On vector-valued functions and the ε-product

Habilitationsschrift

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Abstract

This habilitation thesis centres on linearisation of vector-valued functions which means that vector-valued functions are represented by continuous linear operators. The first question we face is which vector-valued functions may be represented by continuous linear operators. We study this problem in the framework of ε-products and give sufficient conditions in Chapter 3 and 4 when a space $\mathcal{F}(\Omega, E)$ of vector-valued functions on a set Ω coincides (up to an isomorphism) with the ε-product $\mathcal{F}(\Omega)\varepsilon E$ of a corresponding space of scalar-valued functions $\mathcal{F}(\Omega)$ and the codomain E which is usually an infinite-dimensional locally convex Hausdorff space. The ε-product $\mathcal{F}(\Omega)\varepsilon E$ is a space of continuous linear operators from the dual space $\mathcal{F}(\Omega)'$ to E.

Once we have a representation of a space $\mathcal{F}(\Omega, E)$ of vector-valued functions by an ε-product $\mathcal{F}(\Omega)\varepsilon E$, we have access to the rich theory of continuous linear operators which allows us to lift results that are known for the scalar-valued case to the vector-valued case. The whole Chapter 5, which spans more than half of this thesis, is dedicated to this lifting mechanism. But we should point out that this is not only about transferring results from the scalar-valued to the vector-valued case. The results in the vector-valued case encode additional information for the scalar-valued case as well, e.g. we may deduce from the solvability of a linear partial differential equation in the vector-valued case affirmative answers on the parameter dependence of solutions in the scalar-valued case (see Section 5.1).

In Section 5.2 we give a unified approach to handle the problem of extending functions with values in E, which have weak extensions in $\mathcal{F}(\Omega)$, to functions in the vector-valued counterpart $\mathcal{F}(\Omega, E)$ of $\mathcal{F}(\Omega)$. We present different extension theorems depending on the topological properties of the spaces $\mathcal{F}(\Omega)$ and E. These theorems also cover weak-strong principles. In particular, we study weak-strong principles for continuously partially differentiable functions of finite order in Section 5.3 and improve the well-known weak-strong principles of Grothendieck and Schwartz. We use our results on the extension of vector-valued functions to derive Blaschke's convergence theorem for several spaces of vector-valued functions and Wolff's theorem for the description of dual spaces of several function spaces $\mathcal{F}(\Omega)$ in Section 5.4 and 5.5. Starting from the observation that every scalar-valued holomorphic function has a local power series expansion and that this is still true for holomorphic functions with values in E if E is locally complete, we develop a machinery which is based on linearisation and Schauder decomposition to transfer known series expansions from scalar-valued to vector-valued functions in Section 5.6. Especially, we apply this machinery to derive Fourier expansions for E-valued Schwartz functions and $\mathcal{C}^\infty$-smooth functions on $\mathbb{R}^d$ that are 2π-periodic in each variable. The last section of Chapter 5 is devoted to the representation of spaces $\mathcal{F}(\Omega, E)$ of vector-valued functions by sequence spaces, which can be used to identify the coefficient spaces of the series expansions from the preceding section, if one knows the coefficient space in the scalar-valued case. Furthermore, we give several new conditions on the Pettis-integrability of vector-valued functions in Appendix A.2, which are, for instance, needed for the Fourier expansions in Section 5.6.

Kurzfassung

Im Mittelpunkt dieser Habilitationsschrift steht die Linearisierung vektorwertiger Funktionen, d. h. vektorwertige Funktionen sollen durch stetige lineare Operatoren dargestellt werden. Die erste Frage, der man sich stellen muss, ist, welche vektorwertigen Funktionen durch stetige lineare Operatoren dargestellt werden können. Wir untersuchen dieses Problem im Rahmen von ε-Produkten und geben hinreichende Bedingungen in Kapitel 3 und 4 an, wann ein Raum $\mathcal{F}(\Omega, E)$ von vektorwertigen Funktionen auf einer Menge Ω mit dem ε-Produkt $\mathcal{F}(\Omega)\varepsilon E$ eines entsprechenden Raums skalarwertiger Funktionen $\mathcal{F}(\Omega)$ und des Wertebereichs E (bis auf Isomorphie) übereinstimmt. Hierbei ist E üblicherweise ein unendlich-dimensionaler lokalkonvexer Hausdorff Raum. Das ε-Produkt $\mathcal{F}(\Omega)\varepsilon E$ ist ein Raum stetiger linearer Operatoren, die vom Dualraum $\mathcal{F}(\Omega)'$ nach E abbilden.

Sobald wir eine Darstellung eines Raums $\mathcal{F}(\Omega, E)$ von vektorwertigen Funktionen durch ein ε-Produkt $\mathcal{F}(\Omega)\varepsilon E$ gewonnen haben, ist es uns möglich die reichhaltige Theorie der stetigen linearen Operatoren zu nutzen, die es uns erlaubt, Ergebnisse, die für den skalarwertigen Fall bekannt sind, auf den vektorwertigen Fall zu übertragen. Das gesamte Kapitel 5, das mehr als die Hälfte dieser Arbeit einnimmt, widmet sich diesem Übertragungsmechanismus. Es sei jedoch darauf hingewiesen, dass es hier nicht nur um die Übertragung von Ergebnissen aus dem skalarwertigen auf den vektorwertigen Fall geht. Die Ergebnisse im vektorwertigen Fall beinhalten auch zusätzliche Informationen für den skalarwertigen Fall, z. B. können wir aus der Lösbarkeit einer linearen partiellen Differentialgleichung im vektorwertigen Fall Antworten auf die Frage nach der Parameterabhängigkeit der Lösungen im skalarwertigen Fall ableiten (siehe Abschnitt 5.1).

In Abschnitt 5.2 stellen wir einen einheitlichen Ansatz zur Lösung des Fortsetzungsproblems von Funktionen mit Werten in E, die schwache Fortsetzungen in $\mathcal{F}(\Omega)$ haben, zu Funktionen im vektorwertigen Gegenstück $\mathcal{F}(\Omega, E)$ von $\mathcal{F}(\Omega)$ vor. Wir präsentieren verschiedene Fortsetzungssätze in Abhängigkeit von den topologischen Eigenschaften der Räume $\mathcal{F}(\Omega)$ und E. Diese Sätze decken auch schwach-stark Prinzipien ab. Insbesondere untersuchen wir schwach-stark Prinzipien für endlich oft stetig partiell differenzierbare Funktionen in Abschnitt 5.3 und verbessern die bekannten schwach-starken Prinzipien von Grothendieck und Schwartz. Zudem leiten wir von unseren Ergebnissen zur Fortsetzung vektorwertiger Funktionen den Konvergenzsatz von Blaschke für diverse Räume vektorwertiger Funktionen ab und übertragen den Satz von Wolff auf Dualräume mehrerer Funktionenräume $\mathcal{F}(\Omega)$ in den Abschnitten 5.4 und 5.5. Ausgehend von der Beobachtung, dass jede skalarwertige holomorphe Funktion eine lokale Potenzreihenentwicklung hat und dass dies auch für holomorphe Funktionen mit Werten in E gilt, wenn E lokal vollständig ist, entwickeln wir einen Mechanismus, der auf Linearisierung und Schauder-Zerlegung basiert, um in Abschnitt 5.6 bekannte Reihenentwicklungen von skalarwertigen auf vektorwertige Funktionen zu erweitern. Insbesondere wenden wir diesen Mechanismus an, um Fourier-Entwicklungen für E-wertige Schwartz-Funktionen und $\mathcal{C}^{\infty}$-glatte Funktionen auf $\mathbb{R}^d$, die 2π-periodisch in jeder Variablen sind, zu erhalten. Der letzte Abschnitt von Kapitel 5 ist der Darstellung von

Räumen $\mathcal{F}(\Omega, E)$ vektorwertiger Funktionen durch Folgenräume gewidmet, was man dazu nutzen kann, die Koeffizientenräume der Reihenentwicklungen aus dem vorangegangenen Abschnitt zu bestimmen, sofern man den Koeffizientenraum im skalarwertigen Fall kennt. Außerdem legen wir mehrere neue Bedingungen für die Pettis-Integrierbarkeit von vektorwertigen Funktionen in Anhang A.2 dar, die z. B. für die Fourier-Entwicklungen in Abschnitt 5.6 benötigt werden.

Contents

CHAPTER 1

Introduction

This work is dedicated to a classical topic, namely, the linearisation of weighted spaces of vector-valued functions. The setting we are interested in is the following. Let $\mathcal{F}(\Omega)$ be a locally convex Hausdorff space of functions from a non-empty set Ω to a field $\mathbb{K}$ and E be a locally convex Hausdorff space over $\mathbb{K}$. The ε-product of $\mathcal{F}(\Omega)$ and E is defined as the space of linear continuous operators

$$\mathcal{F}(\Omega)\varepsilon E := L_e(\mathcal{F}(\Omega)'_\kappa, E)$$

equipped with the topology of uniform convergence on equicontinuous subsets of the dual $\mathcal{F}(\Omega)'$ which itself is equipped with the topology of uniform convergence on absolutely convex compact subsets of $\mathcal{F}(\Omega)$. Suppose that the point-evaluation functionals δ_x, $x \in \Omega$, belong to $\mathcal{F}(\Omega)'$ and that there is a locally convex Hausdorff space $\mathcal{F}(\Omega, E)$ of E-valued functions on Ω such that the map

$$S\colon \mathcal{F}(\Omega)\varepsilon E \to \mathcal{F}(\Omega, E),\ u \longmapsto [x \mapsto u(\delta_x)], \tag{1}$$

is well-defined. The main question we want to answer reads as follows. When is $\mathcal{F}(\Omega)\varepsilon E$ a linearisation of $\mathcal{F}(\Omega, E)$, i.e. when is S an isomorphism?

In [15, 16, 17] Bierstedt treats the space $\mathcal{CV}(\Omega, E)$ of continuous functions on a completely regular Hausdorff space Ω weighted with a Nachbin-family $\mathcal{V}$ and its topological subspace $\mathcal{CV}_0(\Omega, E)$ of functions that vanish at infinity in the weighted topology. He derives sufficient conditions on Ω, $\mathcal{V}$ and E such that the answer to our question is affirmative, i.e. S is an isomorphism. Schwartz answers this question for several weighted spaces of k-times continuously partially differentiable functions on $\Omega = \mathbb{R}^d$ like the Schwartz space in [158, 159] for quasi-complete E with regard to vector-valued distributions. Grothendieck treats the question in [83], mainly for nuclear $\mathcal{F}(\Omega)$ and complete E. In [99, 100, 101] Komatsu gives a positive answer for ultradifferentiable functions of Beurling or Roumieu type and sequentially complete E with regard to vector-valued ultradistributions. For the space of k-times continuously partially differentiable functions on open subsets Ω of infinite dimensional spaces equipped with the topology of uniform convergence of all partial derivatives up to order k on compact subsets of Ω sufficient conditions for an affirmative answer are deduced by Meise in [129]. For holomorphic functions on open subsets of infinite dimensional spaces a positive answer is given in [52] by Dineen. Bonet, Frerick and Jordá show in [30] that S is an isomorphism for certain closed subsheaves of the sheaf $\mathcal{C}^\infty(\Omega, E)$ of smooth functions on an open subset $\Omega \subset \mathbb{R}^d$ with the topology of uniform convergence of all partial derivatives on compact subsets of Ω and locally complete E which, in particular, covers the spaces of harmonic and holomorphic functions.

An important application of linearisation is within the field of partial differential equations. Let E be a linear space of functions on a set U and $P(\partial)\colon \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$ a linear partial differential operator with $\mathcal{C}^\infty$-smooth coefficients where $\mathcal{C}^\infty(\Omega) := \mathcal{C}^\infty(\Omega, \mathbb{K})$. We call the elements of U parameters and say that a family $(f_\lambda)_{\lambda\in U}$ in $\mathcal{C}^\infty(\Omega)$ depends on a parameter w.r.t. E if the map $\lambda \mapsto f_\lambda(x)$ is an element of E for every $x \in \Omega$. The question of parameter dependence is whether for every family $(f_\lambda)_{\lambda\in U}$ in $\mathcal{C}^\infty(\Omega)$ depending on a parameter w.r.t. E there is a family $(u_\lambda)_{\lambda\in U}$ in $\mathcal{C}^\infty(\Omega)$ with the same kind of parameter dependence which solves the partial differential equation

$$P(\partial)u_\lambda = f_\lambda, \quad \lambda \in U.$$

In particular, it is the question of $\mathcal{C}^k$-smooth (holomorphic, distributional, etc.) parameter dependence if E is the space $\mathcal{C}^k(U)$ of k-times continuously partially differentiable functions on an open set $U \subset \mathbb{R}^d$ (the space $\mathcal{O}(U)$ of holomorphic functions on an open set $U \subset \mathbb{C}$, the space of distributions $\mathcal{D}(V)'$ on an open set $V \subset \mathbb{R}^d$ where $U = \mathcal{D}(V)$, etc.). The question of parameter dependence w.r.t. E has an affirmative answer for several locally convex Hausdorff spaces E due to tensor product techniques and splitting theory. Indeed, the answer is affirmative if the topology of E is stronger than the topology of pointwise convergence on U and

$$P(\partial)^E: \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\Omega, E)$$

is surjective where $P(\partial)^E$ is the version of $P(\partial)$ for E-valued functions. The operator $P(\partial)^E$ is surjective if its version $P(\partial)$ for scalar-valued functions is surjective, for instance, if $P(\partial)$ is elliptic, and E is a Fréchet space. This is a consequence of Grothendieck's theory of tensor products [83], the nuclearity of $\mathcal{C}^\infty(\Omega)$ and the isomorphism $\mathcal{C}^\infty(\Omega, E) \cong \mathcal{C}^\infty(\Omega)\varepsilon E$ for locally complete E. Thanks to the splitting theory of Vogt for Fréchet spaces [173] and of Bonet and Domański for PLS-spaces [54] we even have in case of an elliptic $P(\partial)$ that $P(\partial)^E$ for $d > 1$ is surjective if $E := F_b'$ where F is a Fréchet space satisfying the condition (DN) or if E is an ultrabornological PLS-space having the property (PA) since $\ker P(\partial)$ has the property (Ω). In particular, these three results cover the cases that $E = \mathcal{C}^k(U)$, $\mathcal{O}(U)$ or $\mathcal{D}(V)'$. Of course, this technique to answer the question of parameter dependence is not restricted to linear partial differential operators or the space $\mathcal{C}^\infty(\Omega)$.

Another application of linearisation lies in the problem of extending a vector-valued function $f: \Lambda \to E$ from a subset $\Lambda \subset \Omega$ to a locally convex Hausdorff space E if the scalar-valued functions $e' \circ f$ are extendable for each continuous linear functional e' from certain linear subspaces G of E' under the constraint of preserving the properties, like holomorphy, of the scalar-valued extensions. This problem was considered, among others, by Grothendieck [82, 83], Bierstedt [17], Gramsch [77], Grosse-Erdmann [79, 81], Arendt and Nikolski [6, 7, 8], Bonet, Frerick, Jordá and Wengenroth [30, 69, 70, 92, 93]. Even the simple case $\Lambda = \Omega$ and $G = E'$ is interesting and an affirmative answer is called a weak-strong principle.

Our goal is to give a unified and flexible approach to linearisation which is able to handle new examples and covers the already known examples.

Organisation of this thesis

After fixing some notions and preliminaries on locally convex Hausdorff spaces, continuous linear operators and continuously partially differentiable functions in **Chapter 2**, we study the problem of linearisation in **Chapter 3**. In Section 3.1 we introduce our standard example of spaces $\mathcal{F}(\Omega, E)$ that we consider. Namely, spaces of functions $\mathcal{FV}(\Omega, E)$ from Ω to E which are subspaces of sections of domains of linear operators T^E on E^Ω, and whose topology is generated by a family of weight functions $\mathcal{V}$. These spaces cover many examples of classical spaces of functions appearing in analysis like the mentioned ones and an example of the operators T^E are the partial derivative operators. Then we exploit the structure of our spaces to describe a sufficient condition, which we call consistency, on the interplay of the pairs of operators $(T^E, T^\mathbb{K})$ and the map S such that S becomes an isomorphism into, i.e. an isomorphism to its range (see Theorem 3.1.12).

In Section 3.2 we tackle the problem of surjectivity of S. In our main Theorem 3.2.4 and its Corollary 3.2.5 we give several sufficient conditions on the pairs of operators $(T^E, T^{\mathbb{K}})$ and the spaces involved such that $S: \mathcal{FV}(\Omega)\varepsilon E \to \mathcal{FV}(\Omega, E)$ is an isomorphism. Looking at the pair of partial differential operators $(P(\partial)^E, P(\partial))$ considered above, these conditions allow us to express $P(\partial)^E$ as $P(\partial)^E = S \circ (P(\partial)\varepsilon \operatorname{id}_E) \circ S^{-1}$ where $P(\partial)\varepsilon \operatorname{id}_E$ is the ε-product of $P(\partial)$ and the identity id_E on E. Hence it becomes obvious that the surjectivity of $P(\partial)^E$ is equivalent to the surjectivity of $P(\partial)\varepsilon \operatorname{id}_E$. This is used in [105, 109, 112, 116, 119] in the case of the Cauchy–Riemann operator $P(\partial) = \overline{\partial}$ on spaces of smooth functions with exponential growth.

In **Chapter 4** we take a closer look at the notion of consistency of $(T^E, T^{\mathbb{K}})$. In Section 4.1 we characterise several properties of the functions $S(u)$ for $u \in \mathcal{FV}(\Omega)\varepsilon E$ that are inherited from the elements of $\mathcal{FV}(\Omega)$.

Section 4.2 is devoted to several concrete examples of spaces of vector-valued functions that may be linearised by S and which we use for our applications in the forthcoming sections and chapters.

In Section 4.3 we answer in several cases the question whether given a continuous linear functional $T^{\mathbb{K}}$ on $\mathcal{F}(\Omega)$ there is always a continuous linear operator T^E on $\mathcal{F}(\Omega, E)$ such that $(T^E, T^{\mathbb{K}})$ is consistent. This is closely related to Riesz–Markov–Kakutani theorems for $T^{\mathbb{K}}$, which we transfer to the vector-valued case.

Chapter 5 is dedicated to applications of linearisation. In Section 5.1 we come back to our problem of parameter dependence. We show in our main Theorem 5.1.2 of this section how to use linearisations to transfer properties like injectivity, surjectivity or bijectivity from a map $T^{\mathbb{K}}: \mathcal{F}_1(\Omega_1) \to \mathcal{F}_2(\Omega_2)$ to the corresponding map $T^E: \mathcal{F}_2(\Omega_1, E) \to \mathcal{F}_2(\Omega_2, E)$ if the pair $(T^E, T^{\mathbb{K}})$ is consistent under suitable assumptions on the spaces involved. Besides the problem of parameter dependence for (hypo)elliptic linear partial differential operators (see Corollary 5.1.3), we deduce a vector-valued version of the Borel–Ritt theorem (see Theorem 5.1.4) from this main theorem and give sufficient conditions under which the Fourier transformation $\mathfrak{F}^{\mathbb{C}}: \mathcal{S}_\mu(\mathbb{R}^d) \to \mathcal{S}_\mu(\mathbb{R}^d)$ on the Beurling–Björck space is still an isomorphism in the vector-valued case and may be decomposed as $\mathfrak{F}^E = S \circ (\mathfrak{F}^{\mathbb{C}}\varepsilon \operatorname{id}_E) \circ S^{-1}$ (see Theorem 5.1.5).

In Section 5.2 we present a general approach to the extension problem considered above for a large class of function spaces $\mathcal{F}(\Omega, E)$ if the map S is an isomorphism into. The spaces we treat are of the kind that $\mathcal{F}(\Omega)$ belongs to the class of semi-Montel, Fréchet–Schwartz or Banach spaces, or that E is a semi-Montel space. Apart from linearisation and consistency, the main ingredient of this approach is to view the set $\Lambda \subset \Omega$ from which we want to extend our functions as a set of functionals $\{\delta_x \mid x \in \Lambda\}$. This view allows us to generalise the extension problem in Question 5.2.1 by swapping this set of functionals by other functionals, which opens up new possibilities in applications that we explore in Section 5.3, Section 5.4, Section 5.5 and Section 5.7. In the extension problem we always have to balance the sets Λ from which we extend our functions and the subspaces $G \subset E'$ with which we test. The case of 'thin' sets Λ and 'thick' subspaces G is handled in Section 5.2.1 with main theorems Theorem 5.2.15, Theorem 5.2.20 and Theorem 5.2.29, the converse case of 'thick' sets Λ and 'thin' subspaces G is handled in Section 5.2.2 with main theorems Theorem 5.2.52, Theorem 5.2.63 and Theorem 5.2.69.

In Section 5.3 we consider weak-strong principles for continuously partially differentiable functions of finite order. For locally complete E it is well-known that a function f belongs to $\mathcal{C}^\infty(\Omega, E)$ if and only if $e' \circ f \in \mathcal{C}^\infty(\Omega)$ for all $e' \in E'$ (see e.g. [30, Theorem 9, p. 232]). If $k \in \mathbb{N}_0$, then it is still true that $f \in \mathcal{C}^k(\Omega, E)$ implies $e' \circ f \in \mathcal{C}^k(\Omega)$ for all $e' \in E'$. But the converse is not true anymore. Only a weaker version of this weak-strong principle holds which is due to Grothendieck [82] and Schwartz [158] (see Theorem 5.3.2). Namely, if $k \in \mathbb{N}_0$, E is sequentially complete and $f\colon \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^{k+1}(\Omega)$ for all $e' \in E'$, then $f \in \mathcal{C}^k(\Omega, E)$. Using the results from Section 5.2, we improve this weaker version of the weak-strong principle by allowing E to be locally complete, only testing with less functionals from certain linear subspaces $G \subset E'$ and getting that f does not only belong to $\mathcal{C}^k(\Omega, E)$ but that all partial derivatives of order k are actually locally Lipschitz continuous (see Corollary 5.3.5). If we restrict to semi-Montel spaces E, then even a 'full' weak-strong principle Theorem 5.3.6 holds as in the $\mathcal{C}^\infty$-case.

In Section 5.4 we derive vector-valued Blaschke theorems like Corollary 5.4.2 for several function spaces. This generalises results of Arendt and Nikolski [7] for bounded holomorphic functions and Frerick, Jordá and Wengenroth [70] for bounded functions in the kernel of a hypoelliptic linear partial differential operator. These are results of the form: given a bounded net $(f_\iota)_{\iota \in I}$ in some space $\mathcal{F}_1(\Omega, E)$ of Banach-valued functions which converges pointwise on a certain subset of Ω there is a limit $f \in \mathcal{F}_1(\Omega, E)$ of this net w.r.t. a weaker topology of a linear superspace $\mathcal{F}_2(\Omega, E)$ of $\mathcal{F}_1(\Omega, E)$. In Blaschke's classical convergence theorem [38, Theorem 7.4, p. 219] we have $E = \mathbb{C}$, $\mathcal{F}_1(\Omega, E)$ is the space of bounded holomorphic functions on the open unit disc $\mathbb{D} \subset \mathbb{C}$, $\mathcal{F}_2(\Omega, E)$ is the space of holomorphic functions on $\mathbb{D}$ and the weaker topology is the topology of compact convergence.

In Section 5.5 we present Wolff type descriptions of the dual space of several function spaces $\mathcal{F}(\Omega)$ using linearisation (see Theorem 5.5.1). Wolff's theorem [183, p. 1327] (cf. [81, Theorem (Wolff), p. 402]) phrased in a functional analytic way (see [70, p. 240]) says: if $\Omega \subset \mathbb{C}$ is a domain, then for each $\mu \in \mathcal{O}(\Omega)'$ there are a sequence $(z_n)_{n\in\mathbb{N}}$ which is relatively compact in Ω and a sequence $(a_n)_{n\in\mathbb{N}}$ in the space ℓ^1 of absolutely summable sequences such that $\mu = \sum_{n=1}^{\infty} a_n \delta_{z_n}$.

In Section 5.6 we derive a general result for Schauder decompositions of the ε-product $F\varepsilon E$ for locally convex Hausdorff spaces F and E if F has an equicontinuous Schauder basis (see Theorem 5.6.1). In combination with linearisation and consistency this can be used for $F = \mathcal{F}(\Omega)$ to lift series representations like the power series expansion of holomorphic functions from scalar-valued functions to vector-valued functions (see Corollary 5.6.5). We present several examples in Section 5.6.2, for instance, Fourier expansions in the Schwartz space $\mathcal{S}(\mathbb{R}^d, E)$ and in the space $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E)$ of functions in $\mathcal{C}^\infty(\mathbb{R}^d, E)$ that are 2π-periodic in each variable. In particular, we combine these expansions for locally complete E with the results from Section 5.1 to identify the coefficient spaces of the Fourier expansions in $\mathcal{S}(\mathbb{R}^d, E)$ and $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d, E)$ (see Theorem 5.6.13 and Theorem 5.6.14).

In Section 5.7 an application of our extension results from Section 5.2 is given to represent function spaces $\mathcal{F}(\Omega, E)$ by sequence spaces if one knows such a representation for $\mathcal{F}(\Omega)$ (see Theorem 5.7.1). As examples we treat the space $\mathcal{O}(\mathbb{D}_R(0), E)$ of E-valued holomorphic functions on the disc $\mathbb{D}_R(0) \subset \mathbb{C}$ around 0 with radius $0 < R \leq \infty$ and the multiplier space $\mathcal{O}_M(\mathbb{R}, E)$ of the Schwartz space for locally complete E (see Corollary 5.7.2, Corollary 5.7.3 and Remark 5.7.4).

The first section Appendix A.1 of the **Appendix A** is devoted to the question when the closure of an absolutely convex hull of a set is compact in a locally convex Hausdorff space E and Appendix A.2 to the related question of Pettis-integrability of an E-valued function.

Concerning originality

We note that some parts of chapters or sections are based on our papers and preprints.

- Chapter 3, Section 4.1 and Section 4.2 are based on our paper *Weighted spaces of vector-valued functions and the ε-product* [110] and its extended preprint [106]. Furthermore, Section 4.2 contains results from Sections 3 and 6 of our accepted preprint *Extension of weighted vector-valued functions and sequence space representation* [115] and our paper *Extension of weighted vector-valued functions and weak–strong principles for differentiable functions of finite order* [117] and its extended preprint [120].
- Section 5.1 generalises some results of our papers *Surjectivity of the $\overline{\partial}$-operator between weighted spaces of smooth vector-valued functions* [116] and *Parameter dependence of solutions of the Cauchy–Riemann equation on weighted spaces of smooth functions* [112] and its extended preprint [108].
- Section 5.2, Section 5.3, Section 5.4, Section 5.5 and Section 5.7 are based on our accepted preprint [115] and our paper [117] (and its extended preprint [120]).
- Section 5.6 is based on our paper *Series representations in spaces of vector-valued functions via Schauder decompositions* [114].

Moreover, the introduction Chapter 1 and Chapter 2 on notation and preliminaries are based on the corresponding sections in our papers and preprints [106, 110, 112, 114, 115, 116, 117, 120]. However, not all of the results given in this thesis are already contained in our preprints or papers.

In Chapter 3 the new, i.e. not contained in our preprints or papers, results are Corollary 3.2.5 (ii), Example 3.2.7 e)+f), Example 3.2.9 and Corollary 3.2.10.

In Section 4.2 the new examples and results are Example 4.2.2, Corollary 4.2.3, Example 4.2.11, Example 4.2.13, Proposition 4.2.14, Example 4.2.16, Example 4.2.22 which extends [107, Proposition 3.17 a), p. 244] of our paper *The approximation property for weighted spaces of differentiable function* [107], Proposition 4.2.25 which extends [114, Proposition 4.8, p. 370] from sequentially complete E to locally complete E, Example 4.2.26 and Example 4.2.28 (ii). All the results of Section 4.3 are new except for Definition 4.3.1 which is [115, 2.2 Definition, p. 4] (and also not a result).

The main theorem of Section 5.1, Theorem 5.1.2, is new even though special cases appeared in [112, 116]. Theorem 5.1.4 and Theorem 5.1.5 are new as well. Corollary 5.4.3 extends [120, 7.3 Corollary, p. 22] from metric spaces with finite diameter to arbitrary metric spaces. Theorem 5.6.13 and Theorem 5.6.14 b) extend [114, Theorem 4.9, p. 371–372] and [114, Theorem 4.11, p. 375] from sequentially complete E to locally complete E. Corollary 5.7.2 is new in the sense that there is only a sketch how to prove it in [115, p. 31].

The results of Appendix A are also new except for Proposition A.1.1, Proposition A.1.6, which are contained in [106, 5.2 Proposition, p. 24] and [106, 3.13 Lemma d), p. 10], and Lemma A.2.2 which is [114, Lemma 4.7, p. 369].

CHAPTER 2

Notation and preliminaries

Basics of topology

We equip the spaces $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\mathbb{C}$ with the usual Euclidean norm $|\cdot|$, denote by $\mathbb{B}_r(x) := \{w \in \mathbb{R}^d \mid |w-x| < r\}$ the ball around $x \in \mathbb{R}^d$ and by $\mathbb{D}_r(z) := \{w \in \mathbb{C} \mid |w-z| < r\}$ the disc around $z \in \mathbb{C}$ with radius $r > 0$. Furthermore, for a subset M of a topological space (X,t) we denote the closure of M by $\overline{M}$ and the boundary of M by ∂M. If we want to emphasize that we take the closure in X resp. w.r.t. the topology t, then we write $\overline{M}^X$ resp. $\overline{M}^t$. For a subset M of a vector space X we denote by $\operatorname{ch}(M)$ the circled hull, by $\operatorname{cx}(M)$ the convex hull and by $\operatorname{acx}(M)$ the absolutely convex hull of M. If X is a topological vector space, we write $\overline{\operatorname{acx}}(M)$ for the closure of $\operatorname{acx}(M)$ in X.

Locally convex Hausdorff spaces and continuous linear operators

By E we always denote a non-trivial, i.e. $E \neq \{0\}$, locally convex Hausdorff space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$ and, in short, we write that E is an lcHs. If $E = \mathbb{K}$, then we set $(p_\alpha)_{\alpha\in\mathfrak{A}} := \{|\cdot|\}$.

By X^Ω we denote the set of maps from a non-empty set Ω to a non-empty set X, by χ_K we mean the characteristic function of $K \subset \Omega$, by $\mathcal{C}(\Omega, X)$ the space of continuous functions from a topological space Ω to a topological space X, and by $\mathcal{C}_0(\Omega, X)$ its subspace of continuous functions that vanish at infinity if X is a locally convex Hausdorff space.

We denote by $L(F,E)$ the space of continuous linear operators from F to E where F and E are locally convex Hausdorff spaces. If $E = \mathbb{K}$, we just write $F' := L(F,\mathbb{K})$ for the dual space and G° for the *polar set* of $G \subset F$. If F and E are linearly topologically isomorphic, we just write that F and E are isomorphic, in symbols $F \cong E$. We denote by $L_t(F,E)$ the space $L(F,E)$ equipped with the locally convex topology t of uniform convergence on the finite subsets of F if $t = \sigma$, on the absolutely convex, compact subsets of F if $t = \kappa$, on the absolutely convex, $\sigma(F,F')$-compact subsets of F if $t = \tau$, on the precompact (totally bounded) subsets of F if $t = \gamma$ and on the bounded subsets of F if $t = b$. We use the symbols $t(F',F)$ for the corresponding topology on F' and $t(F)$ for the corresponding bornology on F. We say that a subspace $G \subset F'$ is *separating* (the points of F) if for every $x \in F$ it follows from $y(x) = 0$ for all $y \in G$ that $x = 0$. Clearly, this is equivalent to G being $\sigma(F',F)$-dense in F'. For details and notions on the theory of locally convex spaces not explained in this thesis see [68, 89, 131, 138].

ε-products and tensor products

The so-called *ε-product* of Schwartz is defined by

$$F\varepsilon E := L_e(F'_\kappa, E) \tag{2}$$

where $L(F'_\kappa, E)$ is equipped with the topology of uniform convergence on equicontinuous subsets of F'. This definition of the ε-product coincides with the original one by Schwartz [159, Chap. I, §1, Définition, p. 18]. It is symmetric which means that $F\varepsilon E \cong E\varepsilon F$. In the literature the definition of the ε-product is sometimes done the other way around, i.e. $E\varepsilon F$ is defined by the right-hand side of (2) but due to the symmetry these definitions are equivalent and for our purpose the given definition is more suitable. If we replace F'_κ by F'_γ, we obtain Grothendieck's definition of the ε-product and we remark that the two ε-products coincide if F is quasi-complete because then $F'_\gamma = F'_\kappa$ holds. However, we stick to Schwartz' definition.

For locally convex Hausdorff spaces F_i, E_i and $T_i \in L(F_i, E_i)$, $i = 1, 2$, we define the ε-product $T_1\varepsilon T_2 \in L(F_1\varepsilon F_2, E_1\varepsilon E_2)$ of the operators T_1 and T_2 by

$$(T_1\varepsilon T_2)(u) := T_2 \circ u \circ T_1^t, \quad u \in F_1\varepsilon F_2,$$

where $T_1^t\colon E_1' \to F_1'$, $e' \mapsto e' \circ T_1$, is the *dual map* of T_1. If T_1 is an isomorphism and $F_2 = E_2$, then $T_1\varepsilon\,\mathrm{id}_{E_2}$ is also an isomorphism with inverse $T_1^{-1}\varepsilon\,\mathrm{id}_{E_2}$ by [159, Chap. I, §1, Proposition 1, p. 20] (or [89, 16.2.1 Proposition, p. 347] if the F_i are complete).

As usual we consider the tensor product $F \otimes E$ as a linear subspace of $F\varepsilon E$ for two locally convex Hausdorff spaces F and E by means of the linear injection

$$\Theta\colon F \otimes E \to F\varepsilon E,\ \sum_{n=1}^{k} f_n \otimes e_n \longmapsto \Big[y \mapsto \sum_{n=1}^{k} y(f_n)e_n\Big]. \tag{3}$$

Via Θ the space $F \otimes E$ is identified with the space of operators with finite rank in $F\varepsilon E$ and a locally convex topology is induced on $F \otimes E$. We write $F \otimes_\varepsilon E$ for $F \otimes E$ equipped with this topology and $F \widehat{\otimes}_\varepsilon E$ for the completion of the *injective tensor product* $F \otimes_\varepsilon E$. For more information on the theory of ε-products and tensor products see [49, 89, 94].

Several degrees of completeness

The sufficient conditions for surjectivity of the map $S\colon \mathcal{F}(\Omega)\varepsilon E \to \mathcal{F}(\Omega, E)$ from the introduction, which we derive in the forthcoming, depend on assumptions on different types of completeness of E. For this purpose we recapitulate some definitions which are connected to completeness. We start with local completeness. For a *disk* $D \subset E$, i.e. a bounded, absolutely convex set, the linear space $E_D := \bigcup_{n\in\mathbb{N}} nD$ becomes a normed space if it is equipped with the gauge functional of D as a norm (see [89, p. 151]). The space E is called *locally complete* if E_D is a Banach space for every closed disk $D \subset E$ (see [89, 10.2.1 Proposition, p. 197]). We call a non-empty subset A of an lcHs E *locally closed* if every local limit point of A belongs to A. Here, a point $x \in E$ is called a *local limit point* of A if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A that converges locally to x (see [138, Definition 5.1.14, p. 154–155]), i.e. there is a disk $D \subset E$ such that (x_n) converges to x in E_D (see [138, Definition 5.1.1, p. 151]). The *local closure* of a subset A of E is defined as the smallest locally closed subset of E which contains A (see [138, Definition 5.1.18, p. 155]). Moreover, we note that every locally complete linear subspace of E is locally closed and a locally closed linear subspace of a locally complete space is locally complete by [138, Proposition 5.1.20 (i), p. 155].

Moreover, a locally convex Hausdorff space is locally complete if and only if it is convenient by [104, 2.14 Theorem, p. 20]. In particular, every complete locally convex Hausdorff space is quasi-complete, every quasi-complete space is sequentially complete and every sequentially complete space is locally complete and all these implications are strict. The first two by [89, p. 58] and the third by [138, 5.1.8 Corollary, p. 153] and [138, 5.1.12 Example, p. 154].

Now, let us recall the following definition from [181, 9-2-8 Definition, p. 134] and [175, p. 259]. A locally convex Hausdorff space is said to have the *[metric] convex compactness property* ([metric] ccp) if the closure of the absolutely convex hull of every [metrisable] compact set is compact. Sometimes this condition is phrased with the term convex hull instead of absolutely convex hull but these definitions coincide. Indeed, the first definition implies the second since every convex hull of a set $A \subset E$ is contained in its absolutely convex hull. On the other hand, we have $\operatorname{acx}(A) = \operatorname{cx}(\operatorname{ch}(A))$ by [89, 6.1.4 Proposition, p. 103] and the circled hull $\operatorname{ch}(A)$ of a [metrisable] compact set A is compact by [153, Chap. I, 5.2, p. 26] [and metrisable by [34, Chap. IX, §2.10, Proposition 17, p. 159] since $\mathbb{D} \times A$ is metrisable and $\operatorname{ch}(A) = M_E(\mathbb{D} \times A)$ where $M_E\colon \mathbb{K} \times E \to E$ is the continuous scalar multiplication and $\mathbb{D} := \mathbb{D}_1(0)$ the open unit disc], which yields the other implication.

