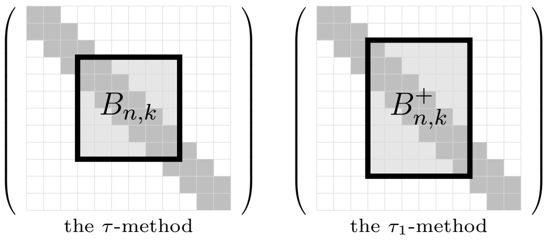

Fig. 1.2. Typical submatrices $B_{n,k}$ and $B^+_{n,k}$ of the block-tridiagonal matrix B.

Remark 1.5 *(Computation).* Computation of $\Sigma^n_\varepsilon(A)$ needs computation of the pseudospectra of finitely many square matrices and the computation of $\|C\|$. We envisage that our new inclusion sets will typically be used in cases where the order, $\#B_{n,k}$, of each $B_{n,k}$, namely $\#B_{n,k} = \sum_{i=k+1}^{k+n} m_i$, is in the range $[1, 1000]$, so that precise computation of pseudospectra of $B_{n,k}$ is feasible (see, e.g., [22, §IX]). On the other hand we expect use in cases where M is so large that exact computation of the 2-norm of C may be unfeasible, in which case $\|C\|$ can be replaced in the definition (1.4), and so in the definition of $\Sigma^n_\varepsilon(A)$, by some more computationally feasible upper bound, at the cost of an increase in the size of the inclusion set, e.g.,

$$\|C\| \le \|C\|_F \qquad \text{or} \qquad \|C\| \le \sqrt{\|C\|_1 \|C\|_\infty}\,,$$

where $\|\cdot\|_F$ denotes the Frobenius norm. In the special case that A is Hermitian, also $A_{n,k}$ is Hermitian for each n and k so that, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$,

$$\mathrm{Spec}_\varepsilon A_{n,k} = \mathrm{Spec}\, A_{n,k} + \varepsilon\overline{\mathbb{D}} = \bigcup_{\lambda \in \mathrm{Spec}\, A_{n,k}} (\lambda + \varepsilon\overline{\mathbb{D}})$$

is the closed ε-neighbourhood of the eigenvalues of $A_{n,k}$, for $\varepsilon > 0$ (see §2). $\square$

1.2. The τ_1 method: rectangular submatrices

Let $b_{0,1}$ and $b_{N+1,N}$ denote zero matrices of dimensions $1 \times m_1$, and $1 \times m_N$, respectively. For $n = 1, \dots, N$ and $k = 0, \dots, N-n$, let $B^+_{n,k}$ be the $(n+2) \times n$ block-tridiagonal matrix (see Fig. 1.2)

$$B^+_{n,k} := \left(\begin{array}{ccccc} b_{k,k+1} & 0 & \cdots & 0 & 0 \\ \hline \multicolumn{5}{c}{B_{n,k}} \\ \hline 0 & 0 & \cdots & 0 & b_{k+n+1,k+n} \end{array}\right), \tag{1.11}$$

with the understanding that the 0's are zero matrices of dimensions consistent with the block structure of $B^+_{n,k}$. For $k = 1, \dots, N-n-1$, $B^+_{n,k}$ consists precisely of the blocks of B whose rows are in the range k through $k+n+1$ and columns in the range $k+1$ to $k+n$. For all $k=0,\dots,N-n$, $B^+_{n,k}$ contains all the non-zero blocks of B in columns $k+1$ through $k+n$.

Our second sequence of inclusion sets, termed the τ_1 method (τ_1 for one-sided truncation), is based on pseudospectra of these *one-sided* or *rectangular truncations*, $B^+_{n,k}$, of the block-tridiagonal matrix B. Our notion of the pseudospectrum of an $(n+2)\times n$ rectangular matrix is essentially that of [24,22]. Let E^+_n and I^+_n be block matrices of the same dimensions and with the same block sizes as $B^+_{n,k}$, with I^+_n taking the form

$$I^+_n := \left(\begin{array}{ccc} 0 & \cdots & 0 \\ \hline & I_n & \\ \hline 0 & \cdots & 0 \end{array}\right), \tag{1.12}$$

where I_n is an order n block identity matrix. Then we define the ε-pseudospectrum of the rectangular matrix E^+_n by

$$\mathrm{Spec}_\varepsilon E^+_n := \{\lambda\in\mathbb{C} : s_{\min}(E^+_n-\lambda I^+_n)\le\varepsilon\}. \tag{1.13}$$

To define the τ_1 inclusion sets, for $n\in\mathbb{N}$ let

$$\varepsilon''_n(A) := 2r(A)\sin(\pi/(2n+2)) + \|C\| \le \frac{\pi r(A)}{n+1} + \|C\|. \tag{1.14}$$

For $\varepsilon\ge 0$ and $n=1,\dots,N-1$, define the τ_1*-method inclusion set* $\Gamma^n_\varepsilon(A)$ by

$$\Gamma^n_\varepsilon(A) \;:=\; \bigcup_{k=0}^{N-n} \mathrm{Spec}_{\varepsilon+\varepsilon''_n(A)} B^+_{n,k}. \tag{1.15}$$

The following is our main result on the τ_1 method. Note that the inclusions here are two-sided, in contrast to Theorem 1.1. Obvious variants of Remarks 1.4 and 1.5 apply in this τ_1 method case.

Theorem 1.6 *(The τ_1 method inclusion sets). For $n=1,\dots,N-1$ and $\varepsilon\ge 0$,*

$$\mathrm{Spec}_\varepsilon A \subset \Gamma^n_\varepsilon(A) \subset \mathrm{Spec}_{\varepsilon+\varepsilon''_n(A)+2\|C\|} A,$$

in particular

$$\mathrm{Spec}\, A \subset \Gamma^n_0(A) \subset \mathrm{Spec}_{\varepsilon''_n(A)+2\|C\|} A.$$

2. Notations and tools

Throughout, $\mathbb{N}=\{1,2,\dots\}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural, integer, real, and complex numbers, respectively, $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$, and $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$.

$\overline{S}$ denotes the closure of a set $S \subset \mathbb{C}$ and $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. For Hilbert spaces H_1 and H_2, $L(H_1, H_2)$ denotes the space of bounded linear operators from H_1 to H_2, with $L(H_1, H_1)$ abbreviated as $L(H_1)$, and H_1' denotes the dual space of H_1 (space of continuous linear functionals). For $E \in L(H_1)$ the spectrum of H_1, $\operatorname{Spec} H_1$, is the set of $\lambda \in \mathbb{C}$ such that $E - \lambda I$ is not invertible, where I is the identity operator. If H_1 is finite-dimensional, this is the set of eigenvalues of E. If E is a complex-valued matrix, E^T denotes its transpose and E^H its conjugate transpose.

Our Hilbert space $Y_{\mathbb{I}}$ Our arguments will feature Hilbert spaces that are finite- or infinite-dimensional, constructed in the following manner. Choose $\mathbb{I} \subset \mathbb{Z}$, finite or infinite. For each $i \in \mathbb{I}$, let X_i denote a complex Hilbert space and set

$$Y_{\mathbb{I}} := \bigoplus_{i \in \mathbb{I}} X_i, \tag{2.1}$$

which (see, e.g., [5, Chapter I, §6]) is a Hilbert space with the norm $\|\cdot\|$ given by

$$\|x\|^2 = \sum_{i \in \mathbb{I}} \|x_i\|^2, \quad \text{for} \quad x = (x_i)_{i \in \mathbb{I}} \in Y_{\mathbb{I}}.$$

In the case $X_i = X$, for $i \in \mathbb{I}$ and some Hilbert space X, we have $Y_{\mathbb{I}} = \ell^2(\mathbb{I}, X)$.

Band operators Let $E \in L(Y_{\mathbb{I}})$. Then (e.g., [16]) E has a matrix representation $[E] = [e_{ij}]_{i,j \in \mathbb{I}}$, denoted again by E, with all $e_{ij} \in L(X_j, X_i)$, such that, for every $x = (x_j)_{j \in \mathbb{I}} \in Y_{\mathbb{I}}$ with finitely many non-zero entries, $Ex = y = (y_i)_{i \in \mathbb{I}}$, where

$$y_i = \sum_{j \in \mathbb{I}} e_{ij} x_j, \quad i \in \mathbb{I}. \tag{2.2}$$

Let $BO(Y_{\mathbb{I}})$ denote the linear subspace of those $E \in L(Y_{\mathbb{I}})$ whose matrix representation is *banded* with some *band-width* w, meaning that $e_{ij} = 0$ for $|i - j| > w$. If $E \in BO(Y_{\mathbb{I}})$, then (2.2) holds (with the sum finite) for all $x \in Y_{\mathbb{I}}$. Conversely, if $[E] = [e_{ij}]_{i,j \in \mathbb{I}}$ is a banded matrix of operators $e_{ij} \in L(X_j, X_i)$ that satisfies

$$\sup_{i,j \in \mathbb{I}} \|e_{ij}\| < \infty, \tag{2.3}$$

then the mapping $E : Y_{\mathbb{I}} \to Y_{\mathbb{I}}$, $x \mapsto y$, with y given by (2.2), satisfies $E \in BO(Y_{\mathbb{I}})$, and $[E]$ is the matrix representation of E. As a special case of this observation the block matrix A given by (1.1) can be identified, in the usual way, with an operator in $L(Y_{\mathbb{I}})$, denoted also by A, with $\mathbb{I} = \{1, \dots, N\}$ and $X_i = \mathbb{C}^{m_i \times 1}$, $i = 1, \dots, N$, and with the mapping $x \mapsto Ax$ given by

$$(Ax)_i = \sum_{j=1}^{N} a_{ij} x_j, \quad \text{for} \quad x = (x_1, \dots, x_N) \in Y_{\mathbb{I}}.$$

Note that $L(Y_{\mathbb{I}}) = BO(Y_{\mathbb{I}})$ if $\mathbb{I}$ is finite.

The dual space $(Y_{\mathbb{I}})'$ *and adjoint operators* For $\mathbb{I} \subset \mathbb{Z}$ and $E \in L(Y_{\mathbb{I}})$, we denote by $E^* \in L(Y_{\mathbb{I}})$ and $E' \in L((Y_{\mathbb{I}})')$ the Hilbert space and Banach space adjoints of E, respectively. We will identify the dual space $(Y_{\mathbb{I}})'$ with $Y_{\mathbb{I}}$ through the mapping $Y_{\mathbb{I}} \to (Y_{\mathbb{I}})'$, $x = (x_i)_{i\in\mathbb{I}} \mapsto \hat{x}$, where $\hat{x}(y) := (y, \bar{x})$, for $y \in Y_{\mathbb{I}}$, where $(\cdot,\cdot)$ is the inner product on $Y_{\mathbb{I}}$, $\bar{x} = (\bar{x}_i)_{i\in\mathbb{Z}}$, and, for $i \in \mathbb{I}$, $\bar{x}_i := J_i x_i$, where $J_i : X_i \to X_i$ is any fixed anti-linear isometric involution on X_i (sometimes called a conjugate map, and easily constructed using an orthonormal basis for X_i, e.g., [19, Conclusion 2.1.18]). In the case that each $X_i = \mathbb{C}^{n_i} = \mathbb{C}^{n_i\times 1}$, for some $n_i \in \mathbb{N}$, we choose $\bar{x}_i = J_i x_i$ to be simply the complex conjugate of x_i. If $[e_{ij}]_{i,j\in\mathbb{I}}$ is the matrix representation of E, then $[e^*_{ji}]_{i,j\in\mathbb{I}}$ is the matrix representation of E^* and, with the above identification of $(Y_{\mathbb{I}})'$ with $Y_{\mathbb{I}}$, $[e'_{ji}]_{i,j\in\mathbb{I}}$ is the matrix representation of E'. In particular, if each $X_i = \mathbb{C}^{n_i} = \mathbb{C}^{n_i\times 1}$, for some $n_i \in \mathbb{N}$, so $e_{ij} \in \mathbb{C}^{n_i\times n_j}$, for $i,j \in \mathbb{I}$, we have $e^*_{ij} = e^H_{ij}$ and $e'_{ij} = e^T_{ij}$, for $i,j \in \mathbb{I}$. Further, E is a block matrix with complex-valued matrix blocks and $E^* = E^H$, $E' = E^T$.

The lower norm, spectrum, and pseudospectrum For $\mathbb{I} \subset \mathbb{Z}$, a set $T \subset \mathbb{I}$ and $E \in L(Y_{\mathbb{I}})$, put

$$\nu_T(E) \;:=\; \inf\{\|Ex\| : \|x\| = 1,\ \mathrm{supp}(x) \subset T\}, \tag{2.4}$$

where, for $x = (x_i)_{i\in\mathbb{I}}$, $\mathrm{supp}(x) := \{i \in \mathbb{I} : x_i \neq 0\}$. Clearly, it holds that

$$\nu_T(E) \;\geq\; \nu_U(E) \quad \text{if} \quad T \subset U \subset \mathbb{I}. \tag{2.5}$$

Another basic result (e.g., [16, Lemma 2.38]) is that, if $E, F \in L(Y_{\mathbb{I}})$,

$$|\nu_T(E) - \nu_T(F)| \;\leq\; \|E - F\|, \qquad \text{for all} \qquad T \subset \mathbb{I}. \tag{2.6}$$

Note also that if $Y_{\mathbb{I}}$ is finite-dimensional, so that $\{x \in Y_{\mathbb{I}} : \|x\| = 1\}$ is compact, then the inf in (2.4) can be replaced by min.

Abbreviate $\nu_{\mathbb{I}}(E) =: \nu(E)$. By analogy to the norm of E, i.e. to $\|E\| := \sup\{\|Ex\| : \|x\| = 1\}$, $\nu(E)$ is sometimes (by abuse of notation) called the *lower norm* of E. Since $Y_{\mathbb{I}}$ is a Hilbert space, and where $s_{\max}(E)$ and $s_{\min}(E)$ denote the largest and smallest singular values[1] of E,

$$\|E\| = s_{\max}(E) \qquad \text{and} \qquad \nu(E) = s_{\min}(E).$$

Another key property of $\nu(\cdot)$ is that

[1] Recall that the singular values of a bounded linear operator F on a Hilbert space Y are the points in the spectrum of $(F^*F)^{1/2}$.

$$\|E^{-1}\|^{-1} \;=\; \min\big(\nu(E), \nu(E')\big) \;=:\; \mu(E), \tag{2.7}$$

where $\|E^{-1}\|^{-1} := 0$ if E is not invertible. In particular, E is invertible if and only if $\nu(E)$ and $\nu(E')$ are both nonzero, i.e., if and only if $\mu(E) \neq 0$, in which case $\nu(E) = \nu(E') = \mu(E)$. Further, if E is Fredholm of index zero, in particular if $Y_{\mathbb{I}}$ is finite-dimensional, then $\nu(E) = 0$ if and only if $\nu(E') = 0$, so that

$$\mu(E) = \nu(E) = s_{\min}(E). \tag{2.8}$$

It follows from (2.6) (since $\mu(E) = \nu(E)$ if $\mu(E) \neq 0$) that, for $E, F \in L(Y_{\mathbb{I}})$, also

$$|\mu(E) - \mu(F)| \leq \|E - F\|. \tag{2.9}$$

We have recalled the definition of $\operatorname{Spec} E$ above. By (2.7),

$$\operatorname{Spec} E \;=\; \{\lambda \in \mathbb{C} \;:\; E - \lambda I \text{ is not invertible}\} \;=\; \{\lambda \in \mathbb{C} \;:\; \mu(E - \lambda I) = 0\}.$$

Generalising (1.2), $\operatorname{Spec}_\varepsilon E$, the *closed ε-pseudospectrum of E*, is defined for $\varepsilon \geq 0$ by

$$\operatorname{Spec}_\varepsilon E := \{\lambda \in \mathbb{C} : \|(E - \lambda I)^{-1}\|^{-1} \leq \varepsilon\} = \{\lambda \in \mathbb{C} \;:\; \mu(E - \lambda I) \leq \varepsilon\}, \tag{2.10}$$

by (2.7). Note that $\operatorname{Spec}_0 E = \operatorname{Spec} E$. We will occasionally mention the *open ε-pseudospectrum of E*, defined for $\varepsilon > 0$ by

$$\operatorname{spec}_\varepsilon E := \{\lambda \in \mathbb{C} : \|(E - \lambda I)^{-1}\|^{-1} < \varepsilon\} = \{\lambda \in \mathbb{C} \;:\; \mu(E - \lambda I) < \varepsilon\}. \tag{2.11}$$

Using (2.9) it is easy to see that the sets $\operatorname{spec}_\varepsilon E$ and $\operatorname{Spec}_\varepsilon E$ are open and closed, respectively. Indeed (see, e.g., [21]), since $Y_{\mathbb{I}}$ is a Hilbert space and $E \in L(Y_{\mathbb{I}})$,

$$\operatorname{Spec}_\varepsilon E = \overline{\operatorname{spec}_\varepsilon E} \quad \text{and} \quad \operatorname{spec}_\varepsilon E = \operatorname{int}(\operatorname{Spec}_\varepsilon E), \tag{2.12}$$

the interior of $\operatorname{Spec}_\varepsilon E$. Suppose that $\mathbb{I} = \cup_{j \in J} \mathbb{J}_j$, with $\mathbb{J}_i \cap \mathbb{J}_j = \emptyset$, for $i \neq j$. In the case that $E \in L(Y_{\mathbb{I}})$ is the direct sum $E = \oplus_{j \in J} E_j$, where $E_j \in L(Y_{\mathbb{J}_j})$ for $j \in J$, a useful result, that can be seen, for example, via (2.11) and (2.12), is that

$$\operatorname{Spec}_\varepsilon E = \overline{\bigcup_{j \in J} \operatorname{Spec}_\varepsilon E_j}, \qquad \text{for} \quad \varepsilon \geq 0. \tag{2.13}$$