In particular, every locally convex Hausdorff space with ccp has obviously metric ccp, every quasi-complete locally convex Hausdorff space has ccp by [181, 9-2-10 Example, p. 134], every sequentially complete locally convex Hausdorff space has metric ccp by [23, A.1.7 Proposition (ii), p. 364] and every locally convex Hausdorff space with metric ccp is locally complete by [175, Remark 4.1, p. 267]. All these implications are strict. The second by [181, 9-2-10 Example, p. 134] and the others by [175, Remark 4.1, p. 267]. For more details on the [metric] convex compactness property and local completeness see [29, 175]. In addition, we remark that every semi-Montel space is semi-reflexive by [89, 11.5.1 Proposition, p. 230] and every semi-reflexive locally convex Hausdorff space is quasi-complete by [153, Chap. IV, 5.5, Corollary 1, p. 144] and these implications are strict as well. Summarizing, we have the following diagram of strict implications:

$$
\begin{array}{ccccccc}
\text{semi-Montel} & \Rightarrow & \text{semi-reflexive} & & & & \\
 & & \Downarrow & & & & \\
\text{complete} \Rightarrow & & \text{quasi-complete} & \Rightarrow & \text{sequentially complete} & \Rightarrow & \text{locally complete} \\
 & & \Downarrow & & \Downarrow & \nearrow\!\!\!\nearrow & \\
 & & \text{ccp} & \Rightarrow & \text{metric ccp} & &
\end{array}
$$

Vector-valued continuously partially differentiable functions

Since weighted spaces of continuously partially differentiable resp. holomorphic vector-valued functions will serve as our standard examples, we recall the definition of the spaces $\mathcal{C}^k(\Omega, E)$ resp. $\mathcal{O}(\Omega, E)$. A function $f\colon \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ to an lcHs E is called *continuously partially differentiable* (f is $\mathcal{C}^1$) if for the n-th unit vector $e_n \in \mathbb{R}^d$ the limit

$$(\partial^{e_n})^E f(x) := \lim_{\substack{h \to 0 \\ h \in \mathbb{R}, h \neq 0}} \frac{f(x + h e_n) - f(x)}{h}$$

exists in E for every $x \in \Omega$ and $(\partial^{e_n})^E f$ is continuous on Ω ($(\partial^{e_n})^E f$ is $\mathcal{C}^0$) for every $1 \leq n \leq d$. For $k \in \mathbb{N}$ a function f is said to be k-times continuously partially differentiable (f is $\mathcal{C}^k$) if f is $\mathcal{C}^1$ and all its first partial derivatives are $\mathcal{C}^{k-1}$. A function f is called infinitely continuously partially differentiable (f is $\mathcal{C}^\infty$) if f is $\mathcal{C}^k$ for every $k \in \mathbb{N}$. For $k \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ the functions $f\colon \Omega \to E$ which are $\mathcal{C}^k$ form a linear space which is denoted by $\mathcal{C}^k(\Omega, E)$. For $\beta \in \mathbb{N}_0^d$ with $|\beta| := \sum_{n=1}^{d} \beta_n \leq k$ and a function $f\colon \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ to an lcHs E we set $(\partial^{\beta_n})^E f := f$ if $\beta_n = 0$, and

$$(\partial^{\beta_n})^E f(x) := \underbrace{(\partial^{e_n})^E \cdots (\partial^{e_n})^E}_{\beta_n\text{-times}} f(x)$$

if $\beta_n \neq 0$ and the right-hand side exists in E for every $x \in \Omega$. Further, we define

$$(\partial^{\beta})^E f(x) := \big((\partial^{\beta_1})^E \cdots (\partial^{\beta_d})^E\big) f(x)$$

if the right-hand side exists in E for every $x \in \Omega$. If $E = \mathbb{K}$, we often just write $\partial^{\beta} f := (\partial^{\beta})^{\mathbb{K}} f$ for $\beta \in \mathbb{N}_0^d$, $|\beta| \leq k$, and $f \in \mathcal{C}^k(\Omega)$. Furthermore, we define the space of bounded continuously partially differentiable functions by

$$\mathcal{C}_b^1(\Omega, E) := \{f \in \mathcal{C}^1(\Omega, E) \mid \forall\, \alpha \in \mathfrak{A}: |f|_{\mathcal{C}_b^1(\Omega),\alpha} := \sup_{\substack{x \in \Omega \\ \beta \in \mathbb{N}_0^d, |\beta| \leq 1}} p_\alpha((\partial^{\beta})^E f(x)) < \infty\}.$$

Vector-valued holomorphic functions

A function $f\colon \Omega \to E$ on an open set $\Omega \subset \mathbb{C}$ to an lcHs E over $\mathbb{C}$ is called *holomorphic* if the limit

$$(\partial_{\mathbb{C}}^1)^E f(z) := \lim_{\substack{h \to 0 \\ h \in \mathbb{C}, h \neq 0}} \frac{f(z+h) - f(z)}{h}, \quad z \in \Omega,$$

exists in E. We denote by $\mathcal{O}(\Omega, E)$ the linear space of holomorphic functions $f\colon \Omega \to E$. Defining the vector-valued version of the *Cauchy–Riemann operator* by

$$\overline{\partial}^E f := \frac{1}{2}((\partial^{e_1})^E + \mathrm{i}(\partial^{e_2})^E) f$$

for $f \in \mathcal{C}(\Omega, E)$ such that the partial derivatives $(\partial^{e_n})^E f$, $n = 1, 2$, exist in E, we remark that

$$\mathcal{O}(\Omega, E) = \{f \in \mathcal{C}(\Omega, E) \mid f \in \ker \overline{\partial}^E\} = \{f \in \mathcal{C}^\infty(\Omega, E) \mid f \in \ker \overline{\partial}^E\} \tag{4}$$

by [113, Theorem 6.1, p. 267] if E is locally complete. Further, we set $(\partial_{\mathbb{C}}^0)^E f := f$ and note that the $(n+1)$-th complex derivative $(\partial_{\mathbb{C}}^{n+1})^E f := (\partial_{\mathbb{C}}^1)^E ((\partial_{\mathbb{C}}^n)^E f)$ exists for all $n \in \mathbb{N}_0$ and $f \in \mathcal{O}(\Omega, E)$ by [79, 2.1 Theorem and Definition, p. 17–18] and [79, 5.2 Theorem, p. 35] if E is locally complete. If $E = \mathbb{C}$, we often just write $f^{(n)} := (\partial_{\mathbb{C}}^n)^{\mathbb{C}} f$ for $n \in \mathbb{N}_0$ and $f \in \mathcal{O}(\Omega) := \mathcal{O}(\Omega, \mathbb{C})$. We note that the real and complex derivatives are related by

$$(\partial^{\beta})^E f(z) = \mathrm{i}^{\beta_2} (\partial_{\mathbb{C}}^{|\beta|})^E f(z), \quad z \in \Omega, \tag{5}$$

for every $f \in \mathcal{O}(\Omega, E)$ and $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$ by [113, Proposition 7.1, p. 270] if E is locally complete.

CHAPTER 3

The ε-product for weighted function spaces

3.1. ε-into-compatibility

In the introduction we already mentioned that linearisations of spaces of vector-valued functions by means of ε-products are essential for our approach. Here, one of the important questions is which spaces of vector-valued functions can be represented by ε-products. Let Ω be a non-empty set and E an lcHs. If $\mathcal{F}(\Omega) \subset \mathbb{K}^\Omega$ is an lcHs such that $\delta_x \in \mathcal{F}(\Omega)'$ for all $x \in \Omega$, then the map

$$S: \mathcal{F}(\Omega)\varepsilon E \to E^\Omega,\ u \longmapsto [x \mapsto u(\delta_x)],$$

is well-defined and linear. This leads to the following definition.

3.1.1. Definition (ε-into-compatible). Let Ω be a non-empty set and E an lcHs. Let $\mathcal{F}(\Omega) \subset \mathbb{K}^\Omega$ and $\mathcal{F}(\Omega, E) \subset E^\Omega$ be lcHs such that $\delta_x \in \mathcal{F}(\Omega)'$ for all $x \in \Omega$. We call the spaces $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ *ε-into-compatible* if the map

$$S: \mathcal{F}(\Omega)\varepsilon E \to \mathcal{F}(\Omega, E),\ u \longmapsto [x \mapsto u(\delta_x)],$$

is a well-defined isomorphism into, i.e. to its range. We call $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ *ε-compatible* if S is an isomorphism. We write $S_{\mathcal{F}(\Omega)}$ if we want to emphasise the dependency on $\mathcal{F}(\Omega)$.

In this section we introduce the weighted space $\mathcal{FV}(\Omega, E)$ of E-valued functions on Ω as a subspace of sections of domains in E^Ω of linear operators T^E_m equipped with a generalised version of a weighted graph topology. This space is the role model for many function spaces and an example for these operators are the partial derivative operators. Then we treat the question whether $\mathcal{FV}(\Omega, E)$ and $\mathcal{FV}(\Omega)\varepsilon E$ are ε-into-compatible. This is deeply connected with the interplay of the pair of operators $(T^E_m, T^\mathbb{K}_m)$ with the map S (see Definition 3.1.7). In our main theorem of this section we give sufficient conditions such that $S: \mathcal{FV}(\Omega)\varepsilon E \to \mathcal{FV}(\Omega, E)$ is an isomorphism into (see Theorem 3.1.12). In the next section we provide conditions such that S becomes surjective (see Theorem 3.2.4). We start with the well-known example $\mathcal{C}^k(\Omega, E)$ of k-times continuously partially differentiable E-valued functions to motivate our definition of $\mathcal{FV}(\Omega, E)$.

3.1.2. Example. Let $k \in \mathbb{N}_\infty$ and $\Omega \subset \mathbb{R}^d$ be open. Consider the space $\mathcal{C}(\Omega, E)$ of continuous functions $f: \Omega \to E$ with the *topology τ_c of compact convergence*, i.e. the topology given by the seminorms

$$\|f\|_{K,\alpha} := \sup_{x \in K} p_\alpha(f(x)), \quad f \in \mathcal{C}(\Omega, E),$$

for compact $K \subset \Omega$ and $\alpha \in \mathfrak{A}$. The usual topology on the space $\mathcal{C}^k(\Omega, E)$ of k-times continuously partially differentiable functions is the graph topology generated by the partial derivative operators $(\partial^\beta)^E: \mathcal{C}^k(\Omega, E) \to \mathcal{C}(\Omega, E)$ for $\beta \in \mathbb{N}^d_0$, $|\beta| \leq k$, i.e. the topology given by the seminorms

$$\|f\|_{K,\beta,\alpha} := \max(\|f\|_{K,\alpha}, \|(\partial^\beta)^E f\|_{K,\alpha}), \quad f \in \mathcal{C}^k(\Omega, E),$$

for compact $K \subset \Omega$, $\beta \in \mathbb{N}_0^d$, $|\beta| \leq k$, and $\alpha \in \mathfrak{A}$. The same topology is induced by the directed system of seminorms given by

$$|f|_{K,m,\alpha} := \sup_{\beta \in \mathbb{N}_0^d, |\beta| \leq m} \|f\|_{K,\beta,\alpha} = \sup_{\substack{x \in K \\ \beta \in \mathbb{N}_0^d, |\beta| \leq m}} p_\alpha\big((\partial^\beta)^E f(x)\big), \quad f \in \mathcal{C}^k(\Omega, E),$$

for compact $K \subset \Omega$, $m \in \mathbb{N}_0$, $m \leq k$, and $\alpha \in \mathfrak{A}$ and may also be seen as a weighted topology induced by the family (χ_K) of characteristic functions of the compact sets $K \subset \Omega$ by writing

$$|f|_{K,m,\alpha} = \sup_{\substack{x \in \Omega \\ \beta \in \mathbb{N}_0^d, |\beta| \leq m}} p_\alpha\big((\partial^\beta)^E f(x)\big)\chi_K(x), \quad f \in \mathcal{C}^k(\Omega, E).$$

This topology is inherited by linear subspaces of functions having additional properties like being holomorphic or harmonic.

We turn to the weight functions which we use to define a kind of weighted graph topology.

3.1.3. Definition (weight function). Let J be a non-empty set and $(\omega_m)_{m \in M}$ a family of non-empty sets. We call $\mathcal{V} := (\nu_{j,m})_{j \in J, m \in M}$ a family of *weight functions* on $(\omega_m)_{m \in M}$ if it fulfils $\nu_{j,m} \colon \omega_m \to [0, \infty)$ for all $j \in J$, $m \in M$ and

$$\forall\, m \in M,\, x \in \omega_m\; \exists\, j \in J: \; 0 < \nu_{j,m}(x). \tag{6}$$

From the structure of Example 3.1.2 we arrive at the following definition of the weighted spaces of vector-valued functions we want to consider.

3.1.4. Definition. Let Ω be a non-empty set, $\mathcal{V} := (\nu_{j,m})_{j \in J, m \in M}$ a family of weight functions on $(\omega_m)_{m \in M}$ and $T^E_m \colon E^\Omega \supset \operatorname{dom} T^E_m \to E^{\omega_m}$ a linear map for every $m \in M$. Let $\operatorname{AP}(\Omega, E)$ be a linear subspace of E^Ω and define the space of intersections

$$F(\Omega, E) := \operatorname{AP}(\Omega, E) \cap \Big(\bigcap_{m \in M} \operatorname{dom} T^E_m\Big)$$

as well as

$$\mathcal{FV}(\Omega, E) := \big\{ f \in F(\Omega, E) \mid \forall\, j \in J,\, m \in M,\, \alpha \in \mathfrak{A}: \; |f|_{j,m,\alpha} < \infty \big\}$$

where

$$|f|_{j,m,\alpha} := \sup_{x \in \omega_m} p_\alpha\big(T^E_m(f)(x)\big)\nu_{j,m}(x) = \sup_{e \in N_{j,m}(f)} p_\alpha(e)$$

with

$$N_{j,m}(f) := \{T^E_m(f)(x)\nu_{j,m}(x) \mid x \in \omega_m\}.$$

Further, we write $F(\Omega) := F(\Omega, \mathbb{K})$ and $\mathcal{FV}(\Omega) := \mathcal{FV}(\Omega, \mathbb{K})$. If we want to emphasise dependencies, we write $M(E)$ instead of M, $\operatorname{AP}_{\mathcal{FV}}(\Omega, E)$ instead of $\operatorname{AP}(\Omega, E)$ and $|f|_{\mathcal{FV}(\Omega),j,m,\alpha}$ instead of $|f|_{j,m,\alpha}$. If J, M or $\mathfrak{A}$ are singletons, we omit the index j, m resp. α in $|f|_{j,m,\alpha}$.

Note that ω_m need not be a subset of Ω. The space $\operatorname{AP}(\Omega, E)$ is a placeholder where we collect *additional properties* (AP) of our functions not being reflected by the operators T^E_m which we integrated in the topology. However, these additional properties might come from being in the domain or kernel of additional operators, e.g. harmonicity means being in the kernel of the Laplacian. But often $\operatorname{AP}(\Omega, E)$ can be chosen as E^Ω or $\mathcal{C}(\Omega, E)$. The space $\mathcal{FV}(\Omega, E)$ is locally convex but need not be Hausdorff. Since it is easier to work with Hausdorff spaces and a directed family of seminorms plus the point evaluation functionals $\delta_x \colon \mathcal{FV}(\Omega) \to \mathbb{K}$, $f \mapsto f(x)$, for $x \in \Omega$ and their continuity play a big role, we introduce the following definition.

3.1.5. DEFINITION (dom-space and $T^E_{m,x}$). We call $\mathcal{FV}(\Omega, E)$ a dom-*space* if it is a locally convex Hausdorff space, the system of seminorms $(|f|_{j,m,\alpha})_{j\in J,m\in M,\alpha\in\mathfrak{A}}$ is directed and, in addition, $\delta_x \in \mathcal{FV}(\Omega)'$ for every $x \in \Omega$ if $E = \mathbb{K}$. We define the point evaluation of T^E_m by $T^E_{m,x}\colon \operatorname{dom} T^E_m \to E$, $T^E_{m,x}(f) := T^E_m(f)(x)$, for $m \in M$ and $x \in \omega_m$.

3.1.6. REMARK. a) It is easy to see that $\mathcal{FV}(\Omega, E)$ is Hausdorff if there is $m \in M$ such that $\omega_m = \Omega$ and $T^E_m = \mathrm{id}_{E^\Omega}$ since E is Hausdorff.

b) If $E = \mathbb{K}$, then $T^{\mathbb{K}}_{m,x} \in \mathcal{FV}(\Omega)'$ for every $m \in M$ and $x \in \omega_m$. Indeed, for $m \in M$ and $x \in \omega_m$ there exists $j \in J$ such that $\nu_{j,m}(x) > 0$ by (6), implying for every $f \in \mathcal{FV}(\Omega)$ that
$$|T^{\mathbb{K}}_{m,x}(f)| = \frac{1}{\nu_{j,m}(x)} |T^{\mathbb{K}}_m(f)(x)| \nu_{j,m}(x) \leq \frac{1}{\nu_{j,m}(x)} |f|_{j,m}.$$
In particular, this implies $\delta_x \in \mathcal{FV}(\Omega)'$ for all $x \in \Omega$ if there is $m \in M$ such that $\omega_m = \Omega$ and $T^{\mathbb{K}}_m = \mathrm{id}_{\mathbb{K}^\Omega}$.

c) Let the family of weight functions $\mathcal{V}$ be *directed*, i.e.
$$\forall\ j_1, j_2 \in J, m_1, m_2 \in M\ \exists\ j_3 \in J,\ m_3 \in M,\ C > 0\ \forall\ i \in \{1, 2\} :$$
$$(\omega_{m_1} \cup \omega_{m_2}) \subset \omega_{m_3} \quad \text{and} \quad \nu_{j_i,m_i} \leq C\nu_{j_3,m_3}.$$
Then the system of seminorms $(|f|_{j,m,\alpha})_{j\in J,m\in M,\alpha\in\mathfrak{A}}$ is directed if $\mathcal{V}$ is directed and additionally it holds with m_i, $i \in \{1, 2, 3\}$, from above that
$$\forall\ f \in \mathcal{FV}(\Omega, E),\ i \in \{1, 2\},\ x \in \omega_{m_i} :\ T^E_{m_i}(f)(x) = T^E_{m_3}(f)(x),$$
since the system $(p_\alpha)_{\alpha\in\mathfrak{A}}$ of E is already directed.

We point out that the additional condition in Remark 3.1.6 c) is missing in [110, Remark 5 c), p. 1516] (resp. [106, 3.5 Remark, p. 6]), which we correct here.

For the lcHs E over $\mathbb{K}$ we want to define a natural E-valued version of a dom-space $\mathcal{FV}(\Omega) = \mathcal{FV}(\Omega, \mathbb{K})$. The natural E-valued version of $\mathcal{FV}(\Omega)$ should be a dom-space $\mathcal{FV}(\Omega, E)$ such that there is a canonical relation between the families $(T^{\mathbb{K}}_m)$ and (T^E_m). This canonical relation will be explained in terms of their interplay with the map
$$S\colon \mathcal{FV}(\Omega)\varepsilon E \to E^\Omega,\ u \longmapsto [x \mapsto u(\delta_x)].$$
Further, the elements of our E-valued version $\mathcal{FV}(\Omega, E)$ of $\mathcal{FV}(\Omega)$ should be compatible with a weak definition in the sense that $e' \circ f \in \mathcal{FV}(\Omega)$ should hold for every $e' \in E'$ and $f \in \mathcal{FV}(\Omega, E)$.

3.1.7. DEFINITION (generator, consistent, strong). Let $\mathcal{FV}(\Omega)$ and $\mathcal{FV}(\Omega, E)$ be dom-spaces such that $M := M(\mathbb{K}) = M(E)$.

a) We call $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ a *generator* for $(\mathcal{FV}(\Omega), E)$, in short, $(\mathcal{FV}, E)$.

b) We call $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ *consistent* if we have for all $u \in \mathcal{FV}(\Omega)\varepsilon E$ that $S(u) \in F(\Omega, E)$ and
$$\forall\ m \in M,\ x \in \omega_m :\ \big(T^E_m S(u)\big)(x) = u(T^{\mathbb{K}}_{m,x}).$$

c) We call $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ *strong* if we have for all $e' \in E'$, $f \in \mathcal{FV}(\Omega, E)$ that $e' \circ f \in F(\Omega)$ and
$$\forall\ m \in M,\ x \in \omega_m :\ T^{\mathbb{K}}_m(e' \circ f)(x) = \big(e' \circ T^E_m(f)\big)(x).$$

More precisely, $T^{\mathbb{K}}_{m,x}$ in b) means the restriction of $T^{\mathbb{K}}_{m,x}$ to $\mathcal{FV}(\Omega)$ and the term $u(T^{\mathbb{K}}_{m,x})$ is well-defined by Remark 3.1.6 b). Consistency will guarantee that the map $S\colon \mathcal{FV}(\Omega)\varepsilon E \to \mathcal{FV}(\Omega, E)$ is a well-defined isomorphism into, i.e. ε-into-compatibility, and strength will help us to prove its surjectivity under some additional assumptions on $\mathcal{FV}(\Omega)$ and E. Let us come to a lemma which describes the topology of $\mathcal{FV}(\Omega)\varepsilon E$ in terms of the operators $T^{\mathbb{K}}_m$ with $m \in M$. It was the motivation for the definition of consistency and allows us to consider $\mathcal{FV}(\Omega)\varepsilon E$ as a topological subspace of $\mathcal{FV}(\Omega, E)$ via S, assuming consistency.

3.1.8. Lemma. *Let $\mathcal{FV}(\Omega)$ be a* dom*-space. Then the topology of $\mathcal{FV}(\Omega)\varepsilon E$ is given by the system of seminorms defined by*

$$\|u\|_{j,m,\alpha} := \sup_{x\in\omega_m} p_\alpha\big(u(T^{\mathbb{K}}_{m,x})\big)\nu_{j,m}(x), \quad u \in \mathcal{FV}(\Omega)\varepsilon E,$$

for $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$.

Proof. We define the sets $D_{j,m} := \{T^{\mathbb{K}}_{m,x}(\cdot)\nu_{j,m}(x) \mid x \in \omega_m\}$ and $B_{j,m} := \{f \in \mathcal{FV}(\Omega) \mid |f|_{j,m} \leq 1\}$ for every $j \in J$ and $m \in M$. We claim that $\operatorname{acx}(D_{j,m})$ is dense in the polar $B^\circ_{j,m}$ with respect to $\kappa(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$. The observation

$$\begin{aligned} D^\circ_{j,m} &= \{T^{\mathbb{K}}_{m,x}(\cdot)\nu_{j,m}(x) \mid x \in \omega_m\}^\circ \\ &= \{f \in \mathcal{FV}(\Omega) \mid \forall x \in \omega_m : |T^{\mathbb{K}}_m(f)(x)|\nu_{j,m}(x) \leq 1\} \\ &= \{f \in \mathcal{FV}(\Omega) \mid |f|_{j,m} \leq 1\} = B_{j,m} \end{aligned}$$

yields

$$\overline{\operatorname{acx}}(D_{j,m})^{\kappa(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))} = (D_{j,m})^{\circ\circ} = B^\circ_{j,m}$$

by the bipolar theorem. By [89, 8.4, p. 152, 8.5, p. 156–157] the system of seminorms defined by

$$q_{j,m,\alpha}(u) := \sup_{y\in B^\circ_{j,m}} p_\alpha\big(u(y)\big), \quad u \in \mathcal{FV}(\Omega)\varepsilon E,$$

for $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$ gives the topology on $\mathcal{FV}(\Omega)\varepsilon E$ (here it is used that the system of seminorms $(|\cdot|_{j,m})$ of $\mathcal{FV}(\Omega)$ is directed). As every $u \in \mathcal{FV}(\Omega)\varepsilon E$ is continuous on $B^\circ_{j,m}$, we may replace $B^\circ_{j,m}$ by a $\kappa(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$-dense subset. Therefore we obtain

$$q_{j,m,\alpha}(u) = \sup\big\{p_\alpha\big(u(y)\big) \mid y \in \operatorname{acx}(D_{j,m})\big\}.$$

For $y \in \operatorname{acx}(D_{j,m})$ there are $n \in \mathbb{N}$, $\lambda_k \in \mathbb{K}$, $x_k \in \omega_m$, $1 \leq k \leq n$, with $\sum_{k=1}^n |\lambda_k| \leq 1$ such that $y = \sum_{k=1}^n \lambda_k T^{\mathbb{K}}_{m,x_k}(\cdot)\nu_{j,m}(x_k)$. Then we have for every $u \in \mathcal{FV}(\Omega)\varepsilon E$

$$p_\alpha\big(u(y)\big) \leq \sum_{k=1}^n |\lambda_k| p_\alpha\big(u(T^{\mathbb{K}}_{m,x_k})\big)\nu_{j,m}(x_k) \leq \|u\|_{j,m,\alpha},$$

thus $q_{j,m,\alpha}(u) \leq \|u\|_{j,m,\alpha}$. On the other hand, we derive

$$q_{j,m,\alpha}(u) \geq \sup_{y\in D_{j,m}} p_\alpha\big(u(y)\big) = \sup_{x\in\omega_m} p_\alpha\big(u(T^{\mathbb{K}}_{m,x})\big)\nu_{j,m}(x) = \|u\|_{j,m,\alpha}. \qquad \square$$

Let us turn to a more general version of Example 3.1.2, namely, to weighted spaces of k-times continuously partially differentiable functions and kernels of linear partial differential operators in these spaces.

3.1.9. Example. Let $k \in \mathbb{N}_\infty$ and $\Omega \subset \mathbb{R}^d$ be open. We consider the cases

(i) $\omega_m := M_m \times \Omega$ with $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq \min(m,k)\}$ for all $m \in \mathbb{N}_0$, or
(ii) $\omega_m := \mathbb{N}_0^d \times \Omega$ for all $m \in \mathbb{N}_0$ and $k = \infty$,

and let $\mathcal{V}^k := (\nu_{j,m})_{j\in J, m\in\mathbb{N}_0}$ be a directed family of weights on $(\omega_m)_{m\in\mathbb{N}_0}$.

a) We define the weighted space of k-times continuously partially differentiable functions with values in an lcHs E as

$$\mathcal{CV}^k(\Omega, E) := \{f \in \mathcal{C}^k(\Omega, E) \mid \forall\, j \in J,\, m \in \mathbb{N}_0,\, \alpha \in \mathfrak{A} :\, |f|_{j,m,\alpha} < \infty\}$$

where

$$|f|_{j,m,\alpha} := \sup_{(\beta,x)\in\omega_m} p_\alpha\big((\partial^\beta)^E f(x)\big)\nu_{j,m}(\beta, x).$$

Setting $\operatorname{dom} T^E_m := \mathcal{C}^k(\Omega, E)$ and

$$T^E_m : \mathcal{C}^k(\Omega, E) \to E^{\omega_m},\ f \longmapsto [(\beta, x) \mapsto (\partial^\beta)^E f(x)],$$

as well as $\operatorname{AP}(\Omega, E) := E^\Omega$, we observe that $\mathcal{CV}^k(\Omega, E)$ is a dom-space by Remark 3.1.6 and

$$|f|_{j,m,\alpha} = \sup_{x \in \omega_m} p_\alpha\big(T^E_m f(x)\big) \nu_{j,m}(x).$$

b) The space $\mathcal{C}^k(\Omega, E)$ with its usual topology given in Example 3.1.2 is a special case of a)(i) with $J := \{K \subset \Omega \mid K \text{ compact}\}$, $\nu_{K,m}(\beta, x) := \chi_K(x)$, $(\beta, x) \in \omega_m$, for all $m \in \mathbb{N}_0$ and $K \in J$ where χ_K is the characteristic function of K. In this case we write $\mathcal{W}^k := \mathcal{V}^k$ for the family of weight functions.

c) The *Schwartz space* is defined by

$$\mathcal{S}(\mathbb{R}^d, E) := \{f \in \mathcal{C}^\infty(\mathbb{R}^d, E) \mid \forall\ m \in \mathbb{N}_0,\ \alpha \in \mathfrak{A} :\ |f|_{m,\alpha} < \infty\}$$

where

$$|f|_{m,\alpha} := \sup_{\substack{x \in \mathbb{R}^d \\ \beta \in \mathbb{N}^d_0, |\beta| \le m}} p_\alpha\big((\partial^\beta)^E f(x)\big)(1 + |x|^2)^{m/2}.$$

This is a special case of a)(i) with $k := \infty$, $\Omega := \mathbb{R}^d$, $J := \{1\}$ and $\nu_{1,m}(\beta, x) := (1 + |x|^2)^{m/2}$, $(\beta, x) \in \omega_m$, for all $m \in \mathbb{N}_0$.

d) The *multiplier space* for the Schwartz space is defined by

$$\mathcal{O}_M(\mathbb{R}^d, E) := \{f \in \mathcal{C}^\infty(\mathbb{R}^d, E) \mid \forall\ g \in \mathcal{S}(\mathbb{R}^d),\ m \in \mathbb{N}_0,\ \alpha \in \mathfrak{A} :\ \|f\|_{g,m,\alpha} < \infty\}$$

where

$$\|f\|_{g,m,\alpha} := \sup_{\substack{x \in \mathbb{R}^d \\ \beta \in \mathbb{N}^d_0, |\beta| \le m}} p_\alpha\big((\partial^\beta)^E f(x)\big)|g(x)|$$

(see [158, 4^0), p. 97]). This is a special case of a)(i) with $k := \infty$, $\Omega := \mathbb{R}^d$, $J := \{j \subset \mathcal{S}(\mathbb{R}^d) \mid j \text{ finite}\}$ and $\nu_{j,1,m}(\beta, x) := \max_{g \in j} |g(x)|$, $(\beta, x) \in \omega_m$, for all $m \in \mathbb{N}_0$. This choice of J guarantees that the family $\mathcal{V}^\infty$ is directed and does not change the topology.

e) Let $\mathfrak{K} := \{K \subset \Omega \mid K \text{ compact}\}$ and $(M_p)_{p \in \mathbb{N}_0}$ be a sequence of positive real numbers. The space $\mathcal{E}^{(M_p)}(\Omega, E)$ of *ultradifferentiable functions of class* (M_p) *of Beurling-type* is defined as

$$\mathcal{E}^{(M_p)}(\Omega, E) := \{f \in \mathcal{C}^\infty(\Omega, E) \mid \forall\ K \in \mathfrak{K},\ h > 0,\ \alpha \in \mathfrak{A} :\ |f|_{(K,h),\alpha} < \infty\}$$

where

$$|f|_{(K,h),\alpha} := \sup_{\substack{x \in K \\ \beta \in \mathbb{N}^d_0}} p_\alpha\big((\partial^\beta)^E f(x)\big) \frac{1}{h^{|\beta|} M_{|\beta|}}.$$

This is a special case of a)(ii) with $J := \mathfrak{K} \times \mathbb{R}_{>0}$ and $\nu_{(K,h),m}(\beta, x) := \chi_K(x) \frac{1}{h^{|\beta|} M_{|\beta|}}$, $(\beta, x) \in \omega_m$, for all $(K, h) \in J$ and $m \in \mathbb{N}_0$ where $\mathbb{R}_{>0} := (0, \infty)$.

f) Let $\mathfrak{K}$ and $(M_p)_{p \in \mathbb{N}_0}$ be as in e). The space $\mathcal{E}^{\{M_p\}}(\Omega, E)$ of *ultradifferentiable functions of class* $\{M_p\}$ *of Roumieu-type* is defined as

$$\mathcal{E}^{\{M_p\}}(\Omega, E) := \{f \in \mathcal{C}^\infty(\Omega, E) \mid \forall\ (K, H) \in J,\ \alpha \in \mathfrak{A} :\ |f|_{(K,H),\alpha} < \infty\}$$

where

$$J := \mathfrak{K} \times \{H = (H_n)_{n \in \mathbb{N}} \mid \exists\ (h_k)_{k \in \mathbb{N}},\ h_k > 0,\ h_k \nearrow \infty\ \forall\ n \in \mathbb{N} :\ H_n = h_1 \cdot \ldots \cdot h_n\}$$

and

$$|f|_{(K,H),\alpha} := \sup_{\substack{x \in K \\ \beta \in \mathbb{N}^d_0}} p_\alpha\big((\partial^\beta)^E f(x)\big) \frac{1}{H_{|\beta|} M_{|\beta|}}$$

(see [101, Proposition 3.5, p. 675]). Again, this is a special case of a)(ii) with $\nu_{(K,H),m}(\beta,x) := \chi_K(x)\frac{1}{H_{|\beta|}M_{|\beta|}}$, $(\beta,x) \in \omega_m$, for all $(K,H) \in J$ and $m \in \mathbb{N}_0$.

g) Let $n \in \mathbb{N}$, $\beta_i \in \mathbb{N}_0^d$ with $|\beta_i| \leq k$ and $a_i\colon \Omega \to \mathbb{K}$ for $1 \leq i \leq n$. We set

$$P(\partial)^E\colon \mathcal{C}^k(\Omega,E) \to E^\Omega,\ P(\partial)^E(f)(x) := \sum_{i=1}^{n} a_i(x)(\partial^{\beta_i})^E(f)(x)$$

and obtain the (topological) subspace of $\mathcal{CV}^k(\Omega,E)$ given by

$$\mathcal{CV}^k_{P(\partial)}(\Omega,E) := \{f \in \mathcal{CV}^k(\Omega,E) \mid f \in \ker P(\partial)^E\}.$$

Choosing $\mathrm{AP}(\Omega,E) := \ker P(\partial)^E$, we see that this is also a dom-space by a). If $P(\partial)^E$ is the Cauchy–Riemann operator (and E locally complete) or the Laplacian, we obtain the weighted space of holomorphic resp. harmonic functions.

Let us show that the generators of these spaces are strong and consistent. In order to obtain consistency for their generators we have to restrict to directed families of weights which are *locally bounded away from zero* on Ω, i.e.

$$\forall\ K \subset \Omega \text{ compact},\ m \in \mathbb{N}_0\ \exists\ j \in J\ \forall\ \beta \in \mathbb{N}_0^d,\ |\beta| \leq \min(m,k) : \inf_{x\in K} \nu_{j,m}(\beta,x) > 0.$$

This condition on $\mathcal{V}^k$ guarantees that the map $I\colon \mathcal{CV}^k(\Omega) \to \mathcal{CW}^k(\Omega)$, $f \mapsto f$, is continuous which is needed for consistency.

3.1.10. Proposition. *Let E be an lcHs, $k \in \mathbb{N}_\infty$, $\mathcal{V}^k$ be a directed family of weights which is locally bounded away from zero on an open set $\Omega \subset \mathbb{R}^d$. The generator of $(\mathcal{CV}^k,E)$ resp. $(\mathcal{CV}^k_{P(\partial)},E)$ from Example 3.1.9 is strong and consistent if $\mathcal{CV}^k(\Omega)$ resp. $\mathcal{CV}^k_{P(\partial)}(\Omega)$ is barrelled.*

Proof. We recall the definitions from Example 3.1.9. We have $\omega_m := M_m \times \Omega$ with $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq \min(m,k)\}$ for all $m \in \mathbb{N}_0$ or $\omega_m := \mathbb{N}_0^d \times \Omega$ for all $m \in \mathbb{N}_0$. Further, $\mathrm{AP}_{\mathcal{CV}^k}(\Omega,E) = E^\Omega$, $\mathrm{AP}_{\mathcal{CV}^k_{P(\partial)}}(\Omega,E) = \ker P(\partial)^E$, $\operatorname{dom} T^E_m := \mathcal{C}^k(\Omega,E)$ and

$$T^E_m\colon \mathcal{C}^k(\Omega,E) \to E^{\omega_m},\ f \longmapsto [(\beta,x) \mapsto (\partial^\beta)^E f(x)],$$

for all $m \in \mathbb{N}_0$ and the same with $\mathbb{K}$ instead of E. The family $(T^E_m,T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ is a strong generator for $(\mathcal{CV}^k,E)$ because

$$(\partial^\beta)^{\mathbb{K}}(e' \circ f)(x) = e'\big((\partial^\beta)^E f(x)\big), \quad (\beta,x) \in \omega_m,$$

for all $e' \in E'$, $f \in \mathcal{CV}^k(\Omega,E)$ and $m \in \mathbb{N}_0$ due to the linearity and continuity of $e' \in E'$. In addition, $e' \circ f \in \ker P(\partial)^{\mathbb{K}}$ for all $e' \in E'$ and $f \in \mathcal{CV}^k_{P(\partial)}(\Omega,E)$, which implies that $(T^E_m,T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ is also a strong generator for $(\mathcal{CV}^k_{P(\partial)},E)$.

For consistency we need to prove that

$$(\partial^\beta)^E S(u)(x) = u(\delta_x \circ (\partial^\beta)^{\mathbb{K}}), \quad (\beta,x) \in \omega_m,$$

for all $u \in \mathcal{CV}^k(\Omega)\varepsilon E$ resp. $u \in \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$. This follows from the subsequent Proposition 3.1.11 b) since $\mathcal{FV}(\Omega) = \mathcal{CV}^k(\Omega)$ resp. $\mathcal{FV}(\Omega) = \mathcal{CV}^k_{P(\partial)}(\Omega)$ is barrelled and $\mathcal{V}^k$ locally bounded away from zero on Ω. Thus $(T^E_m,T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ is a consistent generator for $(\mathcal{CV}^k,E)$. In addition, we have with $P(\partial)^E$ from Example 3.1.9 g) that

$$\begin{aligned} P(\partial)^E(S(u))(x) &= \sum_{i=1}^{n} a_i(x)(\partial^{\beta_i})^E(S(u))(x) = u\Big(\sum_{i=1}^{n} a_i(x)(\delta_x \circ (\partial^{\beta_i})^{\mathbb{K}})\Big) \\ &= u(\delta_x \circ P(\partial)^{\mathbb{K}}) = 0, \quad x \in \Omega, \end{aligned} \tag{7}$$

for every $u \in \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$. This yields $S(u) \in \ker P(\partial)^E$ for all $u \in \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$. Therefore $(T^E_m,T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ is a consistent generator for $(\mathcal{CV}^k_{P(\partial)},E)$ as well. □

Let us turn to the postponed part in the proof of consistency. We denote by $\mathcal{CW}(\Omega)$ the space of scalar-valued continuous functions on an open set $\Omega\subset\mathbb{R}^d$ with the topology of uniform convergence on compact subsets, i.e. the weighted topology given by the family of weights $\mathcal{W}:=\mathcal{W}^0:=\{\chi_K\mid K\subset\Omega\text{ compact}\}$, and we set $\delta(x):=\delta_x$ for $x\in\Omega$.

3.1.11. PROPOSITION. *Let $\Omega\subset\mathbb{R}^d$ be open, $k\in\mathbb{N}_\infty$ and $\mathcal{FV}(\Omega)$ a* dom-*space.*

a) If $T\in L(\mathcal{FV}(\Omega),\mathcal{CW}(\Omega))$, then $\delta\circ T\in\mathcal{C}(\Omega,\mathcal{FV}(\Omega)'_\gamma)$.

b) If $T\in L(\mathcal{FV}(\Omega),\mathcal{CW}^1(\Omega))$ and $\mathcal{FV}(\Omega)$ is barrelled, then

$$\begin{aligned}(\partial^{e_n})^{\mathcal{FV}(\Omega)'_\kappa}(\delta\circ T)(x)&=\lim_{h\to 0}\frac{\delta_{x+he_n}\circ T-\delta_x\circ T}{h}\\&=\delta_x\circ(\partial^{e_n})^{\mathbb{K}}\circ T,\quad x\in\Omega,\ 1\leq n\leq d,\end{aligned}$$

and $\delta\circ T\in\mathcal{C}^1(\Omega,\mathcal{FV}(\Omega)'_\kappa)$.

c) If the inclusion $I\colon\mathcal{FV}(\Omega)\to\mathcal{CW}^k(\Omega)$, $f\mapsto f$, is continuous and $\mathcal{FV}(\Omega)$ barrelled, then $S(u)\in\mathcal{C}^k(\Omega,E)$ and

$$(\partial^\beta)^E S(u)(x)=u(\delta_x\circ(\partial^\beta)^{\mathbb{K}}),\quad \beta\in\mathbb{N}_0^d,\ |\beta|\leq k,\ x\in\Omega,$$

for all $u\in\mathcal{FV}(\Omega)\varepsilon E$.

PROOF. a) First, if $x\in\Omega$ and $(x_\tau)_{\tau\in\mathcal{T}}$ is a net in Ω converging to x, then we observe that

$$(\delta_{x_\tau}\circ T)(f)=T(f)(x_\tau)\to T(f)(x)=(\delta_x\circ T)(f)$$

for every $f\in\mathcal{FV}(\Omega)$ as $T(f)$ is continuous on Ω. Second, let $K\subset\Omega$ be compact. Then there are $j\in J$, $m\in M$ and $C>0$ such that

$$\sup_{x\in K}|(\delta_x\circ T)(f)|=\sup_{x\in K}|T(f)(x)|\leq C|f|_{j,m}$$

for every $f\in\mathcal{FV}(\Omega)$. This means that $\{\delta_x\circ T\mid x\in K\}$ is equicontinuous in $\mathcal{FV}(\Omega)'$. The topologies $\sigma(\mathcal{FV}(\Omega)',\mathcal{FV}(\Omega))$ and $\gamma(\mathcal{FV}(\Omega)',\mathcal{FV}(\Omega))$ coincide on equicontinuous subsets of $\mathcal{FV}(\Omega)'$, implying that the restriction $(\delta\circ T)_{|K}\colon K\to\mathcal{FV}(\Omega)'_\gamma$ is continuous by our first observation. As $\delta\circ T$ is continuous on every compact subset of the open set $\Omega\subset\mathbb{R}^d$, it follows that $\delta\circ T\colon\Omega\to\mathcal{FV}(\Omega)'_\gamma$ is well-defined and continuous.

b) Let $x\in\Omega$ and $1\leq n\leq d$. Then there is $\varepsilon>0$ such that $x+he_n\in\Omega$ for all $h\in\mathbb{R}$ with $0<|h|<\varepsilon$. We note that $\delta\circ T\in\mathcal{C}(\Omega,\mathcal{FV}(\Omega)'_\kappa)$ by part a), which implies $\frac{\delta_{x+he_n}\circ T-\delta_x\circ T}{h}\in\mathcal{FV}(\Omega)'$. For every $f\in\mathcal{FV}(\Omega)$ we have

$$\lim_{h\to 0}\frac{\delta_{x+he_n}\circ T-\delta_x\circ T}{h}(f)=\lim_{h\to 0}\frac{T(f)(x+he_n)-T(f)(x)}{h}=(\partial^{e_n})^{\mathbb{K}}T(f)(x)$$

in $\mathbb{K}$ as $T(f)\in\mathcal{C}^1(\Omega)$. Therefore $\frac{1}{h}(\delta_{x+he_n}\circ T-\delta_x\circ T)$ converges to $\delta_x\circ(\partial^{e_n})^{\mathbb{K}}\circ T$ in $\mathcal{FV}(\Omega)'_\sigma$ and thus in $\mathcal{FV}(\Omega)'_\kappa$ by the Banach–Steinhaus theorem as well. In particular, we obtain

$$\delta_x\circ(\partial^{e_n})^{\mathbb{K}}\circ T=\lim_{h\to 0}\frac{\delta_{x+he_n}\circ T-\delta_x\circ T}{h}=(\partial^{e_n})^{\mathcal{FV}(\Omega)'_\kappa}(\delta\circ T)(x)$$

in $\mathcal{FV}(\Omega)'_\kappa$. Moreover, $\delta\circ(\partial^{e_n})^{\mathbb{K}}\circ T\in\mathcal{C}(\Omega,\mathcal{FV}(\Omega)'_\kappa)$ by part a) as $(\partial^{e_n})^{\mathbb{K}}\circ T\in L(\mathcal{FV}(\Omega),\mathcal{CW}(\Omega))$. Hence we deduce that $\delta\circ T\in\mathcal{C}^1(\Omega,\mathcal{FV}(\Omega)'_\kappa)$.

c) We prove our claim by induction on the order of differentiation. Let $u\in\mathcal{FV}(\Omega)\varepsilon E$. For $\beta\in\mathbb{N}_0^d$ with $|\beta|=0$ we get $S(u)=u\circ\delta\in\mathcal{C}(\Omega,E)$ from part a) with $T=I$. Further,

$$(\partial^\beta)^E S(u)(x)=S(u)(x)=u(\delta_x)=u(\delta_x\circ(\partial^\beta)^{\mathbb{K}}),\quad x\in\Omega.$$

Let $m\in\mathbb{N}_0$, $m<k$, such that $S(u)\in\mathcal{C}^m(\Omega,E)$ and

$$(\partial^{\beta})^{E}S(u)(x)=u(\delta_x\circ(\partial^{\beta})^{\mathbb{K}}),\quad x\in\Omega, \tag{8}$$

for all $\beta\in\mathbb{N}_0^d$ with $|\beta|\leq m$. Let $\beta\in\mathbb{N}_0^d$ with $|\beta|=m+1\leq k$. Then there is $1\leq n\leq d$ and $\widetilde{\beta}\in\mathbb{N}_0^d$ with $|\widetilde{\beta}|=m$ such that $\beta=e_n+\widetilde{\beta}$. The barrelledness of $\mathcal{FV}(\Omega)$ yields that $\frac{1}{h}(\delta_{x+he_n}\circ(\partial^{\widetilde{\beta}})^{\mathbb{K}}-\delta_x\circ(\partial^{\widetilde{\beta}})^{\mathbb{K}})$ converges to $\delta_x\circ(\partial^{e_n})^{\mathbb{K}}\circ(\partial^{\widetilde{\beta}})^{\mathbb{K}}$ in $\mathcal{FV}(\Omega)'_{\kappa}$ for every $x\in\Omega$ by part b) with $T:=(\partial^{\widetilde{\beta}})^{\mathbb{K}}$. Therefore we derive from $\delta_x\circ(\partial^{e_n})^{\mathbb{K}}\circ(\partial^{\widetilde{\beta}})^{\mathbb{K}}=\delta_x\circ(\partial^{\beta})^{\mathbb{K}}$ by Schwarz' theorem that

$$\begin{aligned}u(\delta_x\circ(\partial^{\beta})^{\mathbb{K}})&=\lim_{h\to0}\frac{1}{h}\big(u(\delta_{x+he_n}\circ(\partial^{\widetilde{\beta}})^{\mathbb{K}})-u(\delta_x\circ(\partial^{\widetilde{\beta}})^{\mathbb{K}})\big)\\&\underset{(8)}{=}\lim_{h\to0}\frac{1}{h}\big((\partial^{\widetilde{\beta}})^{E}S(u)(x+he_n)-(\partial^{\widetilde{\beta}})^{E}S(u)(x)\big)\\&=(\partial^{e_n})^{E}(\partial^{\widetilde{\beta}})^{E}S(u)(x)\end{aligned}$$

for every $x\in\Omega$. Moreover, $\delta\circ(\partial^{\beta})^{\mathbb{K}}=(\partial^{e_n})^{\mathcal{FV}(\Omega)'_{\kappa}}(\delta\circ T)\in\mathcal{C}(\Omega,\mathcal{FV}(\Omega)'_{\kappa})$ for $T=(\partial^{\widetilde{\beta}})^{\mathbb{K}}$ by part b). Hence we have $S(u)\in\mathcal{C}^{m+1}(\Omega,E)$ and it follows from Schwarz' theorem again that

$$u(\delta_x\circ(\partial^{\beta})^{\mathbb{K}})=(\partial^{e_n})^{E}(\partial^{\widetilde{\beta}})^{E}S(u)(x)=(\partial^{\beta})^{E}S(u)(x),\quad x\in\Omega.$$ □

Part a) of the preceding proposition is just a modification of [16, 4.1 Lemma, p. 198], where $\mathcal{FV}(\Omega)=\mathcal{CV}(\Omega)$ is the Nachbin-weighted space of continuous functions and $T=\mathrm{id}$, and holds more general for $k_{\mathbb{R}}$-spaces Ω (see Lemma 4.1.2).

3.1.12. THEOREM. *Let $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ be a consistent generator for $(\mathcal{FV},E)$. Then the map $S\colon\mathcal{FV}(\Omega)\varepsilon E\to\mathcal{FV}(\Omega,E)$ is an isomorphism into, i.e. the spaces $\mathcal{FV}(\Omega)$ and $\mathcal{FV}(\Omega,E)$ are ε-into-compatible.*

PROOF. First, we show that $S(\mathcal{FV}(\Omega)\varepsilon E)\subset\mathcal{FV}(\Omega,E)$. Let $u\in\mathcal{FV}(\Omega)\varepsilon E$. Due to the consistency of $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ we have $S(u)\in\mathrm{AP}(\Omega,E)\cap\operatorname{dom}T^E_m$ and

$$\big(T^E_mS(u)\big)(x)=u(T^{\mathbb{K}}_{m,x}),\quad m\in M\ x\in\omega_m.$$

Furthermore, we get by Lemma 3.1.8 for every $j\in J$, $m\in M$ and $\alpha\in\mathfrak{A}$

$$|S(u)|_{j,m,\alpha}=\sup_{x\in\omega_m}p_{\alpha}\big(T^E_m(S(u))(x)\big)\nu_{j,m}(x)=\|u\|_{j,m,\alpha}<\infty, \tag{9}$$

implying $S(u)\in\mathcal{FV}(\Omega,E)$ and the continuity of S. Moreover, we deduce from (9) that S is injective and that the inverse of S on the range of S is also continuous. □

3.1.13. REMARK. If J, M and $\mathfrak{A}$ are countable, then S is an isometry with respect to the induced metrics on $\mathcal{FV}(\Omega,E)$ and $\mathcal{FV}(\Omega)\varepsilon E$ by (9).

The basic idea for Theorem 3.1.12 was derived from analysing the proof of an analogous statement for Bierstedt's weighted spaces $\mathcal{CV}(\Omega,E)$ and $\mathcal{CV}_0(\Omega,E)$ of continuous functions already mentioned in the introduction (see [16, 4.2 Lemma, 4.3 Folgerung, p. 199–200] and [17, 2.1 Satz, p. 137]).

3.2. ε-compatibility

Now, we try to answer the natural question. When is S surjective? The strength of a generator and a weaker concept to define a natural E-valued version of $\mathcal{FV}(\Omega)$ come into play to answer the question on the surjectivity of our key map S. Let $\mathcal{FV}(\Omega)$ be a dom-space. We define the linear space of *E-valued weak $\mathcal{FV}$-functions* by

$$\mathcal{FV}(\Omega,E)_{\sigma}:=\{f\colon\Omega\to E\mid\forall\ e'\in E':\ e'\circ f\in\mathcal{FV}(\Omega)\}.$$

Moreover, for $f \in \mathcal{FV}(\Omega, E)_\sigma$ we define the linear map

$$R_f\colon E' \to \mathcal{FV}(\Omega),\; R_f(e') := e' \circ f,$$

and the dual map

$$R_f^t\colon \mathcal{FV}(\Omega)' \to E'^\star,\; f' \longmapsto \big[e' \mapsto f'\big(R_f(e')\big)\big],$$

where $E'^\star$ is the algebraic dual of E'. Furthermore, we set

$$\mathcal{FV}(\Omega, E)_\kappa := \{f \in \mathcal{FV}(\Omega, E)_\sigma \mid \forall\ \alpha \in \mathfrak{A}:\ R_f(B_\alpha^\circ) \text{ relatively compact in } \mathcal{FV}(\Omega)\}$$

where $B_\alpha := \{x \in E \mid p_\alpha(x) < 1\}$ for $\alpha \in \mathfrak{A}$. Next, we give a sufficient condition for the inclusion $\mathcal{FV}(\Omega, E) \subset \mathcal{FV}(\Omega, E)_\sigma$ by means of the family $(T_m^E, T_m^{\mathbb{K}})_{m\in M}$.

3.2.1. Lemma. *If $(T_m^E, T_m^{\mathbb{K}})_{m\in M}$ is a strong generator for $(\mathcal{FV}, E)$, then we have $\mathcal{FV}(\Omega, E) \subset \mathcal{FV}(\Omega, E)_\sigma$ and*

$$\sup_{e' \in B_\alpha^\circ} |R_f(e')|_{j,m} = |f|_{j,m,\alpha} \tag{10}$$

for every $f \in \mathcal{FV}(\Omega, E)$, $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$.

Proof. Let $f \in \mathcal{FV}(\Omega, E)$. We have $e' \circ f \in \mathcal{F}(\Omega)$ for every $e' \in E'$ since $(T_m^E, T_m^{\mathbb{K}})_{m\in M}$ is a strong generator. Moreover, we have

$$\begin{aligned} |R_f(e')|_{j,m} = |e' \circ f|_{j,m} &= \sup_{x\in\omega_m} \big|T_m^{\mathbb{K}}(e' \circ f)(x)\big|\nu_{j,m}(x) \\ &= \sup_{x\in\omega_m} \big|e'\big(T_m^E(f)(x)\big)\big|\nu_{j,m}(x) = \sup_{x\in N_{j,m}(f)} |e'(x)| \end{aligned} \tag{11}$$

for every $j \in J$ and $m \in M$ with the set $N_{j,m}(f)$ from Definition 3.1.4. We note that $N_{j,m}(f)$ is bounded in E by Definition 3.1.4 and thus weakly bounded, implying that the right-hand side of (11) is finite. Hence we conclude $f \in \mathcal{FV}(\Omega, E)_\sigma$. Further, we observe that

$$\sup_{e' \in B_\alpha^\circ} |R_f(e')|_{j,m} = |f|_{j,m,\alpha}$$

for every $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$ due to [131, Proposition 22.14, p. 256]. □

Now, we phrase some sufficient conditions for $\mathcal{FV}(\Omega, E) \subset \mathcal{FV}(\Omega, E)_\kappa$ to hold which is one of the key points regarding the surjectivity of S.

3.2.2. Lemma. *If $(T_m^E, T_m^{\mathbb{K}})_{m\in M}$ is a strong generator for $(\mathcal{FV}, E)$ and one of the following conditions is fulfilled, then $\mathcal{FV}(\Omega, E) \subset \mathcal{FV}(\Omega, E)_\kappa$.*

a) $\mathcal{FV}(\Omega)$ is a semi-Montel space.

b) E is a semi-Montel or Schwartz space.

c) $\forall\ f \in \mathcal{FV}(\Omega, E), j \in J, m \in M\ \exists\ K \in \gamma(E):\ N_{j,m}(f) \subset K$.

Proof. Let $f \in \mathcal{FV}(\Omega, E)$. By virtue of Lemma 3.2.1 we already have $f \in \mathcal{FV}(\Omega, E)_\sigma$.

a) For every $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$ we derive from

$$\sup_{e' \in B_\alpha^\circ} |R_f(e')|_{j,m} \underset{(10)}{=} |f|_{j,m,\alpha} < \infty$$

that $R_f(B_\alpha^\circ)$ is bounded and thus relatively compact in the semi-Montel space $\mathcal{FV}(\Omega)$.

c) It follows from (11) that $R_f \in L(E'_\gamma, \mathcal{FV}(\Omega))$. Further, the polar B_α° is relatively compact in E'_γ for every $\alpha \in \mathfrak{A}$ by the Alaoğlu–Bourbaki theorem. The continuity of R_f implies that $R_f(B_\alpha^\circ)$ is relatively compact as well.

b) Let $j \in J$ and $m \in M$. The set $K := N_{j,m}(f)$ is bounded in E by Definition 3.1.4. We deduce that K is already precompact in E by [89, 10.4.3 Corollary, p. 202] if E is a Schwartz space resp. since it is relatively compact if E is a semi-Montel space. Hence the statement follows from c). □

Let us turn to sufficient conditions for $\mathcal{FV}(\Omega,E)\cong\mathcal{FV}(\Omega)\varepsilon E$. For the lcHs E we denote by $\mathcal{J}\colon E\to E'^{\star}$, $x\longmapsto[e'\mapsto e'(x)]$, the canonical injection.

3.2.3. CONDITION. Let $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ be a strong generator for $(\mathcal{FV},E)$. Define the following conditions:

a) E is complete.
b) E is quasi-complete and for every $f\in\mathcal{FV}(\Omega,E)$ and $f'\in\mathcal{FV}(\Omega)'$ there is a bounded net $(f'_\tau)_{\tau\in\mathcal{T}}$ in $\mathcal{FV}(\Omega)'$ converging to f' in $\mathcal{FV}(\Omega)'_\kappa$ such that $R^t_f(f'_\tau)\in\mathcal{J}(E)$ for every $\tau\in\mathcal{T}$.
c) E is sequentially complete and for every $f\in\mathcal{FV}(\Omega,E)$ and $f'\in\mathcal{FV}(\Omega)'$ there is a sequence $(f'_n)_{n\in\mathbb{N}}$ in $\mathcal{FV}(\Omega)'$ converging to f' in $\mathcal{FV}(\Omega)'_\kappa$ such that $R^t_f(f'_n)\in\mathcal{J}(E)$ for every $n\in\mathbb{N}$.
d) E is locally complete and for every $f\in\mathcal{FV}(\Omega,E)$ and $f'\in\mathcal{FV}(\Omega)'$ there is a sequence $(f'_n)_{n\in\mathbb{N}}$ in $\mathcal{FV}(\Omega)'$ locally converging to f' in $\mathcal{FV}(\Omega)'_\kappa$ such that $R^t_f(f'_n)\in\mathcal{J}(E)$ for every $n\in\mathbb{N}$.
e) $\forall\, f\in\mathcal{FV}(\Omega,E),\, j\in J,\, m\in M\ \exists\, K\in\tau(E):\ N_{j,m}(f)\subset K$.

3.2.4. THEOREM. *Let $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ be a consistent generator for $(\mathcal{FV},E)$ and let $\mathcal{FV}(\Omega,E)\subset\mathcal{FV}(\Omega,E)_\kappa$. If one of the Conditions 3.2.3 is fulfilled, then the map $S\colon\mathcal{FV}(\Omega)\varepsilon E\to\mathcal{FV}(\Omega,E)$ is an isomorphism, i.e. $\mathcal{FV}(\Omega)$ and $\mathcal{FV}(\Omega,E)$ are ε-compatible. The inverse of S is given by the map*

$$R^t\colon\mathcal{FV}(\Omega,E)\to\mathcal{FV}(\Omega)\varepsilon E,\ f\mapsto\mathcal{J}^{-1}\circ R^t_f,$$

where $\mathcal{J}\colon E\to E'^{\star}$ is the canonical injection and

$$R^t_f\colon\mathcal{FV}(\Omega)'\to E'^{\star},\ f'\longmapsto\big[e'\mapsto f'\big(R_f(e')\big)\big],$$

with $R_f(e')=e'\circ f$.

PROOF. Due to Theorem 3.1.12 we only have to show that S is surjective. We equip $\mathcal{J}(E)$ with the system of seminorms given by

$$p_{B^\circ_\alpha}(\mathcal{J}(x)):=\sup_{e'\in B^\circ_\alpha}|\mathcal{J}(x)(e')|=p_\alpha(x),\quad x\in E, \tag{12}$$

for every $\alpha\in\mathfrak{A}$. Let $f\in\mathcal{FV}(\Omega,E)$. We consider the dual map R^t_f and claim that $R^t_f\in L(\mathcal{FV}(\Omega)'_\kappa,\mathcal{J}(E))$. Indeed, we have

$$p_{B^\circ_\alpha}\big(R^t_f(y)\big)=\sup_{e'\in B^\circ_\alpha}\big|y\big(R_f(e')\big)\big|=\sup_{x\in R_f(B^\circ_\alpha)}|y(x)|\leq\sup_{x\in K_\alpha}|y(x)| \tag{13}$$

for all $y\in\mathcal{FV}(\Omega)'$ where $K_\alpha:=\overline{R_f(B^\circ_\alpha)}$. Since $\mathcal{FV}(\Omega,E)\subset\mathcal{FV}(\Omega,E)_\kappa$, the set $R_f(B^\circ_\alpha)$ is absolutely convex and relatively compact, implying that K_α is absolutely convex and compact in $\mathcal{FV}(\Omega)$ by [89, 6.2.1 Proposition, p. 103]. Further, we have for all $e'\in E'$ and $x\in\Omega$

$$R^t_f(\delta_x)(e')=\delta_x(e'\circ f)=e'\big(f(x)\big)=\mathcal{J}\big(f(x)\big)(e') \tag{14}$$

and thus $R^t_f(\delta_x)\in\mathcal{J}(E)$.

a) Let E be complete and $f'\in\mathcal{FV}(\Omega)'$. Since the span of $\{\delta_x\mid x\in\Omega\}$ is dense in $\mathcal{F}(\Omega)'_\kappa$ by the bipolar theorem, there is a net (f'_τ) converging to f' in $\mathcal{FV}(\Omega)'_\kappa$ with $R^t_f(f'_\tau)\in\mathcal{J}(E)$ by (14). As

$$p_{B^\circ_\alpha}\big(R^t_f(f'_\tau)-R^t_f(f')\big)\underset{(13)}{\leq}\sup_{x\in K_\alpha}|(f'_\tau-f')(x)|\to 0, \tag{15}$$

for all $\alpha\in\mathfrak{A}$, we gain that $(R^t_f(f'_\tau))$ is a Cauchy net in the complete space $\mathcal{J}(E)$. Hence it has a limit $g\in\mathcal{J}(E)$ which coincides with $R^t_f(f')$ since

$$p_{B^\circ_\alpha}\big(g-R^t_f(f')\big)\leq p_{B^\circ_\alpha}\big(g-R^t_f(f'_\tau)\big)+p_{B^\circ_\alpha}\big(R^t_f(f'_\tau)-R^t_f(f')\big)$$

$$\underset{(15)}{\leq} p_{B_\alpha^\circ}\big(g - R_f^t(f_\tau')\big) + \sup_{x\in K_\alpha}\big|(f_\tau' - f')(x)\big| \to 0$$

for all $\alpha \in \mathfrak{A}$. We conclude that $R_f^t(f') \in \mathcal{J}(E)$ for every $f' \in \mathcal{FV}(\Omega)'$.

b) Let Condition 3.2.3 b) hold and $f' \in \mathcal{FV}(\Omega)'$. Then there is a bounded net $(f_\tau')_{\tau\in\mathcal{T}}$ in $\mathcal{FV}(\Omega)'$ converging to f' in $\mathcal{FV}(\Omega)_\kappa'$ such that $R_f^t(f_\tau') \in \mathcal{J}(E)$ for every $\tau \in \mathcal{T}$. Due to (13) we obtain that $(R_f^t(f_\tau'))$ is a bounded Cauchy net in the quasi-complete space $\mathcal{J}(E)$ converging to $R_f^t(f') \in \mathcal{J}(E)$.

c) Let Condition 3.2.3 c) hold and $f' \in \mathcal{FV}(\Omega)'$. Then there is a sequence $(f_n')_{n\in\mathbb{N}}$ in $\mathcal{FV}(\Omega)'$ converging to f' in $\mathcal{FV}(\Omega)_\kappa'$ such that $R_f^t(f_n') \in \mathcal{J}(E)$ for every $n \in \mathbb{N}$. Again (13) implies that $(R_f^t(f_n'))$ is a Cauchy sequence in the sequentially complete space $\mathcal{J}(E)$ which converges to $R_f^t(f') \in \mathcal{J}(E)$.

d) Let Condition 3.2.3 d) hold and $f' \in \mathcal{FV}(\Omega)'$. Then there is an absolutely convex, bounded subset $D \subset \mathcal{FV}(\Omega)_\kappa'$ and a sequence $(f_n')_{n\in\mathbb{N}}$ in $\mathcal{FV}(\Omega)'$ converging to f' in $(\mathcal{FV}(\Omega)_\kappa')_D$ such that $R_f^t(f_n) \in \mathcal{J}(E)$ for every $n \in \mathbb{N}$. Let $r > 0$ and $f_n' - f_k' \in rD$. Then $R_f^t(f_n' - f_k') \in r(R_f^t(D) \cap \mathcal{J}(E))$, implying

$$\{r > 0 \mid f_n' - f_k' \in rD\} \subset \big\{r > 0 \mid R_f^t(f_n' - f_k') \in r\overline{(R_f^t(D) \cap \mathcal{J}(E))}^{\mathcal{J}(E)}\big\}.$$

Setting $B := \overline{R_f^t(D) \cap \mathcal{J}(E)}^{\mathcal{J}(E)}$, we derive

$$q_B(R_f^t(f_n' - f_k')) \leq q_D(f_n' - f_k')$$

where q_B and q_D are the gauge functionals of B resp. D. The set $R_f^t(D) \cap \mathcal{J}(E)$ is absolutely convex as the intersection of two absolutely convex sets and it is bounded by (13) and the boundedness of D. So B, being the closure of a disk, is a disk as well. Since (f_n') is a Cauchy sequence in $(\mathcal{FV}(\Omega)_\kappa')_D$, we conclude that $(R_f'(f_n'))$ is a Cauchy sequence in $\mathcal{J}(E)_B$. The set B is a closed disk in the locally complete space $\mathcal{J}(E)$ and hence a Banach disk by [89, 10.2.1 Proposition, p. 197]. Thus $\mathcal{J}(E)_B$ is a Banach space and $(R_f^t(f_n'))$ has a limit $g \in \mathcal{J}(E)_B$. The continuity of the canonical injection $\mathcal{J}(E)_B \hookrightarrow \mathcal{J}(E)$ implies that $(R_f^t(f_n'))$ converges to g in $\mathcal{J}(E)$ as well. As in a) we obtain that $R_f^t(f') = g \in \mathcal{J}(E)$.

e) Let Condition 3.2.3 e) be fulfilled. Let $f \in \mathcal{FV}(\Omega, E)$ and $e' \in E'$. For every $f' \in \mathcal{FV}(\Omega)'$ there are $j \in J$, $m \in M$ and $C > 0$ such that

$$|R_f^t(f')(e')| \leq C|R_f(e')|_{j,m} \underset{(11)}{=} C \sup_{x\in N_{j,m}(f)} |e'(x)|$$

because $(T_m^E, T_m^{\mathbb{K}})_{m\in M}$ is a strong generator. Since there is $K \in \tau(E)$ such that $N_{j,m}(f) \subset K$, we have

$$|R_f^t(f')(e')| \leq C \sup_{x\in K} |e'(x)|,$$

implying $R_f^t(f') \in (E_\tau')' = \mathcal{J}(E)$ by the Mackey–Arens theorem.

Therefore we obtain that $R_f^t \in L(\mathcal{FV}(\Omega)_\kappa', \mathcal{J}(E))$. So we get for all $\alpha \in \mathfrak{A}$ and $y \in \mathcal{F}(\Omega)'$

$$p_\alpha\big((\mathcal{J}^{-1} \circ R_f^t)(y)\big) \underset{(12)}{=} p_{B_\alpha^\circ}\big(\mathcal{J}((\mathcal{J}^{-1} \circ R_f^t)(y))\big) = p_{B_\alpha^\circ}\big(R_f^t(y)\big) \underset{(13)}{\leq} \sup_{x\in K_\alpha} |y(x)|.$$

This implies $\mathcal{J}^{-1} \circ R_f^t \in L(\mathcal{FV}(\Omega)_\kappa', E) = \mathcal{FV}(\Omega)\varepsilon E$ (as linear spaces) and we gain

$$S(\mathcal{J}^{-1} \circ R_f^t)(x) = \mathcal{J}^{-1}\big(R_f^t(\delta_x)\big) \underset{(14)}{=} \mathcal{J}^{-1}\big(\mathcal{J}(f(x))\big) = f(x)$$

for every $x \in \Omega$. Thus $S(\mathcal{J}^{-1} \circ R_f^t) = f$, proving the surjectivity of S. □

Further sufficient conditions for S being a topological isomorphism can be found in Proposition 5.2.10, Proposition 5.6.6 and Theorem 5.7.1. In particular, we get the following corollary as a special case of Theorem 3.2.4.

3.2.5. COROLLARY. *Let* $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ *be a strong, consistent generator for* $(\mathcal{FV}, E)$. *If*

(i) $\mathcal{FV}(\Omega)$ *is a semi-Montel space and* E *complete, or*
(ii) $\mathcal{FV}(\Omega)$ *is a Fréchet–Schwartz space and* E *locally complete, or*
(iii) E *is a semi-Montel space, or*
(iv) $\forall\, f \in \mathcal{FV}(\Omega, E),\, j \in J,\, m \in M\ \exists\, K \in \kappa(E):\ N_{j,m}(f) \subset K$,

then $\mathcal{FV}(\Omega)$ *and* $\mathcal{FV}(\Omega, E)$ *are* ε*-compatible, in particluar,* $\mathcal{FV}(\Omega, E) \cong \mathcal{FV}(\Omega)\varepsilon E$.

PROOF. (i) Follows from Lemma 3.2.2 a) and Theorem 3.2.4 with Condition 3.2.3 a).

(ii) If $\mathcal{FV}(\Omega)$ is a Fréchet–Schwartz space, then we have

$$\overline{\operatorname{span}\{\delta_x \mid x \in \Omega\}}^{\mathrm{lc}} = \overline{\operatorname{span}\{\delta_x \mid x \in \Omega\}}^{\mathcal{FV}(\Omega)'_b} = \overline{\operatorname{span}\{\delta_x \mid x \in \Omega\}}^{\mathcal{FV}(\Omega)'_\kappa} = \mathcal{FV}(\Omega)'$$

by [30, Lemma 6 (b), p. 231] and the bipolar theorem where $\overline{\operatorname{span}\{\delta_x \mid x \in \Omega\}}^{\mathrm{lc}}$ is the local closure of $\operatorname{span}\{\delta_x \mid x \in \Omega\}$ in $\mathcal{FV}(\Omega)'_b$. Hence for every $f' \in \mathcal{FV}(\Omega)'$ there is a sequence (f'_n) in the span of $\{\delta_x \mid x \in \Omega\}$ which converges locally to f' in $\mathcal{FV}(\Omega)'_\kappa$. Due to (14) we know that $R^t_f(f'_n) \in \mathcal{J}(E)$ for every $f \in \mathcal{FV}(\Omega, E)$ and $n \in \mathbb{N}$. Since Fréchet–Schwartz spaces are also semi-Montel spaces, the statement follows from Lemma 3.2.2 a) and Theorem 3.2.4 with Condition 3.2.3 d).

(iv) Follows from Lemma 3.2.2 c) and Theorem 3.2.4 with Condition 3.2.3 e).

(iii) Is a special case of (iv) since the set $K := \overline{\operatorname{acx}}(N_{j,m}(f))$ is absolutely convex and compact in the semi-Montel space E by [89, 6.2.1 Proposition, p. 103] and [89, 6.7.1 Proposition, p. 112] for every $f \in \mathcal{FV}(\Omega, E)$, $j \in J$ and $m \in M$. □

3.2.6. REMARK. Linearisations of spaces $\mathcal{FV}(\Omega, E)_\sigma$ of weak E-valued functions, where $\mathcal{FV}(\Omega)$ need not be a dom-space, are treated in [118].

Let us apply our preceding results to our weighted spaces of k-times continuously partially differentiable functions on an open set $\Omega \subset \mathbb{R}^d$ with $k \in \mathbb{N}_\infty$.

3.2.7. EXAMPLE. Let E be an lcHs, $k \in \mathbb{N}_\infty$, $\mathcal{V}^k$ be a directed family of weights which is locally bounded away from zero on an open set $\Omega \subset \mathbb{R}^d$.

a) $\mathcal{CV}^k(\Omega, E) \cong \mathcal{CV}^k(\Omega)\varepsilon E$ if E is a semi-Montel space and $\mathcal{CV}^k(\Omega)$ barrelled.
b) $\mathcal{CV}^k_{P(\partial)}(\Omega, E) \cong \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$ if E is a semi-Montel space and $\mathcal{CV}^k_{P(\partial)}(\Omega)$ barrelled.
c) $\mathcal{CV}^k(\Omega, E) \cong \mathcal{CV}^k(\Omega)\varepsilon E$ if E is complete and $\mathcal{CV}^k(\Omega)$ a Montel space.
d) $\mathcal{CV}^k_{P(\partial)}(\Omega, E) \cong \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$ if E is complete and $\mathcal{CV}^k_{P(\partial)}(\Omega)$ a Montel space.
e) $\mathcal{CV}^k(\Omega, E) \cong \mathcal{CV}^k(\Omega)\varepsilon E$ if E is locally complete and $\mathcal{CV}^k(\Omega)$ a Fréchet–Schwartz space.
f) $\mathcal{CV}^k_{P(\partial)}(\Omega, E) \cong \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$ if E is locally complete and $\mathcal{CV}^k_{P(\partial)}(\Omega)$ a Fréchet–Schwartz space.

PROOF. The generator of $(\mathcal{CV}^k, E)$ and $(\mathcal{CV}^k_{P(\partial)}, E)$ is strong and consistent by Proposition 3.1.10. From Corollary 3.2.5 (iii) we deduce part a) and b), from (i) part c) and d) and from (ii) part e) and f). □

Closed subspaces of Fréchet–Schwartz spaces are also Fréchet–Schwartz spaces by [131, Proposition 24.18, p. 284]. The space $\mathcal{CV}^\infty_{P(\partial)}(\Omega)$ is closed in $\mathcal{CV}^\infty(\Omega)$ if there is an lcHs Y such that $P(\partial)_{|\mathcal{CV}^\infty(\Omega)}\colon \mathcal{CV}^\infty(\Omega) \to Y$ is continuous. For example, this is fulfilled if the coefficients of $P(\partial)$ belong to $\mathcal{C}(\Omega)$, in particular, if $P(\partial) := \Delta$ or $\overline{\partial}$, with $Y := (\mathcal{C}(\Omega), \tau_c)$ due to $\mathcal{V}^\infty$ being locally bounded away from zero. The spaces $\mathcal{CV}^k(\Omega)$ from Example 3.1.9 a)(i) with $\omega_m := M_m \times \Omega$ for all $m \in \mathbb{N}_0$, where $M_m := \{\beta \in \mathbb{N}^d_0 \mid |\beta| \leq \min(m, k)\}$, are Fréchet spaces and thus barrelled if the J

in $\mathcal{V}^k := (\nu_{j,m})_{j\in J, m\in\mathbb{N}_0}$ is countable by [107, Proposition 3.7, p. 240]. Sufficient conditions on the weights that guarantee that $\mathcal{CV}^\infty(\Omega)$ is a nuclear Fréchet space and hence a Schwartz space as well can be found in [111, Theorem 3.1, p. 188]. For the case $\omega_m = \mathbb{N}_0^d \times \Omega$ see the references given in [111, p. 1].

If $\mathcal{V}^k = \mathcal{W}^k$, i.e. $\mathcal{C}^k(\Omega, E)$ is equipped with its usual topology of uniform convergence of all partial derivatives up to order k on compact subsets of Ω, Example 3.2.7 c)+d) can be improved to quasi-complete E. For $\Omega = \mathbb{R}^d$ this can be found in [158, Proposition 9, p. 108, Théorème 1, p. 111] and for general open $\Omega \subset \mathbb{R}^d$ it is already mentioned in [94, (9), p. 236] (without a proof) that $\mathcal{CW}^k(\Omega, E) \cong \mathcal{CW}^k(\Omega)\varepsilon E$ for $k \in \mathbb{N}_\infty$ and quasi-complete E. For $k = \infty$ we even have $\mathcal{CW}^\infty(\Omega, E) \cong \mathcal{CW}^\infty(\Omega)\varepsilon E$ for locally complete E by [30, p. 228]. Our technique allows us to generalise the first result and to get back the second result.

3.2.8. Example. Let E be an lcHs, $k \in \mathbb{N}_\infty$ and $\Omega \subset \mathbb{R}^d$ open. If $k < \infty$ and E has metric ccp, or if $k = \infty$ and E is locally complete, then

a) $\mathcal{CW}^k(\Omega, E) \cong \mathcal{CW}^k(\Omega)\varepsilon E$, and
b) $\mathcal{CW}^k_{P(\partial)}(\Omega, E) \cong \mathcal{CW}^k_{P(\partial)}(\Omega)\varepsilon E$ if $\mathcal{CW}^k_{P(\partial)}(\Omega)$ is closed in $\mathcal{CW}^k(\Omega)$.