An informative characterisation of $\operatorname{Spec}_\varepsilon E$ (which holds since $Y_{\mathbb{I}}$ is a Hilbert space; see [15, Theorem 3.27]) is that

$$\operatorname{Spec}_\varepsilon E \;=\; \bigcup_{\|F\| \leq \varepsilon} \operatorname{Spec}(E + F), \qquad \varepsilon > 0. \tag{2.14}$$

It follows from (2.14) that, for $\varepsilon > 0$, $\mathrm{Spec}\, E + \varepsilon\overline{\mathbb{D}} \subset \mathrm{Spec}_\varepsilon E$, with (see [6, p. 247]) equality if E is normal, and that

$$\mathrm{Spec}_\varepsilon(B+T) \subset \mathrm{Spec}_{\varepsilon+\delta} B \quad \text{if} \quad \|T\| \leq \delta. \tag{2.15}$$

We will use the above definitions of $\mathrm{Spec}_\varepsilon E$ and $\mathrm{spec}_\varepsilon E$ also when E is a square complex-valued matrix (cf. (1.2), our norms then the matrix 2-norm), and note that (2.10)-(2.15) hold equally in that case. In particular, in the case that $E = \mathrm{diag}(E_1, E_2)$ is the direct sum of complex-valued square matrices E_1 and E_2, it follows from (2.13) (cf. [22, Thm. 2.4]) that

$$\mathrm{Spec}_\varepsilon E = \mathrm{Spec}_\varepsilon E_1 \cup \mathrm{Spec}_\varepsilon E_2, \quad \text{for} \quad \varepsilon \geq 0. \tag{2.16}$$

In the case when E is an $(n+2)\times n$ matrix, for some $n \in \mathbb{N}$ (with scalar, block, or operator-valued entries), we define the closed ε-pseudospectrum of E, for $\varepsilon \geq 0$, by (1.13). Since $s_{\min}(E - \lambda I_n) = \nu(E - \lambda I_n)$,

$$\mathrm{Spec}_\varepsilon E \;=\; \{\lambda \in \mathbb{C} \;:\; \nu(E - \lambda I_n) \leq \varepsilon\}, \qquad \varepsilon \geq 0. \tag{2.17}$$

The example (5.5) below illustrates that $\mathrm{Spec}_\varepsilon E$ can be empty when E is rectangular, in contrast to the case when E is square when each pseudospectrum contains a neighbourhood of the spectrum.

3. Spectral inclusions for bi-infinite matrices

Our inclusion set results for finite matrices, Theorems 1.1 and 1.6, are derived from generalisations, which we present in this section, of analogous inclusion sets for the spectra and pseudospectra of bi-infinite matrices from [3].

Suppose that $\mathbb{I} = \mathbb{Z}$ and that $E \in L(Y_{\mathbb{I}})$, where $Y_{\mathbb{I}}$ is given by (2.1). It is enough for our purposes to suppose that E has a matrix representation $[e_{ij}]_{i,j\in\mathbb{Z}}$ that is tridiagonal, in which case, since $E \in L(Y_I)$, the coefficients e_{ij} satisfy (2.3). Analogously to (1.3), let

$$\begin{gathered} r_L(E) := \sup\{\|e_{i+1,i}\| : i \in \mathbb{Z}\}, \quad r_U(E) := \sup\{\|e_{i,i+1}\| : i \in \mathbb{Z}\}, \\ r(E) := r_L(E) + r_U(E). \end{gathered} \tag{3.1}$$

For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, let

$$E_{n,k} := [e_{ij}]_{i,j=k+1}^{k+n},$$

and (cf. (1.11)) let

$$E^+_{n,k} := \left(\begin{array}{ccccc} e_{k,k+1} & 0 & \cdots & 0 & 0 \\ \hline & & E_{n,k} & & \\ \hline 0 & 0 & \cdots & 0 & e_{k+n+1,k+n} \end{array}\right),$$
$$(E')^+_{n,k} := \left(\begin{array}{ccccc} e'_{k+1,k} & 0 & \cdots & 0 & 0 \\ \hline & & E'_{n,k} & & \\ \hline 0 & 0 & \cdots & 0 & e'_{k+n,k+n+1} \end{array}\right). \tag{3.2}$$

For $w \in \mathbb{R}^n \setminus \{0\}$, and setting $w_0 := 0$ and $w_{n+1} := 0$, let

$$S_n(w) := \sum_{j=1}^n w_j^2, \quad T_n^-(w) := \sum_{j=1}^n (w_{j-1} - w_j)^2,$$
$$T_n^+(w) := \sum_{j=1}^n (w_{j+1} - w_j)^2, \quad T_n(w) := w_1^2 + w_n^2 + \sum_{j=1}^{n-1} (w_{i+1} - w_i)^2.$$

For $w \in \mathbb{R}^n \setminus \{0\}$ and tridiagonal $E \in L(Y_{\mathbb{I}})$, let

$$\eta_n(E,w) := r_L(E)\sqrt{\frac{T_n^-(w)}{S_n(w)}} + r_U(E)\sqrt{\frac{T_n^+(w)}{S_n(w)}}, \tag{3.3}$$
$$\eta'_n(E,w) := r(E)\sqrt{\frac{T_n(w)}{S_n(w)}}. \tag{3.4}$$

The following is the main result for bi-infinite matrices that we need. The inequalities (3.7) and (3.8) are results regarding the approximation of the lower norm of a bi-infinite matrix by the lower norm of a finite submatrix. (The inequality (3.8) provides an upper as well as a lower bound, the upper bound an immediate consequence of (2.5).) These inequalities generalise results in [3] for the case that, for some Banach space X, $X_i = X$, for $i \in \mathbb{Z}$, so that $Y_{\mathbb{Z}} = \ell^2(\mathbb{Z}, X)$, but the proofs apply to the case we need that $Y_{\mathbb{I}}$ has the form (2.1), with each X_i a Hilbert space. (The proofs of (3.5) and (3.7) are those of [3, Prop. 3.3, Cor. 3.4], of (3.6) and (3.8) are those of [3, Prop. 5.1, Cor. 5.2].)

Theorem 3.1 *(Approximation of $\nu(E)$). Let $n \in \mathbb{N}$, $w = (w_1, \ldots, w_n) \in \mathbb{R}^n \setminus \{0\}$, and $x = (x_i)_{i\in\mathbb{Z}} \in Y_{\mathbb{Z}} \setminus \{0\}$, suppose $E \in L(Y_{\mathbb{Z}})$ is tridiagonal, and let $x_{n,j} := (w_1 x_{j+1}, \ldots, w_n x_{j+n})^T$, for $j \in \mathbb{Z}$. Then, for some $k, \ell, m \in \mathbb{Z}$, $x_{n,k}$, $x_{n,\ell}$, and $x_{n,m}$ are non-zero with*

$$\frac{\|E_{n,k}x_{n,k}\|}{\|x_{n,k}\|} \le \frac{\|Ex\|}{\|x\|} + \eta_n(E,w), \tag{3.5}$$
$$\frac{\|E^+_{n,\ell}x_{n,\ell}\|}{\|x_{n,\ell}\|} \le \frac{\|Ex\|}{\|x\|} + \eta'_n(E,w). \tag{3.6}$$

As a consequence,

$$\inf_{k\in\mathbb{Z}} \nu(E_{n,k}) \le \nu(E) + \eta_n(E,w), \tag{3.7}$$

$$\inf_{k\in\mathbb{Z}} \nu(E^+_{n,k}) \le \nu(E) + \eta'_n(E,w) \;\le\; \inf_{k\in\mathbb{Z}} \nu(E^+_{n,k}) + \eta'_n(E,w). \tag{3.8}$$

We think of the terms $\eta_n(E,w)$ and $\eta'_n(E,w)$ in (3.7) and (3.8) as *penalty terms* accounting for the truncation of E to $E_{n,k}$ and $E^+_{n,k}$, respectively. The following result from [3] minimises these penalty terms as a function of $w \in \mathbb{R}^n \setminus \{0\}$. The formulae (3.9) and (3.10) are proved in [3, Corollaries 3.8 and 5.4].

Theorem 3.2 *(Minimisation of the penalty terms). For each $n \in \mathbb{N}$ and each tridiagonal $E \in L(Y_Z)$, the infimum of each of $\eta_n(E,w)$ and $\eta'_n(E,w)$, as a function of $w \in \mathbb{R}^n \setminus \{0\}$, is achieved for some w with $\|w\| = 1$. Further,*

$$\min_{w\in\mathbb{R}^n,\,\|w\|=1} \eta_n(E,w) \;\le\; \eta_n(E) \;:=\; 2r(E)\sin(\theta_n(E)/2), \tag{3.9}$$

with equality if $n = 1$ or $r_L(E)r_U(E) = 0$, where $\theta_n(E)$ is the unique solution t in the range $\left[\frac{\pi}{2n+1}, \frac{\pi}{n+2}\right]$ of equation (1.5), and

$$\min_{w\in\mathbb{R}^n,\,\|w\|=1} \eta'_n(E,w) = \eta'_n(E) \;:=\; 2r(E)\sin(\pi/(2n+2)). \tag{3.10}$$

Inclusion set results, analogous to Theorems 1.1 and 1.6, follow by application of the above results. Theorem 3.3 below, that follows from the above theorems and that we use to prove our main results in §4, is a variant on results in [3] for the case that $Y_{\mathbb{Z}} = \ell^2(\mathbb{Z}, X)$, for some Banach space X. To state Theorem 3.3 we introduce additional notation. For tridiagonal $E \in L(Y_{\mathbb{Z}})$ and $n \in \mathbb{N}$, let

$$\mu_n(E) := \inf_{k\in\mathbb{Z}} \mu(E_{n,k}) = \inf_{k\in\mathbb{Z}} \min\left(\nu\left(E_{n,k}\right), \nu\left(E'_{n,k}\right)\right), \tag{3.11}$$

$$\mu^+_n(E) := \inf_{k\in\mathbb{Z}} \min\left(\nu\left(E^+_{n,k}\right), \nu\left((E')^+_{n,k}\right)\right), \tag{3.12}$$

and, for $\varepsilon \ge 0$ (cf. (1.6) and (1.15)), let

$$\Sigma^n_\varepsilon(E) := \{\lambda \in \mathbb{C} : \mu_n(E - \lambda I) \le \varepsilon + \eta_n(E)\}, \tag{3.13}$$

$$\Gamma^n_\varepsilon(E) := \left\{\lambda \in \mathbb{C} : \mu^+_n(E - \lambda I) \le \varepsilon + \eta'_n(E)\right\}, \tag{3.14}$$

where $\eta_n(E)$ and $\eta'_n(E)$ are as defined in Theorem 3.2. The following theorem is a bi-infinite version of Theorems 1.1 and 1.6. We omit the proof which is identical to the proofs of this result (in [3, Thm. 3.5, Cor. 3.8] for (3.13) and [3, Thm. 5.3, Cor. 5.4] for (3.14)) for the special case that $X_i = X$, $i \in \mathbb{Z}$.

Theorem 3.3 *(The τ and τ_1 method inclusion sets: bi-infinite matrices). Suppose $E \in L(Y_{\mathbb{Z}})$ is tridiagonal. For $\varepsilon \ge 0$ and $n \in \mathbb{N}$,*

$$\mathrm{Spec}_\varepsilon E \;\subset\; \Sigma^n_\varepsilon(E) \qquad \textit{and} \qquad \mathrm{Spec}_\varepsilon E \;\subset\; \Gamma^n_\varepsilon(E) \subset\; \mathrm{Spec}_{\varepsilon+\eta'_n(E)}.$$

4. The proofs of our main results

We turn now to the proofs of our main results, the inclusion set results for finite matrices, Theorems 1.1 and 1.6. Throughout this section A is the finite complex-valued matrix with block representation (1.1).

Theorem 4.1. *Suppose that A is block-tridiagonal. Then*

$$\min_{0\le k\le N-n} \nu(A^+_{n,k}) \le \nu(A) + \varepsilon'_n(A) \;\le\; \min_{0\le k\le N-n} \nu(A^+_{n,k}) + \varepsilon'_n(A), \tag{4.1}$$

for $n = 1,\dots,N-1$, and

$$\min_{0\le k\le N-n} \nu(A_{n,k}) \le \nu(A) + \varepsilon_{n-2}(A), \qquad \textit{for} \quad n > 2. \tag{4.2}$$

Proof. Let $X_i = \mathbb{C}^{m_i\times 1}$, $i = 1,\dots,N$, so that A given by (1.1) is the matrix representation of a linear operator on $Y_{\mathbb{I}}$ with $\mathbb{I} = \{1,\dots,N\}$. Extend the definition of X_i to $i \in \mathbb{Z}$ by periodicity, i.e. so that $X_{i+N} = X_i$, $i \in \mathbb{Z}$. Define $E = [e_{ij}]_{i,j\in\mathbb{Z}}$ so that $e_{ij} := a_{ij}$, for $1 \le i,j \le N$, and $e_{ij} := 0$, otherwise (see Fig. 4.1). Since A is tridiagonal, $E \in L(Y_{\mathbb{Z}})$ is tridiagonal. Note that, for $n \in \mathbb{N}$,

$$\varepsilon_n(A) = \eta_n(E), \quad \text{and} \quad \varepsilon'_n(A) = \eta'_n(E). \tag{4.3}$$

Suppose now that $n \in \mathbb{I}$. Since $Y_{\mathbb{I}}$ is finite-dimensional we can choose $\tilde{x} \in Y_{\mathbb{I}}$ with $\|\tilde{x}\| = 1$ such that $\nu(A) = \|A\tilde{x}\|$. Extend $\tilde{x}$ by zeros to be an element $x = (x_i)_{i\in\mathbb{Z}}$ of $Y_{\mathbb{Z}}$. Then $x_i = \tilde{x}_i$, for $i \in \mathbb{I}$, $x_i = 0$, otherwise, $\|x\| = 1$, and $\|Ex\| = \|A\tilde{x}\| = \nu(A)$. For $k \in \mathbb{Z}$ let $\widehat{E}_{n,k}$ denote $E_{n,k}$ or $E^+_{n,k}$, and let $\widehat{A}_{n,k}$ denote $A_{n,k}$ or $A^+_{n,k}$, respectively (cf. Fig. 1.2), for $0 \le k \le N-n$. For any $w \in \mathbb{R}^n \setminus \{0\}$, and where $x_{n,j} := (w_1x_{j+1},\dots,w_nx_{j+n})^T$ for $j \in \mathbb{Z}$, it follows by Theorem 3.1 that there exists $k \in \mathbb{Z}$ such that $x_{n,k} \ne 0$ and

$$\nu(\widehat{E}_{n,k}) \le \frac{\|\widehat{E}_{n,k}x_{n,k}\|}{\|x_{n,k}\|} \le \nu(A) + \hat{\eta}_n(E,w), \tag{4.4}$$

where $\hat{\eta}_n(E,w)$ denotes $\eta_n(E,w)$ or $\eta'_n(E,w)$, in the respective cases $\widehat{E}_{n,k} = E_{n,k}$ or $E^+_{n,k}$. Since $x_{n,k} = 0$ if $k+n < 1$ or $k+1 > N$, it follows that $-n < k < N$. To complete the argument we consider the cases $-n < k < 0$, $0 \le k \le N-n$, and $N-n < k < N$ separately (see Fig. 4.1).

Case 1: $0 \le k \le N-n$. If k lies in this range then $\widehat{E}_{n,k} = \widehat{A}_{n,k}$, so that (4.4) implies that

$$\nu(\widehat{A}_{n,k}) \le \nu(A) + \hat{\eta}_n(E,w). \tag{4.5}$$

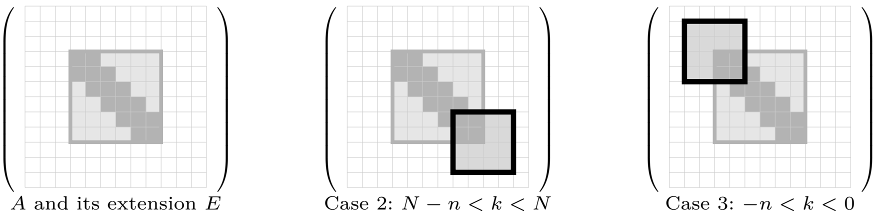

Fig. 4.1. The bi-infinite matrix $E \in L(Y_{\mathbb{Z}})$ that is the extension of $A \in L(Y_{\mathbb{I}})$, and the matrices $E_{n,k}$, outlined in bold, in Cases 2 and 3.