Proof. We recall from Example 3.1.9 b) that $\mathcal{W}^k$ is the family of weights given by $\nu_{K,m}(\beta, x) := \chi_K(x)$, $(\beta, x) \in M_m \times \Omega$, for all $m \in \mathbb{N}_0$ and compact $K \subset \Omega$ where $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq \min(m,k)\}$ and χ_K is the characteristic function of K. We already know that the generator for $(\mathcal{CW}^k, E)$ and $(\mathcal{CW}^k_{P(\partial)}, E)$ is strong and consistent by Proposition 3.1.10 because $\mathcal{W}^k$ is locally bounded away from zero on Ω, and $\mathcal{CW}^k(\Omega)$ and its closed subspace $\mathcal{CW}^k_{P(\partial)}(\Omega)$ are Fréchet spaces. Let $f \in \mathcal{CW}^k(\Omega, E)$, $K \subset \Omega$ be compact, $m \in \mathbb{N}_0$ and consider

$$N_{K,m}(f) = \{(\partial^\beta)^E f(x)\nu_{K,m}(\beta, x) \mid x \in \Omega,\, \beta \in M_m\} = \{0\} \cup \bigcup_{\beta\in M_m} (\partial^\beta)^E f(K).$$

$N_{K,m}(f)$ is compact since it is a finite union of compact sets. Furthermore, the compact sets $\{0\}$ and $(\partial^\beta)^E f(K)$ are metrisable by [34, Chap. IX, §2.10, Proposition 17, p. 159] and thus their finite union $N_{K,m}(f)$ is metrisable as well by [169, Theorem 1, p. 361] since the compact set $N_{K,m}(f)$ is collectionwise normal and locally countably compact by [63, 5.1.18 Theorem, p. 305]. If E has metric ccp, then the set $\overline{\operatorname{acx}}(N_{K,m}(f))$ is absolutely convex and compact. Thus Corollary 3.2.5 (iv) settles the case for $k < \infty$. If $k = \infty$ and E is locally complete, we observe that $K_\beta := \overline{\operatorname{acx}}((\partial^\beta)^E f(K))$ for $f \in \mathcal{CW}^\infty(\Omega, E)$ is absolutely convex and compact by [29, Proposition 2, p. 354]. Then we have

$$N_{K,m}(f) \subset \operatorname{acx}\Big(\bigcup_{\beta\in M_m} K_\beta\Big)$$

and the set on the right-hand side is absolutely convex and compact by [89, 6.7.3 Proposition, p. 113]. Again, the statement follows from Corollary 3.2.5 (iv). □

The statement above for $k = \infty$ follows from Example 3.2.7 e)+f) as well because $\mathcal{CW}^\infty(\Omega)$ and its closed subspaces are Fréchet–Schwartz spaces. In the context of differentiability on infinite dimensional spaces the preceding example a) remains true for an open subset Ω of a Fréchet space or DFM-space and quasi-complete E by [129, 3.2 Corollary, p. 286]. Like here this can be generalised to E with [metric] ccp. A special case of example b) is already known to be a consequence of [30, Theorem 9, p. 232], namely, if $k = \infty$ and $P(\partial)^{\mathbb{K}}$ is hypoelliptic with constant coefficients. In particular, this covers the space of holomorphic functions and the space of harmonic functions. Holomorphy on infinite dimensional spaces is treated in [52, Corollary 6.35, p. 332–333] where $\mathcal{V} = \mathcal{W}^0$, Ω is an open subset of a locally convex Hausdorff k-space and E a quasi-complete locally convex Hausdorff space, both over $\mathbb{C}$, which can be generalised to E with [metric] ccp in a similar way.

For a second improvement of Example 3.2.7 for $k = \infty$ to locally complete E without the condition that $\mathcal{CV}^{\infty}(\Omega)$ resp. $\mathcal{CV}^{\infty}_{P(\partial)}(\Omega)$ is a Fréchet–Schwartz space we introduce the following conditions on the family $\mathcal{V}^{\infty}$ on $(M_m \times \Omega)_{m\in\mathbb{N}_0}$. We say that a family $\mathcal{V}^{\infty}$ of weights on $(M_m \times \Omega)_{m\in\mathbb{N}_0}$ is $\mathcal{C}^1$-*controlled* if

(*i*) $\forall\, j \in J,\, m \in \mathbb{N}_0,\, \beta \in M_m:\ \nu_{j,m}(\beta,\cdot) \in \mathcal{C}^1(\Omega)$,

(*ii*) $\forall\, j \in J,\, m \in \mathbb{N}_0,\, \beta,\gamma \in M_m, x \in \Omega:\ \nu_{j,m}(\beta,x) = \nu_{j,m}(\gamma,x)$,

(*iii*) $\forall\, j \in J,\, m \in \mathbb{N}_0\ \exists\, i \in J,\, k \in \mathbb{N}_0,\, k \geq m,\, C > 0\ \forall\, \beta \in M_m,\, x \in \Omega,\, 1 \leq n \leq d:$
$\left|\partial^{e_n}\nu_{j,m}(\beta,\cdot)\right|(x) \leq C\nu_{i,k}(\beta,x)$.

We say that family $\mathcal{V}^k$, $k \in \mathbb{N}_{\infty}$, fulfils condition (V_{∞}) if

$$\forall\, m \in \mathbb{N}_0,\, j \in J\ \exists\, n \in \mathbb{N}_{\geq m},\, i \in J\ \forall\, \varepsilon > 0\ \exists\, K \subset \Omega \text{ compact } \forall\, \beta \in M_m,\, x \in \Omega \setminus K:$$
$$\nu_{j,m}(\beta,x) \leq \varepsilon\nu_{i,n}(\beta,x)$$

where $\mathbb{N}_{\geq m} := \{n \in \mathbb{N}_0 \mid n \geq m\}$. Here (V_{∞}) stands for *vanishing at infinity* and the condition was introduced in [107, Remark 3.4, p. 239] and for $k = 0$ in [16, 1.3 Bemerkung, p. 189].

3.2.9. Example. Let E be an lcHs and $\mathcal{V}^{\infty}$ a directed $\mathcal{C}^1$-controlled family of weights on an open convex set $\Omega \subset \mathbb{R}^d$ which fulfils (V_{∞}). If E is locally complete, then

a) $\mathcal{CV}^{\infty}(\Omega,E) \cong \mathcal{CV}^{\infty}(\Omega)\varepsilon E$ if $\mathcal{CV}^{\infty}(\Omega)$ is barrelled, and
b) $\mathcal{CV}^{\infty}_{P(\partial)}(\Omega,E) \cong \mathcal{CV}^{\infty}_{P(\partial)}(\Omega)\varepsilon E$ if $\mathcal{CV}^{\infty}_{P(\partial)}(\Omega)$ is barrelled.

Proof. We already know that the generator for $(\mathcal{CV}^{\infty},E)$ and $(\mathcal{CV}^{\infty}_{P(\partial)},E)$ is strong and consistent by Proposition 3.1.10 because $\mathcal{V}^{\infty}$ is locally bounded away from zero on Ω as $\nu_{j,m}(\beta,\cdot)$ is continuous for all $j \in J$, $m \in \mathbb{N}_0$ and $\beta \in M_m$.

Let $f \in \mathcal{CV}^{\infty}(\Omega,E)$, $j \in J$, $m \in \mathbb{N}_0$ and $\beta \in M_m$. We set $g\colon\Omega \to E$, $g(x) := (\partial^{\beta})^E f(x)\nu_{j,m}(\beta,x)$, and note that

$$(\partial^{e_n})^E g(x) = (\partial^{\beta+e_n})^E f(x)\nu_{j,m}(\beta,x) + (\partial^{\beta})^E f(x)\big((\partial^{e_n})^{\mathbb{R}}\nu_{j,m}(\beta,\cdot)\big)(x), \quad x \in \Omega,$$

for all $1 \leq n \leq d$. Since $\mathcal{V}^{\infty}$ is directed and $\mathcal{C}^1$-controlled there are $i_1, i_2 \in J$, $k_1, k_2 \in \mathbb{N}_0$, $k_1 > m$, $k_2 \geq m$, and $C_1, C_2 > 0$ such that

$$\begin{aligned}
&p_{\alpha}((\partial^{e_n})^E g(x))\\
&\leq p_{\alpha}((\partial^{\beta+e_n})^E f(x))\nu_{j,m}(\beta,x) + p_{\alpha}((\partial^{\beta})^E f(x))\left|(\partial^{e_n})^{\mathbb{R}}\nu_{j,m}(\beta,\cdot)\right|(x)\\
&\leq C_1 p_{\alpha}((\partial^{\beta+e_n})^E f(x))\nu_{i_1,k_1}(\beta,x) + C_2 p_{\alpha}((\partial^{\beta})^E f(x))\nu_{i_2,k_2}(\beta,x)\\
&= C_1 p_{\alpha}((\partial^{\beta+e_n})^E f(x))\nu_{i_1,k_1}(\beta+e_n,x) + C_2 p_{\alpha}((\partial^{\beta})^E f(x))\nu_{i_2,k_2}(\beta,x)
\end{aligned}$$

for all $1 \leq n \leq d$ and $\alpha \in \mathfrak{A}$, which implies

$$\sup_{\substack{x\in\Omega\\ \gamma\in\mathbb{N}_0^d,|\gamma|\leq 1}} p_{\alpha}((\partial^{\gamma})^E g(x)) \leq |f|_{j,m,\alpha} + C_1|f|_{i_1,k_1,\alpha} + C_2|f|_{i_2,k_2,\alpha}.$$

Thus g is (weakly) $\mathcal{C}^1_b$.

Due to (V_{∞}) there are $n \in \mathbb{N}_{\geq m}$ and $i \in J$ such that for all $\varepsilon > 0$ there is a compact set $K \subset \Omega$ such that for all $\beta \in M_m$ and $x \in \Omega \setminus K$ we have

$$\nu_{j,m}(\beta,x) \leq \varepsilon\nu_{i,n}(\beta,x).$$

Since $\mathcal{V}^{\infty}$ is directed, we may assume w.l.o.g. that $\nu_{j,m}(\beta,x) \leq \nu_{i,n}(\beta,x)$ for all $x \in \Omega$. This implies that the zeros of $\nu_{i,n}(\beta,\cdot)$ are zeros of $\nu_{j,m}(\beta,\cdot)$. We define $h\colon\Omega \to [0,\infty)$ by $h(x) := \nu_{i,n}(\beta,x)/\nu_{j,m}(\beta,x)$ for $x \in \Omega$ with $\nu_{j,m}(\beta,x) \neq 0$ and

$h(x) := 1$ if $\nu_{j,m}(\beta,x) = 0$. We note that $h(x) > 0$ for all $x \in \Omega$ as the zeros of $\nu_{i,n}(\beta,\cdot)$ are contained in the zeros of $\nu_{j,m}(\beta,\cdot)$. It follows that

$$(\partial^{\beta})^{E} f(x)\nu_{j,m}(\beta,x)h(x) = (\partial^{\beta})^{E} f(x)\nu_{i,n}(\beta,x)$$

for $x \in \Omega$ with $\nu_{j,m}(\beta,x) \neq 0$ and $(\partial^{\beta})^{E} f(x)\nu_{j,m}(\beta,x)h(x) = 0$ for $x \in \Omega$ with $\nu_{j,m}(\beta,x) = 0$. Therefore $(\partial^{\beta})^{E} f\nu_{j,m}(\beta,\cdot)h$ is bounded on Ω. Further,

$$\varepsilon h(x) = \varepsilon\nu_{i,n}(\beta,x)/\nu_{j,m}(\beta,x) \geq 1$$

for $x \in \Omega \smallsetminus K$ with $\nu_{j,m}(\beta,x) \neq 0$ because (V_{∞}) is fulfilled. Further, the zeros of $\nu_{j,m}(\beta,\cdot)$ are contained in $N := \{x \in \Omega \mid (\partial^{\beta})^{E} f(x)\nu_{j,m}(\beta,x) = 0\}$. This yields that $K_{\beta} := \overline{\operatorname{acx}}((\partial^{\beta})^{E} f\nu_{j,m}(\beta,\cdot)(\Omega))$ is absolutely convex and compact by Proposition A.1.4 and A.1.5. Furthermore,

$$N_{j,m}(f) = \{(\partial^{\beta})^{E} f(x)\nu_{j,m}(\beta,x) \mid x \in \Omega,\, \beta \in M_m\} \subset \operatorname{acx}\Big(\bigcup_{\beta\in M_m} K_{\beta}\Big)$$

and the set on the right-hand side is absolutely convex and compact by [89, 6.7.3 Proposition, p. 113]. Finally, our statement follows from Corollary 3.2.5 (iv). □

For the Schwartz space $\mathcal{S}(\mathbb{R}^d,E)$ and the multiplier space $\mathcal{O}_M(\mathbb{R}^d,E)$ from Example 3.1.9 c) and d) an improvement of Example 3.2.7 c) to quasi-complete E is already known, see e.g. [158, Proposition 9, p. 108, Théorème 1, p. 111]. However, due to Example 3.2.9 it is even allowed that E is only locally complete.

3.2.10. COROLLARY. *If E is a locally complete lcHs, then $\mathcal{S}(\mathbb{R}^d,E) \cong \mathcal{S}(\mathbb{R}^d)\varepsilon E$ and $\mathcal{O}_M(\mathbb{R}^d,E) \cong \mathcal{O}_M(\mathbb{R}^d)\varepsilon E$.*

PROOF. We start with the Schwartz space. Due to Example 3.2.9 a) and the barrelledness of the Fréchet space $\mathcal{S}(\mathbb{R}^d)$ we only need to check that its directed family $\mathcal{V}^{\infty} := (\nu_{1,m})_{m\in\mathbb{N}_0}$ of weights given by $\nu_{1,m}(\beta,x) := (1+|x|^2)^{m/2}$, $x \in \mathbb{R}^d$, for $m \in \mathbb{N}_0$ and $\beta \in M_m$ is $\mathcal{C}^1$-controlled and fulfils (V_{∞}). Obviously, condition (i) and (ii) are fulfilled. Since

$$\big|\partial^{e_n}\nu_{1,m}(\beta,\cdot)\big|(x) = (m/2)(1+|x|^2)^{(m/2)-1}2|x_n| \leq m(1+|x|^2)^{m/2} = m\nu_{1,m}(\beta,x)$$

for all $x \in \mathbb{R}^d$ and $1 \leq n \leq d$, condition (iii) is also fulfilled. Thus $\mathcal{V}^{\infty}$ is $\mathcal{C}^1$-controlled. Noting that for every $m \in \mathbb{N}$ and $\varepsilon > 0$ there is $r > 0$ such that

$$\frac{(1+|x|^2)^{m/2}}{(1+|x|^2)^{m}} = (1+|x|^2)^{-m/2} \leq \varepsilon$$

for all $x \notin \overline{\mathbb{B}_r(0)}$, we obtain that

$$\nu_{1,m}(\beta,x) \leq \varepsilon\nu_{1,2m}(\beta,x)$$

for all $x \notin \overline{\mathbb{B}_r(0)}$ and $\beta \in M_m$. Hence $\mathcal{V}^{\infty}$ fulfils condition (V_{∞}).

Now, let us consider the multiplier space. We already know that the generator for $(\mathcal{O}_M,E)$ is strong and consistent by Proposition 3.1.10 because $\mathcal{O}_M(\mathbb{R})$ is a Montel space, thus barrelled, by [83, Chap. II, §4, n°4, Théorème 16, p. 131] and its family of weights is continuous on $\mathbb{R}^d$, thus locally bounded away from zero.

Let $f \in \mathcal{O}_M(\mathbb{R},E)$, $g \in \mathcal{S}(\mathbb{R}^d)$, $m \in \mathbb{N}_0$ and $\beta \in M_m$. Then $(\partial^{\beta})^{E} f \in \mathcal{O}_M(\mathbb{R},E)$ and hence $((\partial^{\beta})^{E} f)g \in \mathcal{S}(\mathbb{R}^d,E)$, which implies that $((\partial^{\beta})^{E} f)g \in \mathcal{C}^1_b(\mathbb{R}^d,E)$. Moreover, we choose $h\colon \mathbb{R}^d \to (0,\infty)$, $h(x) := 1+|x|^2$. Then $((\partial^{\beta})^{E} f)gh$ is bounded on $\mathbb{R}^d$ and for $\varepsilon > 0$ there is $r > 0$ such that $(1+|x|^2)^{-1} \leq \varepsilon$ for all $x \notin \overline{\mathbb{B}_r(0)}$, yielding that $K_{\beta,g} := \overline{\operatorname{acx}}(((\partial^{\beta})^{E} f)g(\mathbb{R}^d))$ is absolutely convex and compact by Proposition A.1.4 and A.1.5. Let $j \subset \mathcal{S}(\mathbb{R}^d)$ be finite. Since for each $x \in \mathbb{R}^d$ we have $(\partial^{\beta})^{E} f(x)\max_{g\in j}|g(x)| = \mathrm{e}^{\mathrm{i}\theta}(\partial^{\beta})^{E} f(x)\widetilde{g}(x)$ for some $\widetilde{g} \in j$ and $\theta \in [0,2\pi)$, we get

$$N_{j,m}(f) = \{(\partial^{\beta})^{E} f(x)\max_{g\in j}|g(x)| \mid x \in \mathbb{R}^d,\, \beta \in M_m\} \subset \operatorname{acx}\Big(\bigcup_{\beta\in M_m, g\in j} K_{\beta,g}\Big).$$

The set on the right-hand side is absolutely convex and compact by [89, 6.7.3 Proposition, p. 113]. Finally, our statement follows from Corollary 3.2.5 (iv). □

For an alternative proof in the case of the Schwartz space we may also use Example 3.2.7 e) since $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet–Schwartz space. Example 3.2.9 can also be used for an alternative proof of Example 3.2.8 if $k=\infty$ by observing that $\mathcal{CW}^\infty(\Omega,E)=\mathcal{CV}^\infty(\Omega,E)$ for any lcHs E where $\mathcal{V}^\infty:=\{\nu\in\mathcal{C}_c^\infty(\Omega)\mid \nu\geq 0\}$ and $\mathcal{C}_c^\infty(\Omega)$ is the space of functions in $\mathcal{C}^\infty(\Omega)$ with compact support.

Now, we improve Example 3.2.7 for the special case of spaces of ultradifferentiable functions $\mathcal{E}^{(M_p)}(\Omega,E)$ and $\mathcal{E}^{\{M_p\}}(\Omega,E)$ from Example 3.1.9 e) and f) where $\omega_m:=\mathbb{N}_0^d\times\Omega$ for all $m\in\mathbb{N}_0$. For this purpose we recall the following conditions of Komatsu for the sequence $(M_p)_{p\in\mathbb{N}_0}$ (see [99, p. 26] and [101, p. 653]):

(M.0) $M_0=M_1=1$,
(M.1) $\forall\, p\in\mathbb{N}:\ M_p^2\leq M_{p-1}M_{p+1}$,
(M.2)' $\exists\, A,C>0\ \forall\, p\in\mathbb{N}_0:\ M_{p+1}\leq AC^{p+1}M_p$,
(M.3)' $\sum_{p=1}^\infty \frac{M_{p-1}}{M_p}<\infty$.

3.2.11. EXAMPLE. Let E be an lcHs, $\Omega\subset\mathbb{R}^d$ open and $(M_p)_{p\in\mathbb{N}_0}$ a sequence of positive real numbers.

a) $\mathcal{E}^{(M_p)}(\Omega,E)\cong\mathcal{E}^{(M_p)}(\Omega)\varepsilon E$ if E is locally complete.
b) $\mathcal{E}^{\{M_p\}}(\Omega,E)\cong\mathcal{E}^{\{M_p\}}(\Omega)\varepsilon E$ if E is complete or semi-Montel and in both cases $(M_p)_{p\in\mathbb{N}_0}$ fulfils (M.1) and (M.3)'.
c) $\mathcal{E}^{\{M_p\}}(\Omega,E)\cong\mathcal{E}^{\{M_p\}}(\Omega)\varepsilon E$ if E is sequentially complete and $(M_p)_{p\in\mathbb{N}_0}$ fulfils (M.0), (M.1), (M.2)' and (M.3)'.

PROOF. The generator is strong and consistent by Proposition 3.1.10 since the family of weights given in Example 3.1.9 e) resp. f) is locally bounded away from zero on Ω and $\mathcal{E}^{(M_p)}(\Omega)$ is a Fréchet–Schwartz space in a) by [99, Theorem 2.6, p. 44] whereas $\mathcal{E}^{\{M_p\}}(\Omega)$ is a Montel space in b) and c) by [99, Theorem 5.12, p. 65–66]. Hence the statements a) and b) follow from Example 3.2.7.

Let us turn to c). We note that $\mathcal{E}^{\{M_p\}}(\Omega,E)\subset\mathcal{E}^{\{M_p\}}(\Omega,E)_\kappa$ by Lemma 3.2.2 a) for any lcHs E. Further, we claim that Condition 3.2.3 c) is fulfilled. Let $f'\in\mathcal{E}^{\{M_p\}}(\Omega)'$. Due to [101, Proposition 3.7, p. 677] there is a sequence $(f_n)_{n\in\mathbb{N}}$ in the space $\mathcal{D}^{\{M_p\}}(\Omega)$ of ultradifferentiable functions of class $\{M_p\}$ of Roumieu-type with compact support which converges to f' in $\mathcal{E}^{\{M_p\}}(\Omega)_b'$. Let $f\in\mathcal{E}^{\{M_p\}}(\Omega,E)$. We observe that for every $e'\in E'$

$$|R_f^t(f_n)(e')|=\Big|\int\limits_\Omega f_n(x)e'(f(x))\mathrm{d}x\Big|\leq\lambda\big(\operatorname{supp}(f_n)\big)\sup_{y\in K_n(f)}|e'(y)|$$

where λ is the Lebesgue measure, $\operatorname{supp}(f_n)$ is the support of f_n and $K_n(f):=\{f_n(x)f(x)\mid x\in\operatorname{supp}(f_n)\}$. The set $K_n(f)$ is compact and metrisable by [34, Chap. IX, §2.10, Proposition 17, p. 159] and thus the closure of its absolutely convex hull is compact in E as the sequentially complete space E has metric ccp. We conclude that $R_f^t(f_n)\in(E_\kappa')'=\mathcal{J}(E)$ for every $n\in\mathbb{N}$. Therefore Condition 3.2.3 c) is fulfilled, implying statement c) for sequentially complete E by Theorem 3.2.4. □

The results a) and b) in this example are new whereas c) is already proved in [101, Theorem 3.10, p. 678] in a different way. In particular, part a) improves [101, Theorem 3.10, p. 678] since Komatsu's conditions (M.0), (M.1), (M.2)' and (M.3)' are not needed and the condition that E is sequentially complete is weakened to local completeness. We included c) to demonstrate an application of Condition 3.2.3 c).

CHAPTER 4

Consistency

4.1. The spaces $\mathrm{AP}(\Omega, E)$ and consistency

This section is dedicated to the properties of functions which are compatible with the ε-product in the sense that the space of functions having these properties can be chosen as the space $\mathrm{AP}(\Omega, E)$ or $\bigcap_{m\in M} \operatorname{dom} T^E_m$ in the Definition 3.1.7 b) of consistency. This is done in a quite general way so that we are not tied to certain spaces and have to redo our argumentation, for example, if we consider the same generator $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ for two different spaces of functions.

Due to the linearity and continuity of $u \in \mathcal{FV}(\Omega)\varepsilon E$ for a dom-space $\mathcal{FV}(\Omega)$ and $S(u) = u \circ \delta$ with $\delta\colon \Omega \to \mathcal{FV}(\Omega)'$, $x \mapsto \delta_x$, these are properties which are purely pointwise or given by pointwise approximation. Among such properties of functions are continuity by Proposition 4.1.1, Cauchy continuity by Proposition 4.1.3, uniform continuity by Proposition 4.1.5, continuous extendability by Proposition 4.1.7, continuous differentiability by Proposition 3.1.10, vanishing at infinity by Proposition 4.1.9 and purely pointwise properties of a function like vanishing on a set by Proposition 4.1.10.

We collect these properties in propositions and in follow-up lemmas we handle properties which can be described by compositions of defining operators $T^E_{m_1} \circ T^E_{m_2}$ like continuous differentiability (of higher order) of Fourier transformations (see Example 4.2.26). We fix the following notation for this section. For a dom-space $\mathcal{FV}(\Omega)$ and linear $T\colon \mathcal{FV}(\Omega) \to \mathbb{K}^\Omega$ we set $(\delta \circ T)(x)(f) := (\delta_x \circ T)(f) := T(f)(x)$ for all $x \in \Omega$ and $f \in \mathcal{FV}(\Omega)$.

4.1.1. Proposition (continuity). *Let Ω be a topological Hausdorff space and $\mathcal{FV}(\Omega)$ a* dom-*space such that $\mathcal{FV}(\Omega) \subset \mathcal{C}(\Omega)$ as a linear subspace. Then $S(u) \in \mathcal{C}(\Omega, E)$ for all $u \in \mathcal{FV}(\Omega)\varepsilon E$ if $\delta \in \mathcal{C}(\Omega, \mathcal{FV}(\Omega)'_\kappa)$.*

Proof. Let $u \in \mathcal{FV}(\Omega)\varepsilon E$. Since $S(u) = u \circ \delta$ and $\delta \in \mathcal{C}(\Omega, \mathcal{FV}(\Omega)'_\kappa)$, we obtain that $S(u)$ is in $\mathcal{C}(\Omega, E)$. □

Now, we tackle the problem of the continuity of $\delta\colon \Omega \to \mathcal{FV}(\Omega)'_\kappa$ in the proposition above and phrase our solution in a way such that it can be applied to show the continuity of the partial derivative $(\partial^\beta)^E(S(u))$ as well (see Proposition 3.1.11). We recall that a topological space Ω is called *completely regular* if for any non-empty closed subset $A \subset \Omega$ and $x \in \Omega \setminus A$ there is $f \in \mathcal{C}(\Omega, [0,1])$ such that $f(x) = 0$ and $f(z) = 1$ for all $z \in A$ (see [88, Definition 11.1, p. 180]). Examples of completely regular spaces are uniformisable, particularly metrisable, spaces by [88, Proposition 11.5, p. 181] and locally convex Hausdorff spaces by [65, Proposition 3.27, p. 95]. A completely regular space Ω is a $k_{\mathbb{R}}$*-space* if for any completely regular space Y and any map $f\colon \Omega \to Y$, whose restriction to each compact $K \subset \Omega$ is continuous, the map is already continuous on Ω (see [37, (2.3.7) Proposition, p. 22]). Examples of $k_{\mathbb{R}}$-spaces are completely regular k-spaces by [63, 3.3.21 Theorem, p. 152]. A topological space Ω is called k*-space* (compactly generated space) if it satisfies the following condition: $A \subset \Omega$ is closed if and only if $A \cap K$ is closed in K for every compact $K \subset \Omega$. Every locally compact Hausdorff space is a completely regular

k-space. Further, every sequential Hausdorff space is a k-space by [63, 3.3.20 Theorem, p. 152], in particular, every first-countable Hausdorff space. Thus metrisable spaces are completely regular Hausdorff k-spaces. Moreover, the dual space (X', τ_c) with the topology of compact convergence τ_c is an example of a completely regular Hausdorff k-space that is neither locally compact nor metrisable by [178, p. 267] if X is an infinite-dimensional Fréchet space.

We denote by $\mathcal{CW}(\Omega)$ the space of scalar-valued continuous functions on a topological Hausdorff space Ω with the topology τ_c of compact convergence, i.e. the topology of uniform convergence on compact subsets, which itself is the weighted topology given by the family of weights $\mathcal{W} := \mathcal{W}^0 := \{\chi_K \mid K \subset \Omega \text{ compact}\}$, and by $\mathcal{C}_b(\Omega)$ the space of scalar-valued bounded, continuous functions on Ω with the topology of uniform convergence on Ω.

4.1.2. LEMMA. *Let Ω be a topological Hausdorff space, $\mathcal{FV}(\Omega)$ a* dom-*space and $T: \mathcal{FV}(\Omega) \to \mathcal{C}(\Omega)$ linear. Then $\delta \circ T \in \mathcal{C}(\Omega, \mathcal{FV}(\Omega)'_\gamma)$ in each of the subsequent cases:*

(i) Ω is a $k_{\mathbb{R}}$-space and $T: \mathcal{FV}(\Omega) \to \mathcal{CW}(\Omega)$ is continuous.
(ii) $T: \mathcal{FV}(\Omega) \to \mathcal{C}_b(\Omega)$ is continuous.

PROOF. First, if $x \in \Omega$ and $(x_\tau)_{\tau \in \mathcal{T}}$ is a net in Ω converging to x, then we observe that

$$(\delta_{x_\tau} \circ T)(f) = T(f)(x_\tau) \to T(f)(x) = (\delta_x \circ T)(f)$$

for every $f \in \mathcal{FV}(\Omega)$ as $T(f)$ is continuous on Ω.

(i) Verbatim as in Proposition 3.1.11 a).

(ii) There are $j \in J$, $m \in M$ and $C > 0$ such that

$$\sup_{x \in \Omega} |(\delta_x \circ T)(f)| = \sup_{x \in \Omega} |T(f)(x)| \leq C |f|_{\mathcal{FV}(\Omega),j,m}$$

for every $f \in \mathcal{FV}(\Omega)$. This means that $\{\delta_x \circ T \mid x \in \Omega\}$ is equicontinuous in $\mathcal{FV}(\Omega)'$, yielding the statement like before. □

The preceding lemma is just a modification of [16, 4.1 Lemma, p. 198] where $\mathcal{FV}(\Omega) = \mathcal{CV}(\Omega)$, the Nachbin-weighted space of continuous functions, and $T = \mathrm{id}$.

Next, we turn to Cauchy continuity. A function $f: \Omega \to E$ from a metric space Ω to an lcHs E is called *Cauchy continuous* if it maps Cauchy sequences to Cauchy sequences. We write $\mathcal{CC}(\Omega, E)$ for the space of Cauchy continuous functions from Ω to E and set $\mathcal{CC}(\Omega) := \mathcal{CC}(\Omega, \mathbb{K})$.

4.1.3. PROPOSITION (Cauchy continuity). *Let Ω be a metric space and $\mathcal{FV}(\Omega)$ a* dom-*space such that $\mathcal{FV}(\Omega) \subset \mathcal{CC}(\Omega)$ as a linear subspace. Then $S(u) \in \mathcal{CC}(\Omega, E)$ for all $u \in \mathcal{FV}(\Omega)\varepsilon E$ if $\delta \in \mathcal{CC}(\Omega, \mathcal{FV}(\Omega)'_\kappa)$.*

PROOF. Let $u \in \mathcal{FV}(\Omega)\varepsilon E$ and (x_n) a Cauchy sequence in Ω. Then (δ_{x_n}) is a Cauchy sequence in $\mathcal{FV}(\Omega)'_\kappa$ since $\delta \in \mathcal{CC}(\Omega, \mathcal{FV}(\Omega)'_\kappa)$. It follows that $(S(u)(x_n))$ is a Cauchy sequence in E because u is uniformly continuous and $u(\delta_{x_n}) = S(u)(x_n)$. Hence we conclude that $S(u) \in \mathcal{CC}(\Omega, E)$. □

For the next lemma we equip the space $\mathcal{CC}(\Omega)$ with the topology of uniform convergence on precompact subsets of Ω.

4.1.4. LEMMA. *Let $\mathcal{FV}(\Omega)$ be a* dom-*space and $T \in L(\mathcal{FV}(\Omega), \mathcal{CC}(\Omega))$ for a metric space Ω. Then $\delta \circ T \in \mathcal{CC}(\Omega, \mathcal{FV}(\Omega)'_\gamma)$.*

PROOF. Let (x_n) be a Cauchy sequence in Ω. We have $(\delta_{x_n} \circ T)(f) = T(f)(x_n)$ for every $f \in \mathcal{FV}(\Omega)$, which implies that $((\delta_{x_n} \circ T)(f))$ is a Cauchy sequence in $\mathbb{K}$ because $T(f) \in \mathcal{CC}(\Omega)$ by assumption. Since $\mathbb{K}$ is complete, it has a unique limit $T_\infty(f) := \lim_{n\to\infty} (\delta_{x_n} \circ T)(f)$ defining a linear functional in f. The set $N :=$

$\{x_n \mid n \in \mathbb{N}\}$ is precompact in Ω since Cauchy sequences are precompact. Hence there are $j \in J$, $m \in M$ and $C > 0$ such that

$$\sup_{n\in\mathbb{N}} |(\delta_{x_n} \circ T)(f)| = \sup_{x\in N} |T(f)(x)| \leq C|f|_{\mathcal{FV}(\Omega),j,m}$$

for every $f \in \mathcal{FV}(\Omega)$. Therefore the set $\{\delta_{x_n} \circ T \mid n \in \mathbb{N}\}$ is equicontinuous in $\mathcal{FV}(\Omega)'$, which implies that $T_\infty \in \mathcal{FV}(\Omega)'$ and the convergence of $(\delta_{x_n} \circ T)$ to T_∞ in $\mathcal{FV}(\Omega)'_\gamma$ due to the observation in the beginning and the fact that $\gamma(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$ and $\sigma(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$ coincide on equicontinuous sets. In particular, $(\delta_{x_n} \circ T)$ is a Cauchy sequence in $\mathcal{FV}(\Omega)'_\gamma$. Furthermore, for every $x \in \Omega$ we obtain from the choice $x_n = x$ for all $n \in \mathbb{N}$ that $\delta_x \circ T \in \mathcal{FV}(\Omega)'$. Thus the map $\delta \circ T \colon \Omega \to \mathcal{FV}(\Omega)'_\gamma$ is well-defined and Cauchy continuous. □

The subsequent proposition and lemma handle the analogous statements for uniform continuity. For a metric space Ω we denote by $\mathcal{C}_u(\Omega, E)$ the space of uniformly continuous functions from Ω to E and set $\mathcal{C}_u(\Omega) := \mathcal{C}_u(\Omega, \mathbb{K})$.

4.1.5. PROPOSITION (uniform continuity). *Let (Ω, d) be a metric space and $\mathcal{FV}(\Omega)$ a* dom*-space such that $\mathcal{FV}(\Omega) \subset \mathcal{C}_u(\Omega)$ as a linear subspace. Then $S(\mathsf{u}) \in \mathcal{C}_u(\Omega, E)$ for all $\mathsf{u} \in \mathcal{FV}(\Omega)\varepsilon E$ if $\delta \in \mathcal{C}_u(\Omega, \mathcal{FV}(\Omega)'_\kappa)$.*[1]

PROOF. Let (z_n), (x_n) be sequences in Ω with $\lim_{n\to\infty} \mathrm{d}(z_n, x_n) = 0$ and $\mathsf{u} \in \mathcal{FV}(\Omega)\varepsilon E$. Then $(\delta_{z_n} - \delta_{x_n})$ converges to 0 in $\mathcal{FV}(\Omega)'_\kappa$ because $\delta \in \mathcal{C}_u(\Omega, \mathcal{FV}(\Omega)'_\kappa)$. As a consequence $(S(\mathsf{u})(z_n) - S(\mathsf{u})(x_n))$ converges to 0 in E since u is uniformly continuous and $\mathsf{u}(\delta_{z_n} - \delta_{x_n}) = S(\mathsf{u})(z_n) - S(\mathsf{u})(x_n)$. Hence we conclude that $S(\mathsf{u}) \in \mathcal{C}_u(\Omega, E)$. □

For the next lemma we mean by $\mathcal{C}_{bu}(\Omega)$ the space of scalar-valued bounded, uniformly continuous functions equipped with the topology of uniform convergence on a metric space Ω.

4.1.6. LEMMA. *Let $\mathcal{FV}(\Omega)$ be a* dom*-space and $T \in L(\mathcal{FV}(\Omega), \mathcal{C}_{bu}(\Omega))$ for a metric space (Ω, d). Then $\delta \circ T \in \mathcal{C}_u(\Omega, \mathcal{FV}(\Omega)'_\gamma)$.*

PROOF. Let (z_n) and (x_n) be sequences in Ω such that $\lim_{n\to\infty} \mathrm{d}(z_n, x_n) = 0$. We have

$$(\delta_{z_n} \circ T - \delta_{x_n} \circ T)(f) = T(f)(z_n) - T(f)(x_n)$$

for every $f \in \mathcal{FV}(\Omega)$, which implies that $(\delta_{z_n} \circ T - \delta_{x_n} \circ T)(f)$ converges to 0 in $\mathbb{K}$ for every $f \in \mathcal{FV}(\Omega)$ because $T(f) \in \mathcal{C}_u(\Omega)$. There exist $j \in J$, $m \in M$ and $C > 0$ such that

$$\sup_{n\in\mathbb{N}} |(\delta_{z_n} \circ T - \delta_{x_n} \circ T)(f)| \leq 2 \sup_{x\in\Omega} |T(f)(x)| \leq 2C|f|_{\mathcal{FV}(\Omega),j,m}$$

for every $f \in \mathcal{FV}(\Omega)$. Therefore the set $\{\delta_{z_n} \circ T - \delta_{x_n} \circ T \mid n \in \mathbb{N}\}$ is equicontinuous in $\mathcal{FV}(\Omega)'$ and we conclude the statement like before. □

Let us turn to continuous extensions. Let X be a metric space and $\Omega \subset X$. We write $\mathcal{C}^{ext}(\Omega, E)$ for the space of functions $f \in \mathcal{C}(\Omega, E)$ which have a continuous extension to $\overline{\Omega}$ and set $\mathcal{C}^{ext}(\Omega) := \mathcal{C}^{ext}(\Omega, \mathbb{K})$.

4.1.7. PROPOSITION (continuous extendability). *Let X be a metric space, $\Omega \subset X$ and $\mathcal{FV}(\Omega)$ a* dom*-space such that $\mathcal{FV}(\Omega) \subset \mathcal{C}^{ext}(\Omega)$ as a linear subspace. Then $S(u) \in \mathcal{C}^{ext}(\Omega, E)$ for all $u \in \mathcal{FV}(\Omega)\varepsilon E$ if $\delta \in \mathcal{C}^{ext}(\Omega, \mathcal{FV}(\Omega)'_\kappa)$.*

[1] Here, we use the symbol u for elements in $\mathcal{FV}(\Omega)\varepsilon E$ instead of the usual u to avoid confusion with the index u of $\mathcal{C}_u(\Omega)$ resp. $\mathcal{C}_u(\Omega, E)$.

Proof. Let $u \in \mathcal{FV}(\Omega)\varepsilon E$. There is $\delta^{ext} \in \mathcal{C}(\overline{\Omega}, \mathcal{FV}(\Omega)'_\kappa)$ such that $\delta^{ext} = \delta$ on Ω since $\delta \in \mathcal{C}^{ext}(\Omega, \mathcal{FV}(\Omega)'_\kappa)$. Moreover, $u \circ \delta^{ext} \in \mathcal{C}(\overline{\Omega}, E)$ and equal to $S(u) = u \circ \delta$ on Ω, yielding $S(u) \in \mathcal{C}^{ext}(\Omega, E)$. □

For the next lemma we equip $\mathcal{C}^{ext}(\Omega)$ with the topology of uniform convergence on compact subsets of Ω.

4.1.8. Lemma. *Let X be a metric space, $\Omega \subset X$, $\mathcal{FV}(\Omega)$ a* dom*-space and $T \in L(\mathcal{FV}(\Omega), \mathcal{C}^{ext}(\Omega))$. Then $\delta \circ T \in \mathcal{C}^{ext}(\Omega, \mathcal{FV}(\Omega)'_\gamma)$ if $\mathcal{FV}(\Omega)$ is barrelled.*

Proof. From Lemma 4.1.2 (i) we derive that $\delta \circ T \in \mathcal{C}(\Omega, \mathcal{FV}(\Omega)'_\gamma)$. Let $x \in \partial\Omega$ and (x_n) be a sequence in Ω with $x_n \to x$. Then $(\delta_{x_n} \circ T)$ is a sequence in $\mathcal{FV}(\Omega)'$ and

$$\lim_{n\to\infty} (\delta_{x_n} \circ T)(f) = \lim_{n\to\infty} T(f)(x_n) =: (\delta_x^{ext} \circ T)(f)$$

in $\mathbb{K}$ for every $f \in \mathcal{FV}(\Omega)$, which implies that $(\delta_{x_n} \circ T)$ converges to $\delta_x^{ext} \circ T$ pointwise on $\mathcal{FV}(\Omega)$ because $T(f) \in \mathcal{C}^{ext}(\Omega)$. As a consequence of the Banach–Steinhaus theorem we get $(\delta_x^{ext} \circ T) \in \mathcal{FV}(\Omega)'$ and the convergence in $\mathcal{FV}(\Omega)'_\gamma$. □

Let $\mathcal{FV}(\Omega, E)$ be a dom-space, X a set, $\mathfrak{K}$ a family of sets and $\pi\colon \bigcup_{m\in M} \omega_m \to X$ such that $\bigcup_{K\in\mathfrak{K}} K \subset X$. We say that a function $f \in \bigcap_{m\in M} \operatorname{dom} T_m^E$ *vanishes at infinity in the weighted topology w.r.t.* $(\pi, \mathfrak{K})$ if

$$\forall\, \varepsilon > 0,\, j \in J,\, m \in M,\, \alpha \in \mathfrak{A}\; \exists\, K \in \mathfrak{K} : \sup_{\substack{x\in\omega_m,\\ \pi(x)\notin K}} p_\alpha\big(T_m^E(f)(x)\big)\nu_{j,m}(x) < \varepsilon. \tag{16}$$

Further, we set

$$\mathrm{AP}_{\pi,\mathfrak{K}}(\Omega, E) := \{f \in \bigcap_{m\in M} \operatorname{dom} T_m^E \mid f \text{ fulfils (16)}\}.$$

4.1.9. Proposition (vanishing at ∞ w.r.t. to $(\pi, \mathfrak{K})$). *Let $(T_m^E, T_m^{\mathbb{K}})_{m\in M}$ be the generator for $(\mathcal{FV}, E)$, let $\mathcal{FV}(\Omega, Y) \subset \mathrm{AP}_{\pi,\mathfrak{K}}(\Omega, Y)$ as a linear subspace for $Y \in \{\mathbb{K}, E\}$ and $\mathfrak{K}$ be closed under taking finite unions.*

(i) If for all $u \in \mathcal{FV}(\Omega)\varepsilon E$ it holds that $S(u) \in \bigcap_{m\in M} \operatorname{dom}(T_m^E)$ and

$$\forall\, m \in M,\, x \in \omega_m : \big(T_m^E S(u)\big)(x) = u(T_{m,x}^{\mathbb{K}}), \tag{17}$$

then $S(u) \in \mathrm{AP}_{\pi,\mathfrak{K}}(\Omega, E)$ for all $u \in \mathcal{FV}(\Omega)\varepsilon E$.

(ii) If for all $e' \in E'$ and $f \in \mathcal{FV}(\Omega, E)$ it holds that $e' \circ f \in \bigcap_{m\in M} \operatorname{dom}(T_m^{\mathbb{K}})$ and

$$\forall\, m \in M,\, x \in \omega_m : T_m^{\mathbb{K}}(e' \circ f)(x) = \big(e' \circ T_m^E(f)\big)(x), \tag{18}$$

then $e' \circ f \in \mathrm{AP}_{\pi,\mathfrak{K}}(\Omega)$ for all $e' \in E'$ and $f \in \mathcal{FV}(\Omega, E)$.

Proof. (i) We set $B_{j,m} := \{f \in \mathcal{FV}(\Omega) \mid |f|_{j,m} \leq 1\}$ for $j \in J$ and $m \in M$. Let $u \in \mathcal{FV}(\Omega)\varepsilon E$. The topologies $\sigma(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$ and $\kappa(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$ coincide on the equicontinuous set $B_{j,m}^\circ$ and we deduce that the restriction of u to $B_{j,m}^\circ$ is $\sigma(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$-continuous.

Let $\varepsilon > 0$, $j \in J$, $m \in M$, $\alpha \in \mathfrak{A}$ and set $U_{\alpha,\varepsilon} := \{x \in E \mid p_\alpha(x) < \varepsilon\}$. Then there are a finite set $N \subset \mathcal{FV}(\Omega)$ and $\eta > 0$ such that $u(f') \in U_{\alpha,\varepsilon}$ for all $f' \in V_{N,\eta}$ where

$$V_{N,\eta} := \{f' \in \mathcal{FV}(\Omega)' \mid \sup_{f\in N} |f'(f)| < \eta\} \cap B_{j,m}^\circ$$

because the restriction of u to $B_{j,m}^\circ$ is $\sigma(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$-continuous. Since $N \subset \mathcal{FV}(\Omega)$ is finite, $\mathcal{FV}(\Omega) \subset \mathrm{AP}_{\pi,\mathfrak{K}}(\Omega)$ and $\mathfrak{K}$ is closed under taking finite unions, there is $K \in \mathfrak{K}$ such that

$$\sup_{\substack{x\in\omega_m\\ \pi(x)\notin K}} |T_m^{\mathbb{K}}(f)(x)|\nu_{j,m}(x) < \eta \tag{19}$$

for every $f \in N$. It follows from (19) and (the proof of) Lemma 3.1.8 that

$$D_{\pi\notin K,j,m} := \{T^{\mathbb{K}}_{m,x}(\cdot)\nu_{j,m}(x) \mid x \in \omega_m,\, \pi(x) \notin K\} \subset V_{N,\eta}$$

and thus $u(D_{\pi\notin K,j,m}) \subset U_{\alpha,\varepsilon}$. Therefore we have

$$\sup_{\substack{x\in\omega_m\\ \pi(x)\notin K}} p_\alpha\big(T^E_m(S(u))(x)\big)\nu_{j,m}(x) \underset{(17)}{=} \sup_{\substack{x\in\omega_m\\ \pi(x)\notin K}} p_\alpha\big(u(T^{\mathbb{K}}_{m,x})\big)\nu_{j,m}(x) < \varepsilon.$$

Hence we conclude that $S(u) \in \mathrm{AP}_{\pi,\mathfrak{K}}(\Omega, E)$.

(ii) Let $\varepsilon > 0$, $f \in \mathcal{FV}(\Omega, E)$ and $e' \in E'$. Then there exist $\alpha \in \mathfrak{A}$ and $C > 0$ such that $|e'(x)| \leq C p_\alpha(x)$ for every $x \in E$. For $j \in J$ and $m \in M$ there is $K \in \mathfrak{K}$ such that

$$\sup_{\substack{x\in\omega_m\\ \pi(x)\notin K}} p_\alpha\big(T^E_m(f)(x)\big)\nu_{j,m}(x) < \frac{\varepsilon}{C}$$

since $\mathcal{FV}(\Omega, E) \subset \mathrm{AP}_{\pi,\mathfrak{K}}(\Omega, E)$. It follows that

$$\sup_{\substack{x\in\omega_m\\ \pi(x)\notin K}} |T^{\mathbb{K}}_m(e' \circ f)(x)|\nu_{j,m}(x) \underset{(18)}{=} \sup_{\substack{x\in\omega_m\\ \pi(x)\notin K}} \big|e'\big(T^E_m(f)(x)\big)\big|\nu_{j,m}(x) < C\frac{\varepsilon}{C} = \varepsilon,$$

yielding $e' \circ f \in \mathrm{AP}_{\pi,\mathfrak{K}}(\Omega)$. □

The first part of the proof above adapts an idea in the proof of [16, 4.4 Theorem, p. 199–200] where $(T^E_m, T^{\mathbb{K}}_m)_{m\in M} = (\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ which is a special case of our proposition.

Our last proposition of this section is immediate. For $\omega \subset \Omega$ we set $\mathrm{AP}_\omega(\Omega, E) := \{f \in E^\Omega \mid \forall\, x \in \omega :\ f(x) = 0\}$ and $\mathrm{AP}_\omega(\Omega) := \mathrm{AP}_\omega(\Omega, \mathbb{K})$.

4.1.10. Proposition (vanishing on a subset). *Let $\omega \subset \Omega$ and $\mathcal{FV}(\Omega)$ a dom-space such that $\mathcal{FV}(\Omega) \subset \mathrm{AP}_\omega(\Omega)$ as a linear subspace. Then $S(u) \in \mathrm{AP}_\omega(\Omega, E)$ for all $u \in \mathcal{FV}(\Omega)\varepsilon E$.*

4.2. Further examples of ε-products

In Chapter 3 we dealt with weighted spaces of continuously partially differentiable functions. Now, we treat many examples of weighted spaces $\mathcal{FV}(\Omega, E)$ of functions with less regularity on a set Ω with values in a locally convex Hausdorff space E over the field $\mathbb{K}$. Applying the results of the preceding sections, we give conditions on E such that $\mathcal{FV}(\Omega)$ and $\mathcal{FV}(\Omega, E)$ are ε-compatible, in particular, that

$$\mathcal{FV}(\Omega, E) \cong \mathcal{FV}(\Omega)\varepsilon E$$

holds. We start with the simplest example of all. Let Ω be a non-empty set and equip the space E^Ω with the topology of pointwise convergence, i.e. the locally convex topology given by the seminorms

$$|f|_{K,\alpha} := \sup_{x\in K} p_\alpha(f(x))\chi_K(x), \quad f \in E^\Omega,$$

for finite $K \subset \Omega$ and $\alpha \in \mathfrak{A}$. To prove $E^{\mathbb{N}_0} \cong \mathbb{K}^{\mathbb{N}_0}\varepsilon E$ for complete E is given as an exercise in [94, Aufgabe 10.5, p. 259], which we generalise now.

4.2.1. Example. Let Ω be a non-empty set and E an lcHs. Then $E^\Omega \cong \mathbb{K}^\Omega\varepsilon E$.

Proof. The strength and consistency of the generator $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ is obvious. Let $f \in E^\Omega$, $K \subset \Omega$ be finite and set $N_K(f) := f(\Omega)\chi_K(\Omega)$. Then we have $N_K(f) = f(K) \cup \{0\}$ if $K \neq \Omega$, and $N_K(f) = f(K)$ if $K = \Omega$. Thus $N_K(f)$ is finite, hence compact, $N_K(f) \subset \mathrm{acx}(f(K))$ and $\mathrm{acx}(f(K))$ is a subset of the finite dimensional subspace $\mathrm{span}(f(K))$ of E. It follows that $\mathrm{acx}(f(K))$ is compact by [89, 6.7.4 Proposition, p. 113], implying our statement by virtue of Corollary 3.2.5 (iv). □

The next example will give us the counterpart of Example 3.2.9 a) on the level of sequence spaces. Let Ω be a set, E an lcHs and $\mathcal{V} := (\nu_j)_{j\in J}$ a directed family of weights $\nu_j\colon \Omega \to [0,\infty)$ on Ω. We set

$$\ell\mathcal{V}(\Omega, E) := \{f \in E^{\Omega} \mid \forall\ j \in J,\ \alpha \in \mathfrak{A} :\ |f|_{j,\alpha} := \sup_{x\in\Omega} p_\alpha(f(x))\nu_j(x) < \infty\}$$

and $\ell\mathcal{V}(\Omega) := \ell\mathcal{V}(\Omega, \mathbb{K})$.

4.2.2. Example. Let E be an lcHs, (Ω, d) a *uniformly discrete metric space*, i.e. there is $r > 0$ such that $\mathrm{d}(x,y) \geq r$ for all $x, y \in \Omega$, $x \neq y$, and $\mathcal{V} := (\nu_j)_{j\in J}$ a directed family of weights on Ω such that

$$\forall\ j \in J\ \exists\ i \in J\ \forall\ \varepsilon > 0\ \exists\ K \subset \Omega \text{ compact } \forall\ x \in \Omega \smallsetminus K :\ \nu_j(x) \leq \varepsilon \nu_i(x). \tag{20}$$

If E is locally complete, then $\ell\mathcal{V}(\Omega, E) \cong \ell\mathcal{V}(\Omega)\varepsilon E$.

Proof. Let $f \in \ell\mathcal{V}(\Omega, E)$ and $j \in J$. Then $f\nu_j$ is bounded on Ω by definition of $\ell\mathcal{V}(\Omega, E)$. Since (Ω, d) is uniformly discrete, there is $r > 0$ such that

$$\frac{p_\alpha(f(x)\nu_j(x) - f(y)\nu_j(y))}{\mathrm{d}(x,y)} \leq \frac{2}{r}|f|_{j,\alpha} < \infty, \quad x, y \in \Omega,\ x \neq y,$$

for every $\alpha \in \mathfrak{A}$. Therefore $f\nu_j \in \mathcal{C}_b^{[1]}(\Omega, E)$ where

$$\mathcal{C}_b^{[1]}(\Omega, E) := \Big\{g \in E^{\Omega} \mid \forall \alpha \in \mathfrak{A} :\ \sup_{x\in\Omega} p_\alpha(g(x)) < \infty \text{ and } \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{p_\alpha(g(x) - g(y))}{\mathrm{d}(x,y)} < \infty\Big\}.$$

Due to (20) there is $i \in J$ such that for all $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $\nu_j(x) \leq \varepsilon\nu_i(x)$ for all $x \in \Omega \smallsetminus K$. As $\mathcal{V}$ is directed, we may assume w.l.o.g. that $\nu_j(x) \leq \nu_i(x)$ for all $x \in \Omega$. This implies that the zeros of ν_i are zeros of ν_j. We define $h\colon \Omega \to [0,\infty)$ by $h(x) := \nu_i(x)/\nu_j(x)$ for $x \in \Omega$ with $\nu_j(x) \neq 0$ and $h(x) := 1$ if $\nu_j(x) = 0$. We observe that $h(x) > 0$ for all $x \in \Omega$ as the zeros of ν_i are contained in the zeros of ν_j. It follows that

$$f(x)\nu_j(x)h(x) = f(x)\nu_i(x)$$

for $x \in \Omega$ with $\nu_j(x) \neq 0$ and $f(x)\nu_j(x)h(x) = 0$ for $x \in \Omega$ with $\nu_j(x) = 0$. Hence $f\nu_j h$ is bounded on Ω. Further,

$$\varepsilon h(x) = \varepsilon\nu_i(x)/\nu_j(x) \geq 1,$$

for $x \in \Omega \smallsetminus K$ with $\nu_j(x) \neq 0$ because (20) is fulfilled. Moreover, the zeros of ν_j are contained in $N := \{x \in \Omega \mid f(x)\nu_j(x) = 0\}$. This yields that $\overline{\operatorname{acx}}(f\nu_j(\Omega))$ is absolutely convex and compact by Proposition A.1.4. So our statement follows from Corollary 3.2.5 (iv). □

Let us apply the preceding result to some known sequence spaces. We recall that a matrix $A := (a_{k,j})_{k,j\in\mathbb{N}}$ of non-negative numbers is called *Köthe matrix* if it fulfils:

(1) $\forall\ k \in \mathbb{N}\ \exists\ j \in \mathbb{N} :\ a_{k,j} > 0$,
(2) $\forall\ k, j \in \mathbb{N} :\ a_{k,j} \leq a_{k,j+1}$.

We note that what we call k is usually called j and vice-versa (see e.g. [131, Definition, p. 326]). But the notation we chose is more in line with the meaning of j in our Definition 3.1.3 of a weight function and therefore we prefer to keep our notation consistent. For an lcHs E we define the *Köthe space*

$$\lambda^\infty(A, E) := \{x = (x_k) \in E^{\mathbb{N}} \mid \forall\ j \in \mathbb{N},\ \alpha \in \mathfrak{A} :\ |x|_{j,\alpha} := \sup_{k\in\mathbb{N}} p_\alpha(x_k)a_{k,j} < \infty\}$$

and the *spaces of E-valued rapidly decreasing sequences* which we need for some theorems on Fourier expansions (see Theorem 5.6.13, Theorem 5.6.14) by

$$s(\Omega, E) := \{x = (x_k) \in E^{\Omega} \mid \forall\, j \in \mathbb{N},\, \alpha \in \mathfrak{A} : \ |x|_{j,\alpha} := \sup_{k\in\Omega} p_\alpha(x_k)(1+|k|^2)^{j/2} < \infty\}$$

with $\Omega = \mathbb{N}^d$, $\mathbb{N}_0^d$, $\mathbb{Z}^d$. Further, we set $\lambda^\infty(A) := \lambda^\infty(A,\mathbb{K})$ and $s(\Omega) := s(\Omega,\mathbb{K})$.

4.2.3. Corollary. *Let E be a locally complete lcHs.*

a) If $A := (a_{k,j})_{k,j\in\mathbb{N}}$ is a Köthe matrix such that

$$\forall\, j \in \mathbb{N}\ \exists\, i \in \mathbb{N}\ \forall\, \varepsilon > 0\ \exists\, K \in \mathbb{N}\ \forall\, k \in \mathbb{N},\, k > K : \ a_{k,j} \leq \varepsilon a_{k,i}, \tag{21}$$

then $\lambda^\infty(A,E) \cong \lambda^\infty(A)\varepsilon E$.

b) $s(\Omega, E) \cong s(\Omega)\varepsilon E$ for $\Omega = \mathbb{N}^d$, $\mathbb{N}_0^d$, $\mathbb{Z}^d$.

Proof. We observe that $\mathbb{N}$ and Ω are uniformly discrete metric spaces if they are equipped with the metric induced by the absolute value. Further, a set in a discrete space is compact if and only if it is finite. In case b) we set $\nu_j\colon \Omega \to (0,\infty)$, $\nu_j(k) := (1+|k|^2)^{j/2}$ for $j \in \mathbb{N}$. Then for $\varepsilon > 0$ there is $K \in \mathbb{N}$ such that

$$\frac{(1+|k|^2)^{j/2}}{(1+|k|^2)^{j}} = (1+|k|^2)^{-j/2} \leq \varepsilon$$

for all $k \in \Omega$ with $|k| > K$. In both cases the family of weights are directed, in case a) due to condition (2) of the definition of a Köthe matrix. Hence we can apply Example 4.2.2 in both cases. □

Due to [131, Proposition 27.10, p. 330–331] condition (21) is equivalent to $\lambda^\infty(A)$ being a Schwartz space. Since $\lambda^\infty(A)$ is also a Fréchet space by [131, Lemma 27.1, p. 326], another way to prove Corollary 4.2.3 a) (and b) as well) is given by Corollary 3.2.5 (ii).

Our next examples are *Favard-spaces*. Let E be an lcHs, $0 < \gamma \leq 1$, Ω a compact Hausdorff space, $\varphi\colon [0,\infty) \times \Omega \to \Omega$ a continuous *semiflow*, i.e.

$$\varphi(t+r, s) = \varphi(t, \varphi(r,s)) \quad \text{and} \quad \varphi(0,s) = s, \quad t, r \in [0,\infty),\, s \in \Omega,$$

and $(\widetilde{T}_t^E)_{t\geq 0}$ the induced semigroup given by $\widetilde{T}_t^E\colon \mathcal{C}(\Omega,E) \to \mathcal{C}(\Omega,E)$, $\widetilde{T}_t^E(f) := f(\varphi(t,\cdot))$. The semigroup $(\widetilde{T}_t^{\mathbb{K}})_{t\geq 0}$ is (equi-)bounded and strongly continuous by [62, Chap. II, 3.31 Exercises (1), p. 95]. The vector-valued *Favard space* of order γ of the semigroup $(\widetilde{T}_t^E)_{t\geq 0}$ is defined by

$$F_\gamma(\Omega, E) := \{f \in \mathcal{C}(\Omega, E) \mid \forall\, \alpha \in \mathfrak{A} : \sup_{x\in\Omega, t>0} p_\alpha(\widetilde{T}_t^E(f)(x) - f(x))t^{-\gamma} < \infty\}$$

equipped with the system of seminorms given by

$$|f|_\alpha := \max\bigl(\sup_{x\in\Omega} p_\alpha(f(x)),\ \sup_{x\in\Omega, t>0} p_\alpha(\widetilde{T}_t^E(f)(x) - f(x))t^{-\gamma}\bigr), \quad f \in F_\gamma(\Omega, E),$$

for $\alpha \in \mathfrak{A}$ (see [39, Definition 3.1.2, p. 160] and [39, Proposition 3.1.3, p. 160]). Further, we set $F_\gamma(\Omega) := F_\gamma(\Omega,\mathbb{K})$. $F_\gamma(\Omega,E)$ is a dom-space, which follows from the setting $\omega := [0,\infty) \times \Omega$, $\operatorname{dom} T^E := \mathcal{C}(\Omega, E)$ and $T^E\colon \mathcal{C}(\Omega,E) \to E^\omega$ given by

$$T^E(f)(0,x) := f(x) \ \text{ and } \ T^E(f)(t,x) := \widetilde{T}_t^E(f)(x) - f(x), \quad t > 0,\, x \in \Omega,$$

as well as $\mathrm{AP}(\Omega, E) := E^\Omega$ and the weight given by $\nu(0,x) := 1$ and $\nu(t,x) := t^{-\gamma}$ for $t > 0$ and $x \in \Omega$.

4.2.4. Example. Let E be a semi-Montel space, $0 < \gamma \leq 1$, Ω a compact Hausdorff space, $\varphi\colon [0,\infty) \times \Omega \to \Omega$ a continuous semiflow. Then $F_\gamma(\Omega, E) \cong F_\gamma(\Omega)\varepsilon E$ holds for the Favard space of order γ of the induced semigroup $(\widetilde{T}_t^E)_{t\geq 0}$.

PROOF. The generator $(T^E, T^{\mathbb{K}})$ for $(\mathcal{F}_\gamma, E)$ is consistent by Proposition 4.1.1 and Lemma 4.1.2 b)(ii). Its strength is clear. Thus our statement follows from Corollary 3.2.5 (iii). □

The *space of càdlàg functions* on a set $\Omega \subset \mathbb{R}$ with values in an lcHs E is defined by

$$D(\Omega, E) := \{f \in E^\Omega \mid \forall\, x \in \Omega : \lim_{w\to x+} f(w) = f(x) \text{ and } \lim_{w\to x-} f(w) \text{ exists}\}.^2$$

Further, we set $D(\Omega) := D(\Omega, \mathbb{K})$. Due to Proposition A.1.1 the maps given by

$$|f|_{K,\alpha} := \sup_{x\in\Omega} p_\alpha(f(x))\chi_K(x), \quad f \in D(\Omega, E),$$

for compact $K \subset \Omega$ and $\alpha \in \mathfrak{A}$ form a system of seminorms inducing a locally convex Hausdorff topology on $D(\Omega, E)$.

4.2.5. EXAMPLE. Let E be an lcHs and $\Omega \subset \mathbb{R}$ locally compact. If E is quasi-complete, then $D(\Omega)\varepsilon E \cong D(\Omega, E)$.

PROOF. First, we show that the generator $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ for (D, E) is strong and consistent. The strength is a consequence of a simple calculation, so we only prove the consistency explicitly. We have to show that $S(u) \in D(\Omega, E)$ for all $u \in D(\Omega)\varepsilon E$. Let $x \in \Omega$ be an accumulation point of $[x, \infty) \cap \Omega$ resp. $(-\infty, x] \cap \Omega$, (x_n) be a sequence in Ω such that $x_n \to x+$ resp. $x_n \to x-$. We have

$$\delta_{x_n}(f) = f(x_n) \to f(x) = \delta_x(f), \quad x_n \to x+,$$

and

$$\delta_{x_n}(f) = f(x_n) \to \lim_{n\to\infty} f(x_n) =: T(f)(x), \quad x_n \to x-,$$

for every $f \in D(\Omega)$, which implies that (δ_{x_n}) converges to δ_x if $x_n \to x+$, and to $\delta_x \circ T$ if $x_n \to x-$ in $D(\Omega)'_\sigma$. Since Ω is locally compact, there are a compact neighbourhood $U(x) \subset \Omega$ of x and $n_0 \in \mathbb{N}$ such that $x_n \in U(x)$ for all $n \geq n_0$. Hence we deduce

$$\sup_{n\geq n_0} |\delta_{x_n}(f)| \leq |f|_{U(x)}$$

for every $f \in D(\Omega)$. Therefore the set $\{\delta_{x_n} \mid n \geq n_0\}$ is equicontinuous in $D(\Omega)'$, which implies that (δ_{x_n}) converges to δ_x if $x_n \to x+$ and to $\delta_x \circ T$ if $x_n \to x-$ in $D(\Omega)'_\gamma$ and thus in $D(\Omega)'_\kappa$. From

$$S(u)(x) = u(\delta_x) = \lim_{n\to\infty} u(\delta_{x_n}) = \lim_{n\to\infty} S(u)(x_n), \quad x_n \to x+,$$

and

$$u(\delta_x \circ T) = \lim_{n\to\infty} u(\delta_{x_n}) = \lim_{n\to\infty} S(u)(x_n), \quad x_n \to x-,$$

for every $u \in D(\Omega)\varepsilon E$ follows the consistency. Second, let $f \in D(\Omega, E)$, $K \subset \Omega$ be compact and consider $N_K(f) = f(\Omega)\chi_K(\Omega)$. We observe that $N_K(f) = \overline{f(K)} \cup \{0\}$ if $K \neq \Omega$, and $N_K(f) = f(K)$ if $K = \Omega$. We note that $N_K(f) \subset \overline{\mathrm{acx}}(\overline{f(K)})$ and $\overline{\mathrm{acx}}(\overline{f(K)})$ is absolutely convex and compact by Proposition A.1.1 because E is quasi-complete. Thus we derive our statement from Corollary 3.2.5 (iv). □

We turn to Cauchy continuous functions. Let Ω be a metric space, E an lcHs and the space $\mathcal{CC}(\Omega, E)$ of Cauchy continuous functions from Ω to E be equipped with the system of seminorms given by

$$|f|_{K,\alpha} := \sup_{x\in K} p_\alpha(f(x))\chi_K(x), \quad f \in \mathcal{CC}(\Omega, E),$$

for $K \subset \Omega$ precompact and $\alpha \in \mathfrak{A}$.

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[2] We note that for $x \in \Omega$ we only demand $\lim_{w\to x+} f(w) = f(x)$ if x is an accumulation point of $[x, \infty) \cap \Omega$, and the existence of the limit $\lim_{w\to x-} f(w)$ if x is an accumulation point of $(-\infty, x] \cap \Omega$.

4.2.6. Example. Let E be an lcHs and Ω a metric space. If E is a Fréchet space or a semi-Montel space, then $\mathcal{CC}(\Omega, E) \cong \mathcal{CC}(\Omega)\varepsilon E$.

Proof. The generator $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ for $(\mathcal{CC}, E)$ is consistent by Proposition 4.1.3 with Lemma 4.1.4. Its strength follows from the uniform continuity of every $e' \in E'$. First, we consider the case that E is a Fréchet space. Let $f \in \mathcal{CC}(\Omega, E)$, $K \subset \Omega$ be precompact and consider $N_K(f) = f(\Omega)\chi_K(\Omega)$. Then $N_K(f) = f(K) \cup \{0\}$ if $K \neq \Omega$, and $N_K(f) = f(K)$ if $K = \Omega$. The set $f(K)$ is precompact in the metrisable space E by [13, Proposition 4.11, p. 576]. Thus we obtain $\mathcal{CC}(\Omega, E) \subset \mathcal{CC}(\Omega, E)_\kappa$ by virtue of Lemma 3.2.2 c). Since E is complete, the first part of the statement follows from Theorem 3.2.4 with Condition 3.2.3 a). If E is a semi-Montel space, then it is a consequence of Corollary 3.2.5 (iii). □

Let (Ω, d) be a metric space, E an lcHs and the space $\mathcal{C}_{bu}(\Omega, E)$ of bounded uniformly continuous functions from Ω to E be equipped with the system of seminorms given by

$$|f|_\alpha := \sup_{x \in \Omega} p_\alpha(f(x)), \quad f \in \mathcal{C}_{bu}(\Omega, E),$$

for $\alpha \in \mathfrak{A}$.

4.2.7. Example. Let E be an lcHs and (Ω, d) a metric space. If E is a semi-Montel space, then $\mathcal{C}_{bu}(\Omega, E) \cong \mathcal{C}_{bu}(\Omega)\varepsilon E$.

Proof. The generator $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ for $(\mathcal{C}_{bu}, E)$ is consistent by Proposition 4.1.5 with Lemma 4.1.6. It is also strong due to the uniform continuity of every $e' \in E'$, yielding our statement by Corollary 3.2.5 (iii). □

4.2.8. Remark. If $\mathbb{N}$ is equipped with the metric induced by the absolut value, then $\mathcal{C}_{bu}(\mathbb{N}, E) = \ell^\infty(\mathbb{N}, E)$ where $\ell^\infty(\mathbb{N}, E)$ is the space of bounded E-valued sequences. If E is a separable infinite-dimensional Hilbert space, then the map $S: \mathcal{C}_{bu}(\mathbb{N})\varepsilon E \to \mathcal{C}_{bu}(\mathbb{N}, E)$ is not surjective by [17, 2.8 Beispiel, p. 140] and [94, Satz 10.5, p. 235–236]. Hence one cannot drop the condition that E is a semi-Montel space in Example 4.2.7.

Let (Ω, d) be a metric space, $z \in \Omega$, E an lcHs, $0 < \gamma \leq 1$ and define the space of E-valued γ-Hölder continuous functions on Ω that vanish at z by

$$\mathcal{C}_z^{[\gamma]}(\Omega, E) := \{f \in E^\Omega \mid f(z) = 0 \text{ and } \forall\ \alpha \in \mathfrak{A}:\ |f|_\alpha < \infty\}$$

where

$$|f|_\alpha := \sup_{\substack{x,w \in \Omega \\ x \neq w}} \frac{p_\alpha\big(f(x) - f(w)\big)}{\mathrm{d}(x,w)^\gamma}.$$

The topological subspace $\mathcal{C}_{z,0}^{[\gamma]}(\Omega, E)$ of γ-Hölder continuous functions that vanish at infinity consists of all $f \in \mathcal{C}_z^{[\gamma]}(\Omega, E)$ such that for all $\varepsilon > 0$ there is $\delta > 0$ with

$$\sup_{\substack{x,w \in \Omega \\ 0 < \mathrm{d}(x,w) < \delta}} \frac{p_\alpha\big(f(x) - f(w)\big)}{\mathrm{d}(x,w)^\gamma} < \varepsilon.$$

Further, we set $\mathcal{C}_z^{[\gamma]}(\Omega) := \mathcal{C}_z^{[\gamma]}(\Omega, \mathbb{K})$ and $\mathcal{C}_{z,0}^{[\gamma]}(\Omega) := \mathcal{C}_{z,0}^{[\gamma]}(\Omega, \mathbb{K})$. Moreover, we define $M := J := \{1\}$, $\omega_1 := \Omega^2 \setminus \{(x,x) \mid x \in \Omega\}$ and $T_1^E: E^\Omega \to E^{\omega_1}$, $T_1^E(f)(x,w) := f(x) - f(w)$, and

$$\nu_{1,1}: \omega_1 \to [0, \infty),\ \nu_{1,1}(x,w) := \frac{1}{\mathrm{d}(x,w)^\gamma}.$$

Then we have for every $\alpha \in \mathfrak{A}$ that

$$|f|_\alpha = \sup_{(x,w) \in \omega_1} p_\alpha\big(T_1^E(f)(x,w)\big)\nu_{1,1}(x,w), \quad f \in \mathcal{C}_z^{[\gamma]}(\Omega, E).$$

4.2.9. EXAMPLE. Let E be an lcHs, (Ω, d) a metric space, $z \in \Omega$ and $0 < \gamma \leq 1$. Then

a) $\mathcal{C}_z^{[\gamma]}(\Omega, E) \cong \mathcal{C}_z^{[\gamma]}(\Omega)\varepsilon E$ if E is a semi-Montel space,
b) $\mathcal{C}_{z,0}^{[\gamma]}(\Omega, E) \cong \mathcal{C}_{z,0}^{[\gamma]}(\Omega)\varepsilon E$ if Ω is precompact and E quasi-complete.

PROOF. Let us start with a). From Proposition 4.1.10 for vanishing at z and a simple calculation follows that $(T_1^E, T_1^{\mathbb{K}})$ is a strong and consistent generator for $(\mathcal{C}_z^{[\gamma]}, E)$. This proves part a) by Corollary 3.2.5 (iii). Concerning part b), we set $\mathfrak{K} := \big\{\{(x,w) \in \Omega^2 \mid \mathrm{d}(x,w) \geq \delta\} \mid \delta > 0\big\}$, and let $\pi\colon \omega_1 \to \omega_1$ be the identity. Then $\mathcal{C}_{z,0}^{[\gamma]}(\Omega, E) = \mathcal{C}_z^{[\gamma]}(\Omega, E) \cap \mathrm{AP}_{\pi,\mathfrak{K}}(\Omega, E)$ with $\mathrm{AP}_{\pi,\mathfrak{K}}(\Omega, E)$ from Proposition 4.1.9 and the generator $(T_1^E, T_1^{\mathbb{K}})$ for $(\mathcal{C}_{z,0}^{[\gamma]}, E)$ is strong and consistent by Proposition 4.1.10 for vanishing at z and Proposition 4.1.9 for vanishing at infinity w.r.t. $(\pi, \mathfrak{K})$.

Let $f \in \mathcal{C}_{z,0}^{[\gamma]}(\Omega, E)$ and $K_\delta := \{(x,w) \in \Omega^2 \mid \mathrm{d}(x,w) \geq \delta\}$ for $\delta > 0$. For

$$N_{\pi \subset K_\delta,1,1}(f) = \{T_1^E(f)(x,w)\nu_{1,1}(x,w) \mid (x,w) \in K_\delta\} = \Big\{\tfrac{f(x)-f(w)}{\mathrm{d}(x,w)^\gamma} \mid (x,w) \in K_\delta\Big\}$$

we have

$$\begin{aligned} N_{\pi \subset K_\delta,1,1}(f) &\subset \delta^{-\gamma}\{c(f(x) - f(w)) \mid x, w \in \Omega,\ |c| \leq 1\} \\ &= \delta^{-\gamma} \operatorname{ch}\big(f(\Omega) - f(\Omega)\big). \end{aligned}$$

The set $f(\Omega)$ is precompact because Ω is precompact and the γ-Hölder continuous function f is uniformly continuous. It follows that the linear combination $f(\Omega) - f(\Omega)$ is precompact and the circled hull of a precompact set is still precompact by [153, Chap. I, 5.1, p. 25]. Therefore $N_{\pi \subset K_\delta,1,1}(f)$ is precompact for every $\delta > 0$, giving the precompactness of

$$N_{1,1}(f) = \{T_1^E(f)(x,w)\nu_{1,1}(x,w) \mid (x,w) \in \omega_1\}$$

by Proposition A.1.6. Hence statement b) is a consequence of Corollary 3.2.5 (iv), Proposition A.1.6 and the quasi-completeness of E. □

Let Ω be a topological Hausdorff space and $\mathcal{V} := (\nu_j)_{j \in J}$ a directed family of weights $\nu_j\colon \Omega \to [0, \infty)$. The weighted space of continuous functions on Ω with values in an lcHs E is given by

$$\mathcal{CV}(\Omega, E) := \{f \in \mathcal{C}(\Omega, E) \mid \forall\ j \in J,\ \alpha \in \mathfrak{A}:\ |f|_{j,\alpha} < \infty\}$$

where

$$|f|_{j,\alpha} := \sup_{x \in \Omega} p_\alpha(f(x))\nu_j(x).$$

Its topological subspace of functions that vanish at infinity in the weighted topology is defined by

$$\begin{aligned} \mathcal{CV}_0(\Omega, E) := \{f \in \mathcal{CV}(\Omega, E) \mid \forall\ j \in J,\ \alpha \in \mathfrak{A},\ \varepsilon > 0 \\ \exists\ K \subset \Omega \text{ compact}:\ |f|_{\Omega \smallsetminus K,j,\alpha} < \varepsilon\} \end{aligned}$$

where

$$|f|_{\Omega \smallsetminus K,j,\alpha} := \sup_{x \in \Omega \smallsetminus K} p_\alpha(f(x))\nu_j(x).$$

Further, we define $\mathcal{CV}(\Omega) := \mathcal{CV}(\Omega, \mathbb{K})$ and $\mathcal{CV}_0(\Omega) := \mathcal{CV}(\Omega, \mathbb{K})$. In particular, we set $\mathcal{C}_b(\Omega, E) := \mathcal{CV}(\Omega, E)$, i.e. the space of bounded continuous functions, and have $\mathcal{CV}_0(\Omega, E) = \mathcal{C}_0(\Omega, E)$ if $\mathcal{V} := \{1\}$. In [15, 16, 17] Bierstedt studies these spaces in the case that $\mathcal{V}$ is a *Nachbin-family* which means that the functions ν_j are upper semi-continuous for all $j \in J$ and directed in the sense that for $j_1, j_2 \in J$ and $\lambda \geq 0$ there is $j_3 \in J$ such that $\lambda\nu_{j_1}, \lambda\nu_{j_2} \leq \nu_{j_3}$. Formally this is stronger than our definition of being directed in Remark 3.1.6 c). The notion $\mathcal{U} \leq \mathcal{V}$ for two Nachbin-families means that for every $\mu \in \mathcal{U}$ there is $\nu \in \mathcal{V}$ such that $\mu \leq \nu$. One of his main results from [17] is the following theorem.