Case 2: $N - n < k < N$. If k lies in this range put

$$x_{n,k}^- := (w_1 x_{k+1}, \dots, w_{N-k} x_N)^T \in \bigoplus_{i=k+1}^{N} X_i$$

and

$$\hat{x}_{n,k} := \begin{pmatrix} 0_{n+k-N} \\ x_{n,k}^- \end{pmatrix} \in \bigoplus_{i=N+1-n}^{N} X_i,$$

where 0_ℓ denotes an appropriate null vector of length ℓ, and put

$$\alpha := a_{k,k+1} \quad \text{and} \quad \beta := w_1 \alpha x_{k+1}.$$

Then $\|x_{n,k}\| = \|x_{n,k}^-\| = \|\hat{x}_{n,k}\|$ and, in the cases $\widehat{E}_{n,k} = E_{n,k}$ and $E_{n,k}^+$ we have

$$\widehat{E}_{n,k} x_{n,k} = \begin{pmatrix} A_{N-k,N+1-n}\, x_{n,k}^- \\ 0_{n+k-N} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_{N-k,N+1-n}^+\, x_{n,k}^- \\ 0_{n+k-N} \end{pmatrix},$$

respectively, and

$$\widehat{A}_{n,N-n}\, \hat{x}_{n,k} = \begin{pmatrix} 0_{n+k-N-1} \\ w_1 \alpha x_{k+1} \\ A_{N-k,N+1-n}\, x_{n,k}^- \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0_{n+k-N} \\ A_{N-k,N+1-n}^+\, x_{n,k}^- \end{pmatrix},$$

respectively. Thus, provided that $w_1 = 0$ in the case that $\widehat{E}_{n,k} = E_{n,k}$ and $\widehat{A}_{n,k} = A_{n,k}$ (which implies that $n > 1$), we have that $\|\widehat{E}_{n,k} x_{n,k}\| = \|\widehat{A}_{n,N-n}\, \hat{x}_{n,k}\|$, so that (4.4) implies that (4.5) holds for $k = N - n$.

Case 3: $-n < k < 0$. If k lies in this range a similar argument to that of Case 2 shows that (4.5) holds for $k = 0$, provided that $w_n = 0$ in the case that $\widehat{E}_{n,k} = E_{n,k}$ and $\widehat{A}_{n,k} = A_{n,k}$.

Thus, in every case, (4.5) holds for some k in the range $0 \le k \le N - n$, provided that $w_1 = w_n = 0$ in the case that $\widehat{A}_{n,k} = A_{n,k}$ (which implies that $n > 2$). Thus

$$\min_{0\le k\le N-n} \nu(\widehat{A}_{n,k}) \le \nu(A) + \hat{\eta}_n(E,w).$$

Taking the infimum of $\hat{\eta}_n(E,w)$ in the above equation over all $w \in \mathbb{R}^n \setminus \{0\}$ (over all w with $w_1 = w_n = 0$ in the case that $n > 2$ and $\widehat{A}_{n,k} = A_{n,k}$, noting that this infimum is the same as the infimum of $\eta_{n-2}(E,\tilde{w})$ over all $\tilde{w} \in \mathbb{R}^{n-2} \setminus \{0\}$), we see that (4.1) and (4.2) follow from Theorem 3.2 and (4.3) and, in the case of (4.1), by application of (2.5). □

We come now to the proofs of our main results, establishing inclusion sets for $\operatorname{Spec} A$ and $\operatorname{Spec}_\varepsilon A$. Many authors, for example Davies [6] and Trefethen & Embree [22], prefer to work with open pseudospectra, defined by (2.11), rather than closed pseudospectra given by (2.10). We can derive τ-method inclusion sets for the open pseudospectra $\operatorname{spec}_\varepsilon A$ by combining Theorem 1.1 with (2.12).

Proof of Theorem 1.6. Note that it is enough to consider the case that A is block-tridiagonal. For if $n \in \{1,\dots,N-1\}$ and $\operatorname{Spec}_\varepsilon A \subset \Gamma^n_\varepsilon(A) \subset \operatorname{Spec}_{\varepsilon+\varepsilon'_n(A)} A$ whenever A is block-tridiagonal and $\varepsilon \ge 0$, then, in the case that A is not block-tridiagonal, $\operatorname{Spec}_\varepsilon B \subset \Gamma^n_\varepsilon(B) \subset \operatorname{Spec}_{\varepsilon+\varepsilon'_n(A)} B$, where B is the tridiagonal part of A, so that, by (2.15),

$$\operatorname{Spec}_\varepsilon A \subset \operatorname{Spec}_{\varepsilon+\|C\|}(B) \subset \Gamma^n_{\varepsilon+\|C\|}(B) = \Gamma^n_\varepsilon(A),$$

and also

$$\Gamma^n_\varepsilon(A) = \Gamma^n_{\varepsilon+\|C\|}(B) \subset \operatorname{Spec}_{\varepsilon+\|C\|+\varepsilon'_n(A)} B \subset \operatorname{Spec}_{\varepsilon+2\|C\|+\varepsilon'_n(A)} A.$$

So suppose that A is block-tridiagonal, so that A coincides with B, its tridiagonal part, and its remaining part is $C = 0$. Let $1 \le n < N$ and $\varepsilon \ge 0$, and suppose that $\lambda \in \operatorname{Spec}_\varepsilon A$. Then, by (2.10) and (2.8), $\nu(A - \lambda I) \le \varepsilon$. But this implies, by Theorem 4.1, that, for some $k \in \{0,\dots,N-n\}$, $\nu(A^+_{n,k} - \lambda I) \le \varepsilon + \varepsilon'_n(A)$, so that, by (2.17), $\lambda \in \operatorname{Spec}_{\varepsilon+\varepsilon'_n(A)} A^+_{n,k} \subset \Gamma^n_\varepsilon(A)$. On the other hand, if $\lambda \in \Gamma^n_\varepsilon(A)$, then $\lambda \in \operatorname{Spec}_{\varepsilon+\varepsilon'_n(A)} A^+_{n,k}$, for some $k \in \{0,\dots,N-n\}$, so that $\nu(A^+_{n,k} - \lambda I) \le \varepsilon + \varepsilon'_n(A)$, so that, by (2.5), $\nu(A - \lambda I) \le \varepsilon + \varepsilon'_n(A)$, so that $\lambda \in \operatorname{Spec}_{\varepsilon+\varepsilon'_n(A)}(A)$. □

Proof of Theorem 1.1. As in the above proof it is enough to consider the case that A is block-tridiagonal.

Let $X_i = \mathbb{C}^{m_i\times 1}$, for $i = 1,\dots,N$, so that A given by (1.1) is the matrix representation of a linear operator on $Y_{\mathbb{I}}$ with $\mathbb{I} = \{1,\dots,N\}$. Extend the definition of X_i to $i \in \mathbb{Z}$ by periodicity, i.e. so that $X_{i+N} = X_i$, $i \in \mathbb{Z}$. Suppose that A is block-tridiagonal, so that A coincides with B, its tridiagonal part, and its remaining part is $C = 0$. Let $1 \le n < N$ and $\varepsilon \ge 0$. To show that $\operatorname{Spec}_\varepsilon A \subset \Sigma^n_\varepsilon(A)$, we need to show that $\operatorname{Spec}_\varepsilon A \subset \sigma^n_\varepsilon(A)$ and that, for $n > 2$, $\operatorname{Spec}_\varepsilon A \subset \hat{\sigma}^n_\varepsilon(A)$.

To see that $\mathrm{Spec}_\varepsilon A \subset \sigma_\varepsilon^n(A)$, define $E = [e_{ij}]_{i,j\in\mathbb{Z}} = \mathrm{diag}(\ldots, A, A, \ldots)$, with $e_{1,1} = a_{1,1}$. Note that $E \in L(Y_\mathbb{Z})$, with $Y_\mathbb{Z}$ given by (2.1), E is tridiagonal, and $\mathrm{Spec}_\varepsilon E = \mathrm{Spec}_\varepsilon A$ by (2.13). Thus, for $\varepsilon \ge 0$ and $n = 1, \ldots, N$, $\mathrm{Spec}_\varepsilon A = \mathrm{Spec}_\varepsilon E \subset \Sigma_\varepsilon^n(E)$ by Theorem 3.3. With our definition of E, $E_{n,N+k} = E_{n,k}$, for $k \in \mathbb{Z}$. Note also that $\eta_n(E) = \varepsilon_n(A)$, since $\|C\| = 0$. Thus

$$\Sigma_\varepsilon^n(E) = \{\lambda \in \mathbb{C} : \mu_n(E - \lambda I) \le \varepsilon + \varepsilon_n(A)\}, \quad \text{where} \quad \mu_n(E) := \min_{0\le k\le N-1} \mu(E_{n,k}),$$

so that, by (2.10),

$$\Sigma_\varepsilon^n(E) = \bigcup_{k=0}^{N-1} \mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} E_{n,k}.$$

Now $E_{n,k} = A_{n,k}$, for $k = 0, \ldots, N-n$. Further, if $n > 1$, then, for $k = N-n+1, \ldots, N-1$, $E_{n,k} = \mathrm{diag}(A_{N-k,k}, A_{k+n-N,0})$, so that

$$\mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} E_{n,k} = \mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} A_{N-k,k} \cup \mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} A_{k+n-N,0},$$

by (2.16), so that $\Sigma_\varepsilon^n(E) = \sigma_\varepsilon^n(A)$. Thus $\mathrm{Spec}_\varepsilon A = \mathrm{Spec}_\varepsilon E \subset \sigma_\varepsilon^n(E)$.

It follows that also $\mathrm{Spec}_\varepsilon A \subset \hat{\sigma}_\varepsilon^n(A)$, if $N > n \ge 3$, by arguing by application of Theorem 4.1 (i.e., (4.2)) as in the last part of the proof of Theorem 1.6. □

5. Examples

In this section we illustrate the inclusion set bounds that we have proposed in Theorems 1.1 and 1.6, with some focus on cases where the matrix A is banded, so that Remark 1.4 applies.

5.1. The discrete Laplacian and Jordan block

We start with two simple tridiagonal examples, one real symmetric ($A = L_M$), the other non-normal ($A = V_M$), both Toeplitz so that we know the spectrum explicitly. Here L_M and V_M, both order M, are a discrete Laplacian and a Jordan block, respectively. Precisely, $L_1 = V_1 = 0$ and, for $M > 1$,

$$L_M := \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}_{M\times M} \quad \text{and} \quad V_M := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}_{M\times M}.$$

For $M \in \mathbb{N}$, $\mathrm{Spec}\, V_M = \{0\}$ and (e.g., [1])

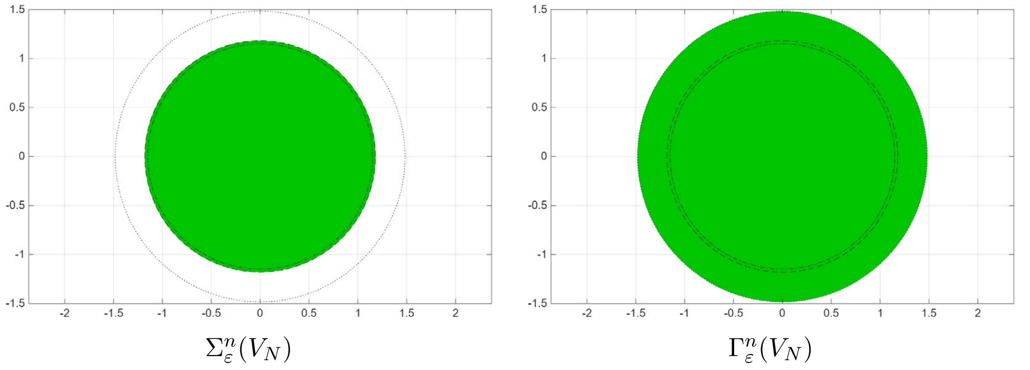

Fig. 5.1. The τ and τ_1 inclusion sets, $\Sigma_\varepsilon^n(V_N)$ and $\Gamma_\varepsilon^n(V_N)$, for $n = 4$ and $\varepsilon = 0.15$. Each is an inclusion set for $\mathrm{Spec}_\varepsilon V_N$ if $N \geq n$. Also shown in each panel are circles of radii $1 + \varepsilon = 1.15$ (−.), $\alpha_n(\varepsilon_n + \varepsilon) \approx 1.18$ (− −), and $\alpha_n(\varepsilon'_n + \varepsilon) \approx 1.48$ ($\cdots$). The circles of radii $\alpha_n(\varepsilon_n + \varepsilon)$ and $\alpha_n(\varepsilon'_n + \varepsilon)$ are the boundaries of $\Sigma_\varepsilon^n(V_N)$ and $\Gamma_\varepsilon^n(V_N)$, respectively, by (5.11), (5.12), and (5.8). In the supplementary materials we provide Matlab codes to reproduce this figure and Figs. 5.2 and 5.3 below, and to produce similar figures for other values of the parameters n and ε.

$$\mathrm{Spec}\, L_M = \{2\cos(j\pi/(M+1)) : j \in \{1, \ldots, M\}\}. \tag{5.1}$$

5.1.1. *The Jordan block*

Let's write down the τ and τ_1 inclusion sets of §1 for the Jordan block case $A = V_M$, choosing for simplicity $N = M$, so that the block structure (1.1) is trivial, each block a single complex number. These inclusion sets are illustrated, for particular values of n and ε, in Fig. 5.1. In the case $A = V_N$ Theorem 1.1 tells us that, for $\varepsilon \geq 0$ and $n < N$,

$$\mathrm{Spec}_\varepsilon V_N \subset \Sigma_\varepsilon^n(V_N) = \begin{cases} \bigcup_{m=1}^n \mathrm{Spec}_{\varepsilon+\varepsilon_n} V_m, & n = 1, 2, \\ \mathrm{Spec}_{\varepsilon+\varepsilon_{n-2}} V_n \cap \bigcup_{m=1}^n \mathrm{Spec}_{\varepsilon+\varepsilon_n} V_m, & n > 2, \end{cases} \tag{5.2}$$

where

$$\varepsilon_n := \varepsilon_n(V_N) = 2\sin(\pi/(4n+2)).$$

Let $V_1^+ := (1, 0, 0)^T$, $\widetilde{V}_1^+ := (0, 0, 0)^T$, and, for $n > 1$, let

$$V_n^+ := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \hline & V_n & & \\ \hline 0 & 0 & \cdots & 0 \end{pmatrix}_{(n+2)\times n}, \quad \widetilde{V}_n^+ := \begin{pmatrix} 0 & \cdots & 0 \\ \hline & V_n & \\ \hline 0 & \cdots & 0 \end{pmatrix}_{(n+2)\times n}.$$

Then $\{B_{n,k}^+ : 0 \leq k \leq N - n\} = \{V_n^+, \widetilde{V}_n^+\}$ so that, by Theorem 1.6,

$$\mathrm{Spec}_\varepsilon V_N \subset \Gamma_\varepsilon^n(V_N) = \mathrm{Spec}_{\varepsilon+\varepsilon'_n} V_n^+ \cup \mathrm{Spec}_{\varepsilon+\varepsilon'_n} \widetilde{V}_n^+ \subset \mathrm{Spec}_{\varepsilon+\varepsilon'_n} V_N, \tag{5.3}$$

for $\varepsilon \geq 0$ and $1 \leq n < N$, where

$$\varepsilon_n' := \varepsilon_n'(V_N) = 2\sin(\pi/(2n+2)).$$

Our next proposition characterises the inclusion sets (5.2) and (5.3) more precisely, and uses the notations v_n and v_n^+, where

$$v_n(s) := s_{\min}(V_n - sI_n), \qquad v_n^+(s) := s_{\min}(V_n^+ - sI_n^+), \qquad \text{for } s \geq 0, \quad n \in \mathbb{N}.$$

It is shown in [3, §8.1] that

$$s_{\min}(V_n - \lambda I_n) = v_n(|\lambda|) \quad \text{and} \quad s_{\min}(V_n^+ - \lambda I_n^+) = v_n^+(|\lambda|), \qquad \lambda \in \mathbb{C},$$

that v_n is continuous and strictly monotonically increasing on $[0,\infty)$, with $v_n(0) = 0$, $v_n(1) = \varepsilon_n$, and $v_n(s) \to +\infty$ as $s \to +\infty$, and that

$$v_n^+(s) = \sqrt{1 + s^2 - 2sc_n}, \quad s \geq 0, \text{ where } c_n := \cos(\pi/(n+1)) = 1 - (\varepsilon_n')^2/2. \tag{5.4}$$

Noting that v_n^+ has a unique minimum on $(0,\infty)$ at $s = c_n$, with $v_n^+(c_n) = \sin(\pi/(n+1))$, it follows that, for $\varepsilon \geq 0$,

$$\begin{aligned} \mathrm{Spec}_\varepsilon V_n^+ &= \begin{cases} \emptyset, & 0 \leq \varepsilon < \sin(\pi/(n+1)), \\ \{\lambda \in \mathbb{C} : a_n^-(\varepsilon) \leq |\lambda| \leq a_n^+(\varepsilon)\}, & \varepsilon \geq \sin(\pi/(n+1)), \end{cases} \\ \mathrm{Spec}_\varepsilon V_n &= \alpha_n(\varepsilon)\overline{\mathbb{D}}, \end{aligned} \tag{5.5}$$

where

$$a_n^\pm(\varepsilon) := \cos(\pi/(n+1)) \pm \sqrt{\varepsilon^2 - \sin^2(\pi/(n+1))},$$

are the solutions of $v_n^+(s) = \varepsilon$, and $\alpha_n(\varepsilon)$ is the unique solution of $v_n(s) = \varepsilon$. We will show in the proof of the following proposition that, analogously to (5.4),

$$v_n(s) = \sqrt{1 + s^2 - 2s\cos(\phi_n(s))}, \quad s \geq 1, \tag{5.6}$$

where $\phi_n(s)$ is the unique solution t in the range $[\pi/(2n+1), \pi/(n+1))$ of the equation

$$s\sin((n+1)t) = \sin(nt). \tag{5.7}$$

Proposition 5.1. *For $n \in \mathbb{N}$ and $\varepsilon \geq 0$,*

$$\mathrm{Spec}_\varepsilon \widetilde{V}_n^+ = \mathrm{Spec}_\varepsilon V_n = \alpha_n(\varepsilon)\overline{\mathbb{D}}. \tag{5.8}$$

Further, for $n \in \mathbb{N}$, $\alpha_n(\varepsilon) < \alpha_n(\varepsilon')$, for $0 \leq \varepsilon < \varepsilon'$,

$$\max\{a_n^+(\varepsilon_n + \varepsilon), \alpha_j(\varepsilon_n + \varepsilon)\} < \alpha_n(\varepsilon_n + \varepsilon), \quad \textit{for} \quad \varepsilon \geq 0 \;\; \textit{and} \;\; 1 \leq j < n, \tag{5.9}$$

and

$$1+\varepsilon \le \alpha_n(\varepsilon_n+\varepsilon) \le 1+\varepsilon+\min\{\varepsilon_n, \sqrt{2\varepsilon_n\varepsilon}\}, \quad \textit{for} \quad \varepsilon \ge 0. \tag{5.10}$$

Thus, for $N > 1$*,* $n = 1, \ldots, N-1$*, and* $\varepsilon \ge 0$*,*

$$\begin{aligned}(1+\varepsilon)\overline{\mathbb{D}} \subset \Sigma_\varepsilon^n(V_N) = \operatorname{Spec}_{\varepsilon+\varepsilon_n} V_n = \alpha_n(\varepsilon+\varepsilon_n)\overline{\mathbb{D}} \\ \subset \left(1+\varepsilon+\min\{\varepsilon_n, \sqrt{2\varepsilon_n\varepsilon}\}\right)\overline{\mathbb{D}},\end{aligned} \tag{5.11}$$

in particular $\Sigma_0^n(V_N) = \overline{\mathbb{D}}$*, and*

$$\Sigma_\varepsilon^n(V_N) \subsetneq \Gamma_\varepsilon^n(V_N) = \operatorname{Spec}_{\varepsilon+\varepsilon_n'} V_n = \alpha_n(\varepsilon+\varepsilon_n')\overline{\mathbb{D}}. \tag{5.12}$$

We prove this proposition below. But let us first make two observations:

1. By (5.11) and (5.12), the τ-method inclusions sets $\Sigma_\varepsilon^n(V_N)$ are, for this example, sharper than the τ_1-method inclusion sets $\Gamma_\varepsilon^n(V_N)$. Both are closed discs centred on the origin, the τ_1 discs having larger radius, since $\varepsilon_n' > \varepsilon_n$ so that $\alpha_n(\varepsilon+\varepsilon_n') > \alpha_n(\varepsilon+\varepsilon_n)$. See Fig. 5.1, which provides an illustration of (5.11) and (5.12) for $n = 4$ and $\varepsilon = 0.15$.
2. The τ-method inclusion set for $\operatorname{Spec} V_N$ is $\Sigma_0^n(V_N) = \overline{\mathbb{D}}$, the same inclusion set as provided by the Gerschgorin theorem (1.10). This is a poor approximation to $\operatorname{Spec}_0 V_N = \operatorname{Spec} V_N = \{0\}$. On the other hand, for every $\varepsilon > 0$, $\Sigma_\varepsilon^n(V_N)$ and $\Gamma_\varepsilon^n(V_N)$ are both good approximations to $\operatorname{Spec}_\varepsilon V_N$ if N and n are sufficiently large, in the sense that, where S_n denotes $\Sigma_\varepsilon^n(V_N)$ or $\Gamma_\varepsilon^n(V_N)$, the Hausdorff distance between S_n and $\operatorname{Spec}_\varepsilon V_N$ (see (5.13) below) satisfies $d_H(S_n, \operatorname{Spec}_\varepsilon V_N) \to 0$ as $n, N \to \infty$ with $n < N$.

The last sentence of 2 is the content of Corollary 5.2, a special case of general results for Toeplitz matrices, Theorems 5.5 and 5.9 below. This corollary (and the later theorems) reference the standard Hausdorff distance between compact subsets $T_1, T_2 \subset \mathbb{C}$, defined by

$$d_H(T_1, T_2) \;:=\; \max\left\{\sup_{t_1\in T_1} \operatorname{dist}(t_1, T_2)\,,\; \sup_{t_2\in T_2} \operatorname{dist}(t_2, T_1)\right\}. \tag{5.13}$$

For a sequence $(T_n)_{n\in\mathbb{N}}$ of compact subsets of $\mathbb{C}$ and a compact $T \subset \mathbb{C}$ we will write $T_n \stackrel{\mathrm{H}}{\to} T$ as $n \to \infty$ if $d_H(T_n, T) \to 0$ as $n \to \infty$. We recall (e.g., [15]) that $d_H(\cdot,\cdot)$ is a metric on the set of compact subsets of $\mathbb{C}$.

Corollary 5.2. *Suppose that* $\varepsilon > 0$*. As* $N \to \infty$*,* $\operatorname{Spec}_\varepsilon V_N \stackrel{\mathrm{H}}{\to} (1+\varepsilon)\overline{\mathbb{D}}$*. Further, where* S_n *denotes* $\Sigma_\varepsilon^n(V_N)$ *or* $\Gamma_\varepsilon^n(V_N)$*,* $S_n \stackrel{\mathrm{H}}{\to} (1+\varepsilon)\overline{\mathbb{D}}$*, as* $n, N \to \infty$ *with* $n < N$*, so that* $d_H(S_n, \operatorname{Spec}_\varepsilon V_N) \to 0$ *as* $n, N \to \infty$ *with* $n < N$*.*

Proof. By (5.11), if N is large enough so that $\varepsilon_N < \varepsilon$, then $(1+\varepsilon-\varepsilon_N)\overline{\mathbb{D}} \subset \mathrm{Spec}_\varepsilon V_N \subset (1+\varepsilon)\overline{\mathbb{D}}$, so that $\mathrm{Spec}_\varepsilon V_N \xrightarrow{\mathrm{H}} (1+\varepsilon)\overline{\mathbb{D}}$ as $N \to \infty$. The rest of the corollary follows from (5.11) and (5.12). □

Proof of Proposition 5.1. Let us first show the claimed (5.6). For $n \in \mathbb{N}$ and $s \geq 0$, $(V_n - sI_n)^H(V_n - sI_n) = (1+s^2)I_n - s(V_n + V_n^T) - O_n$, where $O_n = [o_{ij}]_{i,j=1}^n$ and $o_{ij} = \delta_{1,i}\delta_{1,j}$, $i,j = 1,\ldots,n$, so that

$$\begin{aligned}[s_{\min}(V_n - sI_n)]^2 &= \min \mathrm{Spec}\,((V_n - sI_n)^H(V_n - sI_n)) \\ &= 1 + s^2 - s \min \mathrm{Spec}\,(-V_n - V_n^T - s^{-1}O_n),\end{aligned}$$

for $s > 0$. Further, for $s \geq 1$, $\mathrm{Spec}\,(-V_n - V_n^T - s^{-1}O_n) \subset [-2,2]$ by (1.10). Thus, arguing, e.g., as in [3, §8.1], if $s \geq 1$, $\lambda \in \mathrm{Spec}\,(-V_n - V_n^T - s^{-1}O_n)$ is an eigenvalue with corresponding eigenvector $(v_n,\ldots,v_1)^T$ if and only if $\lambda = -2\cos(t)$, for some $t \in (0,\pi)$, with $v_j = \sin(jt)$, $j = 1,\ldots,n$, and t satisfying (5.7). It is easy to see that the smallest positive solution of (5.7) is the unique solution in $[\pi/(2n+1), \pi/(n+1))$, so that $1 + s^2 - s\min \mathrm{Spec}\,(-V_n - V_n^T - s^{-1}O_n) = [v_n(s)]^2$, and (5.6) follows. Suppose now that $n \in \mathbb{N}$ and $\varepsilon \geq 0$. It is clear that, for $\lambda \in \mathbb{C}$, $\nu(\widetilde{V}_n^+ - \lambda I_n^+) = \nu(V_n - \lambda I_n)$, so that $\mathrm{Spec}_\varepsilon \widetilde{V}_n^+ = \mathrm{Spec}_\varepsilon V_n$, by (2.8), (2.10), and (2.17). Further, $\mathrm{Spec}_\varepsilon V_n = \alpha_n(\varepsilon)\overline{\mathbb{D}}$ by (5.5), so (5.8) follows.

For $n \in \mathbb{N}$, $\alpha_n(\varepsilon_1) < \alpha_n(\varepsilon_2)$, for $0 \leq \varepsilon_1 < \varepsilon_2$, since v_n is strictly monotonic increasing. Let us show that, for $s \geq 1$, $\phi_n(s) > \phi_{n+1}(s)$, for $n \in \mathbb{N}$. This is clear for $s = 1$ as $\phi_n(1) = \pi/(2n+1)$, $n \in \mathbb{N}$. To see that this holds also for $s > 1$, let $\psi(s,p)$ denote the unique solution t of $s\sin((p+1)t) = \sin(pt)$ in $(\pi/(2p+1), \pi/(p+1))$, for $s > 1$, $p \geq 1$, so that $\psi(s,n) = \phi_n(s)$, $n \in \mathbb{N}$. Differentiating this equation with respect to p, we see that $\partial_p\psi(s,p) < 0$, for $s > 1$, $p \geq 1$, so that $\psi(s,p)$ is strictly decreasing as a function of p, and the required result follows. Thus, for $s \geq 1$, $n \in \mathbb{N}$, and $1 \leq j < n$, $v_j(s) > v_n(s)$, and note that also $v_n^+(s) > v_n(s)$. Since, moreover, $v_n(1) = \varepsilon_n$, $n \in \mathbb{N}$, so that $\alpha_n(\varepsilon_n + \varepsilon) \geq 1$, for $\varepsilon \geq 0$, it follows that $\alpha_j(\varepsilon_n + \varepsilon) < \alpha_n(\varepsilon_n + \varepsilon)$, for $1 \leq j < n$ and $\varepsilon \geq 0$, and also $a_n^+(\varepsilon_n + \varepsilon) < \alpha_n(\varepsilon_n + \varepsilon)$, so that (5.9) holds.

Suppose that $n \in \mathbb{N}$ and $\varepsilon \geq 0$. To see (5.10), note first that $v_n(s) = \nu(V_n - sI_n)$, for $s \geq 0$, so that$|v_n(1+\varepsilon) - v_n(1)| \leq \varepsilon$, by (2.6). Since also $v_n(1) = \varepsilon_n$, it follows that $v_n(1+\varepsilon) \leq \varepsilon_n + \varepsilon$, so $1 + \varepsilon \leq \alpha_n(\varepsilon_n + \varepsilon)$. On the other hand, let $s = \alpha_n(\varepsilon_n + \varepsilon) \geq 1$. Then $v_n(s) = \varepsilon_n + \varepsilon$ and, since $\phi_n(s) \geq \pi/(2n+1)$ and $\cos(\pi/(2n+1)) = 1 - \varepsilon_n^2/2$, it follows from (5.6) that

$$1 + s^2 - 2s + \varepsilon_n^2 s \leq (\varepsilon_n + \varepsilon)^2,$$

so that, since $\sqrt{a^2+b^2} \leq a + b$, for $a, b \in \mathbb{R}$,

$$s \leq 1 - \frac{\varepsilon_n^2}{2} + \sqrt{\frac{\varepsilon_n^4}{4} + 2\varepsilon_n\varepsilon + \varepsilon^2} \leq 1 + \sqrt{2\varepsilon_n\varepsilon + \varepsilon^2}$$

$$= 1 + \varepsilon + \frac{2\varepsilon_n\varepsilon}{\varepsilon + \sqrt{2\varepsilon_n\varepsilon + \varepsilon^2}} \le 1 + \varepsilon + \min\{\varepsilon_n, \sqrt{2\varepsilon_n\varepsilon}\},$$

establishing (5.10).

Since $\alpha_n(\varepsilon_1) < \alpha_n(\varepsilon_2)$, for $0 \le \varepsilon_1 < \varepsilon_2$ and $n \in \mathbb{N}$, it follows from (5.8) that $\mathrm{Spec}_{\varepsilon+\varepsilon_{n-2}} V_n \cap \mathrm{Spec}_{\varepsilon+\varepsilon_n} V_n = \mathrm{Spec}_{\varepsilon+\varepsilon_n} V_n$, for $n > 2$, and that

$$\mathrm{Spec}_{\varepsilon+\varepsilon_n} V_n \subsetneq \mathrm{Spec}_{\varepsilon+\varepsilon'_n} V_n. \tag{5.14}$$

Further, (5.9) implies that $\mathrm{Spec}_{\varepsilon+\varepsilon_n} V_m \subset \mathrm{Spec}_{\varepsilon+\varepsilon_n} V_n$, for $m = 1, \ldots, n$ and that $\mathrm{Spec}_{\varepsilon+\varepsilon'_n} V_n^+ \subset \mathrm{Spec}_{\varepsilon+\varepsilon'_n} V_n$. Thus, and by (5.2), (5.3), and (5.8),

$$\Sigma_\varepsilon^n(V_N) = \mathrm{Spec}_{\varepsilon+\varepsilon_n} V_n \quad \text{and} \quad \Gamma_\varepsilon^n(V_N) = \mathrm{Spec}_{\varepsilon+\varepsilon'_n} V_n. \tag{5.15}$$

Equation (5.11) follows from (5.15), (5.8), and (5.10). The inclusions (5.12) follow from (5.15), (5.8), (5.14), and (5.12). □

5.1.2. The discrete Laplacian

Let us turn now to the case $A = L_M$, again choosing for simplicity $N = M$ so that the block structure (1.1) is trivial. By Theorem 1.1 (cf. (5.2)) the τ-method inclusion set is, for $\varepsilon \ge 0$ and $n < N$,

$$\mathrm{Spec}_\varepsilon L_N \subset \Sigma_\varepsilon^n(L_N) = \begin{cases} \bigcup_{m=1}^n \mathrm{Spec}_{\varepsilon+\varepsilon_n} L_m, & n = 1, 2, \\ \mathrm{Spec}_{\varepsilon+\varepsilon_{n-2}} L_n \cap \bigcup_{m=1}^n \mathrm{Spec}_{\varepsilon+\varepsilon_n} L_m, & n > 2, \end{cases} \tag{5.16}$$

with

$$\varepsilon_n := \varepsilon_n(L_N) = 4\sin(\theta_n^*/2),$$

where $\theta_n^* = \theta_n(L_N)$ denotes the unique solution in the range $[\pi/(2n+1), \pi/(n+2)]$ of the equation

$$8\sin(t/2)\cos((n+1/2)t) + \sin((n-1)t) = 0.$$

Equivalently (see [3, §3.2]), θ_n^* is the unique solution in the range $(\pi/(n+3), \pi/(n+2)]$ of the equation

$$2\cos((n+1)t/2) = \cos((n-1)t/2), \tag{5.17}$$

in particular, $\theta_1^* = \pi/3$ and $\theta_2^* = 2\cos^{-1}(\sqrt{7/8})$. Thus $\varepsilon_1 = 2$, $\varepsilon_2 = \sqrt{2}$, and

$$4\sin(\pi/(2n+6)) < \varepsilon_n < 4\sin(\pi/(2n+4)), \qquad n > 1,$$

so that $\varepsilon_n \sim 2\pi/n$ as $n \to \infty$. As noted in Remark 1.5, since each L_n is Hermitian,

$$\mathrm{Spec}_\varepsilon L_n = \mathrm{Spec}\, L_n + \varepsilon\overline{\mathbb{D}}, \qquad n \in \mathbb{N}, \quad \varepsilon > 0, \tag{5.18}$$

with $\mathrm{Spec}\, L_n$ given by (5.1). Note that $\Sigma_0^1(L_N) = \mathrm{Spec}_{\varepsilon_1} L_1 = 2\overline{\mathbb{D}}$, the same inclusion set G as provided by Gershgorin's theorem (1.10) in the case that $N > 2$.

In this example, in contrast to the Jordan block example (see (5.11)), $\Sigma_\varepsilon^n(A)$, for $n > 1$, cannot be characterised as the pseudospectrum of a single submatrix of A. In particular, for $n > 1$, the inclusion set $\Sigma_\varepsilon^n(L_N)$ is strictly larger than $\mathrm{Spec}_{\varepsilon_n+\varepsilon} L_n$. But, importantly, $\mathrm{Spec}_{\varepsilon_n+\varepsilon} L_n$ is also an inclusion set for $\mathrm{Spec}_\varepsilon L_N$, for every $N \in \mathbb{N}$. This follows by applying Theorem 3.3 to the bi-infinite version of L_N, i.e. to the operator $E = V + V'$, where V and its adjoint V' are the left and right shift, respectively. To see this, note that $\mathrm{Spec}\, E = [-2, 2] \supset \mathrm{Spec}\, L_N$ and $\Sigma_\varepsilon^n(E) = \mathrm{Spec}_{\varepsilon_n+\varepsilon} L_n$, by (2.10) and (3.13), so that Theorem 3.3 implies that $\mathrm{Spec}_{\varepsilon_n+\varepsilon} L_n = \Sigma_\varepsilon^n(E) \supset [-2, 2] + \varepsilon\overline{\mathbb{D}} \supset \mathrm{Spec}_\varepsilon L_N$.