4.2.10. THEOREM ([17, 2.4 Theorem (2), p. 138–139]). *Let E be a quasi-complete lcHs, Ω a completely regular Hausdorff space and $\mathcal{V}$ a Nachbin-family on Ω. If*

(i) $\mathcal{Z} := \{v{:}\,\Omega \to \mathbb{R} \mid v$ *constant*, $v \geq 0\} \leq \mathcal{V}$, *or*
(ii) $\widetilde{\mathcal{W}} := \{\mu\chi_K \mid \mu > 0,\ K \subset \Omega$ *compact*$\} \leq \mathcal{V}$ *and Ω is a $k_{\mathbb{R}}$-space,*

then $\mathcal{CV}_0(\Omega, E) \cong \mathcal{CV}_0(\Omega)\varepsilon E$.

We note that $\mathcal{C}\widetilde{\mathcal{W}}(\Omega, E) = \mathcal{CW}(\Omega, E)$ with our definition of $\mathcal{W} = \{\chi_K \mid K \subset \Omega$ compact$\}$ from above Lemma 4.1.2. The only difference is that $\mathcal{W}$ is not a Nachbin-family because it is not directed in the sense of Nachbin-families but in the sense of Remark 3.1.6 c). We improve this result by strengthening the conditions on Ω and $\mathcal{V}$ which allows us to weaken the assumptions on E.

4.2.11. EXAMPLE. Let E be an lcHs, Ω a locally compact topological Hausdorff space and $\mathcal{V}$ a directed family of continuous weights on Ω.

(i) If E has ccp, or
(ii) if E has metric ccp and Ω is second-countable,

then $\mathcal{CV}_0(\Omega, E) \cong \mathcal{CV}_0(\Omega)\varepsilon E$.

PROOF. We set $\mathfrak{K} := \{K \subset \Omega \mid K$ compact$\}$ and $\pi{:}\,\Omega \to \Omega$, $\pi(x) := x$. It follows from Proposition 4.1.1 combined with Lemma 4.1.2 (i) (continuity) and Proposition 4.1.9 (vanish at infinity w.r.t. $(\pi, \mathfrak{K})$) that the generator $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ is strong and consistent since $\mathcal{V}$ is a family of continuous weights and Ω a $k_{\mathbb{R}}$-space due to local compactness.

Let $f \in \mathcal{CV}_0(\Omega, E)$, $j \in J$ and consider $N_j(f) = (f\nu_j)(\Omega)$. By Proposition A.1.3 the set $K := \overline{\mathrm{acx}}(N_j(f))$ is absolutely convex and compact as $f\nu_j \in \mathcal{C}_0(\Omega, E)$, implying our statement by Corollary 3.2.5 (iv). □

4.2.12. EXAMPLE. Let E be an lcHs and Ω a [metrisable] $k_{\mathbb{R}}$-space. If E has [metric] ccp, then $\mathcal{CW}(\Omega, E) \cong \mathcal{CW}(\Omega)\varepsilon E$.

PROOF. First, we observe that the generator $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ for $(\mathcal{CW}, E)$ is consistent by Proposition 4.1.1 and Lemma 4.1.2 b)(i). Its strength is obvious. Let $f \in \mathcal{CW}(\Omega, E)$, $K \subset \Omega$ be compact and consider $N_K(f) = f(\Omega)\nu_K(\Omega)$. Then $N_K(f) = f(K) \cup \{0\}$ if $K \neq \Omega$, and $N_K(f) = f(K)$ if $K = \Omega$, which yields that $N_K(f)$ is compact in E. If Ω is even metrisable, then $f(K)$ is also metrisable by [34, Chap. IX, §2.10, Proposition 17, p. 159] and thus the finite union $N_K(f)$ as well by [169, Theorem 1, p. 361] since the compact set $N_K(f)$ is collectionwise normal and locally countably compact by [63, 5.1.18 Theorem, p. 305]. Further, $\overline{\mathrm{acx}}(N_K(f))$ is absolutely convex and compact in E if E has ccp resp. if Ω is metrisable and E has metric ccp. We conclude that $\mathcal{CW}(\Omega, E) \cong \mathcal{CW}(\Omega)\varepsilon E$ if E has ccp resp. if Ω is metrisable and E has metric ccp by Corollary 3.2.5 (iv). □

Bierstedt also considers closed subspaces of $\mathcal{CV}(\Omega)$ and $\mathcal{CV}_0(\Omega)$, for instance subspaces of holomorpic functions on open Ω, and of holomorpic functions on the inner points of Ω which are continuous on the boundary in [17, 3.1 Bemerkung, p. 141] and [17, 3.7 Satz, p. 144].

Let $\Omega \subset \mathbb{C}$ be open and bounded and E an lcHs over $\mathbb{C}$. We denote by $\mathcal{A}(\overline{\Omega}, E)$ the space of continuous functions from $\overline{\Omega}$ to an lcHs E which are holomorphic on Ω and equip $\mathcal{A}(\overline{\Omega}, E)$ with the system of seminorms given by

$$|f|_\alpha := \sup_{x \in \overline{\Omega}} p_\alpha(f(x)), \quad f \in \mathcal{A}(\overline{\Omega}, E),$$

for $\alpha \in \mathfrak{A}$. We set $\mathcal{A}(\overline{\Omega}) := \mathcal{A}(\overline{\Omega}, \mathbb{C})$, $J := M := \{1\}$ and $\nu_{1,1} := 1$ on $\overline{\Omega}$.

4.2.13. Example. Let E be an lcHs and $\Omega \subset \mathbb{C}$ open and bounded. Then $\mathcal{A}(\overline{\Omega}, E) \cong \mathcal{A}(\overline{\Omega})\varepsilon E$ if E has metric ccp.

Proof. The space $\mathcal{A}(\overline{\Omega})$ is a Banach space and hence barrelled. The inclusion $I\colon \mathcal{A}(\overline{\Omega}) \to \mathcal{CW}^{\infty}_{\overline{\partial}}(\Omega)$ is continuous due to the Cauchy inequality (I is an inclusion due to the identity theorem). It follows from Proposition 4.1.1, Lemma 4.1.2 b)(i), Proposition 3.1.11 c) and (4) that the generator $(\mathrm{id}_{E^{\overline{\Omega}}}, \mathrm{id}_{\mathbb{C}^{\overline{\Omega}}})$ is consistent and as in Proposition 3.1.10 that it is strong, too.

Let $f \in \mathcal{A}(\overline{\Omega}, E)$ and $N_{1,1}(f) = f(\overline{\Omega})$. The set $K := \overline{\mathrm{acx}}(N_{1,1}(f))$ is absolutely convex and compact by Proposition A.1.3 since $f \in \mathcal{C}(\overline{\Omega}, E) = \mathcal{C}_0(\overline{\Omega}, E)$, implying our statement by Corollary 3.2.5 (iv). □

For quasi-complete E this is already covered by [17, 3.1 Bemerkung, p. 141]. More general than holomorphic functions, we may also consider kernels of hypoelliptic linear partial differential operators in $\mathcal{CV}(\Omega)$ and $\mathcal{CV}_0(\Omega)$. For an open set $\Omega \subset \mathbb{R}^d$, a directed family $\mathcal{V} := (\nu_j)_{j\in\mathbb{N}}$ of weights $\nu_j\colon \Omega \to [0, \infty)$, an lcHs E and a linear partial differential operator $P(\partial)^E$ which is hypoelliptic if $E = \mathbb{K}$ we define the space of zero solutions

$$\mathcal{CV}_{P(\partial)}(\Omega, E) := \{f \in \mathcal{C}^{\infty}_{P(\partial)}(\Omega, E) \mid \forall\ j \in \mathbb{N},\ \alpha \in \mathfrak{A} :\ |f|_{j,\alpha} < \infty\},$$

where $\mathcal{C}^{\infty}_{P(\partial)}(\Omega, E)$ is the kernel of $P(\partial)^E$ in $\mathcal{C}^{\infty}(\Omega, E)$,

$$|f|_{j,\alpha} := \sup_{x\in\Omega} p_\alpha(f(x))\nu_j(x),$$

and its topological subspace

$$\mathcal{CV}_{0,P(\partial)}(\Omega, E) := \mathcal{CV}_{P(\partial)}(\Omega, E) \cap \mathcal{CV}_0(\Omega, E).$$

Further, we set $\mathcal{CV}_{P(\partial)}(\Omega) := \mathcal{CV}_{P(\partial)}(\Omega, \mathbb{K})$ and $\mathcal{CV}_{0,P(\partial)}(\Omega) := \mathcal{CV}_{0,P(\partial)}(\Omega, \mathbb{K})$. We say that $\mathcal{V}$ is *locally bounded away from zero* on Ω if

$$\forall\ K \subset \Omega \text{ compact } \exists\ j \in \mathbb{N} :\ \inf_{x\in K} \nu_j(x) > 0.$$

This is an extension of the definition of being *locally bounded away from zero* from $\mathcal{V}^k$ with $k \in \mathbb{N}_\infty$ to the case $k = 0$ (see Proposition 3.1.10). If $\mathcal{V}$ is a Nachbin-family, this means that $\widetilde{\mathcal{W}} \leq \mathcal{V}$ (see Theorem 4.2.10 (ii)).

4.2.14. Proposition. *Let $\Omega \subset \mathbb{R}^d$ be open, $\mathcal{V} := (\nu_j)_{j\in\mathbb{N}}$ an increasing family of weights which is locally bounded away from zero on Ω and $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator. Then $\mathcal{CV}_{P(\partial)}(\Omega)$ and $\mathcal{CV}_{0,P(\partial)}(\Omega)$ are Fréchet spaces.*

Proof. We note that $\mathcal{CV}_{P(\partial)}(\Omega)$ is metrisable as $\mathcal{V}$ is countable. Let (f_n) be a Cauchy sequence in $\mathcal{CV}_{P(\partial)}(\Omega)$. From $\mathcal{V}$ being locally bounded away from zero it follows that for every compact $K \subset \Omega$ there is $j \in \mathbb{N}$ such that

$$\sup_{x\in K} |f(x)| \leq \sup_{z\in K} \nu_j(z)^{-1} \sup_{x\in K} |f(x)|\nu_j(x) \leq \sup_{z\in K} \nu_j(z)^{-1} |f|_j, \quad f \in \mathcal{CV}_{P(\partial)}(\Omega), \tag{22}$$

which means that the inclusion $I\colon \mathcal{CV}_{P(\partial)}(\Omega) \to \mathcal{CW}_{P(\partial)}(\Omega)$ is continuous. Thus (f_n) is also a Cauchy sequence in $\mathcal{CW}_{P(\partial)}(\Omega)$ and has a limit f there as $\mathcal{CW}_{P(\partial)}(\Omega)$ is complete due to the hypoellipticity of $P(\partial)^{\mathbb{K}}$. Let $j \in \mathbb{N}$, $\varepsilon > 0$ and $x \in \Omega$. Then there is $m_{j,\varepsilon,x} \in \mathbb{N}$ such that for all $m \geq m_{j,\varepsilon,x}$ it holds that

$$|f_m(x) - f(x)| < \frac{\varepsilon}{2\nu_j(x)}$$

if $\nu_j(x) \neq 0$. Further, there is $m_{j,\varepsilon} \in \mathbb{N}$ such that for all $n, m \geq m_{j,\varepsilon}$ it holds that

$$|f_n - f_m|_j < \frac{\varepsilon}{2}.$$

Hence for $n \geq m_{j,\varepsilon}$ we choose $m \geq \max(m_{j,\varepsilon}, m_{j,\varepsilon,x})$ and derive

$$|f_n(x)-f(x)|\nu_j(x) \leq |f_n(x)-f_m(x)|\nu_j(x)+|f_m(x)-f(x)|\nu_j(x) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2\nu_j(x)}\nu_j(x) = \varepsilon.$$

It follows that $|f_n - f|_j \leq \varepsilon$ and $|f|_j \leq \varepsilon + |f_n|_j$ for all $n \geq m_{j,\varepsilon}$, implying the convergence of (f_n) to f in $\mathcal{CV}_{P(\partial)}(\Omega)$. Therefore $\mathcal{CV}_{P(\partial)}(\Omega)$ is a Fréchet space. $\mathcal{CV}_{0,P(\partial)}(\Omega)$ is a closed subspace of $\mathcal{CV}_{P(\partial)}(\Omega)$ and so a Fréchet space as well. □

Due to the proposition above the spaces $\mathcal{CV}_{P(\partial)}(\Omega)$ and $\mathcal{CV}_{0,P(\partial)}(\Omega)$ are closed subspaces of $\mathcal{CV}(\Omega)$ resp. $\mathcal{CV}_0(\Omega)$. Hence we have the following consequence of Theorem 4.2.10 (ii), [17, 2.12 Satz (1), p. 141] and [17, 3.1 Bemerkung, p. 141].

4.2.15. COROLLARY. *Let E be an lcHs, $\Omega \subset \mathbb{R}^d$ open, $\mathcal{V}$ a Nachbin-family on Ω which is locally bounded away from zero and $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator.*

a) $\mathcal{CV}_{P(\partial)}(\Omega, E) \cong \mathcal{CV}_{P(\partial)}(\Omega)\varepsilon E$ if E is a semi-Montel space.
b) $\mathcal{CV}_{0,P(\partial)}(\Omega, E) \cong \mathcal{CV}_{0,P(\partial)}(\Omega)\varepsilon E$ if E is quasi-complete.

Like before we may improve this result by strengthening the conditions on $\mathcal{V}$ and $\mathcal{CV}_{P(\partial)}(\Omega)$ resp. $\mathcal{CV}_{0,P(\partial)}(\Omega)$ which allows us to weaken the assumptions on E.

4.2.16. EXAMPLE. Let E be an lcHs, $\Omega \subset \mathbb{R}^d$ open, $\mathcal{V} := (\nu_j)_{j\in\mathbb{N}}$ an increasing family of weights which is locally bounded away from zero on Ω and $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator.

a) $\mathcal{CV}_{P(\partial)}(\Omega, E) \cong \mathcal{CV}_{P(\partial)}(\Omega)\varepsilon E$ if E is complete and $\mathcal{CV}_{P(\partial)}(\Omega)$ a semi-Montel space.
b) $\mathcal{CV}_{P(\partial)}(\Omega, E) \cong \mathcal{CV}_{P(\partial)}(\Omega)\varepsilon E$ if E is locally complete and $\mathcal{CV}_{P(\partial)}(\Omega)$ a Schwartz space.
c) $\mathcal{CV}_{0,P(\partial)}(\Omega, E) \cong \mathcal{CV}_{0,P(\partial)}(\Omega)\varepsilon E$ if E has metric ccp and $\nu_j \in \mathcal{C}(\Omega)$ for all $j \in \mathbb{N}$.
d) $\mathcal{CV}_{0,P(\partial)}(\Omega, E) \cong \mathcal{CV}_{0,P(\partial)}(\Omega)\varepsilon E$ if E is locally complete and $\mathcal{CV}_{0,P(\partial)}(\Omega)$ a Schwartz space.

PROOF. Let $\mathcal{F}$ stand for $\mathcal{CV}_{P(\partial)}$ or $\mathcal{CV}_{0,P(\partial)}$. The space $\mathcal{F}(\Omega)$ is a Fréchet space and hence barrelled by Proposition 4.2.14. The inclusion $I\colon \mathcal{F}(\Omega) \to \mathcal{CW}_{P(\partial)}(\Omega)$ is continuous since $\mathcal{V}$ is locally bounded away from zero on Ω. The hypoellipticity of $P(\partial)^{\mathbb{K}}$ (see e.g. [70, p. 690]) yields that $\mathcal{CW}_{P(\partial)}(\Omega) = \mathcal{CW}^{\infty}_{P(\partial)}(\Omega)$ as locally convex spaces. Thus the inclusion $I\colon \mathcal{F}(\Omega) \to \mathcal{CW}^{\infty}_{P(\partial)}(\Omega)$ is continuous. It follows from Proposition 3.1.11 c) that the generator $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ is consistent if $\mathcal{F} = \mathcal{CV}_{P(\partial)}$, and combined with Proposition 4.1.9 (vanish at infinity w.r.t. $(\pi, \mathfrak{K})$) if $\mathcal{F} = \mathcal{CV}_{0,P(\partial)}$ where $\mathfrak{K}$ and π are chosen as in Example 4.2.11. The strength of the generator follows as in Proposition 3.1.10 and, if $\mathcal{F} = \mathcal{CV}_{0,P(\partial)}$, in combination with Proposition 4.1.9 b). This proves part a), b) and d) due to Corollary 3.2.5 (i) and (ii).

Let us turn to part c). Let $f \in \mathcal{CV}_{0,P(\partial)}(\Omega, E)$, $j \in \mathbb{N}$ and $N_j(f) := (f\nu_j)(\Omega)$. The set $K := \overline{\mathrm{acx}}(N_j(f))$ is absolutely convex compact by Proposition A.1.3 as $f\nu_j \in \mathcal{C}_0(\Omega, E)$, implying our statement by Corollary 3.2.5 (iv). □

At least for some weights and operators $P(\partial)$ we can show that $\mathcal{CV}_{P(\partial)}(\Omega, E)$ coincides with a corresponding space $\mathcal{CV}^{\infty}_{P(\partial)}(\Omega, E)$ from Example 3.1.9 if E is locally complete.

4.2.17. Proposition. *Let E be a locally complete lcHs, $\Omega\subset\mathbb{R}^d$ and $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator. Then we have $\mathcal{CW}_{P(\partial)}(\Omega)\varepsilon E\cong\mathcal{CW}_{P(\partial)}(\Omega,E)$ and $\mathcal{CW}_{P(\partial)}(\Omega,E)=\mathcal{CW}^{\infty}_{P(\partial)}(\Omega,E)$ as locally convex spaces.*

Proof. We already know that

$$S_{\mathcal{CW}^{\infty}_{P(\partial)}(\Omega)}\colon\mathcal{CW}^{\infty}_{P(\partial)}(\Omega)\varepsilon E\to\mathcal{CW}^{\infty}_{P(\partial)}(\Omega,E)$$

is an isomorphism by Example 3.2.8 b). The hypoellipticity of $P(\partial)^{\mathbb{K}}$ (see e.g. [70, p. 690]) yields that $\mathcal{CW}_{P(\partial)}(\Omega)\varepsilon E=\mathcal{CW}^{\infty}_{P(\partial)}(\Omega)\varepsilon E$. Thus $S_{\mathcal{CW}_{P(\partial)}(\Omega)}(u)=S_{\mathcal{CW}^{\infty}_{P(\partial)}(\Omega)}(u)\in\mathcal{C}^{\infty}_{P(\partial)}(\Omega,E)$ for all $u\in\mathcal{CW}_{P(\partial)}(\Omega)\varepsilon E$. In particular, we obtain that

$$S_{\mathcal{CW}_{P(\partial)}(\Omega)}\colon\mathcal{CW}_{P(\partial)}(\Omega)\varepsilon E\to\mathcal{CW}^{\infty}_{P(\partial)}(\Omega,E)$$

is an isomorphism. From Proposition 3.1.11 c) and Theorem 3.1.12 with $(T^E,T^{\mathbb{K}}):=(\mathrm{id}_{E^\Omega},\mathrm{id}_{\mathbb{K}^\Omega})$ we deduce that

$$S_{\mathcal{CW}_{P(\partial)}(\Omega)}\colon\mathcal{CW}_{P(\partial)}(\Omega)\varepsilon E\to\mathcal{CW}_{P(\partial)}(\Omega,E)$$

is an isomorphism into, and from

$$S_{\mathcal{CW}_{P(\partial)}(\Omega)}\big(\mathcal{CW}_{P(\partial)}(\Omega)\varepsilon E\big)=\mathcal{C}^{\infty}_{P(\partial)}(\Omega,E)$$

that $\mathcal{CW}_{P(\partial)}(\Omega,E)=\mathcal{CW}^{\infty}_{P(\partial)}(\Omega,E)$ as locally convex spaces, which proves our statement. □

Hence the topology τ_c of compact convergence induced by $\mathcal{C}(\Omega,E)$ and the usual topology from Example 3.1.2 induced by $\mathcal{C}^\infty(\Omega,E)$ coincide on $\mathcal{C}_{P(\partial)}(\Omega,E)$ if $P(\partial)^{\mathbb{K}}$ is hypoelliptic and E locally complete by Proposition 4.2.17. In particular, we have

$$(\mathcal{O}(\Omega,E),\tau_c)\underset{(4)}{=}\mathcal{CW}_{\overline{\partial}}(\Omega,E)=\mathcal{CW}^{\infty}_{\overline{\partial}}(\Omega,E)\tag{23}$$

if E is locally complete. For more interesting weights than $\mathcal{W}$ we introduce the following condition.

4.2.18. Condition. Let $\mathcal{V}:=(\nu_j)_{j\in\mathbb{N}}$ be an increasing family of continuous weights on $\mathbb{R}^d$. Let there be $r\colon\mathbb{R}^d\to(0,1]$ and for any $j\in\mathbb{N}$ let there be $\psi_j\in L^1(\mathbb{R}^d)$, $\psi_j>0$, and $\mathbb{N}\ni I_m(j)\geq j$ and $A_m(j)>0$, $m\in\{1,2,3\}$, such that for any $x\in\mathbb{R}^d$:

(α.1) $\sup_{\zeta\in\mathbb{R}^d,\,\|\zeta\|_\infty\leq r(x)}\nu_j(x+\zeta)\leq A_1(j)\inf_{\zeta\in\mathbb{R}^d,\,\|\zeta\|_\infty\leq r(x)}\nu_{I_1(j)}(x+\zeta)$,
(α.2) $\nu_j(x)\leq A_2(j)\psi_j(x)\nu_{I_2(j)}(x)$,
(α.3) $\nu_j(x)\leq A_3(j)r(x)\nu_{I_3(j)}(x)$.

Here, $\|\zeta\|_\infty:=\sup_{1\leq n\leq d}|\zeta_n|$ for $\zeta=(\zeta_n)\in\mathbb{R}^d$. The preceding condition is a special case of [111, Condition 2.1, p. 176] with $\Omega:=\Omega_n:=\mathbb{R}^d$ for all $n\in\mathbb{N}$. If $\mathcal{V}$ fulfils Condition 4.2.18 and we set $\mathcal{V}^\infty:=(\nu_{j,m})_{j\in\mathbb{N},m\in\mathbb{N}_0}$ where $\nu_{j,m}\colon\{\beta\in\mathbb{N}_0^d\mid|\beta|\leq m\}\times\mathbb{R}^d\to[0,\infty)$, $\nu_{j,m}(\beta,x):=\nu_j(x)$, then $\mathcal{CV}^\infty(\mathbb{R}^d)$ and its closed subspace $\mathcal{CV}^{\infty}_{P(\partial)}(\mathbb{R}^d)$ for $P(\partial)$ with continuous coefficients are nuclear by [111, Theorem 3.1, p. 188] in combination with [111, Remark 2.7, p. 178–179] and Fréchet spaces by [107, Proposition 3.7, p. 240].

4.2.19. Proposition. *Let E be a locally complete lcHs, $\mathcal{V}:=(\nu_j)_{j\in\mathbb{N}}$ an increasing family of continuous weights on $\mathbb{R}^d$ and $\mathcal{V}^\infty$ defined as above. If $\mathcal{V}$ fulfils Condition 4.2.18, then $\mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ and $\mathcal{CV}_{\Delta}(\mathbb{R}^d)$ are nuclear Fréchet spaces and $\mathcal{CV}_{\overline{\partial}}(\mathbb{C},E)=\mathcal{CV}^{\infty}_{\overline{\partial}}(\mathbb{C},E)$ and $\mathcal{CV}_{\Delta}(\mathbb{R}^d,E)=\mathcal{CV}^{\infty}_{\Delta}(\mathbb{R}^d,E)$ as locally convex spaces.*

Proof. Let $P(\partial):=\overline{\partial}$ ($d:=2$ and $\mathbb{K}:=\mathbb{C}$) or $P(\partial):=\Delta$. First, we show that $\mathcal{CV}_{P(\partial)}(\mathbb{R}^d)=\mathcal{CV}^{\infty}_{P(\partial)}(\mathbb{R}^d)$ as locally convex spaces, which implies that $\mathcal{CV}_{P(\partial)}(\mathbb{R}^d)$ is a nuclear Fréchet space as $\mathcal{CV}^{\infty}_{P(\partial)}(\mathbb{R}^d)$ is such a space. Let $f\in\mathcal{CV}_{\overline{\partial}}(\mathbb{C})$, $j\in\mathbb{N}$,

$m \in \mathbb{N}_0$, $z \in \mathbb{C}$ and $\beta := (\beta_1, \beta_2) \in \mathbb{N}_0^2$. Then it follows from $\|\cdot\|_\infty \leq |\cdot|$ and Cauchy's inequality that

$$\begin{aligned}|\partial^\beta f(z)|\nu_j(z) &\underset{(5)}{=} |\mathrm{i}^{\beta_2}\partial_{\mathbb{C}}^{|\beta|} f(z)|\nu_j(z) \leq \frac{|\beta|!}{r(z)^{|\beta|}} \sup_{|w-z|=r(z)} |f(w)|\nu_j(z) \\ &\underset{(\alpha.3)}{\leq} |\beta|! C(j,|\beta|) \sup_{|w-z|=r(z)} |f(w)|\nu_{B_3(j)}(z) \\ &\underset{(\alpha.1)}{\leq} |\beta|! C(j,|\beta|) A_1(B_3(j)) \sup_{|w-z|=r(z)} |f(w)|\nu_{I_1 B_3(j)}(w) \\ &\leq |\beta|! C(j,|\beta|) A_1(B_3(j)) |f|_{\mathcal{CV}_{\overline{\partial}}(\mathbb{C}), I_1 B_3(j)}\end{aligned}$$

where $C(j,|\beta|) := A_3(j)A_3(I_3(j))\cdots A_3((B_3-1)(j))$ and $B_3 - 1$ is the $(|\beta|-1)$-fold composition of I_3. Choosing $k := \max_{|\beta|\leq m} I_1 B_3(j)$, it follows that

$$|f|_{\mathcal{CV}^\infty_{\overline{\partial}}(\mathbb{C}),j,m} \leq \sup_{|\beta|\leq m} |\beta|! C(j,|\beta|) A_1(B_3(j)) |f|_{\mathcal{CV}_{\overline{\partial}}(\mathbb{C}),k} < \infty$$

and thus $f \in \mathcal{CV}^\infty_{\overline{\partial}}(\mathbb{C})$ and $\mathcal{CV}_{\overline{\partial}}(\mathbb{C}) = \mathcal{CV}^\infty_{\overline{\partial}}(\mathbb{C})$ as locally convex spaces. In the case $P(\partial) = \Delta$ an analogous proof works due to Cauchy's inequality for harmonic functions, i.e. for all $f \in \mathcal{CV}_\Delta(\mathbb{R}^d)$, $j \in \mathbb{N}$, $x \in \mathbb{R}^d$ and $\beta \in \mathbb{N}_0^d$ it holds that

$$|\partial^\beta f(x)|\nu_j(x) \leq \Big(\frac{d|\beta|}{r(x)}\Big)^{|\beta|} \sup_{|w-x|<r(x)} |f(w)|\nu_j(x)$$

(see e.g. [74, Theorem 2.10, p. 23]).

Nuclear Fréchet spaces are Fréchet–Schwartz spaces and hence we have

$$\mathcal{CV}^\infty_{P(\partial)}(\mathbb{R}^d, E) \cong \mathcal{CV}^\infty_{P(\partial)}(\mathbb{R}^d)\varepsilon E \cong \mathcal{CV}_{P(\partial)}(\mathbb{R}^d)\varepsilon E \cong \mathcal{CV}_{P(\partial)}(\mathbb{R}^d, E)$$

by Example 3.2.7 f) and Example 4.2.16 b). The isomorphism $\mathcal{CV}^\infty_{P(\partial)}(\mathbb{R}^d, E) \cong \mathcal{CV}_{P(\partial)}(\mathbb{R}^d, E)$ is

$$S^{-1}_{\mathcal{CV}^\infty_{P(\partial)}(\mathbb{R}^d)} \circ \mathrm{id}_{\mathcal{CV}^\infty_{P(\partial)}(\mathbb{R}^d)\varepsilon E} \circ S_{\mathcal{CV}_{P(\partial)}(\mathbb{R}^d)} = S^{-1}_{\mathcal{CV}^\infty_{P(\partial)}(\mathbb{R}^d)} \circ S_{\mathcal{CV}_{P(\partial)}(\mathbb{R}^d)} = \mathrm{id}_{\mathcal{CV}^\infty_{P(\partial)}(\mathbb{R}^d,E)}$$

by Theorem 3.2.4. □

4.2.20. REMARK. Let E be a locally complete lcHs and $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator.

a) Let $0 \leq \tau < \infty$. Then $\mathcal{V} := (\nu_j)_{j\in\mathbb{N}}$ given by $\nu_j(x) := \exp(-(\tau + \frac{1}{j})|x|)$, $x \in \mathbb{R}^d$, fulfils Condition 4.2.18 by [111, Example 2.8 (iii), p. 179]. Then $\mathcal{CV}_{P(\partial)}(\mathbb{R}^d, E)$ is the space of smooth functions of *exponential type* τ in the kernel of $P(\partial)$. If $\tau = 0$, then the elements of these spaces are also called functions of *infra-exponential type*. In particular, if $P(\partial) = \overline{\partial}$, $d = 2$ and $\mathbb{K} = \mathbb{C}$, or $P(\partial) = \Delta$, then $A^\tau_{\overline{\partial}}(\mathbb{C}, E) := \mathcal{CV}_{\overline{\partial}}(\mathbb{C}, E)$ is the space of entire and $A^\tau_\Delta(\mathbb{R}^d, E) := \mathcal{CV}_\Delta(\mathbb{R}^d, E)$ the space of harmonic functions of exponential type τ.
b) Further examples of families of weights fulfilling Condition 4.2.18 can be found in [111, Example 2.8, p. 179] and [130, 1.5 Examples, p. 205].

Next, we take a look at k-times continuously partially differentiable functions that vanish with all their derivatives when weighted at infinity. Let $k \in \mathbb{N}_\infty$, $\Omega \subset \mathbb{R}^d$ be open, $\omega_m := M_m \times \Omega$ with $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq \min(m,k)\}$ for all $m \in \mathbb{N}_0$ and $\mathcal{V}^k := (\nu_{j,m})_{j\in J, m\in\mathbb{N}_0}$ be a directed family of weights on $(\omega_m)_{m\in\mathbb{N}_0}$. We define the topological subspace of $\mathcal{CV}^k(\Omega, E)$ from Example 3.1.9 a)(i) consisting of the functions that vanishs with all their derivatives when weighted at infinity by

$$\begin{aligned}\mathcal{CV}^k_0(\Omega, E) := \{f \in \mathcal{CV}^k(\Omega, E) \mid &\forall\, j \in J,\, m \in \mathbb{N}_0,\, \alpha \in \mathfrak{A},\, \varepsilon > 0 \\ &\exists\, K \subset \Omega \text{ compact}:\ |f|_{\Omega\setminus K, j, m, \alpha} < \varepsilon\}\end{aligned}$$

where

$$|f|_{\Omega\smallsetminus K,j,m,\alpha} := \sup_{\substack{x\in\Omega\smallsetminus K\\ \beta\in M_m}} p_\alpha\big((\partial^\beta)^E f(x)\big)\nu_{j,m}(\beta,x).$$

Further, we define its subspace $\mathcal{CV}^k_{0,P(\partial)}(\Omega,E) := \{f\in\mathcal{CV}^k_0(\Omega,E) \mid f\in\ker P(\partial)^E\}$ where

$$P(\partial)^E\colon \mathcal{C}^k(\Omega,E)\to E^\Omega,\; P(\partial)^E(f)(x) := \sum_{i=1}^n a_i(x)(\partial^{\beta_i})^E(f)(x),$$

with $n\in\mathbb{N}$, $\beta_i\in\mathbb{N}_0^d$ such that $|\beta_i|\leq k$ and $a_i\colon\Omega\to\mathbb{K}$ for $1\leq i\leq n$.

4.2.21. Remark. If $\mathcal{V}^k$ fulfils condition (V_∞) from Example 3.2.9, then we have $\mathcal{CV}^k_0(\Omega,E)=\mathcal{CV}^k(\Omega,E)$ (see [107, Remark 3.4, p. 239]).

So $\mathcal{CW}^k(\Omega,E)$, $\mathcal{S}(\mathbb{R}^d,E)$ and $\mathcal{O}_M(\mathbb{R}^d,E)$ are concrete examples of spaces $\mathcal{CV}^k_0(\Omega,E)$ (see Corollary 3.2.10). We present the counterpart for differentiable functions to Bierstedt's Theorem 4.2.10 for the space $\mathcal{CV}_0(\Omega,E)$ of continuous functions from a completely regular Hausdorff space Ω to an lcHs E weighted with a Nachbin-family $\mathcal{V}$ that vanish at infinity in the weighted topology. For this purpose we need the following definition. We call $\mathcal{V}^k$ *locally bounded* on Ω if

$$\forall\, K\subset\Omega \text{ compact},\; j\in J,\; m\in\mathbb{N}_0,\; \beta\in M_m:\; \sup_{x\in K}\nu_{j,m}(\beta,x)<\infty.$$

4.2.22. Example. Let E be an lcHs, $k\in\mathbb{N}_\infty$, $\mathcal{V}^k$ be a directed family of weights which is locally bounded away from zero on an open set $\Omega\subset\mathbb{R}^d$.

a) $\mathcal{CV}^k_0(\Omega,E)\cong\mathcal{CV}^k_0(\Omega)\varepsilon E$ if E is quasi-complete, $\mathcal{V}^k$ locally bounded and $\mathcal{CV}^k_0(\Omega)$ barrelled.
b) $\mathcal{CV}^k_0(\Omega,E)\cong\mathcal{CV}^k_0(\Omega)\varepsilon E$ if E has metric ccp, $\mathcal{CV}^k_0(\Omega)$ is barrelled and $\nu_{j,m}(\beta,\cdot)\in\mathcal{C}(\Omega)$ for all $j\in J$, $m\in\mathbb{N}_0$, $\beta\in\mathbb{N}_0^d$, $|\beta|\leq\min(m,k)$.
c) $\mathcal{CV}^k_0(\Omega,E)\cong\mathcal{CV}^k_0(\Omega)\varepsilon E$ if E is locally complete and $\mathcal{CV}^k_0(\Omega)$ a Fréchet–Schwartz space.
d) $\mathcal{CV}^k_{0,P(\partial)}(\Omega,E)\cong\mathcal{CV}^k_{0,P(\partial)}(\Omega)\varepsilon E$ if E is quasi-complete, $\mathcal{V}^k$ loc. bounded and $\mathcal{CV}^k_{0,P(\partial)}(\Omega)$ barrelled.
e) $\mathcal{CV}^k_{0,P(\partial)}(\Omega,E)\cong\mathcal{CV}^k_{0,P(\partial)}(\Omega)\varepsilon E$ if E has metric ccp, $\mathcal{CV}^k_{0,P(\partial)}(\Omega)$ is barrelled and $\nu_{j,m}(\beta,\cdot)\in\mathcal{C}(\Omega)$ for all $j\in J$, $m\in\mathbb{N}_0$, $\beta\in\mathbb{N}_0^d$, $|\beta|\leq\min(m,k)$.
f) $\mathcal{CV}^k_{0,P(\partial)}(\Omega,E)\cong\mathcal{CV}^k_{0,P(\partial)}(\Omega)\varepsilon E$ if E is locally complete and $\mathcal{CV}^k_{0,P(\partial)}(\Omega)$ a Fréchet–Schwartz space.

Proof. The generator $(T^E_m,T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ for $(\mathcal{CV}^k_0,E)$ and $(\mathcal{CV}^k_{0,P(\partial)},E)$ is given by $\operatorname{dom}T^E_m := \mathcal{C}^k(\Omega,E)$ and

$$T^E_m\colon\mathcal{C}^k(\Omega,E)\to E^{\omega_m},\; f\longmapsto[(\beta,x)\mapsto(\partial^\beta)^E f(x)],$$

for all $m\in\mathbb{N}_0$ and the same with $\mathbb{K}$ instead of E.