This last observation, in particular that $[-2, 2] \subset \Sigma_0^n(L_N)$, leads to an explicit characterisation of $\mathbb{R} \cap \Sigma_0^n(L_N)$, which, since L_N is Hermitian, is also an inclusion set for $\mathrm{Spec}\, L_N$. Since, for $m \in \mathbb{N}$ and $\varepsilon \geq 0$, $\mathbb{R} \cap \mathrm{Spec}_\varepsilon L_m = \mathrm{Spec}\, L_m + [-\varepsilon, \varepsilon]$, with $\mathrm{Spec}\, L_m$ given by (5.1), it follows that

$$[-2, 2] \cup (\mathbb{R} \cap \mathrm{Spec}_\varepsilon L_m) = [-S_{\max}(m) - \varepsilon, S_{\max}(m) + \varepsilon], \tag{5.19}$$

with $S_{\max}(m) := \max \mathrm{Spec}\, L_m = -\min \mathrm{Spec}\, L_m = 2\cos(\pi/(m+1))$, so that, by (5.16),

$$\mathrm{Spec}\, L_N \subset \mathbb{R} \cap \Sigma_0^n(L_N) = [-2 - \varepsilon_n^*, 2 + \varepsilon_n^*], \tag{5.20}$$

where, for $n \in \mathbb{N}$,

$$\begin{aligned} \varepsilon_n^* &:= S_{\max}(n) - 2 + \varepsilon_n = 4(\sin(\theta_n^*/2) - \sin^2(\pi/(2n+2))) \\ &\geq 4(\sin(\pi/(4n+2)) - \sin^2(\pi/(2n+2))) \geq 0, \end{aligned}$$

with equality in the above inequalities if and only if $n = 1$. Note that $\varepsilon_n^* \sim \varepsilon_n \sim 2\pi/n$ as $n \to \infty$.

Let us turn now to the τ_1 inclusion sets. Let $L_1^+ := (1, 0, 1)^T$, $\widetilde{L}_1^+ := (1, 0, 0)^T$, $\widehat{L}_1^+ := (0, 0, 1)^T$, and, for $n > 1$, let

$$L_n^+ := \left(\begin{array}{ccccc} 1 & 0 & \cdots & 0 & 0 \\ \hline & & V_n & & \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right)_{(n+2)\times n}, \quad \widetilde{L}_n^+ := \left(\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ \hline & & V_n & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right)_{(n+2)\times n},$$

$$\widehat{L}_n^+ := \left(\begin{array}{cccc} 0 & \cdots & 0 & 0 \\ \hline & V_n & & \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)_{(n+2)\times n}.$$

Then $\{B_{n,k}^+ : 0 \leq k \leq N - n\} = \{\widetilde{L}_n^+, \widehat{L}_n^+\}$, if $n = N - 1$, $= \{L_n^+, \widetilde{L}_n^+, \widehat{L}_n^+\}$, if $n < N - 1$. Recall that a permutation of rows of a matrix leaves its singular values unchanged. Thus

$s_{\min}(\widetilde{L}_n^+ - \lambda I_N^+) = s_{\min}(\widehat{L}_n^+ - \lambda I_N^+)$, for $\lambda \in \mathbb{C}$, so that $\mathrm{Spec}_\varepsilon \widetilde{L}_n^+ = \mathrm{Spec}_\varepsilon \widehat{L}_n^+$, for $\varepsilon \geq 0$. Thus, for $\varepsilon \geq 0$, by application of Theorem 1.6,

$$\mathrm{Spec}_\varepsilon L_N \subset \Gamma_\varepsilon^n(L_N) = \begin{cases} \mathrm{Spec}_{\varepsilon+\varepsilon_n'} \widetilde{L}_n^+, & n = N-1, \\ \mathrm{Spec}_{\varepsilon+\varepsilon_n'} L_n^+ \cup \mathrm{Spec}_{\varepsilon+\varepsilon_n'} \widetilde{L}_n^+, & n < N-1, \end{cases} \tag{5.21}$$

and also $\Gamma_\varepsilon^n(L_N) \subset \mathrm{Spec}_{\varepsilon+\varepsilon_n'} L_N$, where

$$\varepsilon_n' := \varepsilon_n'(L_N) = 4\sin(\pi/(2n+2)).$$

We note that, for this example, because $A = L_N$ is Hermitian, the computation of $\Sigma_\varepsilon^n(L_N)$ is straightforward in view of (5.18), while determining $\Gamma_\varepsilon^n(L_N)$ requires the computation of pseudospectra of rectangular matrices.

Equations (5.1) and (5.20) suggest that $\Sigma_0^n(L_N)$ is an increasingly good approximation of $\mathrm{Spec}\, L_N$ as n and N increase. This, and the analogous result for $\Gamma_\varepsilon^n(L_N)$, is confirmed by the following lemma (cf. Corollary 5.2).

Lemma 5.3. *Suppose that* $\varepsilon \geq 0$. *As* $N \to \infty$, $\mathrm{Spec}_\varepsilon L_N \xrightarrow{\mathrm{H}} [-2,2] + \varepsilon\overline{\mathbb{D}}$. *Further, where* S_n *denotes* $\Sigma_\varepsilon^n(L_N)$ *or* $\Gamma_\varepsilon^n(L_N)$, $S_n \xrightarrow{\mathrm{H}} [-2,2] + \varepsilon\overline{\mathbb{D}}$, *as* $n, N \to \infty$ *with* $n < N$, *so that* $d_H(S_n, \mathrm{Spec}_\varepsilon L_N) \to 0$ *as* $n, N \to \infty$ *with* $n < N$.

Proof. It is clear from (5.1) that $\mathrm{Spec}\, L_N \xrightarrow{\mathrm{H}} [-2,2]$ as $N \to \infty$, so that $\mathrm{Spec}_\varepsilon L_n = \mathrm{Spec}\, L_N + \varepsilon\overline{\mathbb{D}} \xrightarrow{\mathrm{H}} [-2,2] + \varepsilon\overline{\mathbb{D}}$. By (5.16), $\mathrm{Spec}_\varepsilon L_N \subset \Sigma_\varepsilon^n(L_N) \subset \mathrm{Spec}_{\varepsilon_{n-2}+\varepsilon} L_n = \mathrm{Spec}\, L_n + (\varepsilon + \varepsilon_{n-2})\overline{\mathbb{D}}$, so that also $\Sigma_\varepsilon^n(L_N) \xrightarrow{\mathrm{H}} [-2,2] + \varepsilon\overline{\mathbb{D}}$ as $n, N \to \infty$. Similarly, by the discussion around (5.21), $\mathrm{Spec}_\varepsilon L_N \subset \Gamma_\varepsilon^n(L_N) \subset \mathrm{Spec}_{\varepsilon_n''+\varepsilon} L_N$, so also $\Gamma_\varepsilon^n(L_N) \xrightarrow{\mathrm{H}} [-2,2] + \varepsilon\overline{\mathbb{D}}$ as $n, N \to \infty$. $\square$

The above lemma tells us that the inclusion set $\Sigma_0^n(L_N)$ is an increasingly good approximation to $\mathrm{Spec}\, L_N$ as $n, N \to \infty$, whereas the inclusion set provided by the standard Gershgorin theorem (1.10) is $G = 2\overline{\mathbb{D}}$ if $N > 2$. Of course, where the matrix is Hermitian so the spectrum is real, it makes sense to use this fact, taking the intersection of inclusion sets with $\mathbb{R}$. While G is a poor approximation to $\mathrm{Spec}\, L_N$, $\mathbb{R} \cap G = [-2,2]$ is a sharp bound when N is large, by Lemma 5.3, and, except when $n = 1$, sharper than $\mathbb{R} \cap \Sigma_0^n(L_N)$ given by (5.20). Indeed, where G is the Gershgorin inclusion set (1.10), we will see in Remark 5.8 that $G \cap \mathbb{R}$ is a sharp inclusion set for the eigenvalues of large tridiagonal Hermitian Toeplitz matrices in general; our τ and τ_1 sets offer no improvement.

Thus the example L_N makes the point that our new inclusion sets need not improve on Gershgorin; that, for some Hermitian matrices, $\mathbb{R} \cap G$ already provides sharp eigenvalue enclosures. But we will see in §5.2 that, for large Hermitian Toeplitz matrices in general, $\mathbb{R} \cap G$ overestimates the spectrum while our τ- and τ_1-inclusion sets continue to provide sharp approximations (see Theorem 5.6, Remark 5.8, and Theorem 5.9). Similarly, we will see in §5.3 that, within the class of tridiagonal matrices, if L_N is perturbed by addition of a real diagonal matrix, $\mathbb{R} \cap G$ can significantly overestimate the spectrum while $\Sigma_0^n(L_N)$ can, again, provide a sharp inclusion set.

Remark 5.4 *(Comparison with existing block-matrix Gershgorin).* We have just made comparison with the classical Gershgorin theorem (1.10). But how does our new inclusion set family compare with the existing block matrix versions of the Gershgorin theorem, discussed in Remark 1.2? Suppose that $A = L_M$ for some $M > 1$ and write A in the block form (1.1) for some $1 < N \le M$, so that $a_{ij} \in \mathbb{C}^{m_i \times m_j}$, for $i, j = 1, \dots, N$. Let us compute the inclusion set (1.9) given by the block-matrix version of Gershgorin in the case that the matrix norms are the 2-norm. It is clear that A is block-tridiagonal, that $a_{ii} = L_{m_i}$, $i = 1, \dots, N$, and that each non-zero off-diagonal block a_{ij} has a single non-zero entry (taking the value 1), so that $\|a_{i,i+1}\| = \|a_{i+1,i}\| = 1$, $i = 1, \dots, N-1$. Thus (1.9) reads in this case

$$\begin{aligned}\operatorname{Spec} L_M \;\subset\; G^N &:= \operatorname{Spec}_1 L_{m_1} \cup \operatorname{Spec}_1 L_{m_N} \cup \bigcup_{k=2}^{N-1} \operatorname{Spec}_2 L_{m_k} \\ &= \left(\overline{\mathbb{D}} + (\operatorname{Spec} L_{m_1} \cup \operatorname{Spec} L_{m_N})\right) \cup \left(2\overline{\mathbb{D}} + \bigcup_{k=2}^{N-1} \operatorname{Spec} L_{m_k}\right).\end{aligned}$$

This is no better than the inclusion set provided by the standard Gershgorin theorem, which it reduces to if $N = M$. Indeed, if $M > 2$ it is immediate from the above and (5.1) that $[-2, 2] \subset G^N$. Thus, arguing as we did to get (5.20), using (5.19), we see that, where $m_- := \max(m_1, m_N)$ and $m_+ := \max_{k=2,\dots,N-1} m_k$,

$$\mathbb{R} \cap G^N = [-S_{\max}(m_-) - 1, S_{\max}(m_-) + 1] \cup [-S_{\max}(m_+) - 2, S_{\max}(m_+) + 2], \quad (5.22)$$

with the second interval present only for $N > 2$. Thus $\mathbb{R} \cap G^N$ is strictly larger than $\mathbb{R} \cap G = [-2, 2]$ if $m_- > 2$, or if $N > 2$ and $m_+ > 1$. Indeed, as $M \to \infty$ with $N > 2$ and $m_+ \to \infty$, $G^N \xrightarrow{\text{H}} [-2, 2] + 2\overline{\mathbb{D}}$ and $\mathbb{R} \cap G^N \xrightarrow{\text{H}} [-4, 4]$. □

5.2. General Toeplitz matrices

The matrices in §5.1 are both examples of tridiagonal Toeplitz matrices. In this section we consider general Toeplitz matrices, i.e., the case $A = A_M$, for some $M > 1$, where, for some complex coefficients $(a_j)_{j \in \mathbb{Z}}$,

$$A_M := \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{1-M} \\ a_1 & a_0 & \cdots & a_{2-M} \\ \vdots & \vdots & & \vdots \\ a_{M-2} & a_{M-3} & \cdots & a_{-1} \\ a_{M-1} & a_{M-2} & \cdots & a_0 \end{pmatrix}_{M \times M}, \quad M \in \mathbb{N}.$$

We are particularly interested in the case when M is large, seeking to generalise Corollary 5.2 and Lemma 5.3 as they relate to the τ and τ_1 methods. The study of large Toeplitz matrices and the asymptotics of their spectra and pseudospectra has a substantial literature that we will draw on; see, for example, [18] and the monographs [1,2,22].

Let a denote the symbol associated to the coefficients $(a_j)_{j\in\mathbb{Z}}$, defined formally by

$$a(t) := \sum_{j\in\mathbb{Z}} a_j t^j, \qquad t \in \mathbb{T}. \tag{5.23}$$

As a substantial illustration of Remark 1.4, we will first study, focusing on the τ method, the case that, for some $w \in \mathbb{N}$, $a_j = 0$ if $|j| > w$, so that $(A_M)_{M>1}$ is a family of banded Toeplitz matrices, each with band-width $\le w$. In that case the sum (5.23) is finite and a is rational. As our only example in this paper where A does not coincide with its tridiagonal part B, so that the remaining part $C = A - B$ is non-zero, we will also consider, focusing on the τ_1 method, the case where infinitely many of the coefficients a_j are non-zero, but a is in the so-called *W*iener algebra (e.g., [2, Example 1.5]), meaning that

$$\|a\|_{\mathcal{W}} := \sum_{j\in\mathbb{Z}} |a_j| < \infty. \tag{5.24}$$

In that case the sum (5.23) is absolutely and uniformly convergent, and $a \in C(\mathbb{T})$, the symbol is continuous. A key result is that if a is in the Wiener algebra then[2] [2, Corollary 3.18], for all $\varepsilon > 0$,

$$\mathrm{Spec}_\varepsilon A_M \xrightarrow{\mathrm{H}} \mathrm{Spec}_\varepsilon A_\infty^+, \qquad \text{as} \qquad M \to \infty. \tag{5.25}$$

Here A_∞^+ is the infinite Toeplitz matrix given by

$$A_\infty^+ := \begin{pmatrix} a_0 & a_{-1} & \cdots \\ a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix},$$

which is a bounded operator on $\ell^2(\mathbb{N})$ for a in the Wiener algebra, with

$$\|A_\infty^+\| \le \|a\|_{\mathcal{W}}.$$

$\mathrm{Spec}_\varepsilon A_\infty^+$, the ε-pseudospectrum of A_∞^+ as an operator on $\ell^2(\mathbb{N})$, is characterised in [2, §3.6].

[2] This result from [2] needs, in fact, only that a is piecewise continuous. The notion of convergence of sets used in [2, Corollary 3.18] coincides with Hausdorff convergence by standard arguments, e.g., [15, Proposition 3.6].

If A_∞^+ is self-adjoint, so that $a_{-j} = \bar{a}_j$, $j \in \mathbb{Z}$, then A_M is Hermitian, so that, for $\varepsilon > 0$, $\mathrm{Spec}_\varepsilon A_\infty^+ = \mathrm{Spec}\, A_\infty^+ + \varepsilon\overline{\mathbb{D}}$, $\mathrm{Spec}_\varepsilon A_M = \mathrm{Spec}\, A_M + \varepsilon\overline{\mathbb{D}}$, and $d_H(\mathrm{Spec}\, A_M, \mathrm{Spec}\, A_\infty^+) \leq 2\varepsilon + d_H(\mathrm{Spec}_\varepsilon A_M, \mathrm{Spec}_\varepsilon A_\infty^+)$. Thus, if A_∞^+ is self-adjoint and a is in the Wiener algebra, (5.25) holds also for $\varepsilon = 0$, i.e., $\mathrm{Spec}\, A_M \xrightarrow{\mathrm{H}} \mathrm{Spec}\, A_\infty^+$; further, a is real-valued and (e.g., [2, Theorem 1.17]) $\mathrm{Spec}\, A_\infty^+ = a(\mathbb{T})$, i.e.

$$\mathrm{Spec}\, A_\infty^+ = [a_{\min}, a_{\max}], \quad \text{where} \quad a_{\min} := \min_{t\in\mathbb{T}} a(t), \quad a_{\max} := \max_{t\in\mathbb{T}} a(t). \tag{5.26}$$

5.2.1. The banded case

Suppose that, for some $w \in \mathbb{N}$, $a_j = 0$ if $|j| > w$, so that, for each $M > 1$, A_M is banded with band-width $w_M \leq w$. Let us proceed as in Remark 1.4, writing $A = A_M$ in the block form (1.1), with $m_i \geq w_M$, for $i = 1, \ldots, N$. Then A, written in the block form (1.1), is tridiagonal, coinciding with its tridiagonal part B. Further, for $n = 1, \ldots, N$ and $k = 0, \ldots, N-n$,

$$B_{n,k} = A_{M_{n,k}}, \qquad \text{with} \qquad M_{n,k} := \sum_{i=k+1}^{k+n} m_i,$$

so that, by Theorem 1.1,

$$\mathrm{Spec}_\varepsilon A \subset \Sigma_\varepsilon^n(A) = \begin{cases} \sigma_\varepsilon^n(A), & n = 1, 2, \\ \sigma_\varepsilon^n(A) \cap \widehat{\sigma}_\varepsilon^n(A), & n > 2, \end{cases} \tag{5.27}$$

for $\varepsilon \geq 0$, where

$$\begin{aligned} \widehat{\sigma}_\varepsilon^n(A) &:= \bigcup_{k=0}^{N-n} \mathrm{Spec}_{\varepsilon+\varepsilon_{n-2}(A)} A_{M_{n,k}}, \\ \sigma_\varepsilon^n(A) &:= \bigcup_{k=0}^{N-n} \mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} A_{M_{n,k}} \cup \\ &\qquad \bigcup_{m=1}^{n-1} \left(\mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} A_{M_{m,0}} \cup \mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} A_{M_{m,N-m}} \right), \end{aligned} \tag{5.28}$$

and with $\varepsilon_n(A)$ given by (1.4) (with $C = 0$).

The following result is essentially a corollary of (5.25). Its proof uses standard properties of Hausdorff convergence and pseudospectra (see, e.g., [3, §1.3, Eqn. (2.8)]), namely: i) that, if L, U, S, and T are compact subsets of $\mathbb{C}$ and $L \subset S \subset U$, then

$$d_H(S, T) \leq \max\{d_H(L, T), d_H(U, T)\},$$

in particular $d_H(S, U) \leq d_H(L, U)$; and ii) that if $F \in L(H)$, for some Hilbert space H, then, for $\varepsilon \geq 0$, $\mathrm{Spec}_\eta F \xrightarrow{\mathrm{H}} \mathrm{Spec}_\varepsilon F$ as $\eta \to \varepsilon^+$.

Theorem 5.5. *Suppose $\varepsilon > 0$ and $A = A_M$ and $m_i \geq w_M$, for $i = 1, \dots, N$. Then $\mathrm{Spec}_\varepsilon A \overset{\mathrm{H}}{\to} \mathrm{Spec}_\varepsilon A_\infty^+$ as $M \to \infty$ and $\Sigma_\varepsilon^n(A) \overset{\mathrm{H}}{\to} \mathrm{Spec}_\varepsilon A_\infty^+$ as $n \to \infty$ (with $M \geq w_M N$ and $N > n$), so that $d_H(\Sigma_\varepsilon^n(A), \mathrm{Spec}_\varepsilon A) \to 0$ as $M, n \to \infty$. If $a_{-j} = \bar{a}_j$, $j \in \mathbb{Z}$, then these results hold also for $\varepsilon = 0$, in particular $d_H(\Sigma_0^n(A), \mathrm{Spec}\, A) \to 0$ as $M, n \to \infty$.*

Proof. Suppose $\varepsilon > 0$. That $\mathrm{Spec}_\varepsilon A \overset{\mathrm{H}}{\to} \mathrm{Spec}_\varepsilon A_\infty^+$ as $M \to \infty$ is (5.25). To see the rest of the theorem, note that $r_L(A) \leq \|A\|_\infty \leq \|a\|_{\mathcal{W}}$ and $r_U(A) \leq \|a\|_{\mathcal{W}}$, so that $r(A) \leq 2\|a\|_{\mathcal{W}}$ and, by (1.4),

$$\varepsilon_{n-2}(A) \leq \frac{2\pi\|a\|_{\mathcal{W}}}{n} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.29}$$

Suppose now that $K > 1$. Then, for every $\eta > \varepsilon$, provided n is sufficiently large so that $\varepsilon + \varepsilon_{n-2}(A) \leq \eta$,

$$\begin{aligned} d_H(\mathrm{Spec}_{\varepsilon+\varepsilon_{n-2}(A)} A_K, \mathrm{Spec}_\varepsilon A_K) &\leq d_H(\mathrm{Spec}_\eta A_K, \mathrm{Spec}_\varepsilon A_K) \\ &\leq d_H(\mathrm{Spec}_\eta A_K, \mathrm{Spec}_\eta A_\infty^+) \\ &\qquad + d_H(\mathrm{Spec}_\eta A_\infty^+, \mathrm{Spec}_\varepsilon A_\infty^+) \\ &\qquad + d_H(\mathrm{Spec}_\varepsilon A_\infty^+, \mathrm{Spec}_\varepsilon A_K). \end{aligned}$$

The second term on the right-hand side of this last expression can be made arbitrarily small by choosing η sufficiently close to ε, and, for every $\eta > \varepsilon$, the other terms $\to 0$ as $K \to \infty$ by (5.25). Thus, and by (5.25), for every $\delta > 0$ there exists $n_0, K_0 \in \mathbb{N}$ such that

$$d_H(\mathrm{Spec}_{\varepsilon+\varepsilon_{n-2}(A)} A_K, \mathrm{Spec}_\varepsilon A_\infty^+) \leq \delta$$

if $n > n_0$ and $K > K_0$. Since $M_{n,k} \geq n$, for each $k \in \{0, \dots, N-n\}$, it follows that $\widehat{\sigma}_\varepsilon^n(A) \overset{\mathrm{H}}{\to} \mathrm{Spec}_\varepsilon A_\infty^+$ as $n \to \infty$ with $N > n$. Since $\mathrm{Spec}_\varepsilon A \subset \Sigma_\varepsilon^n(A) \subset \widehat{\sigma}_\varepsilon^n(A)$, also $\Sigma_\varepsilon^n(A) \overset{\mathrm{H}}{\to} \mathrm{Spec}_\varepsilon A_\infty^+$ as $n \to \infty$ with $N > n$. If A_∞^+ is self-adjoint then (5.25) holds also for $\varepsilon = 0$, so the above arguments work also in that case. □

Where B is a square matrix or $B \in L(H)$, for a Hilbert space H, let $W(B)$ denote the numerical range of B. Where $\mathrm{conv}(S)$ denotes the convex hull of a set $S \subset \mathbb{C}$, recall that $\mathrm{conv}(\mathrm{Spec}\, B) \subset \overline{W(B)}$, with equality if B is normal (e.g., [14]), and note that $W(PB|_{\widetilde{H}}) \subset W(B)$ if $B \in L(H)$, $\widetilde{H} \subset H$ is a closed subspace, and $P : H \to \widetilde{H}$ is orthogonal projection. Thus, and recalling (5.26), we see that, for M, $M' \in \mathbb{N}$ with $M < M'$,

$$\mathrm{conv}(\mathrm{Spec}\, A_M) \subset \mathrm{conv}(\mathrm{Spec}\, A_{M'}) = W(A_{M'}) \subset W(A_\infty^+) = \mathrm{Spec}\, A_\infty^+, \tag{5.30}$$

in the Hermitian case $a_{-j} = \bar{a}_j$, $j \in \mathbb{Z}$, so that

$$\mathrm{Spec}_\varepsilon A = \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}} \subset \mathrm{Spec}\, A_\infty^+ + \varepsilon\overline{\mathbb{D}} = \mathrm{Spec}_\varepsilon A_\infty^+, \qquad \varepsilon \geq 0. \tag{5.31}$$

In the Hermitian case it is convenient to introduce $\rho(a) := \|a - a_0\|_{\mathcal{W}} = 2\sum_{j=1}^\infty |a_j|$, noting that (cf. the proof of Theorem 5.5)

$$r(A) \leq \rho(a). \tag{5.32}$$

Recall also θ_n^* introduced above (5.17), so that $\theta_n^* \in (\pi/(n+3), \pi/(n+2]$ is the solution of (1.5) in the Hermitian case that $r_L(A) = r_U(A) = r(A)/2$.

Theorem 5.6. *Suppose $a_{-j} = \bar{a}_j$, $j \in \mathbb{Z}$, $\varepsilon \geq 0$, $A = A_M$, and $m_i \geq w$, for $i = 1, \ldots, N$. Then*

$$d_H(\Sigma_\varepsilon^n(A), \mathrm{Spec}_\varepsilon A) \leq \Theta_{n,N}\, \rho(a), \tag{5.33}$$

for $1 \leq n < N$ and $\varepsilon \geq 0$, where

$$\Theta_{n,N} := \Theta_n + \Theta_N \text{ and } \Theta_m := 2\sin(\theta_m^*/2) \leq 2\sin\left(\frac{\pi}{2m+4}\right) \leq \frac{\pi}{m+2},\ m \in \mathbb{N}.$$

Further,

$$\mathbb{R} \cap \Sigma_0^n(A) \subset [a_{\min} - \Theta_n\rho(a), a_{\max} + \Theta_n\rho(A)] \tag{5.34}$$

and

$$\mathrm{Spec}\, A \subset \mathbb{R} \cap \Sigma_0^n(A) \subset \mathrm{Spec}\, A + [-\Theta_{n,N}\,\rho(a), \Theta_{n,N}\,\rho(a)]. \tag{5.35}$$

Proof. Let A_∞ denote the bi-infinite Laurent matrix with symbol a, defined as in [2, §1.2], and recall that $\mathrm{Spec}\, A_\infty = a(\mathbb{T})$ (e.g., [2, Theorem 1.2]), so that, by (5.26),

$$\mathrm{Spec}_\varepsilon A_\infty^+ = \mathrm{Spec}\, A_\infty^+ + \varepsilon\overline{\mathbb{D}} = \mathrm{Spec}\, A_\infty + \varepsilon\overline{\mathbb{D}} = \mathrm{Spec}_\varepsilon A_\infty, \quad \varepsilon \geq 0, \tag{5.36}$$

in this Hermitian case. Write A_∞ in block-tridiagonal form as $A_\infty = [e_{ij}]_{i,j\in\mathbb{Z}}$, choosing the blocks so that $e_{ij} = a_{ij}$, $1 \leq i,j \leq N$, and so that $e_{i+N,j+N} = e_{ij}$, $i,j \in \mathbb{Z}$. Then, where $E_{n,k}$ is as defined below (3.1) and $\eta_N(A_\infty)$ by (3.9), $E_{N,k} = A$, for $k \in \mathbb{Z}$. Thus, and by Theorem 3.3, $\mathrm{Spec}_\varepsilon A_\infty \subset \Sigma_\varepsilon^N(A_\infty) = \mathrm{Spec}_{\varepsilon+\eta_N(A_\infty)} A$, for $\varepsilon \geq 0$, with $\eta_N(A_\infty) = 2r(A_\infty)\sin(\theta_N^*/2) \leq 2\rho(a)\sin(\theta_N^*/2)$ (cf. (5.32)). Thus, and by (5.31) and (5.36),

$$\mathrm{Spec}_\varepsilon A \subset \mathrm{Spec}_\varepsilon A_\infty^+ \subset \mathrm{Spec}_{\varepsilon+\eta_N(A_\infty)} A = \mathrm{Spec}_\varepsilon A + \eta_N(A_\infty)\overline{\mathbb{D}}, \qquad \varepsilon \geq 0. \tag{5.37}$$

By (5.27), $\mathrm{Spec}_\varepsilon A \subset \Sigma_\varepsilon^n(A) \subset \sigma_\varepsilon^n(A)$. Further, $\sigma_\varepsilon^n(A) \subset \mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} A_\infty^+$ by (5.31), where $\varepsilon_n(A) = 2r(A)\sin(\theta_n^*/2) \leq 2\rho(a)\sin(\theta_n^*/2)$, by (5.32), so that, applying (5.37),

$$\mathrm{Spec}_\varepsilon A \subset \Sigma_\varepsilon^n(A) \subset \mathrm{Spec}_\varepsilon A + (\varepsilon_n(A) + \eta_N(A_\infty))\overline{\mathbb{D}} \subset \mathrm{Spec}_\varepsilon A + \Theta_{n,N}\,\rho(a)\overline{\mathbb{D}},$$

and (5.33) and (5.35) follow. The inclusion (5.34) holds since, by (5.31) and (5.26), $\mathrm{Spec}\, A_J \subset [a_{\min}, a_{\max}]$ and $\mathbb{R} \cap \mathrm{Spec}_\varepsilon A_J = \mathrm{Spec}\, A_J + [-\varepsilon, \varepsilon]$, for $\varepsilon \geq 0$ and $J \in \mathbb{N}$. □

It follows from [2, Theorem 3.19] that $\mathrm{Spec}_\varepsilon A_\infty^+ = (1+\varepsilon)\overline{\mathbb{D}}$ and $\mathrm{Spec}_\varepsilon A_\infty^+ = [-2,2] + \varepsilon\overline{\mathbb{D}}$ for the examples in §5.1.1 and §5.1.2, respectively. Thus Theorem 5.5, applied with $m_i = 1$, $i = 1, \ldots, N$, so that $N = M$, implies Corollary 5.2 and Lemma 5.3, as they relate to the τ method, as special cases.

Remark 5.7 *(Approximating* $\mathrm{Spec}_\varepsilon A$ *using the above results).* Given a large Toeplitz matrix A of order M and band-width w one might proceed as follows, informed by the above results, to obtain an inclusion set for $\mathrm{Spec}_\varepsilon A$ that is also a good approximation to $\mathrm{Spec}_\varepsilon A$.

1. Assuming that $M \geq w^2$, set $N := \lfloor M/w \rfloor$, the largest integer $\leq M/w$, set $r := M - wN \in \{0, \ldots, w-1\}$, and note that $N - r \geq 1$.
2. Set $m_i := w$, for $i = 1, \ldots, N-r$, $m_i := w+1$, if $N-r < i \leq N$. Then $\sum_{i=1}^N m_i = M$ and also $nw \leq M_{n,k} \leq n(w+1)$, for $1 \leq n < N$, $0 \leq k \leq N-n$, so that there are at most $n+1$ different values of $M_{n,k}$.
3. For $i, j = 1, \ldots, N$, let a_{ij} be the appropriate $m_i \times m_j$ submatrix of A such that A is (1.1) in block form. Note that this block matrix is tridiagonal by Remark 1.4.
4. Define $\Sigma_\varepsilon^n(A)$ by (5.27), for $n = 1, \ldots, N-1$. The observations in the above steps regarding the size of $M_{n,k}$ imply that, no matter how large the value of M, computing $\Sigma_\varepsilon^n(A)$ requires the computation of at most $4n$ pseudospectra of square matrices, and each of these matrices has order $\leq n(w+1)$. Of course, $\mathrm{Spec}_\varepsilon A \subset \Sigma_\varepsilon^n(A)$, for each n. Further, given that M is large, Theorem 5.5 suggests that, if $\varepsilon > 0$, $\Sigma_\varepsilon^n(A)$ will be an increasingly good approximation to $\mathrm{Spec}_\varepsilon A$ as n increases. Indeed, if A is Hermitian, this is true also for $\varepsilon = 0$ and the error is quantified, for all $\varepsilon \geq 0$, in Theorem 5.6.

The above algorithm is most straightforward when M is a multiple of the band-width w, i.e., $M = Nw$, so that $m_i = w$, $i = 1, \ldots, N$. The expression for $\Sigma_\varepsilon^N(A)$ simplifies in that case to

$$\Sigma_\varepsilon^n(A) := \mathrm{Spec}_{\varepsilon+\varepsilon_{n-2}(A)} A_{nw} \cap \bigcup_{m=1}^{n} \mathrm{Spec}_{\varepsilon+\varepsilon_n(A)} A_{mw}, \tag{5.38}$$

with the term $\mathrm{Spec}_{\varepsilon+\varepsilon_{n-2}(A)} A_{nw}$ absent for $n = 1, 2$. This is an inclusion set for $\mathrm{Spec}_\varepsilon A = \mathrm{Spec}_\varepsilon A_{Nw}$, for $N > n$. Since $\mathrm{Spec}_\varepsilon A_{Nw} \xrightarrow{\mathrm{H}} \mathrm{Spec}_\varepsilon A_\infty^+$ as $N \to \infty$, by (5.25), $\Sigma_\varepsilon^n(A)$, defined by (5.38), is also an inclusion set for $\mathrm{Spec}_\varepsilon A_\infty^+$, i.e.,

$$\mathrm{Spec}_\varepsilon A_\infty^+ \subset \Sigma_\varepsilon^n(A), \qquad n \in \mathbb{N}. \tag{5.39}$$

This holds, because it uses (5.25), only for $\varepsilon > 0$ in the first instance, but then, by taking the intersection of (5.39) over all $\varepsilon > 0$, also for $\varepsilon = 0$. This in turn implies, in the Hermitian case $a_{-j} = \bar{a}_j$, $j \in \mathbb{Z}$, that

$$\mathbb{R} \cap \Sigma_0^n(A) = [a_-(nw) - \varepsilon_n(A), a_+(nw) + \varepsilon_n(A)] \supset [a_{\min}, a_{\max}], \tag{5.40}$$

where, for $J \in \mathbb{N}$, $a_-(J) := \min \operatorname{Spec} A_J$ and $a_+(J) := \max \operatorname{Spec} A_J$. For (5.39) and (5.26) imply in the Hermitian case that $[a_{\min}, a_{\max}] \subset \Sigma_0^n(A)$, and then (5.40) follows by arguing as we did to obtain (5.20), using (5.30). □

Remark 5.8 *(Gershgorin versus τ method: Hermitian Toeplitz).* As noted in and above (5.26) and in (5.31), $\operatorname{Spec} A \subset \operatorname{Spec} A_\infty^+$ and $\operatorname{Spec} A \overset{\mathrm{H}}{\to} \operatorname{Spec} A_\infty^+ = [a_{\min}, a_{\max}]$ as $M \to \infty$, so that the eigenvalues of A are an increasingly dense subset of $[a_{\min}, a_{\max}]$ as M increases. Thus, and by (5.34), as quantified already in (5.35), $\mathbb{R} \cap \Sigma_0^n(A)$ is a sharp inclusion set for $\operatorname{Spec} A$ when n and N are large.

Where G is the Gershgorin inclusion set given by (1.10), we have $G = a_0 + \rho(a)\overline{\mathbb{D}}$ if $M > w$, so that

$$\operatorname{Spec} A \subset \mathbb{R} \cap G = [a_0 - \rho(a), a_0 + \rho(a)] \supset [a_{\min}, a_{\max}].$$

Thus $\mathbb{R} \cap G$ is a sharp inclusion set, asymptotically in the limit $M \to \infty$, if and only if $[a_0 - \rho(a), a_0 + \rho(a)] = [a_{\min}, a_{\max}]$. This holds if A has only three non-zero diagonals, in particular if A is tridiagonal, and in certain other cases. (As an example, suppose that $a_2, a_6 > 0$, but $a_j = 0$ if $j \in \mathbb{N} \setminus \{2, 6\}$. Then $a_{\max} = a(0) = a_0 + \rho(a)$ and $a_{\min} = a(\mathrm{i}) = a_0 - \rho(a)$.) But generically, when A has more than three non-zero diagonals, $\mathbb{R} \cap G$ is strictly larger than $[a_{\min}, a_{\max}]$.