Set $X:=\Omega$, $\mathfrak{K}:=\{K\subset\Omega\mid K\text{ compact}\}$ and $\pi\colon\bigcup_{m\in\mathbb{N}_0}\omega_m\to X$, $\pi(\beta,x):=x$. We have

$$|f|_{\Omega\smallsetminus K,j,m,\alpha} = \sup_{\substack{x\in\omega_m\\ \pi(x)\notin K}} p_\alpha\big(T^E_m(f)(x)\big)\nu_{j,m}(x),$$

for $f\in\mathcal{CV}^k_0(\Omega,E)$, $K\in\mathfrak{K}$, $j\in J$ and $m\in\mathbb{N}_0$, implying that (16) is satisfied. With $\mathrm{AP}_{\pi,\mathfrak{K}}(\Omega,E)$ from Proposition 4.1.9 we note that

$$\mathcal{CV}^k_0(\Omega,E)=\mathcal{CV}^k(\Omega,E)\cap\mathrm{AP}_{\pi,\mathfrak{K}}(\Omega,E).$$

As in Proposition 3.1.10 it follows that the generator $(T^E_m,T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ fulfils (17) and (18) where we use Proposition 3.1.11, the barrelledness of $\mathcal{CV}^k_0(\Omega)$ resp. $\mathcal{CV}^k_{0,P(\partial)}(\Omega)$

and the assumption that $\mathcal{V}^k$ is locally bounded away from zero on Ω. Therefore the generator is strong and consistent by virtue of Proposition 4.1.9.

a)+d) Let $f \in \mathcal{CV}_0^k(\Omega, E)$, $K \in \mathfrak{K}$, $j \in J$ and $m \in \mathbb{N}_0$. We claim that the set

$$N_{j,m}(f) = \{\partial^\beta)^E f(x)\nu_{j,m}(\beta, x) \mid x \in \Omega,\, \beta \in M_m\}$$

is precompact in E by Proposition A.1.6. Since f vanishes at infinity in the weighted topology, condition (i) of Proposition A.1.6 is fulfilled. Hence we only need to show that condition (ii) is satisfied as well, i.e. we have to show that

$$N_{\pi\subset K,j,m}(f) = \bigcup_{\beta\in M_m} (\partial^\beta)^E f\nu_{j,m}(\beta, \cdot)(K)$$

is precompact in E. Thus we only have to prove that the sets $(\partial^\beta)^E f\nu_{j,m}(\beta,\cdot)(K)$ are precompact since $N_{\pi\subset K,j,m}(f)$ is a finite union of these sets. But this is a consequence of the proof of [15, §1, 16. Lemma, p. 15] using the continuity of $(\partial^\beta)^E f$ and the boundedness of $\nu_{j,m}(\beta, K)$, which follows from $\mathcal{V}^k$ being locally bounded. So we deduce statements a) and d) from Corollary 3.2.5 (iv), Proposition A.1.6 and the quasi-completeness of E.

b)+e) The set $K_\beta := \overline{\operatorname{acx}}((\partial^\beta)^E f\nu_{j,m}(\beta,\cdot)(\Omega))$ is absolutely convex and compact by Proposition A.1.3 (ii) for every $f \in \mathcal{CV}_0^k(\Omega, E)$, $j \in J$, $m \in \mathbb{N}_0$ and $\beta \in M_m$ as E has metric ccp and $\nu_{j,m}(\beta,\cdot) \in \mathcal{C}(\Omega)$. We have

$$N_{j,m}(f) = \{(\partial^\beta)^E f(x)\nu_{j,m}(\beta, x) \mid x \in \Omega,\, \beta \in M_m\} \subset \operatorname{acx}\big(\bigcup_{\beta\in M_m} K_\beta\big)$$

and the set on the right-hand side is absolutely convex and compact by [89, 6.7.3 Proposition, p. 113]. Now, statements b)+e) follow from Corollary 3.2.5 (iv).

c)+f) They follow from Corollary 3.2.5 (ii). □

The spaces $\mathcal{CV}_0^k(\Omega)$ are Fréchet spaces and thus barrelled if J is countable by [107, Proposition 3.7, p. 240]. In [107, Theorem 5.2, p. 255] the question is answered when they have the approximation property. The spaces $\mathcal{CV}_0^\infty(\Omega)$ and $\mathcal{CV}_{P(\partial),0}^\infty(\Omega)$ are closed subspaces of $\mathcal{CV}^\infty(\Omega)$ and $\mathcal{CV}_{P(\partial)}^\infty(\Omega)$, respectively. For conditions that they are Fréchet–Schwartz spaces see the remarks below Example 3.2.7.

We already saw different choices for $\mathfrak{K}$ in Example 4.2.9 b) and Example 4.2.22. For holomorphic functions on an open subset Ω of an infinite dimensional Banach space X the family $\mathfrak{K}$ of Ω-*bounded* sets, i.e. bounded sets $K \subset \Omega$ with positive distance to $X \setminus \Omega$, is used in [71, p. 2] and [93, p. 2]. This family is clearly closed under taking finite unions, so Proposition 4.1.9 is applicable as well.

Now, we consider an example of weighted smooth functions where the corresponding space of scalar-valued functions may not be barrelled. For an open set $\Omega \subset \mathbb{R}^d$, an lcHs E and a linear partial differential operator $P(\partial)^E$ which is hypoelliptic if $E = \mathbb{K}$ we define the space of bounded zero solutions

$$\mathcal{C}_{P(\partial),b}^\infty(\Omega, E) := \{f \in \mathcal{C}_{P(\partial)}^\infty(\Omega, E) \mid \forall\ \alpha \in \mathfrak{A}:\ \|f\|_{\infty,\alpha} := \sup_{x\in\Omega} p_\alpha(f(x)) < \infty\}$$

where $\mathcal{C}_{P(\partial)}^\infty(\Omega, E)$ is the kernel of $P(\partial)^E$ in $\mathcal{C}^\infty(\Omega, E)$. Further, we set $\mathcal{C}_{P(\partial),b}^\infty(\Omega) := \mathcal{C}_{P(\partial),b}^\infty(\Omega, \mathbb{K})$. Apart from the topology given by $(\|\cdot\|_{\infty,\alpha})_{\alpha\in\mathfrak{A}}$ there is another weighted locally convex topology on $\mathcal{C}_{P(\partial),b}^\infty(\Omega, E)$ which is of interest, namely, the one induced by the seminorms

$$|f|_{\nu,\alpha} := \sup_{x\in\Omega} p_\alpha(f(x))|\nu(x)|, \quad f \in \mathcal{C}_{P(\partial),b}^\infty(\Omega, E),$$

for $\nu \in \mathcal{C}_0(\Omega)$ and $\alpha \in \mathfrak{A}$. We denote by $(\mathcal{C}_{P(\partial),b}^\infty(\Omega, E), \beta)$ the space $\mathcal{C}_{P(\partial),b}^\infty(\Omega, E)$ equipped with the topology β induced by the seminorms $(|\cdot|_{\nu,\alpha})_{\nu\in\mathcal{C}_0(\Omega),\alpha\in\mathfrak{A}}$. The topology β is called the *strict topology*. It is a bit tricky to prove the ε-compatibility of $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)$ and $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\beta)$ because $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)$ may not be barrelled.

4.2.23. Remark. Let $\Omega\subset\mathbb{R}^d$ be open and $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator. Then $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)$ is non-barrelled if τ_c does not coincide with the $\|\cdot\|_{\infty}$-topology by [46, Section I.1, 1.15 Proposition, p. 12], e.g. $(\mathcal{C}^{\infty}_{\overline{\partial},b}(\mathbb{D}),\beta)$ is non-barrelled.

Hence we cannot use Proposition 3.1.11 c) directly.

4.2.24. Proposition. *Let $\Omega\subset\mathbb{R}^d$ be open, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator and E an lcHs. Then $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)\varepsilon E\cong(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\beta)$ if E has metric ccp.*

Proof. We set $\mathrm{AP}(\Omega,E):=\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E)$ and observe that $(\mathrm{id}_{E^{\Omega}},\mathrm{id}_{\Omega^{\mathbb{K}}})$ is the generator of $((\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta),E)$. First, we prove that the generator is consistent. Clearly, we only need to show that $S(u)\in\mathrm{AP}(\Omega,E)$ for every $u\in(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)\varepsilon E$. Let $u\in(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)\varepsilon E$. Next, we show that $u\in\mathcal{CW}^{\infty}_{P(\partial)}(\Omega)\varepsilon E$ with $\mathcal{CW}^{\infty}_{P(\partial)}(\Omega)$ from Example 3.1.9 b). For $\alpha\in\mathfrak{A}$ there are an absolutely convex, compact $K\subset(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)$ and $C>0$ such that for all $f'\in(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)'$ it holds that

$$p_{\alpha}(u(f'))\leq C\sup_{f\in K}|f'(f)|. \tag{24}$$

We denote by τ_c the topology of compact convergence on $\mathcal{C}^{\infty}_{P(\partial),b}(\Omega)$, i.e. the topology of uniform convergence on compact subsets of Ω. From the compactness of K in $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)$ it follows that K is $\|\cdot\|_{\infty}$-bounded and τ_c-compact by [45, Proposition 1 (viii), p. 586] since $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)$ carries the induced topology of $(\mathcal{C}_b(\Omega),\beta)$ and the strict topology β is the mixed topology $\gamma(\tau_c,\|\cdot\|_{\infty})$ by [45, Proposition 3, p. 590]. Let $f'\in(\mathcal{C}^{\infty}_{P(\partial)}(\Omega),\tau_c)'$. Then there are $M\subset\Omega$ compact and $C_0>0$ such that

$$|f'(f)|\leq C_0\sup_{x\in M}|f(x)|$$

for all $f\in\mathcal{C}^{\infty}_{P(\partial)}(\Omega)$. Choosing a compactly supported cut-off function $\nu\in\mathcal{C}^{\infty}_c(\Omega)$ with $\nu=1$ near M, we obtain

$$|f'(f)|\leq C_0\sup_{x\in\Omega}|f(x)||\nu(x)|=C_0|f|_{\nu}$$

for all $f\in\mathcal{C}^{\infty}_{P(\partial)}(\Omega)$. Therefore $f'\in(\mathcal{C}^{\infty}_{P(\partial)}(\Omega),\beta)'$. In combination with the τ_c-compactness of K it follows from (24) that $u\in(\mathcal{C}^{\infty}_{P(\partial)}(\Omega),\tau_c)\varepsilon E$. Using that $(\mathcal{C}^{\infty}_{P(\partial)}(\Omega),\tau_c)=\mathcal{CW}^{\infty}_{P(\partial)}(\Omega)$ as locally convex spaces by the hypoellipticity of $P(\partial)^{\mathbb{K}}$ (see e.g. [70, p. 690]), we obtain that $u\in\mathcal{CW}^{\infty}_{P(\partial)}(\Omega)\varepsilon E$. Due to Proposition 3.1.11 c) this yields that $S(u)\in\mathcal{C}^{\infty}_{P(\partial)}(\Omega,E)$. Furthermore, we note that

$$\begin{aligned}\|S(u)\|_{\infty,\alpha}&=\sup_{x\in\Omega}p_{\alpha}(S(u)(x))=\sup_{x\in\Omega}p_{\alpha}(u(\delta_x))\underset{(24)}{\leq}C\sup_{x\in\Omega}\sup_{f\in K}|\delta_x(f)|\\&=C\sup_{f\in K}\|f\|_{\infty}<\infty\end{aligned}$$

as K is $\|\cdot\|_{\infty}$-bounded, implying that $S(u)\in\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E)=\mathrm{AP}(\Omega,E)$. Hence the generator $(\mathrm{id}_{E^{\Omega}},\mathrm{id}_{\Omega^{\mathbb{K}}})$ is consistent.

It is easily seen that $e'\circ f\in\mathcal{C}^{\infty}_{P(\partial),b}(\Omega)=\mathrm{AP}(\Omega)$ for all $e'\in E'$ and $f\in\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E)$ (see the proof of Proposition 3.1.10), which proves that the generator is strong as well. Moreover, we define $N_{\nu}(f):=\{f(x)|\nu(x)|\mid x\in\Omega\}$ for $f\in(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\beta)$ and $\nu\in\mathcal{C}_0(\Omega)$. The set $K:=\overline{\mathrm{acx}}(N_{\nu}(f))$ is absolutely convex and compact in E by Proposition A.1.3 (ii) because $f|\nu|\in\mathcal{C}_0(\Omega,E)$ and Ω second-countable, yielding our statement by Corollary 3.2.5 (iv). $\square$

If $\Omega\subset\mathbb{C}$ is an open, simply connected set, $P(\partial)=\overline{\partial}$ and E is complete, then the preceding result is also a consequence of [17, 3.10 Satz, p. 146].

Next, we consider the vector-valued *Beurling–Björck space* $\mathcal{S}_\mu(\mathbb{R}^d,E)$ which generalises the Schwartz space and whose scalar-valued counterpart was studied by Björck in [20], by Schmeisser and Triebel in [155] (see [20, Definition 1.8.1, p. 375], [155, 1.2.1.2 Definition, p. 15]) whereas semigroups on its toplogical dual space were treated by Alvarez et al. in [5]. Since Fourier transformation is involved in the definition of $\mathcal{S}_\mu(\mathbb{R}^d,E)$, we start with the following statement.

4.2.25. PROPOSITION. *Let E be a locally complete lcHs over $\mathbb{C}$, $f\in\mathcal{S}(\mathbb{R}^d,E)$ and $x\in\mathbb{R}^d$. Then $f\mathrm{e}^{-\mathrm{i}\langle x,\cdot\rangle}$ is Pettis-integrable on $\mathbb{R}^d$ where $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbb{R}^d$.*

PROOF. We choose $m:=d+1$ and set $\psi\colon\mathbb{R}^d\to[0,\infty)$, $\psi(\zeta):=(1+|\zeta|^2)^{-m/2}$, as well as $g\colon\mathbb{R}^d\to[0,\infty)$, $g(\zeta):=\psi(\zeta)^{-1}$. Then $\psi\in\mathcal{L}^1(\mathbb{R}^d,\lambda)$ and $\psi g=1$. Moreover, let $x=(x_i)\in\mathbb{R}^d$ and set $u\colon\mathbb{R}^d\to E$, $u(\zeta):=f(\zeta)\mathrm{e}^{-\mathrm{i}\langle x,\zeta\rangle}g(\zeta)$. We note that

$$\begin{aligned}&(\partial^{e_n})^E u(\zeta)\\&=(\partial^{e_n})^E f(\zeta)\mathrm{e}^{-\mathrm{i}\langle x,\zeta\rangle}g(\zeta)-\mathrm{i}x_n f(\zeta)\mathrm{e}^{-\mathrm{i}\langle x,\zeta\rangle}g(\zeta)+mf(\zeta)\mathrm{e}^{-\mathrm{i}\langle x,\zeta\rangle}(1+|\zeta|^2)^{(m/2)-1}\zeta_n\end{aligned}$$

for all $\zeta=(\zeta_i)\in\mathbb{R}^d$ and $1\leq n\leq d$, which implies

$$p_\alpha((\partial^{e_n})^E u(\zeta))\leq p_\alpha((\partial^{e_n})^E f(\zeta))g(\zeta)+|x_n|p_\alpha(f(\zeta))g(\zeta)+mp_\alpha(f(\zeta))g(\zeta)$$

for all $\alpha\in\mathfrak{A}$ and hence

$$\sup_{\substack{\zeta\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d,|\beta|\leq 1}} p_\alpha((\partial^\beta)^E u(\zeta))\leq(1+|x_n|+m)|f|_{\mathcal{S}(\mathbb{R}^d),m,\alpha}.$$

Therefore $u=f\mathrm{e}^{-\mathrm{i}\langle x,\cdot\rangle}g$ is (weakly) $\mathcal{C}^1_b$, which yields $u\in\mathcal{C}^{[1]}_b(\mathbb{R}^d,E)$ by Proposition A.1.5. Now, we choose $h\colon\mathbb{R}^d\to(0,\infty)$, $h(\zeta):=1+|\zeta|^2$. Then

$$\sup_{\zeta\in\mathbb{R}^d}p_\alpha(u(\zeta)h(\zeta))\leq\sup_{\zeta\in\mathbb{R}^d}p_\alpha(f(\zeta))(1+|\zeta|^2)^{(m+2)/2}\leq|f|_{\mathcal{S}(\mathbb{R}^d),m+2,\alpha}<\infty$$

for all $\alpha\in\mathfrak{A}$, and for every $\varepsilon>0$ there is $r>0$ such that $1\leq\varepsilon h(\zeta)$ for all $\zeta\notin\overline{\mathbb{B}_r(0)}=:K$. We deduce from Proposition A.2.7 (iii) that $f\mathrm{e}^{-\mathrm{i}\langle x,\cdot\rangle}$ is Pettis-integrable on $\mathbb{R}^d$. □

Thus, for $f\in\mathcal{S}(\mathbb{R}^d,E)$ with locally complete E the *Fourier transformation*

$$\mathfrak{F}^E(f)\colon\mathbb{R}^d\to E,\ \mathfrak{F}^E(f)(x):=(2\pi)^{-d/2}\int_{\mathbb{R}^d}f(\zeta)\mathrm{e}^{-\mathrm{i}\langle x,\zeta\rangle}\mathrm{d}\zeta,$$

is defined. From the Pettis-integrability we get $(e'\circ\mathfrak{F}^E)(f)=\mathfrak{F}^{\mathbb{C}}(e'\circ f)$ for every $e'\in E'$. As $\mathfrak{F}^{\mathbb{C}}(e'\circ f)\in\mathcal{S}(\mathbb{R}^d)$ for every $e'\in E'$ by [20, Proposition 1.8.2, p. 375], we obtain from the weak-strong principle Corollary 5.2.21 (or [30, Theorem 9, p. 232] and [131, Mackey's theorem 23.15, p. 268]) that $\mathfrak{F}^E(f)\in\mathcal{S}(\mathbb{R}^d,E)$.

For a locally complete lcHs E over $\mathbb{C}$ and a continuous function $\mu\colon\mathbb{R}^d\to[0,\infty)$ such that

(γ) there are $a\in\mathbb{R}$, $b>0$ with $\mu(x)\geq a+b\ln(1+|x|)$ for all $x\in\mathbb{R}^d$,

we set

$$\mathcal{S}_\mu(\mathbb{R}^d,E):=\{f\in\mathcal{C}^\infty(\mathbb{R}^d,E)\mid\forall\ m,j\in\mathbb{N}_0,\ \alpha\in\mathfrak{A}:\ |f|_{j,m,\alpha}<\infty\}$$

where $|f|_{m,j,\alpha}:=\max(q_{m,j,\alpha}(f),q_{m,j,\alpha}(\mathfrak{F}^E(f)))$ with

$$q_{m,j,\alpha}(f):=\sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d,|\beta|\leq m}}p_\alpha((\partial^\beta)^E f(x))\mathrm{e}^{j\mu(x)}.$$

We note that from $q_{m,j,\alpha}(f) < \infty$ for all $m, j \in \mathbb{N}_0$, $\alpha \in \mathfrak{A}$ and condition (γ) it follows that $f \in \mathcal{S}(\mathbb{R}^d, E)$ and hence $q_{m,j,\alpha}(\mathfrak{F}^E(f))$ is defined. Further, we set $\mathcal{S}_\mu(\mathbb{R}^d) := \mathcal{S}_\mu(\mathbb{R}^d, \mathbb{C})$. We observe that $\mathcal{S}_\mu(\mathbb{R}^d, E)$ is a dom-space. Indeed, let $\omega_m := \widetilde{\omega}_m \cup \widetilde{\omega}_{m,1}$ where $\widetilde{\omega}_m := M_m \times \mathbb{R}^d$ with $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq m\}$ and $\widetilde{\omega}_{m,1} := \widetilde{\omega}_m \times \{1\}$ for all $m \in \mathbb{N}_0$. Setting $\operatorname{dom} T_m^E := \mathcal{S}(\mathbb{R}^d, E)$ and $T_m^E : \mathcal{S}(\mathbb{R}^d, E) \to E^{\omega_m}$ by

$$T_m^E(f)(\beta, x) := (\partial^\beta)^E f(x) \text{ and } T_m^E(f)(\beta, x, 1) := ((\partial^\beta)^E \circ \mathfrak{F}^E) f(x), \quad (\beta, x) \in \widetilde{\omega}_m,$$

for every $m \in \mathbb{N}_0$ as well as $\operatorname{AP}(\mathbb{R}^d, E) := E^{\mathbb{R}^d}$, we have that $\mathcal{S}_\mu(\mathbb{R}^d, E)$ is a dom-space with weights given by $\nu_{j,m}(\beta, x) := \nu_{j,m}(\beta, x, 1) := \mathrm{e}^{j\mu(x)}$ for all $(\beta, x) \in \widetilde{\omega}_m$ and $m, j \in \mathbb{N}_0$.

The condition (γ) is introduced in [20, p. 363]. Choosing $\mu(x) := \ln(1 + |x|)$, $x \in \mathbb{R}^d$, we get the Schwartz space $\mathcal{S}_\mu(\mathbb{R}^d, E) = \mathcal{S}(\mathbb{R}^d, E)$ back.

4.2.26. Example. Let E be a locally complete lcHs over $\mathbb{C}$ and $\mu : \mathbb{R}^d \to [0, \infty)$ continuous such that condition (γ) is fulfilled.

(i) If E has metric ccp, or
(ii) if $\mu \in \mathcal{C}^1(\mathbb{R}^d)$ and there are $k \in \mathbb{N}_0$, $C > 0$ such that $|\partial^{e_n} \mu(x)| \leq C\mathrm{e}^{k\mu(x)}$ for all $x \in \mathbb{R}^d$ and $1 \leq n \leq d$,

then $\mathcal{S}_\mu(\mathbb{R}^d, E) \cong \mathcal{S}_\mu(\mathbb{R}^d) \varepsilon E$.

Proof. First, we show that the generator $(T_m^E, T_m^{\mathbb{C}})_{m \in \mathbb{N}_0}$ for $(\mathcal{S}_\mu, E)$ is strong and consistent. From

$$(\partial^\beta)^{\mathbb{C}}(e' \circ f)(x) = e'\big((\partial^\beta)^E f(x)\big), \quad (\beta, x) \in \widetilde{\omega}_m,$$

where $\widetilde{\omega}_m = \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq m\} \times \mathbb{R}^d$, we get in combination with the Pettis-integrability by Proposition 4.2.25 that

$$\big((\partial^\beta)^{\mathbb{C}} \circ \mathfrak{F}^{\mathbb{C}}\big)(e' \circ f)(x) = e'\big(\big((\partial^\beta)^E \circ \mathfrak{F}^E\big) f(x)\big), \quad (\beta, x) \in \widetilde{\omega}_m \tag{25}$$

for all $e' \in E'$, $f \in \mathcal{S}_\mu(\mathbb{R}^d, E)$ and $m \in \mathbb{N}_0$, which means that the generator is strong. For consistency we consider the case $\mu(x) = \ln(1 + |x|)$, $x \in \mathbb{R}^d$, i.e. the Schwartz space, first. Due to Corollary 3.2.10 the map $S : \mathcal{S}(\mathbb{R}^d) \varepsilon E \to \mathcal{S}(\mathbb{R}^d, E)$ is an isomorphism and according to Theorem 3.2.4 its inverse is given by

$$R^t : \mathcal{S}(\mathbb{R}^d, E) \to \mathcal{S}(\mathbb{R}^d) \varepsilon E, \ f \mapsto \mathcal{J}^{-1} \circ R_f^t.$$

Let $u \in \mathcal{S}(\mathbb{R}^d) \varepsilon E$. Thanks to the proof of Corollary 3.2.10 we only need to show that

$$u(\delta_x \circ (\partial^\beta \circ \mathfrak{F}^{\mathbb{C}})) = (\partial^\beta)^E \mathfrak{F}^E(S(u))(x), \quad x \in \mathbb{R}^d.$$

We set $f := S(u) \in \mathcal{S}(\mathbb{R}^d, E)$ and from (25) we obtain

$$R_f^t\big(\delta_x \circ ((\partial^\beta)^{\mathbb{C}} \circ \mathfrak{F}^{\mathbb{C}})\big)(e') = (\partial^\beta)^{\mathbb{C}} \mathfrak{F}^{\mathbb{C}}(e' \circ f)(x) = e'\big((\partial^\beta)^E \mathfrak{F}^E(f)(x)\big), \quad e' \in E',$$

for all $x \in \mathbb{R}^d$ and $\beta \in \mathbb{N}_0^d$, which results in

$$\begin{aligned} u\big(\delta_x \circ ((\partial^\beta)^{\mathbb{C}} \circ \mathfrak{F}^{\mathbb{C}})\big) &= S^{-1}(f)\big(\delta_x \circ ((\partial^\beta)^{\mathbb{C}} \circ \mathfrak{F}^{\mathbb{C}})\big) = \mathcal{J}^{-1}\big(R_f^t(\delta_x \circ ((\partial^\beta)^{\mathbb{C}} \circ \mathfrak{F}^{\mathbb{C}}))\big) \\ &= (\partial^\beta)^E \mathfrak{F}^E(f)(x) = (\partial^\beta)^E \mathfrak{F}^E(S(u))(x). \end{aligned} \tag{26}$$

Thus $(T_m^E, T_m^{\mathbb{C}})_{m \in \mathbb{N}_0}$ is a consistent generator for $(\mathcal{S}, E)$.

Let us turn to general μ. Let $u \in \mathcal{S}_\mu(\mathbb{R}^d) \varepsilon E$. We show that $u \in \mathcal{S}(\mathbb{R}^d) \varepsilon E$. Then it follows from the first part of the proof that $(T_m^E, T_m^{\mathbb{C}})_{m \in \mathbb{N}_0}$ is a consistent generator for $(\mathcal{S}_\mu, E)$. For $\alpha \in \mathfrak{A}$ there are an absolutely convex compact set $K \subset \mathcal{S}_\mu(\mathbb{R}^d)$ and $C > 0$ such that for all $f' \in \mathcal{S}_\mu(\mathbb{R}^d)'$ it holds

$$p_\alpha(u(f')) \leq C \sup_{f \in K} |f'(f)|. \tag{27}$$

The compactness of K in $\mathcal{S}_\mu(\mathbb{R}^d)$ and the estimate

$$\sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d,|\beta|\leq m}} |(\partial^\beta)^{\mathbb{C}} f(x)|(1+|x|^2)^{j/2} \leq \sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d,|\beta|\leq m}} |(\partial^\beta)^{\mathbb{C}} f(x)|\mathrm{e}^{(j/2)(2/b)(\mu(x)-a)}$$
$$\leq \mathrm{e}^{-(aj)/b}|f|_{j,m}, \quad f\in\mathcal{S}_\mu(\mathbb{R}^d),$$

for all $j,m\in\mathbb{N}_0$ by condition (γ) imply that the inclusion $\mathcal{S}_\mu(\mathbb{R}^d)\hookrightarrow\mathcal{S}(\mathbb{R}^d)$ is continuous and thus that K is compact in $\mathcal{S}(\mathbb{R}^d)$. Let $f'\in\mathcal{S}(\mathbb{R}^d)'$. Then there are $j,m\in\mathbb{N}_0$ and $C_0>0$ such that

$$|f'(f)|\leq C_0 \sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d,|\beta|\leq m}} |(\partial^\beta)^{\mathbb{C}} f(x)|(1+|x|^2)^{j/2} \leq C_0\mathrm{e}^{-(aj)/b}|f|_{j,m}$$

for all $f\in\mathcal{S}_\mu(\mathbb{R}^d)$. Hence $f'\in\mathcal{S}_\mu(\mathbb{R}^d)'$ and from (27) we obtain that $u\in\mathcal{S}(\mathbb{R}^d)\varepsilon E$ because K is absolutely convex and compact in $\mathcal{S}(\mathbb{R}^d)$.

Condition (γ) implies that $\mu(x)\to\infty$ for $|x|\to\infty$. Noting that for every $j\in\mathbb{N}$ and $\varepsilon>0$ there is $r>0$ such that

$$\frac{\mathrm{e}^{j\mu(x)}}{\mathrm{e}^{2j\mu(x)}} = \mathrm{e}^{-j\mu(x)} < \varepsilon \tag{28}$$

for all $x\notin\overline{\mathbb{B}_r(0)}$, we deduce $|f|_{\mathbb{R}^d\smallsetminus\overline{\mathbb{B}_r(0)},m,j,\alpha}\leq\varepsilon|f|_{m,2j,\alpha}$ for every $f\in\mathcal{S}_\mu(\mathbb{R}^d,E)$, $m\in\mathbb{N}_0$ and $\alpha\in\mathfrak{A}$.

(i) Thus, if E has metric ccp, then the sets $K_\beta := \overline{\operatorname{acx}}((\partial^\beta)^E f\mathrm{e}^{j\mu}(\mathbb{R}^d))$ and $K_{\beta,1} := \overline{\operatorname{acx}}((\partial^\beta)^E\mathfrak{F}^E(f)\mathrm{e}^{j\mu}(\mathbb{R}^d))$ are absolutely convex and compact by Proposition A.1.3 (ii) for every $f\in\mathcal{S}_\mu(\mathbb{R}^d,E)$, $j,m\in\mathbb{N}_0$ and $\beta\in M_m$ as $(\partial^\beta)^E f\mathrm{e}^{j\mu}\in\mathcal{C}_0(\mathbb{R}^d,E)$ and $(\partial^\beta)^E\mathfrak{F}^E(f)\mathrm{e}^{j\mu}\in\mathcal{C}_0(\mathbb{R}^d,E)$.

(ii) We set $g_0\colon\mathbb{R}^d\to E$, $g_0(x) := (\partial^\beta)^E f(x)\mathrm{e}^{j\mu(x)}$, and $g_1\colon\mathbb{R}^d\to E$, $g_1(x) := (\partial^\beta)^E\mathfrak{F}^E(f)(x)\mathrm{e}^{j\mu(x)}$, for $j,m\in\mathbb{N}_0$ and $\beta\in M_m$. We observe that

$$(\partial^{e_n})^E g_0(x) = (\partial^{\beta+e_n})^E f(x)\mathrm{e}^{j\mu(x)} + j(\partial^\beta)^E f(x)\mathrm{e}^{j\mu(x)}\partial^{e_n}\mu(x)$$

and

$$(\partial^{e_n})^E g_1(x) = (\partial^{\beta+e_n})^E \mathfrak{F}^E(f)(x)\mathrm{e}^{j\mu(x)} + j(\partial^\beta)^E \mathfrak{F}^E(f)(x)\mathrm{e}^{j\mu(x)}\partial^{e_n}\mu(x)$$

for all $x\in\mathbb{R}^d$ and $1\leq n\leq d$. As in Example 3.2.9 it follows from condition (ii) that there are $k\in\mathbb{N}_0$, $C>0$ such that

$$\sup_{\substack{x\in\mathbb{R}^d\\ \gamma\in\mathbb{N}_0^d,|\gamma|\leq 1}} p_\alpha((\partial^\gamma)^E g_i(x)) \leq |f|_{m+1,j,\alpha} + Cj|f|_{m,j+k,\alpha}$$

for all $\alpha\in\mathfrak{A}$ and $i=0,1$. Thus g_0 and g_1 are (weakly) $\mathcal{C}^1_b$. We set $h := \mathrm{e}^{j\mu}$ and note that

$$\sup_{x\in\mathbb{R}^d} p_\alpha(g_i(x)h(x)) \leq |f|_{m,2j,\alpha} < \infty$$

for all $\alpha\in\mathfrak{A}$ and $i=0,1$. This yields that $K_\beta = \overline{\operatorname{acx}}(g_0(\mathbb{R}^d))$ and $K_{\beta,1} = \overline{\operatorname{acx}}(g_1(\mathbb{R}^d))$ are absolutely convex and compact by Proposition A.1.4 with (28) and Proposition A.1.5.

Then we have in both cases

$$\begin{aligned} N_{j,m}(f) = \ & \big(\{(\partial^\beta)^E f(x)\mathrm{e}^{j\mu(x)} \mid x\in\mathbb{R}^d,\ \beta\in M_m\} \\ & \cup \{(\partial^\beta)^E\mathfrak{F}^E(f)(x)\mathrm{e}^{j\mu(x)} \mid x\in\mathbb{R}^d,\ \beta\in M_m\}\big) \\ & \subset \operatorname{acx}\Big(\bigcup_{\beta\in M_m}(K_\beta\cup K_{\beta,1})\Big) \end{aligned}$$

and the set on the right-hand side is absolutely convex and compact by [89, 6.7.3 Proposition, p. 113], which implies that $\mathcal{S}_{\mu}(\mathbb{R}^d, E) \cong \mathcal{S}_{\mu}(\mathbb{R}^d)\varepsilon E$ by Corollary 3.2.5 (iv). □

We come back to these spaces in Theorem 5.1.5. Another example that is related to Fourier transformation is the space of vector-valued smooth functions that are 2π-periodic in each variable. We equip the space $\mathcal{C}^{\infty}(\mathbb{R}^d, E)$ for an lcHs E with the system of seminorms generated by

$$|f|_{K,m,\alpha} := \sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d, |\beta|\leq m}} p_{\alpha}((\partial^{\beta})^E f(x))\chi_K(x) = \sup_{\substack{x\in K\\ \beta\in\mathbb{N}_0^d, |\beta|\leq m}} p_{\alpha}((\partial^{\beta})^E f(x)), \quad f \in \mathcal{C}^{\infty}(\mathbb{R}^d, E),$$

for $K \subset \mathbb{R}^d$ compact, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$, i.e. we consider $\mathcal{CW}^{\infty}(\mathbb{R}^d, E)$. By $\mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d, E)$ we denote the topological subspace of $\mathcal{C}^{\infty}(\mathbb{R}^d, E)$ consisting of the functions which are 2π-periodic in each variable. Further, we set $\mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d) := \mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d, \mathbb{K})$.

4.2.27. Example. If E is a locally complete lcHs, then $\mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d, E) \cong \mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d)\varepsilon E$.

Proof. First, we note that for each $x \in \mathbb{R}^d$ and $1 \leq n \leq d$ we have $\delta_x = \delta_{x+2\pi e_n}$ in $\mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d)'$ and thus

$$S_{\mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d)}(u)(x) - S_{\mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d)}(u)(x + 2\pi e_n) = u(\delta_x - \delta_{x+2\pi e_n}) = 0, \quad u \in \mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d)\varepsilon E,$$

implying that $S_{\mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d)}(u)$ is 2π-periodic in each variable. In addition, we observe that $e' \circ f$ is 2π-periodic in each variable for all $e' \in E'$ and $f \in \mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d, E)$. Now, we obtain as in Example 3.2.8 a) for $k = \infty$ that $S_{\mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d)} \colon \mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d)\varepsilon E \to \mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d, E)$ is an isomorphism. □

We return to $\mathcal{C}^{\infty}_{2\pi}(\mathbb{R}^d, E)$ in Theorem 5.1.4 and Theorem 5.6.14. Now, we direct our attention to spaces of continuously partially differentiable functions on an open bounded set such that all derivatives can be continuously extended to the boundary. Let E be an lcHs, $k \in \mathbb{N}_{\infty}$ and $\Omega \subset \mathbb{R}^d$ open and bounded. The space $\mathcal{C}^k(\overline{\Omega}, E)$ is given by

$$\mathcal{C}^k(\overline{\Omega}, E) := \{f \in \mathcal{C}^k(\Omega, E) \mid (\partial^{\beta})^E f \text{ cont. extendable on } \overline{\Omega} \text{ for all } \beta \in \mathbb{N}_0^d,\ |\beta| \leq k\}$$

and equipped with the system of seminorms given by

$$|f|_{\alpha} := \sup_{\substack{x\in\Omega\\ \beta\in\mathbb{N}_0^d, |\beta|\leq k}} p_{\alpha}\big((\partial^{\beta})^E f(x)\big), \quad f \in \mathcal{C}^k(\overline{\Omega}, E),$$

for $\alpha \in \mathfrak{A}$ if $k < \infty$, and by

$$|f|_{m,\alpha} := \sup_{\substack{x\in\Omega\\ \beta\in\mathbb{N}_0^d, |\beta|\leq m}} p_{\alpha}\big((\partial^{\beta})^E f(x)\big), \quad f \in \mathcal{C}^{\infty}(\overline{\Omega}, E),$$

for $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$ if $k = \infty$. Further, we set $\mathcal{C}^k(\overline{\Omega}) := \mathcal{C}^k(\overline{\Omega}, \mathbb{K})$.

4.2.28. Example. Let E be an lcHs, $k \in \mathbb{N}_{\infty}$ and $\Omega \subset \mathbb{R}^d$ open and bounded.

(i) If E has metric ccp, or
(ii) if E is locally complete, $k = \infty$ and there exists $C > 0$ such that for each $x, y \in \Omega$ there is a continuous path from x to y in Ω whose length is bounded by $C|x - y|$,

then $\mathcal{C}^k(\overline{\Omega}, E) \cong \mathcal{C}^k(\overline{\Omega})\varepsilon E$.

Proof. The generator coincides with the one of Example 4.2.22. Due to Proposition 3.1.11 we have $S(u) \in \mathcal{C}^k(\Omega, E)$ and

$$(\partial^{\beta})^E S(u)(x) = u(\delta_x \circ (\partial^{\beta})^{\mathbb{K}}), \quad \beta \in \mathbb{N}_0^d,\ |\beta| \leq k,\ x \in \Omega,$$

for all $u \in \mathcal{C}^k(\overline{\Omega})\varepsilon E$ since $\mathcal{C}^k(\overline{\Omega})$ is a Banach space if $k < \infty$, and a Fréchet space if $k = \infty$, in particular, both are barrelled. As a consequence of Proposition 4.1.7 and Lemma 4.1.8 with $T = (\partial^\beta)^{\mathbb{K}}$ for $\beta \in \mathbb{N}_0^d$, $|\beta| \leq k$, we obtain that $(\partial^\beta)^E S(u) \in \mathcal{C}^{ext}(\Omega, E)$ for all $u \in \mathcal{C}^k(\overline{\Omega})\varepsilon E$. Thus the generator is consistent. It is easy to check that it is strong, too. This yields (ii) by Corollary 3.2.5 (ii) since $\mathcal{C}^\infty(\overline{\Omega})$ is a nuclear Fréchet space by [131, Examples 28.9 (5), p. 350] under the conditions on Ω.

Let us turn to part (i). Let $f \in \mathcal{C}^k(\overline{\Omega}, E)$, $J := \{1\}$, $m \in \mathbb{N}_0$ and set $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq k\}$ if $k < \infty$, and $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq m\}$ if $k = \infty$. We denote by f_β the continuous extension of $(\partial^\beta)^E f$ on the compact metrisable set $\overline{\Omega}$. The set

$$N_{1,m}(f) = \{(\partial^\beta)^E f(x) \mid x \in \Omega,\, \beta \in M_m\} \subset \bigcup_{\beta \in M_m} f_\beta(\overline{\Omega})$$

is relatively compact and metrisable since it is a subset of a finite union of the compact metrisable sets $f_\beta(\overline{\Omega})$ as in Example 3.2.8. Due to Corollary 3.2.5 (iv) we obtain our statement (i) as E has metric ccp. □

We close this section by an examination of the topological subspace

$$\mathcal{E}_0(E) := \{f \in \mathcal{C}^\infty([0,1], E) \mid \forall\, k \in \mathbb{N}_0 : \ (\partial^k)^E f(1) = 0\}$$

where $(\partial^k)^E f(1) := \lim_{x \to 1+} (\partial^k)^E f(x)$. Further, we set $\mathcal{E}_0 := \mathcal{E}_0(\mathbb{K})$.