As a concrete example, suppose that A is pentadiagonal, with $a_0 = 0$, $a_1 = a_2 = 1$. Then $\mathbb{R} \cap G = [-\rho(a), \rho(a)] = [-4, 4]$ while elementary calculations give that $[a_{\min}, a_{\max}] = [a(-1/4 \pm \mathrm{i}\sqrt{15}/4), a(0)] = [-9/4, 4]$. In the case $M = 110$ we plot $\operatorname{Spec} A$ and $\mathbb{R} \cap \Sigma_0^n(A)$, given by (5.40) with $w = 2$, in Fig. 5.2. As n increases $\mathbb{R} \cap \Sigma_0^n(A) \overset{\mathrm{H}}{\to} [a_{\min}, a_{\max}]$, as predicted. In this example a sharper bound than $\mathbb{R} \cap \Sigma_0^n(A)$ is $\mathbb{R} \cap \Sigma_0^n(A) \cap G$, taking the intersection of our τ-method inclusions sets with classical Gershgorin. □

5.2.2. The Wiener algebra case

Suppose now that a is in the Wiener algebra, so that (5.24) holds. As in the previous subsection we suppose that $A = A_M$, for some $M > 1$, and write A in block form as (1.1). In contrast to the previous section A does not, in general, coincide with its tridiagonal part, so its remaining part $C = A - B$ is non-zero. If, for some $w \in \mathbb{N}$, we choose the block form of A so that $m_i \geq w$, $i = 1, \ldots, N$, then $C = [c_{ij}]_{i,j=1}^M$ satisfies $c_{ij} = 0$ if $|i - j| \leq w$, so that

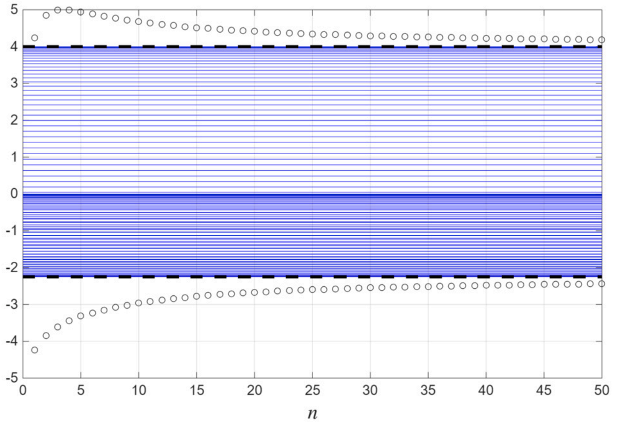

Fig. 5.2. Plot of Spec A, where $A = A_M$ is the order M pentadiagonal Toeplitz example at the end of Remark 5.8. Also plotted are the τ-method inclusions sets, $\mathbb{R} \cap \Sigma_0^n(A)$. The horizontal solid lines indicate the eigenvalues of A when $M = 110$, and the dashed lines the endpoints of $\operatorname{Spec} A_\infty^+ = [-9/4, 4]$. The circles are the upper and lower limits of the interval $\mathbb{R} \cap \Sigma_0^n(A)$, given by (5.40). Where G is the standard Gershgorin inclusion set, $\mathbb{R} \cap G = [-4, 4]$; see Remark 5.8.

$$\|C\| \leq \max\{\|C\|_1, \|C\|_\infty\} \leq a_w := \sum_{j=w+1}^{\infty} (|a_j| + |a_{-j}|). \tag{5.41}$$

The argument we made in the banded case to show $\Sigma_\varepsilon^n(A) \xrightarrow{\text{H}} \operatorname{Spec}_\varepsilon A_\infty^+$ does not apply here because $\widehat{\sigma}_\varepsilon^n(A)$ is no longer a union over k of pseudospectra of Toeplitz matrices. Instead, we prove a version of Theorem 5.5 for the τ_1 inclusion sets $\Gamma_\varepsilon^n(A) \supset \operatorname{Spec}_\varepsilon A$ given by (1.15). Note that the two-sided inclusions for the τ_1 set $\Gamma_\varepsilon^n(A)$, captured in Theorem 1.6, play a crucial role in the proof.

Theorem 5.9. *Suppose $\varepsilon > 0$ and $A = A_M$. Then $\operatorname{Spec}_\varepsilon A \xrightarrow{\text{H}} \operatorname{Spec}_\varepsilon A_\infty^+$ as $M \to \infty$. Further, $\Gamma_\varepsilon^n(A) \xrightarrow{\text{H}} \operatorname{Spec}_\varepsilon A_\infty^+$ as $n, M, w \to \infty$, where*

$$w := \min_{1 \leq i \leq N} m_i.$$

Thus $d_H(\Gamma_\varepsilon^n(A), \operatorname{Spec}_\varepsilon A) \to 0$ as $M, n, w \to \infty$. If $a_{-j} = \bar{a}_j$, $j \in \mathbb{Z}$, then these results hold also for $\varepsilon = 0$, in particular $d_H(\Gamma_0^n(A), \operatorname{Spec} A) \to 0$ as $M, n, w \to \infty$.

Proof. Suppose that $\varepsilon > 0$. Again, the first result is just (5.25). Analogously to (5.29), we have by (1.14) that

$$\varepsilon_n''(A) \leq \frac{2\pi \|a\|_{\mathcal{W}}}{n+1} + \|C\|,$$

so that $\varepsilon_n''(A) \to 0$ as $n, w \to \infty$, by (5.41). By Theorem 1.6,

$$\operatorname{Spec}_\varepsilon A \subset \Gamma_\varepsilon^n(A) \subset \operatorname{Spec}_{\varepsilon + \varepsilon_n''(A) + 2\|C\|} A.$$

Thus, arguing as in the proof of Theorem 5.5, using (5.25) and (5.41), we see that, as $n, M, w \to \infty$, $\mathrm{Spec}_{\varepsilon+\varepsilon''_n(A)+2\|C\|} A \xrightarrow{\mathrm{H}} \mathrm{Spec}_\varepsilon A_\infty^+$ so that also $\Gamma_\varepsilon^n(A) \xrightarrow{\mathrm{H}} \mathrm{Spec}_\varepsilon A_\infty^+$. If A_∞^+ is self-adjoint then (5.25) holds also for $\varepsilon = 0$, so the above arguments work also in that case. □

5.3. A non-Toeplitz example

We finish by applying our τ method to a tridiagonal Hermitian matrix A that is a perturbation, by a real diagonal matrix $D_M = \mathrm{diag}(d_1, \ldots, d_M)$, of the discrete Laplacian L_M introduced in §5.1, i.e., $A = D_M + L_M$, for some $M \in \mathbb{N}$. We choose $N = M$ so that the block structure (1.1), used in the construction of our τ method inclusion set $\Sigma_0^n(A)$, is trivial, each block a single complex number, as in §5.1.2. To obtain an example that has features beyond the Toeplitz case, but is simple enough that everything is known explicitly, we assume that the diagonal is 2-periodic, i.e., $d_{i+2} = d_i$, $i = 1, \ldots, M-2$, so that A is r-Toeplitz with $r = 2$ in the sense of [12], i.e., $a_{i+2,j+2} = a_{ij}$, $i, j = 1, \ldots, M-2$. In the case that M is even, this means that A is block Toeplitz, with 2×2 blocks. Noting that

$$\mathrm{Spec}\,(\lambda I + A) = \lambda + \mathrm{Spec} A \quad \text{and} \quad \Sigma_0^n(\lambda I + A) = \lambda + \Sigma_0^n(A), \qquad n = 1, \ldots, N-1,$$

without loss of generality we focus on the case that, for some $\Delta \in \mathbb{R}$,

$$a_{ii} = d_i = (-1)^{i-1}\Delta, \quad i = 1, \ldots, M,$$

denoting the matrix A in that case by $A_M(\Delta)$ when we want to make explicit the dependence on M and Δ.

The study of the eigenvalues of tridiagonal 2-Toeplitz matrices was initiated in [13]. An explicit formula, for the effect of a 2-periodic diagonal perturbation on the eigenvalues of any tridiagonal matrix with zero main diagonal, was presented recently in [7]. To apply the results of [7] we note that the eigenvalues of L_M are given by (5.1) and, for $M \in \mathbb{N}$ with $M \geq 2$, let $M^+ := \lfloor M/2 \rfloor$,

$$\mathrm{Spec}^+ L_M := \{\lambda \in \mathrm{Spec}\, L_M : \lambda > 0\} = \{2\cos(j\pi/(M+1)) : j \in \{1, \ldots, M^+\}\},$$

and

$$S_M^+(\Delta) := \{\sqrt{\lambda^2 + \Delta^2} : \lambda \in \mathrm{Spec}^+ L_M\} \subset \left(|\Delta|, \sqrt{4 + \Delta^2}\right). \tag{5.42}$$

Then [7, Theorem 3.1] gives that, for $\Delta \in \mathbb{R}$ and $M \in \mathbb{N}$, with $S_1^+(\Delta) := \emptyset$,

$$\mathrm{Spec}\, A = \mathrm{Spec}\, A_M(\Delta) = \begin{cases} -S_M^+(\Delta) \cup S_M^+(\Delta), & \text{if } M \text{ is even,} \\ -S_M^+(\Delta) \cup S_M^+(\Delta) \cup \{\Delta\}, & \text{if } M \text{ is odd.} \end{cases} \tag{5.43}$$

It is informative to relate $\operatorname{Spec} A$ to the spectra of the block Laurent matrix $A_\infty(\Delta) = [a_{ij}]_{i,j\in\mathbb{Z}}$ and the block Toeplitz matrix $A_\infty^+(\Delta) = [a_{ij}]_{i,j\in\mathbb{N}}$, where $a_{ij} := 0$ if $|i-j| > 1$, $:= 1$ if $|i-j| = 1$, and $a_{ii} := (-1)^{i-1}\Delta$, $i \in \mathbb{Z}$. $A_\infty(\Delta)$ and $A_\infty^+(\Delta)$ are examples of discrete Schrödinger operators, so that $A_N(\Delta)$ is a finite section of a discrete Schrödinger operator, e.g. [10]. We have (see [10, Examples 3.5(c), 3.10(a)]) that, for $\Delta \in \mathbb{R}$,

$$\operatorname{Spec} A_\infty^+(\Delta) = \operatorname{Spec} A_\infty(\Delta) = \left[-\sqrt{4+\Delta^2}, -|\Delta|\right] \cup \left[|\Delta|, \sqrt{4+\Delta^2}\right]. \tag{5.44}$$

Thus, for $\Delta \in \mathbb{R}$, $\operatorname{Spec} A_M(\Delta) \subset \operatorname{Spec} A_\infty^+(\Delta)$, for $M \in \mathbb{N}$, and $\operatorname{Spec} A_M(\Delta) \xrightarrow{\mathrm{H}} \operatorname{Spec} A_\infty^+(\Delta)$ as $M \to \infty$ (cf. Lemma 5.3, for the special case $\Delta = 0$).

5.3.1. The τ method inclusion sets

Let us write down the τ method inclusion sets, as defined in §1.1, for this case. For $n = 1, \ldots, N-1$ and $k = 0, \ldots, N-n$, we have that $B_{n,k} = A_n(\Delta)$ if k is even, $= A_n(-\Delta)$ if k is odd. Thus, for $N > n \geq 3$ and $\varepsilon \geq 0$,

$$\widehat{\sigma}_\varepsilon^n(A) = \operatorname{Spec}_{\varepsilon+\varepsilon_{n-2}} A_n(\Delta) \cup \operatorname{Spec}_{\varepsilon+\varepsilon_{n-2}} A_n(-\Delta),$$

where θ_n^* and $\varepsilon_n := 4\sin(\theta_n^*/2)$ are defined as in the discrete Laplacian case of §5.1.2. Since $A_n(\pm\Delta)$ are Hermitian, $\operatorname{Spec}_\varepsilon A_n(\pm\Delta) = \operatorname{Spec} A_n(\pm\Delta) + \varepsilon\overline{\mathbb{D}}$, for $n \in \mathbb{N}$ and $\varepsilon \geq 0$, and note that, by (5.43),

$$\operatorname{Spec} A_n(\Delta) \cup \operatorname{Spec} A_n(-\Delta) = \begin{cases} -S_n^+(\Delta) \cup S_n^+(\Delta), & \text{if } n \text{ is even}, \\ -S_n^+(\Delta) \cup S_n^+(\Delta) \cup \{-\Delta, \Delta\}, & \text{if } n \text{ is odd}, \end{cases}$$

so that

$$\widehat{\sigma}_\varepsilon^n(A) = \begin{cases} -S_n^+(\Delta) \cup S_n^+(\Delta) + (\varepsilon+\varepsilon_{n-2})\overline{\mathbb{D}}, & \text{if } n \text{ is even}, \\ -S_n^+(\Delta) \cup S_n^+(\Delta) \cup \{-\Delta, \Delta\} + (\varepsilon+\varepsilon_{n-2})\overline{\mathbb{D}}, & \text{if } n \text{ is odd}. \end{cases} \tag{5.45}$$

Similarly, for $N > n \geq 1$ and $\varepsilon \geq 0$,

$$\sigma_\varepsilon^n(A) = \begin{cases} \{\Delta\} \cup \bigcup_{m=1}^n \left(-S_m^+(\Delta) \cup S_m^+(\Delta)\right) + (\varepsilon+\varepsilon_n)\overline{\mathbb{D}}, & n \text{ even and } N \text{ odd}, \\ \{-\Delta, \Delta\} \cup \bigcup_{m=1}^n \left(-S_m^+(\Delta) \cup S_m^+(\Delta)\right) + (\varepsilon+\varepsilon_n)\overline{\mathbb{D}}, & \text{otherwise}. \end{cases} \tag{5.46}$$

Recall that, for $\varepsilon \geq 0$, the τ-method inclusion sets for $\operatorname{Spec}_\varepsilon A$ are $\Sigma_\varepsilon^n(A)$, where $\Sigma_\varepsilon^n(A) := \sigma_\varepsilon^n(A)$, for $n = 1, 2$, $:= \widehat{\sigma}_\varepsilon^n(A) \cap \sigma_\varepsilon^n(A)$, for $n > 2$.

Similarly to (5.45), it follows from Theorem 3.3 that

$$\operatorname{Spec}_\varepsilon A_\infty \subset \Sigma_\varepsilon^n(A_\infty) = \begin{cases} -S_n^+(\Delta) \cup S_n^+(\Delta) + (\varepsilon+\varepsilon_n)\overline{\mathbb{D}}, & n \text{ even}, \\ -S_n^+(\Delta) \cup S_n^+(\Delta) \cup \{-\Delta, \Delta\} + (\varepsilon+\varepsilon_n)\overline{\mathbb{D}}, & n \text{ odd}. \end{cases}$$

Note that $\Sigma^n_\varepsilon(A_\infty) \subset \Sigma^n_\varepsilon(A)$, for $\varepsilon \ge 0$ and $N > n \ge 2$, so that, recalling the characterisation above of $\operatorname{Spec} A_\infty$ and (5.42),

$$-S^+_m(\Delta) \cup S^+_m(\Delta) \subset \left[-\sqrt{4+\Delta^2}, -|\Delta|\right] \cup \left[|\Delta|, \sqrt{4+\Delta^2}\right] \subset \Sigma^n_0(A),$$

for $1 \le m \le n$. From these inclusions, arguing as in the proof of (5.20), we get that, for $N > n \ge 1$ and $\Delta \in \mathbb{R}$,

$$\mathbb{R} \cap \Sigma^n_0(A) = -\widehat{S}^+_{n,N}(-\Delta) \cup \widehat{S}^+_{n,N}(\Delta), \tag{5.47}$$

where

$$\widehat{S}^+_{n,N}(\Delta) := \left[c^-_{n,N}(\Delta), c^+_n(\Delta)\right] \supset \left[|\Delta|, \sqrt{4+\Delta^2}\right],$$

with $c^+_1(\Delta) := |\Delta| + \varepsilon_1$, $c^+_n(\Delta) := \max S^+_n(|\Delta|) + \varepsilon_n$, for $n \ge 2$, and

$$c^-_{n,N}(\Delta) := \begin{cases} |\Delta| - \varepsilon_n, & \text{if } n \text{ is odd or} \\ & n = 2 \text{ and } (N \text{ even or } \Delta > 0), \\ \min S^+_n(|\Delta|) - \varepsilon_n, & \text{if } n \text{ even, } N \text{ odd, and } \Delta \le 0, \\ \max(\min S^+_n(|\Delta|) - \varepsilon_{n-2}, |\Delta| - \varepsilon_n), & \text{otherwise.} \end{cases}$$

Note that $\max S^+_n(\Delta) = \sqrt{4\cos^2(\pi/(n+1)) + \Delta^2}$ and, in the case that n is even, $\min S^+_n(\Delta) = \sqrt{4\cos^2(n\pi/(2n+2)) + \Delta^2}$, so that, as $n \to \infty$,

$$c^+_n(\Delta) = \sqrt{4+\Delta^2} + \frac{2\pi}{n} + O(n^{-2}), \quad c^-_{n,N}(\Delta) = |\Delta| - \frac{2\pi}{n} + O(n^{-2}). \tag{5.48}$$

5.3.2. Comparison with Gershgorin

$\mathbb{R} \cap \Sigma^n_0(A)$, given by (5.47), is the family of τ-method inclusion sets. Since, as discussed above, $\operatorname{Spec} A \xrightarrow{\text{H}} -\widehat{S}^+_\infty(|\Delta|) \cup \widehat{S}^+_\infty(|\Delta|)$ as $M \to \infty$, where

$$\widehat{S}^+_\infty(\Delta) := [\Delta, \sqrt{4+\Delta^2}], \qquad \Delta \ge 0,$$

the above asymptotics for $c^+_n(\Delta)$ and $c^-_{n,N}(\Delta)$ make clear that $\mathbb{R} \cap \Sigma^n_0(A)$ is a sharp inclusion set for $\operatorname{Spec} A$ in the limit $M, n \to \infty$. By contrast, where G is the Gershgorin theorem inclusion set (1.10),

$$\mathbb{R} \cap G = \{-\Delta, \Delta\} + [-2, 2], \tag{5.49}$$

if $M \ge 3$. This is a sharp bound for $\operatorname{Spec} A$ only in the discrete Laplacian case $\Delta = 0$. In particular, $\operatorname{Spec} A \cap (-|\Delta|, |\Delta|) = \emptyset$, for $\Delta \ne 0$, while $(-|\Delta|, |\Delta|) \subset \mathbb{R} \cap G$ for $|\Delta| \le 2$, $(-|\Delta| + 2] \cup [-2 + |\Delta|) \subset \mathbb{R} \cap G$ for $|\Delta| \ge 2$.

As we did in Remark 5.4 for $\Delta = 0$, let us make comparison also with the block matrix Gershgorin theorem, discussed in Remark 1.2. Recall that $A = A_M(\Delta)$, for some

$M > 1$ and $\Delta \in \mathbb{R}$, and write A in the block form (1.1) for some $1 < N \le M$, so that $a_{ij} \in \mathbb{C}^{m_i \times m_j}$, for $i, j = 1, \dots, N$. As in Remark 5.4, we compute the inclusion set (1.9) given by the block-matrix Gershgorin when the matrix norms are the 2-norm. A is block-tridiagonal, with $a_{ii} = A_{m_i}((-1)^{r_i}\Delta)$, for $i = 1, \dots, N$, where $r_1 = 0$ and $r_i = \sum_{j=1}^{i-1} m_j$, $i > 1$. Each non-zero off-diagonal block a_{ij} has a single non-zero entry with value 1, so $\|a_{i,i+1}\| = \|a_{i+1,i}\| = 1$, $i = 1, \dots, N-1$. Thus (1.9) reads in this case (cf. Remark 5.4)

$$\operatorname{Spec} A \ \subset \ G^N = \big(\overline{\mathbb{D}} + (\operatorname{Spec} a_{1,1} \cup \operatorname{Spec} a_{N,N})\big) \cup \left(2\overline{\mathbb{D}} + \bigcup_{k=2}^{N-1} \operatorname{Spec} a_{k,k}\right).$$

As noted for the case $\Delta = 0$ in Remark 5.4, this seems no better than the inclusion set G provided by the standard Gershgorin theorem, which it reduces to if $N = M$. In particular, as $M \to \infty$ with $N > 2$ and $\min_{1 \le k \le N} m_k \to \infty$, it is clear from (5.42) and (5.43) that

$$G^N \overset{\mathrm{H}}{\to} -\widehat{S}^+_\infty(|\Delta|) \cup \widehat{S}^+_\infty(|\Delta|) + 2\overline{\mathbb{D}} \quad \text{so} \quad \mathbb{R} \cap G^N \overset{\mathrm{H}}{\to} -\widehat{S}^+_\infty(|\Delta|) \cup \widehat{S}^+_\infty(|\Delta|) + [-2, 2].$$

For example, if $\Delta = 1.5$, $\operatorname{Spec} A \overset{\mathrm{H}}{\to} [-2.5, -1.5] \cup [1.5, 2.5]$ as $M \to \infty$, the τ method inclusion set $\Sigma^n_0(A)$ tends to the same limit as $n \to \infty$, while the $\min_{1 \le k \le N} m_k \to \infty$ limit of the block-Gershgorin inclusion set is $-\widehat{S}^+_\infty(|\Delta|) \cup \widehat{S}^+_\infty(|\Delta|) + [-2, 2] = [-4.5, 4.5]$ and the standard Gershgorin inclusion is $\mathbb{R} \cap G = [-3.5, 3.5]$.

In more detail, consider the case that $n \in \mathbb{N}$ divides M and $M \ge 3$, and set $N = M/n$ and $m_i = n$, $i = 1, \dots, N$. Then $\mathbb{R} \cap G^N = \mathbb{R} \cap G = \{-\Delta, \Delta\} + [-2, 2]$ if $n = 1$. If $n \ge 2$ and $N > 2$ it follows from the above representation for $\mathbb{R} \cap G^N$ and (5.43) that, where $\operatorname{Spec} A^+_\infty$ is given by (5.44), $\operatorname{Spec} A^+_\infty \subset G^N$. Thus, arguing as we did to get (5.22), and restricting attention to the case $\Delta \ge 0$ and $N > 3$, we see that

$$\mathbb{R} \cap G^N = -G^r_n \cup G^r_n \tag{5.50}$$

where

$$G^r_n := \begin{cases} [\min S^+_n(\Delta) - 2, \max S^+_n(\Delta) + 2]\,, & n \text{ even}, \\ [\Delta - 2, \max S^+_n(\Delta) + 2]\,, & n \text{ odd}. \end{cases}$$

In Fig. 5.3 we show these inclusion sets for $\operatorname{Spec} A = \operatorname{Spec} A_M(\Delta)$ in the case $\Delta = 1.5$, plotting $\mathbb{R} \cap \Sigma^n_0(A)$, given by (5.47), and $\mathbb{R} \cap G^N$, given by (5.50), against n. These are inclusion sets for $\operatorname{Spec} A$ provided $n < M$ for $\mathbb{R} \cap \Sigma^n_0(A)$ (Theorem 1.1), provided $N = M/n$ is an integer ≥ 4 for $\mathbb{R} \cap G^N$ (see above discussion). For fixed n, $\mathbb{R} \cap G^N = -G^r_n \cap G^r_n$ and $\mathbb{R} \cap \Sigma^n_0(A)$ are both based on computing eigenvalues of principal submatrices of A of order n. But, while $\mathbb{R} \cap \Sigma^n_0(A)$ is monotonically decreasing, converging to a sharp inclusion set for $\operatorname{Spec} A$, $\mathbb{R} \cap G^N$ becomes larger as n increases, approaching the limit $[-4.5, 4.5]$.

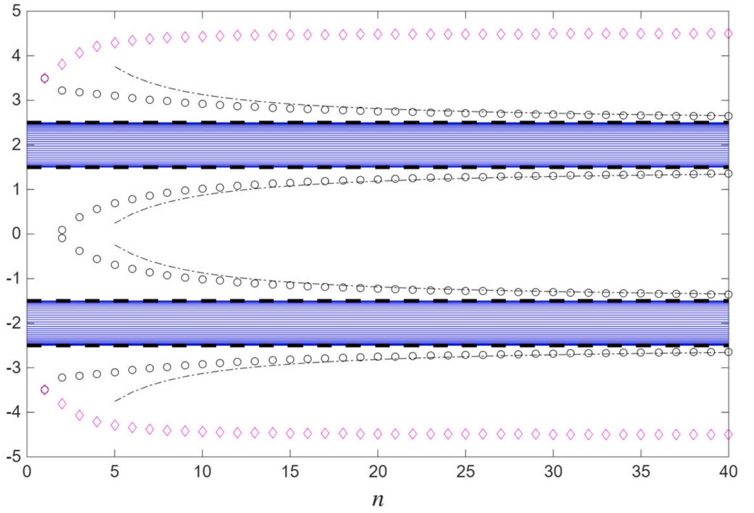

Fig. 5.3. Spec A, for $A = A_M(\Delta)$ with $\Delta = 1.5$ and $M = 60$; the Gershgorin inclusion $\mathbb{R} \cap G$ and its block matrix versions; and the τ-method inclusions sets, $\mathbb{R} \cap \Sigma_0^n(A)$. The horizontal solid lines are the eigenvalues of A and the dashed lines the boundary of Spec $A_\infty^+(\Delta) = [-3.5, -1.5] \cup [1.5, 3.5]$. The circles are $\pm c_n^+(\pm\Delta)$ (top and bottom lines) and $\pm c_{n,M}^-(\pm\Delta)$ (middle two lines); for each $M > n$, Spec $A \subset \mathbb{R} \cap \Sigma_0^n(A) = \left[-c_n^+(-\Delta), -c_{n,M}^-(-\Delta)\right] \cup \left[c_{n,M}^-(\Delta), c_n^+(\Delta)\right]$; see §5.3.1. The values of $c_{1,M}^-(\pm\Delta)$ (not plotted) are negative, so $\mathbb{R} \cap \Sigma_0^1(A) = [-c_1^+(-\Delta), c_1^+(\Delta)] = [-3.5, 3.5] = \mathbb{R} \cap G$, where G is the Gershgorin inclusion set (1.10). The magenta diamonds are $\pm B_n$, where $B_n := \max S_n^+(\Delta) + 2$. By (5.50), $\mathbb{R} \cap G^N = [-B_n, B_n]$ for this value of Δ, where G^N is the block matrix Gershgorin inclusion set (1.8) with blocks of size n. The dash-dot lines are the first two terms of the asymptotic expansions (5.48) for $\pm c_n^+(\pm\Delta)$ and $\pm c_{n,M}^-(\pm\Delta)$.

6. Conclusion and possible extensions

In this paper we have, extending recent bi-infinite matrix results [3], derived sequences of inclusion sets (our τ and τ_1 methods) for the spectra and pseudospectra of a finite matrix A (Theorems 1.1 and 1.6). Each inclusion set in each sequence is expressed as a union of pseudospectra of finite submatrices of what we term the tridiagonal part of A when A is written in the block form (1.1), square submatrices for the τ method, rectangular for the τ_1 method. In §5 we have explored the sharpness of these inclusion sets for the case when the finite matrix is a large Toeplitz matrix, showing that both sequences produce sharp inclusion sets for the pseudospectrum of A in the case that A is large (if A is banded for the τ method, has symbol in the Wiener class for the τ_1 method). If A is Hermitian, then our inclusion sets also provide sharp inclusions for the spectrum. Importantly, the inclusions in Theorem 1.6 are two-sided, associated with the use of rectangular rather than square submatrices in the τ_1 method. This is a key proof ingredient for the τ_1-method results for general Toeplitz matrices.

In §5.3 we have extended our convergence results for the τ-method to a class of tridiagonal 2-Toeplitz Hermitian matrices, and used this class and a simple Toeplitz example (Remark 5.4) to demonstrate that our τ-method sequence can be superior to existing block-matrix Gershgorin generalisations. Where G is the classical Gershgorin set, we have also shown for the classes of large Hermitian matrices studied that, while $\mathbb{R} \cap G$ can be a sharp spectrum bound in particular cases, our τ and τ_1 sequences generate sharp inclusions in every case (Remark 5.8, §5.3.2).

There are many directions for further work. Theorems 1.1 and 1.6, and Theorem 3.3 and [3], provide inclusion sets for finite matrices and for bi-infinite matrices, respectively. In work in progress we are constructing similar inclusion set sequences for the semi-infinite case (exemplified by the infinite Toeplitz matrix A^+_∞ in §5.2). Our results for that case, like the results in this paper for the finite matrix case, depend on inclusion sets from [3] for the bi-infinite case, extended in §3. The arguments for the bi-infinite case use, implicitly, that $\mathbb{Z}$ is a group under addition. In other work in progress, with Christian Seifert, we are extending these arguments to general groups; while our results in [3] apply for bi-infinite matrices acting on $\ell^2(\mathbb{Z})$, our new results apply to matrices acting on $\ell^2(G)$, for some Abelian group G. When G is finite, this provides a route to spectral inclusion sets for certain finite matrices, complementing the work in this paper.

Other possible directions for future research include:

1. Work on the (efficient) implementation of our inclusion sets for general finite matrices, the development of associated open source codes, and large-scale computational experiments investigating the sharpness of our inclusion sets, and the relative merits of our τ and τ_1 families.
2. Complementing 1, further theoretical studies related to the sharpness of our inclusion sets for large matrices. A key tool in our §5.2 investigations for large Toeplitz matrices was (5.25), capturing the asymptotics of pseudospectra for large matrix size. Another matrix class, of interest for applications in mathematical physics (see, e.g., [22, §VIII] and [3, §8.4]) and where we understand the asymptotics of pseudospectra, is the class of random tridiagonal matrices (see [4, Remark 4.17]).
3. Our results in [3] and in Theorem 3.3 for bi-infinite matrices allow operator-valued matrix entries. Such matrices arise, for example, in the study of integral operators on $L^2(\mathbb{R})$ via a standard *discretisation* (e.g., [16, §1.2.3]) which replaces the operator on $L^2(\mathbb{R})$ by a unitarily equivalent operator on $\ell^2(\mathbb{Z}, L^2[0,1])$, to which Theorem 3.3 applies. Similarly, for any $N \in \mathbb{N}$, an integral operator on $L^2[a,b]$, for some finite interval $[a,b]$, is unitarily equivalent to an operator on $\ell^2(\mathbb{I}, L^2[0,h])$, where $\mathbb{I} = \{1,\dots,N\}$ and $h = (b-a)/N$, to which versions of Theorems 1.1 and 1.6 could be applied that allowed operator-valued entries in (1.1). Extensions to such cases should be relatively straightforward, starting from Theorem 3.3 and the results in [3]. Similarly, the block-matrix versions of Gershgorin's theorem, (1.8) and (1.9), have been extended to matrices with operator-valued entries in [20].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors acknowledge very useful feedback from the two anonymous referees that has led to significant improvements to the paper.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.laa.2025.11.017.

Data availability

The Matlab codes used to produce the data displayed in Figs. 5.1-5.3, and to produce the figures, are provided in the supplementary materials.

References

[1] A. Böttcher, S.M. Grudsky, Spectral Properties of Banded Toeplitz Matrices, SIAM, Philadelphia, 2005.
[2] A. Böttcher, B. Silbermann, Introduction to Large Truncated Toeplitz Matrices, Springer Verlag, New York, 1999.
[3] S.N. Chandler-Wilde, R. Chonchaiya, M. Lindner, On spectral inclusion sets and computing the spectra and pseudospectra of bounded linear operators, J. Spectr. Theory 14 (2024) 719–804.
[4] S.N. Chandler-Wilde, M. Lindner, Coburn's lemma and the finite section method for random Jacobi operators, J. Funct. Anal. 270 (2016) 802–841.
[5] J.B. Conway, A Course in Functional Analysis, Springer Verlag, New York, 1985.
[6] E.B. Davies, Linear Operators and Their Spectra, CUP, 2007.
[7] A. Dyachenko, M. Tyaglov, On the spectrum of tridiagonal matrices with two-periodic main diagonal, Spec. Matrices 12 (2024) 20240009.
[8] D.G. Feingold, R.S. Varga, Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem, Pac. J. Math. 12 (1962) 1241–1250.
[9] M. Fiedler, V. Pták, Generalized norms of matrices and the localisation of the spectrum, Czechoslov. Math. J. 12 (1962) 558–571.
[10] F. Gabel, D. Gallaun, J. Großmann, M. Lindner, R. Ukena, Finite sections of periodic Schrödinger operators, in: Y. Choi, M. Daws, G. Blower (Eds.), Operators, Semigroups, Algebras and Function Theory. IWOTA 2021, in: Operator Theory: Advances and Applications, vol. 292, Birkhäuser, Cham, 2023.
[11] S. Gershgorin, Über die Abgrenzung der Eigenwerte einer Matrix, Izv. Akad. Nauk. Uzb. SSR Otd. Fiz.-Mat. Nauk 7 (1931) 749–754.
[12] M.J.C. Gover, S. Barnett, Inversion of Toeplitz matrices which are not strongly nonsingular, IMA J. Numer. Anal. 5 (1994) 101–110.
[13] M.J.C. Gover, The eigenproblem of a tridiagonal 2-Toeplitz matrix, Linear Algebra Appl. 197–198 (1994) 63–78.
[14] K.E. Gustafson, D.K.M. Rao, Numerical Range: the Field of Values of Linear Operators and Matrices, Springer-Verlag, New York, 1997.
[15] R. Hagen, S. Roch, B. Silbermann, C^*-Algebras and Numerical Analysis, Marcel Dekker, Inc., New York, Basel, 2001.
[16] M. Lindner, Infinite Matrices and Their Finite Sections: An Introduction to the Limit Operator Method, Frontiers in Mathematics, Birkhäuser, 2006.
[17] A.M. Ostrowski, On some metrical properties of operator matrices and matrices partitioned into blocks, J. Math. Anal. Appl. 2 (1961) 161–209.

[18] L. Reichel, L.N. Trefethen, Eigenvalues and pseudo-eigenvalues of Toeplitz matrices, Linear Algebra Appl. 162–164 (1992) 153–185.
[19] S.A. Sauter, C. Schwab, Boundary Element Methods, Springer Verlag, Berlin, 2011.
[20] H.N. Salas, Gershgorin's theorem for matrices of operators, Linear Algebra Appl. 291 (1999) 15–36.
[21] E. Shargorodsky, On the level sets of the resolvent norm of a linear operator, Bull. Lond. Math. Soc. 40 (2008) 493–504.
[22] L.N. Trefethen, M. Embree, Spectra and Pseudospectra: the Behavior of Nonnormal Matrices and Operators, Princeton University Press, Princeton, NJ, 2005.
[23] R.S. Varga, Geršgorin and His Circles, Springer, New York, 2004.
[24] T.G. Wright, L.N. Trefethen, Pseudospectra of rectangular matrices, IMA J. Numer. Anal. 22 (2002) 501–519.