4.2.29. Example. Let E be a locally complete lcHs. Then $\mathcal{E}_0 \varepsilon E \cong \mathcal{E}_0(E)$.

Proof. We note that $\Omega := (0,1)$ satisfies the condition on Ω in Example 4.2.28 (ii) with $C := 1$ and thus $\mathcal{C}^\infty([0,1])$ and its closed subspace $\mathcal{E}_0$ are nuclear Fréchet spaces. The generator coincides with the one of Example 4.2.28. From the proof of Example 4.2.28 we know that

$$\lim_{x \to 1+} (\partial^k)^E S(u)(x) = u(\delta_1 \circ (\partial^k)^{\mathbb{K}}) = u(0) = 0, \quad k \in \mathbb{N}_0,$$

for all $u \in \mathcal{E}_0 \varepsilon E$. In combination with Example 4.2.28 this yields the consistency of the generator. Again, its strength is easy to check. Therefore our statement is valid by Corollary 3.2.5 (ii). □

4.3. Riesz–Markov–Kakutani representation theorems

In this subsection we generalise the concept of strength and consistency such that it is not strictly bounded to dom-spaces and their generators anymore. This allows us to answer the question: Given $T_m^{\mathbb{K}} \in \mathcal{F}(\Omega)'$ is there $T_m^E \in L(\mathcal{F}(\Omega, E), E)$ such that $(T_m^E, T_m^{\mathbb{K}})$ is strong and consistent? Furthermore, we will see that the operators T_m^E are usually the ones that can be obtained from integral representations of $T_m^{\mathbb{K}}$, i.e. we transfer Riesz–Markov–Kakutani theorems from the scalar-valued to the vector-valued case. We recall that the *Riesz–Markov–Kakutani theorem* for compact topological Hausdorff spaces Ω says that for every $T^{\mathbb{R}} \in \mathcal{C}_b(\Omega)'$ there is a unique regular $\mathbb{R}$-valued Borel measure μ on Ω such that

$$T^{\mathbb{R}}(f) = \int_\Omega f(x) \mathrm{d}\mu(x), \quad f \in \mathcal{C}_b(\Omega), \tag{29}$$

which was proved by Riesz [146, p. 976] in the case $\Omega := [0,1]$ and by Kakutani [95, Theorem 9, p. 1009] for general compact Hausdorff Ω (see Saks [152, Eq. (1.1), 6., p. 408, 411] for compact metric Ω). Markov treated the case where Ω is a normal (not necessarily Hausdorff) topological space and the $T^{\mathbb{R}}$ are positive linear functionals on $\mathcal{C}_b(\Omega)$ such that $T^{\mathbb{R}}(1) = 1$ [127, Definition 2, p. 167]. In this case, for every such $T^{\mathbb{R}}$ there is a unique exterior density μ on Ω in the sense of [127, Definition 3, p. 167] such that (29) holds by [127, Theorem 22, p. 184] and the right-hand side is read in the sense of [127, Eq. (71), (72), (80), p. 180–181] (see also [60, IV.6.2 Theorem, p. 262] for a more familiar version with regular (finitely) additive bounded Borel measures μ).

4.3.1. Definition (strong, consistent). Let E be an lcHs and Ω a non-empty set. Let $\mathcal{F}(\Omega)\subset\mathbb{K}^{\Omega}$ and $\mathcal{F}(\Omega,E)\subset E^{\Omega}$ be lcHs such that $\delta_x\in\mathcal{F}(\Omega)'$ for all $x\in\Omega$. Let $(\omega_m)_{m\in M}$ be a family of non-empty sets, $T^{\mathbb{K}}_m\colon \operatorname{dom}T^{\mathbb{K}}_m\to\mathbb{K}^{\omega_m}$ and $T^{E}_m\colon \operatorname{dom}T^{E}_m\to E^{\omega_m}$ be linear with $\mathcal{F}(\Omega)\subset\operatorname{dom}T^{\mathbb{K}}_m\subset\mathbb{K}^{\Omega}$ and $\mathcal{F}(\Omega,E)\subset\operatorname{dom}T^{E}_m\subset E^{\Omega}$ for all $m\in M$.

a) We call $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ a *consistent family* for $(\mathcal{F}(\Omega),E)$, in short $(\mathcal{F},E)$, if we have for every $u\in\mathcal{F}(\Omega)\varepsilon E$, $m\in M$ and $x\in\omega_m$ that
   (i) $S(u)\in\mathcal{F}(\Omega,E)$ and $T^{\mathbb{K}}_{m,x}:=\delta_x\circ T^{\mathbb{K}}_m\in\mathcal{F}(\Omega)'$,
   (ii) $T^E_mS(u)(x)=u(T^{\mathbb{K}}_{m,x})$.
b) We call $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ a *strong family* for $(\mathcal{F}(\Omega),E)$, in short $(\mathcal{F},E)$, if we have for every $e'\in E'$, $f\in\mathcal{F}(\Omega,E)$, $m\in M$ and $x\in\omega_m$ that
   (i) $e'\circ f\in\mathcal{F}(\Omega)$,
   (ii) $T^{\mathbb{K}}_m(e'\circ f)(x)=\big(e'\circ T^E_m(f)\big)(x)$.

Note that ω_m need not be a subset of Ω. As a convention we omit the index m of the set ω_m, the operators T^E_m and $T^{\mathbb{K}}_m$ if M is a singleton. The following remark shows that the preceding definition of a consistent resp. strong family coincides with the usual definition in the case of generators of dom-spaces (see Definition 3.1.7).

4.3.2. Remark. Let $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ be a generator for $(\mathcal{FV},E)$. We note that the condition $T^{\mathbb{K}}_{m,x}\in\mathcal{FV}(\Omega)'$ for all $m\in M$ and $x\in\omega_m$ in a)(i) of Definition 4.3.1 is always satisfied for generators by Remark 3.1.6 b). Moreover, if $S(u)\in\operatorname{AP}(\Omega,E)\cap\operatorname{dom}T^E_m$ for $u\in\mathcal{FV}(\Omega)\varepsilon E$ and all $m\in M$ and a)(ii) of Definition 4.3.1 is fulfilled, then $S(u)\in\mathcal{FV}(\Omega,E)$ by Lemma 3.1.8, implying that a)(i) is satisfied. Further, if $f\in\mathcal{FV}(\Omega,E)$ and $e'\circ f\in\operatorname{AP}(\Omega)\cap\operatorname{dom}T^{\mathbb{K}}_m$ for all $e'\in E'$ and $m\in M$ and b)(ii) of Definition 4.3.1 is fulfilled, then $e'\circ f\in\mathcal{FV}(\Omega)$ by Lemma 3.2.1, implying that b)(i) is satisfied.

The next proposition is the key result in transferring Riesz–Markov–Kakutani theorems from the scalar-valued to the vector-valued case. To state this proposition we need that our map $S\colon\mathcal{F}(\Omega)\varepsilon E\to\mathcal{F}(\Omega,E)$ is an isomorphism and that its inverse is given as in Theorem 3.2.4, i.e. that

$$R^t\colon\mathcal{F}(\Omega,E)\to\mathcal{F}(\Omega)\varepsilon E,\ f\mapsto\mathcal{J}^{-1}\circ R^t_f,$$

is the inverse of S where $R^t_f(f')(e')=f'(e'\circ f)$, for $f'\in\mathcal{F}(\Omega)'$ and $e'\in E'$, and $\mathcal{J}\colon E\to E'^{\star}$ is the canonical injection in the algebraic dual $E'^{\star}$ of E'.

4.3.3. Proposition. *Let E be an lcHs, (Ω,Σ,μ) a measure space and $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ ε-compatible with inverse R^t of S and $(T^E_0,T^{\mathbb{K}}_0)$ a strong family for $(\mathcal{F},E)$ with $\omega_0:=\Omega$. If $T^{\mathbb{K}}_0(f)$ is integrable for every $f\in\mathcal{F}(\Omega)$ and $T^E_0(f)$ is Pettis-integrable on Ω for every $f\in\mathcal{F}(\Omega,E)$ and*

$$T^{\mathbb{K}}\colon\mathcal{F}(\Omega)\to\mathbb{K},\ T^{\mathbb{K}}(f):=\int_{\Omega}T^{\mathbb{K}}_0(f)(x)\,\mathrm{d}\mu(x),$$

is continuous, then

$$u(T^{\mathbb{K}})=\int_{\Omega}T^E_0S(u)(x)\,\mathrm{d}\mu(x),\quad u\in\mathcal{F}(\Omega)\varepsilon E.$$

Proof. Let $u\in\mathcal{F}(\Omega)\varepsilon E$ and set $f:=S(u)\in\mathcal{F}(\Omega,E)$. We have

$$R^t_f(T^{\mathbb{K}})(e')=T^{\mathbb{K}}(e'\circ f)=\int_{\Omega}T^{\mathbb{K}}_0(e'\circ f)(x)\,\mathrm{d}\mu(x)=\langle e',\int_{\Omega}T^E_0f(x)\,\mathrm{d}\mu(x)\rangle,\quad e'\in E',$$

by the strength of $(T_0^E, T_0^{\mathbb{K}})$ and the Pettis-integrability of $T_0^E(f)$, which yields

$$u(T^{\mathbb{K}}) = S^{-1}(f)(T^{\mathbb{K}}) = \mathcal{J}^{-1}\big(R_f^t(T^{\mathbb{K}})\big) = \int_{\Omega} T_0^E f(x)\mathrm{d}\mu(x) = \int_{\Omega} T_0^E S(u)(x)\mathrm{d}\mu(x)$$

due to R^t being the inverse of S. □

4.3.4. Proposition. *Let E be an lcHs, (Ω,Σ,μ) a measure space, $(T_0^E, T_0^{\mathbb{K}})$ a strong family for $(\mathcal{F},E)$ with $\omega_0 := \Omega$ such that $T_0^E(f)$ is Pettis-integrable on Ω for every $f\in\mathcal{F}(\Omega,E)$, and $(T^E,T^{\mathbb{K}})$ a consistent family for $(\mathcal{F},E)$ such that*

$$T^E(f) = \int_{\Omega} T_0^E f(x)\mathrm{d}\mu(x), \quad f\in\mathcal{F}(\Omega,E).$$

Then $(T^E,T^{\mathbb{K}})$ is a strong family for $(\mathcal{F},E)$, $T_0^{\mathbb{K}}(f)$ is integrable for every $f\in\mathcal{F}(\Omega)$ and

$$T^{\mathbb{K}} = \int_{\Omega} T_0^{\mathbb{K}}(f)(x)\mathrm{d}\mu(x), \quad f\in\mathcal{F}(\Omega).$$

Proof. We set $f\cdot e\colon\Omega\to E$, $(f\cdot e)(x) := f(x)e$, for $e\in E$ and $f\in\mathcal{F}(\Omega)$. Since $(T^E,T^{\mathbb{K}})$ is a consistent family for $(\mathcal{F},E)$, we get $f\cdot e = S(\Theta(e\otimes f))\in\mathcal{F}(\Omega,E)$ and

$$T^E(f\cdot e) = \Theta(e\otimes f)(T^{\mathbb{K}}) = T^{\mathbb{K}}(f)e \tag{30}$$

with the map Θ from (3). From the strength of $(T_0^E, T_0^{\mathbb{K}})$ we deduce that

$$e'\circ T_0^E(f\cdot e) = T_0^{\mathbb{K}}(e'\circ(f\cdot e)) = T_0^{\mathbb{K}}(e'(e)f) = e'(e)T_0^{\mathbb{K}}(f)$$

and from the Pettis-integrability of $T_0^E(f\cdot e)$ that

$$T^{\mathbb{K}}(f)e'(e) \underset{(30)}{=} \langle e', T^E(f\cdot e)\rangle = \int_{\Omega} e'(e)T_0^{\mathbb{K}}(f)(x)\mathrm{d}\mu(x)$$

for all $e'\in E'$. This implies that $e'(e)T_0^{\mathbb{K}}(f)$ is integrable for all $e'\in E'$. Further, since E is non-trivial by our assumptions in Chapter 2, there is some $e_0\in E$, $e_0\neq 0$. By the Hahn–Banach theorem there is some $e_0'\in E'$ with $e_0'(e_0)\neq 0$, which yields that $T_0^{\mathbb{K}}(f)$ is integrable and

$$T^{\mathbb{K}}(f) = \int_{\Omega} T_0^{\mathbb{K}}(f)(x)\mathrm{d}\mu(x).$$

Furthermore, we conclude in combination with the strength of $(T_0^E, T_0^{\mathbb{K}})$ and the Pettis-integrability of $T_0^E(f)$ for all $f\in\mathcal{F}(\Omega,E)$ that

$$\langle e', T^E(f)\rangle = \int_{\Omega} T_0^{\mathbb{K}}(e'\circ f)(x)\mathrm{d}\mu(x) = T^{\mathbb{K}}(e'\circ f)$$

for all $f\in\mathcal{F}(\Omega,E)$ and $e'\in E'$, which means that $(T^E,T^{\mathbb{K}})$ is a strong family for $(\mathcal{F},E)$. □

Let us apply the preceding propositions to the space $D([0,1],E)$ of E-valued càdlàg functions on $[0,1]$. For $f\in D([0,1],E)$ we set $f(x-) := \lim_{w\to x-} f(w)$ if $x\in(0,1]$, and $f(0-) := 0$.

4.3.5. Proposition. *Let E be a quasi-complete lcHs. Then for every $T^{\mathbb{K}}\in D([0,1])'$ there is $T^E\in L(D([0,1],E),E)$ such that $(T^E,T^{\mathbb{K}})$ is a consistent family for (D,E) and there are a unique regular $\mathbb{K}$-valued Borel measure μ on $[0,1]$ and a unique $\varphi\in\ell^1([0,1],\mathbb{K})$ such that*

$$T^E(f) = \int_{[0,1]} f(x)\mathrm{d}\mu(x) + \sum_{x\in[0,1]} (f(x)-f(x-))\overline{\varphi(x)}, \quad f\in D([0,1],E). \tag{31}$$

On the other hand, if $(T^E,T^{\mathbb{K}})$ *is a consistent family, there is a unique regular* $\mathbb{K}$*-valued Borel measure* μ *on* $[0,1]$ *such that* (31) *holds and* $T^E\in L(D([0,1],E),E)$.

Proof. Due to the representation theorem [139, Theorem 1, p. 383] there are a unique regular $\mathbb{K}$-valued Borel measure μ on $[0,1]$ and a unique $\varphi\in\ell^1([0,1],\mathbb{K})$ such that

$$T^{\mathbb{K}}(f)=\int\limits_{[0,1]}f(x)\mathrm{d}\mu(x)+\sum_{x\in[0,1]}(f(x)-f(x-))\overline{\varphi(x)},\quad f\in D([0,1],\mathbb{K}). \tag{32}$$

By Example 4.2.5 $S\colon D([0,1])\varepsilon E\to D([0,1],E)$ is an isomorphism with inverse $R^t\colon f\mapsto \mathcal{J}\circ R^t_f$.

The next part is the analogon of Proposition 4.3.3 for $\sum_{x\in[0,1]}$. We set

$$T_1^{\mathbb{K}}\colon D([0,1])\to\mathbb{K},\ T_1^{\mathbb{K}}(f):=\sum_{x\in[0,1]}(f(x)-f(x-))\overline{\varphi(x)},$$

and note that $T_1^{\mathbb{K}}\in D([0,1])'$. Let $u\in D([0,1])\varepsilon E$ and set $f:=S(u)\in D([0,1],E)$. We have

$$\begin{aligned}R^t_f(T_1^{\mathbb{K}})(e')=T_1^{\mathbb{K}}(e'\circ f)&=\sum_{x\in[0,1]}((e'\circ f)(x)-(e'\circ f)(x-))\overline{\varphi(x)}\\&=\langle e',\sum_{x\in[0,1]}(f(x)-f(x-))\overline{\varphi(x)}\rangle,\quad e'\in E',\end{aligned}$$

due to the Pettis-summability of $x\mapsto(f(x)-f(x-))\overline{\varphi(x)}$ on $[0,1]$ by Proposition A.2.6, which yields

$$\begin{aligned}u(T_1^{\mathbb{K}})=S^{-1}(f)(T_1^{\mathbb{K}})=\mathcal{J}^{-1}\big(R^t_f(T_1^{\mathbb{K}})\big)&=\sum_{x\in[0,1]}(f(x)-f(x-))\overline{\varphi(x)}\\&=\sum_{x\in[0,1]}(S(u)(x)-S(u)(x-))\overline{\varphi(x)}.\end{aligned} \tag{33}$$

We note that every $f\in D([0,1],E)$ is Pettis-integrable and that $x\mapsto(f(x)-f(x-))\overline{\varphi(x)}$ is Pettis-summable on $[0,1]$ by Proposition A.2.6. Further,

$$p_\alpha\Big(\int\limits_{[0,1]}f(x)\mathrm{d}\mu(x)+\sum_{x\in[0,1]}(f(x)-f(x-))\overline{\varphi(x)}\Big)\leq(|\mu|([0,1])+2\|\varphi\|_{\ell^1})\sup_{x\in[0,1]}p_\alpha(f(x))$$

for all $f\in D([0,1],E)$ and $\alpha\in\mathfrak{A}$. The rest follows from Proposition 4.3.3 with $(T_0^E,T_0^{\mathbb{K}}):=(\mathrm{id}_{E^{[0,1]}},\mathrm{id}_{\mathbb{K}^{[0,1]}})$ combined with (33). For the uniqueness of μ in (31) use that the μ in (32) is unique and Proposition 4.3.4 (and for the uniqueness of φ use an analogon of Proposition 4.3.4 for $(T_1^E,T_1^{\mathbb{K}})$). □

Let us turn to continuous functions that vanish at infinity.

4.3.6. Proposition. *Let* Ω *be a locally compact [second countable] topological Hausdorff space and* E *an lcHs with [metric] ccp. Then for every* $T^{\mathbb{K}}\in\mathcal{C}_0(\Omega)'$ *there is* $T^E\in L(\mathcal{C}_0(\Omega,E),E)$ *such that* $(T^E,T^{\mathbb{K}})$ *is a consistent family for* $(\mathcal{C}_0,E)$ *and there is a unique regular* $\mathbb{K}$*-valued Borel measure* μ *on* Ω *such that*

$$T^E(f)=\int\limits_{\Omega}f(x)\mathrm{d}\mu(x),\quad f\in\mathcal{C}_0(\Omega,E). \tag{34}$$

On the other hand, if $(T^E,T^{\mathbb{K}})$ *is a consistent family, then there is a unique regular* $\mathbb{K}$*-valued Borel measure* μ *on* Ω *such that* (34) *holds and* $T^E\in L(\mathcal{C}_0(\Omega,E),E)$.

PROOF. Due to the Riesz–Markov–Kakutani representation theorem (see [149, 6.19 Theorem, p. 130]) there is a unique regular $\mathbb{K}$-valued Borel measure μ on Ω such that

$$T^{\mathbb{K}}(f)=\int_{\Omega} f(x)\mathrm{d}\mu(x),\quad f\in\mathcal{C}_0(\Omega). \tag{35}$$

By Example 4.2.11 $S\colon\mathcal{C}_0(\Omega)\varepsilon E\to\mathcal{C}_0(\Omega,E)$ is an isomorphism with inverse $R^t\colon f\mapsto\mathcal{J}\circ R^t_f$. We note that every $f\in\mathcal{C}_0(\Omega,E)$ is Pettis-integrable by Proposition A.2.7 (i) resp. (ii) with $\psi:=g:=1$ since

$$\int_{\Omega}|\psi(x)|\mathrm{d}|\mu|(x)=|\mu|(\Omega)<\infty$$

and

$$p_\alpha\Big(\int_{\Omega} f(x)\mathrm{d}\mu(x)\Big)\leq|\mu|(\Omega)\sup_{x\in\Omega}p_\alpha(f(x)),\quad f\in\mathcal{C}_0(\Omega,E),$$

for all $\alpha\in\mathfrak{A}$. The rest follows from Proposition 4.3.3 with $(T^E_0,T^{\mathbb{K}}_0):=(\mathrm{id}_{E^\Omega},\mathrm{id}_{\mathbb{K}^\Omega})$. For the uniqueness of μ in (34) use that the μ in (35) is unique and Proposition 4.3.4. □

Next, we consider the space of bounded continuous E-valued functions on a locally compact topological Hausdorff space Ω, i.e.

$$\mathcal{C}_b(\Omega,E)=\{f\in\mathcal{C}(\Omega,E)\mid\forall\ \alpha\in\mathfrak{A}:\ \sup_{x\in\Omega}p_\alpha(f(x))<\infty\},$$

but equipped with the strict topology β (see Remark 4.2.23) which is induced by the seminorms

$$|f|_{\nu,\alpha}:=\sup_{x\in\Omega}p_\alpha(f(x))|\nu(x)|,\quad f\in\mathcal{C}_b(\Omega,E),$$

for $\nu\in\mathcal{C}_0(\Omega)$ and $\alpha\in\mathfrak{A}$.

4.3.7. PROPOSITION. *Let Ω be a locally compact [second countable] topological Hausdorff space and E an lcHs with [metric] ccp. Then for every $T^{\mathbb{K}}\in(\mathcal{C}_b(\Omega),\beta)'$ there is $T^E\in L((\mathcal{C}_b(\Omega,E),\beta),E)$ such that $(T^E,T^{\mathbb{K}})$ is a consistent family for $((\mathcal{C}_b(\Omega),\beta),E)$ and there is a unique regular $\mathbb{K}$-valued Borel measure μ on Ω such that*

$$T^E(f)=\int_{\Omega} f(x)\mathrm{d}\mu(x),\quad f\in\mathcal{C}_b(\Omega,E). \tag{36}$$

On the other hand, if $(T^E,T^{\mathbb{K}})$ is a consistent family, then there is a unique regular $\mathbb{K}$-valued Borel measure μ on Ω such that (36) holds and $T^E\in L((\mathcal{C}_b(\Omega,E),\beta),E)$.

PROOF. Due to the Riesz–Markov–Kakutani representation theorem [89, 7.6.3 Theorem, p. 141] for the strict topology there is a unique regular $\mathbb{K}$-valued Borel measure μ on Ω such that

$$T^{\mathbb{K}}(f)=\int_{\Omega} f(x)\mathrm{d}\mu(x),\quad f\in\mathcal{C}_b(\Omega).$$

Since $T^{\mathbb{K}}$ is continuous, there are $\nu\in\mathcal{C}_0(\Omega)$ and $C>0$ such that

$$\Big|\int_{\Omega}\langle e',f(x)\rangle\mathrm{d}\mu(x)\Big|=|T^{\mathbb{K}}(e'\circ f)|\leq C\sup_{x\in\Omega}|(e'\circ f)(x)\nu(x)|\leq C\sup_{x\in K}|e'(x)|,\quad e'\in E',$$

for $f\in\mathcal{C}_b(\Omega,E)$ with $K:=\overline{\mathrm{acx}}(f\nu(\Omega))$. As K is absolutely convex and compact by Proposition A.1.3, f is Pettis-integrable on Ω w.r.t. μ by the Mackey–Arens theorem. The remaining parts of the proof follow from Example 4.2.11 and Proposition 4.3.3 as in Proposition 4.3.6. □

4.3.8. Proposition. *Let Ω be a locally compact [second countable] topological Hausdorff space and E an lcHs with [metric] ccp. Then for every $T^{\mathbb{K}}\in\mathcal{CW}(\Omega)'$ there is $T^{E}\in L(\mathcal{CW}(\Omega,E),E)$ such that $(T^{E},T^{\mathbb{K}})$ is a consistent family for $(\mathcal{CW},E)$ and there is a unique regular $\mathbb{K}$-valued Borel measure μ on Ω with compact support such that*

$$T^{E}(f)=\int_{\Omega}f(x)\mathrm{d}\mu(x),\quad f\in\mathcal{C}(\Omega,E).\tag{37}$$

On the other hand, if $(T^{E},T^{\mathbb{K}})$ is a consistent family, then there is a unique regular $\mathbb{K}$-valued Borel measure μ on Ω with compact support such that (37) *holds and $T^{E}\in L(\mathcal{CW}(\Omega,E),E)$.*

Proof. By the Riesz–Markov–Kakutani representation theorem given in the remark after [35, Chap. 4, §4.8, Proposition 14, p. INT IV.48] for the topology of compact convergence there is a unique regular $\mathbb{K}$-valued Borel measure μ on Ω with compact support such that

$$T^{\mathbb{K}}(f)=\int_{\Omega}f(x)\mathrm{d}\mu(x),\quad f\in\mathcal{C}(\Omega).$$

Since $T^{\mathbb{K}}$ is continuous, there are a compact set $M\subset\Omega$ and $C>0$ such that

$$\Big|\int_{\Omega}\langle e',f(x)\rangle\mathrm{d}\mu(x)\Big|=|T^{\mathbb{K}}(e'\circ f)|\leq C\sup_{x\in M}|(e'\circ f)(x)|\leq C\sup_{x\in K}|e'(x)|,\quad e'\in E',$$

for $f\in\mathcal{C}(\Omega,E)$ with the absolutely convex and compact set $K:=\overline{\operatorname{acx}}(f(M))$, implying that f is Pettis-integrable on Ω w.r.t. μ by the Mackey–Arens theorem. The rest of the proof is identical to the one of Proposition 4.3.7. □

4.3.9. Proposition. *Let $\Omega\subset\mathbb{R}^{d}$ be open and E a locally complete lcHs. Then for every $T^{\mathbb{K}}\in\mathcal{CW}^{\infty}(\Omega)'$ there is $T^{E}\in L(\mathcal{CW}^{\infty}(\Omega,E),E)$ such that $(T^{E},T^{\mathbb{K}})$ is a consistent family for $(\mathcal{CW}^{\infty},E)$. Given any open neighbourhood $U\subset\Omega$ of the compact distributional support $\operatorname{supp}T^{\mathbb{K}}$ of $T^{\mathbb{K}}$ there are $m\in\mathbb{N}_0$ and a family of $\mathbb{K}$-valued Radon measures $(\mu_{\beta})_{\beta\in\mathbb{N}_0^{d},|\beta|\leq m}$ on Ω such that $\operatorname{supp}\mu_{\beta}\subset U$ for all $\beta\in\mathbb{N}_0^{d}$, $|\beta|\leq m$, and*

$$T^{E}(f)=\sum_{|\beta|\leq m}\int_{\Omega}(\partial^{\beta})^{E}f(x)\mathrm{d}\mu_{\beta}(x),\quad f\in\mathcal{C}^{\infty}(\Omega,E).\tag{38}$$

On the other hand, if $(T^{E},T^{\mathbb{K}})$ is a consistent family, then given any open neighbourhood $U\subset\Omega$ of the compact distributional support $\operatorname{supp}T^{\mathbb{K}}$ of $T^{\mathbb{K}}$ there are $m\in\mathbb{N}_0$ and a family of $\mathbb{K}$-valued Radon measures $(\mu_{\beta})_{\beta\in\mathbb{N}_0^{d},|\beta|\leq m}$ on Ω such that (38) *holds, $\operatorname{supp}\mu_{\beta}\subset U$ for all $\beta\in\mathbb{N}_0^{d}$, $|\beta|\leq m$ and $T^{E}\in L(\mathcal{CW}^{\infty}(\Omega,E),E)$.*

Proof. $T^{\mathbb{K}}\in\mathcal{CW}^{\infty}(\Omega)'$ is a distribution with compact support and thus has finite order by [171, Corollary, p. 259]. Denote by $m\in\mathbb{N}_0$ the order of $T^{\mathbb{K}}$. Given any open neighbourhood $U\subset\Omega$ of $\operatorname{supp}T^{\mathbb{K}}$ there is a family of $\mathbb{K}$-valued Radon measures $(\mu_{\beta})_{\beta\in\mathbb{N}_0^{d},|\beta|\leq m}$ on Ω such that

$$T^{\mathbb{K}}(f)=\sum_{|\beta|\leq m}\int_{\Omega}(\partial^{\beta})^{\mathbb{K}}f(x)\mathrm{d}\mu_{\beta}(x),\quad f\in\mathcal{C}^{\infty}(\Omega),$$

and $\operatorname{supp}\mu_{\beta}\subset U$ for all $\beta\in\mathbb{N}_0^{d}$, $|\beta|\leq m$ by [171, Theorem 24.4, p. 259]. Since the support $K_{\beta}:=\operatorname{supp}\mu_{\beta}$ of μ_{β} is compact

$$\int_{\Omega}(\partial^{\beta})^{\mathbb{K}}f(x)\mathrm{d}\mu_{\beta}(x)=\int_{K_{\beta}}(\partial^{\beta})^{\mathbb{K}}f(x)\mathrm{d}\mu_{\beta}(x),\quad f\in\mathcal{C}^{\infty}(\Omega),$$

and $(\partial^{\beta})^{E}f\in\mathcal{C}^{1}(\Omega,E)$ for $f\in\mathcal{C}^{\infty}(\Omega,E)$, it follows from Lemma A.2.2 and Remark A.2.4 that $(\partial^{\beta})^{E}f$ is Pettis-integrable on Ω w.r.t. μ_{β} for all β and that

$$p_{\alpha}\Big(\int_{\Omega}(\partial^{\beta})^{E}f(x)\mathrm{d}\mu_{\beta}(x)\Big)\leq|\mu_{\beta}|(K_{\beta})\sup_{x\in K_{\beta}}p_{\alpha}((\partial^{\beta})^{E}f(x)),\quad \alpha\in\mathfrak{A}.$$

By Example 3.2.8 a) the map $S\colon\mathcal{CW}^{\infty}(\Omega)\varepsilon E\to\mathcal{CW}^{\infty}(\Omega,E)$ is an isomorphism with inverse $R^{t}\colon f\mapsto\mathcal{J}\circ R^{t}_{f}$. The remaining parts of the proof follow from Proposition 4.3.3 with $(T^{E}_{0},T^{\mathbb{K}}_{0}):=((\partial^{\beta})^{E},(\partial^{\beta})^{\mathbb{K}})$. □

4.3.10. Proposition. *Let E be a locally complete lcHs. Then for every $T^{\mathbb{K}}\in\mathcal{S}(\mathbb{R}^{d})'$ there is $T^{E}\in L(\mathcal{S}(\mathbb{R}^{d},E),E)$ such that $(T^{E},T^{\mathbb{K}})$ is a consistent family for $(\mathcal{S},E)$ and there are $m\in\mathbb{N}_{0}$ and a family of continuous functions $(g_{\beta})_{\beta\in\mathbb{N}_{0}^{d},|\beta|\leq m}$ on $\mathbb{R}^{d}$ growing at infinity slower than some polynomial such that*

$$T^{E}(f)=\sum_{|\beta|\leq m}\int_{\mathbb{R}^{d}}g_{\beta}(x)(\partial^{\beta})^{E}f(x)\mathrm{d}x,\quad f\in\mathcal{S}(\mathbb{R}^{d},E).\tag{39}$$

On the other hand, if $(T^{E},T^{\mathbb{K}})$ is a consistent family, then there are $m\in\mathbb{N}_{0}$ and a family of continuous functions $(g_{\beta})_{\beta\in\mathbb{N}_{0}^{d},|\beta|\leq m}$ on $\mathbb{R}^{d}$ growing at infinity slower than some polynomial such that (39) *holds and $T^{E}\in L(\mathcal{S}(\mathbb{R}^{d},E),E)$.*

Proof. Let $T^{\mathbb{K}}\in\mathcal{S}(\mathbb{R}^{d})'$. Then there are $m\in\mathbb{N}_{0}$ and a family of continuous functions $(g_{\beta})_{\beta\in\mathbb{N}_{0}^{d},|\beta|\leq m}$ on $\mathbb{R}^{d}$ growing at infinity slower than some polynomial such that

$$T^{\mathbb{K}}(f)=\sum_{|\beta|\leq m}\int_{\mathbb{R}^{d}}g_{\beta}(x)(\partial^{\beta})^{\mathbb{K}}f(x)\mathrm{d}x,\quad f\in\mathcal{S}(\mathbb{R}^{d}),$$

by [171, Theorem 25.4, p. 272]. Here, g_{β} *growing at infinity slower than some polynomial* means that there are $k\in\mathbb{N}_{0}$ and $C>0$ such that $|g_{\beta}(x)|\leq C(1+|x|^{2})^{k/2}$ for all $x\in\mathbb{R}^{d}$. Since the family (g_{β}) is finite, we can take one k and one C for all β. Due to the proof of Example 3.2.9 and Corollary 3.2.10 we know that $K_{\beta}:=\overline{\operatorname{acx}}(((\partial^{\beta})^{E}f)(1+|\cdot|^{2})^{k/2}(\mathbb{R}^{d}))$ is absolutely convex and compact for $f\in\mathcal{S}(\mathbb{R}^{d},E)$. The estimate

$$\Big|\int_{\mathbb{R}^{d}}\langle e',g_{\beta}(x)(\partial^{\beta})^{E}f(x)\rangle\mathrm{d}x\Big|\leq C\sup_{x\in\mathbb{R}^{d}}|e'((\partial^{\beta})^{E}f(x))|(1+|x|^{2})^{k/2}=C\sup_{x\in K_{\beta}}|e'(x)|$$

for all $e'\in E'$ and $f\in\mathcal{S}(\mathbb{R}^{d},E)$ yields that $g_{\beta}(\partial^{\beta})^{E}f$ is Pettis-integrable on $\mathbb{R}^{d}$ w.r.t. the Lebesgue measure by the Mackey–Arens theorem. Further, it implies that

$$p_{\alpha}\Big(\int_{\mathbb{R}^{d}}g_{\beta}(x)(\partial^{\beta})^{E}f(x)\mathrm{d}x\Big)\leq C\sup_{x\in\mathbb{R}^{d}}p_{\alpha}((\partial^{\beta})^{E}f(x))(1+|x|^{2})^{k/2},\quad \alpha\in\mathfrak{A},$$

as in Lemma A.2.2. By Corollary 3.2.10 the map $S\colon\mathcal{S}(\mathbb{R}^{d})\varepsilon E\to\mathcal{S}(\mathbb{R}^{d},E)$ is an isomorphism with inverse $R^{t}\colon f\mapsto\mathcal{J}\circ R^{t}_{f}$. The remaining parts of the proof follow from Proposition 4.3.3 with $(T^{E}_{0},T^{\mathbb{K}}_{0}):=(g_{\beta}(\partial^{\beta})^{E},g_{\beta}(\partial^{\beta})^{\mathbb{K}})$. □

4.3.11. Remark. a) Let $\Omega\subset\mathbb{R}^{d}$ be open and E a locally complete lcHs. Then Proposition 4.3.9 is still valid with $\mathcal{CW}^{\infty}$ replaced by $\mathcal{CW}^{\infty}_{P(\partial)}$ due to the Hahn–Banach theorem and Example 3.2.8 b). If $P(\partial)^{\mathbb{K}}$ is a hypoelliptic linear partial differential operator, then one can represent T^{E} as in (37) due to Proposition 4.2.17 but the measure μ need not be unique anymore.

b) Let $\Omega \subset \mathbb{R}^d$ be open and E an lcHs with metric ccp. Then Proposition 4.3.7 is still valid with $\mathcal{C}_b$ replaced by $\mathcal{C}^{\infty}_{P(\partial),b}$ for a hypoelliptic linear partial differential operator $P(\partial)^{\mathbb{K}}$ due to the Hahn–Banach theorem and Proposition 4.2.24 but the measure μ need not be unique anymore.

c) All families $(T^E, T^{\mathbb{K}})$ considered in this section are strong which is a consequence of Proposition 4.3.4 (and of Pettis-summability in Proposition 4.3.5).

CHAPTER 5

Applications

5.1. Lifting the properties of maps from the scalar-valued case

In this section we briefly show how to use the ε-compatibility of spaces $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ to lift properties like injectivity, surjectivity, bijectivity and continuity from a map $T^{\mathbb{K}}$ to a map T^{E} if $(T^{E}, T^{\mathbb{K}})$ forms a consistent family. Especially, we pay attention to surjectivity whose transfer to the vector-valued case is accomplished by Grothendieck's classical theory of tensor products of Fréchet spaces [83] and by the splitting theory of Vogt for Fréchet spaces [173] and of Bonet and Domański for PLS-spaces [54]. In order to apply splitting theory, we recall the definitions of the topological invariants (Ω), (DN) and (PA).

Let us recall that a Fréchet space F with an increasing fundamental system of seminorms $(|||\cdot|||_{k})_{k\in\mathbb{N}}$ satisfies (Ω) if

$$\forall\, p\in\mathbb{N}\;\exists\, q\in\mathbb{N}\;\forall\, k\in\mathbb{N}\;\exists\, n\in\mathbb{N},\, C>0\;\forall\, r>0:\; U_{q}\subset Cr^{n}U_{k}+\frac{1}{r}U_{p}$$

where $U_{k}:=\{x\in F\mid |||x|||_{k}\leq 1\}$ (see [131, Chap. 29, Definition, p. 367]).

We recall that a Fréchet space $(F,(|||\cdot|||_{k})_{k\in\mathbb{N}})$ satisfies (DN) by [131, Chap. 29, Definition, p. 359] if

$$\exists\, p\in\mathbb{N}\;\forall\, k\in\mathbb{N}\;\exists\, n\in\mathbb{N},\, C>0\;\forall\, x\in F:\; |||x|||_{k}^{2}\leq C|||x|||_{p}|||x|||_{n}.$$

A *PLS-space* is a projective limit $X=\lim\limits_{\substack{\longleftarrow\\ N\in\mathbb{N}}} X_{N}$, where the X_{N} given by inductive limits $X_{N}=\lim\limits_{\substack{\longrightarrow\\ n\in\mathbb{N}}}(X_{N,n},|||\cdot|||_{N,n})$ are DFS-spaces (which are also called LS-spaces), and it satisfies (PA) if

$$\forall\, N\;\exists\, M\;\forall\, K\;\exists\, n\;\forall\, m\;\forall\,\eta>0\;\exists\, k,C,r_{0}>0\;\forall\, r>r_{0}\;\forall\, x'\in X_{N}':$$

$$|||x'\circ i_{N}^{M}|||_{M,m}^{*}\leq C\big(r^{\eta}|||x'\circ i_{N}^{K}|||_{K,k}^{*}+\frac{1}{r}|||x'|||_{N,n}^{*}\big)$$

where $|||\cdot|||^{*}$ denotes the dual norm of $|||\cdot|||$ and i_{N}^{M}, i_{N}^{K} the linking maps (see [26, Section 4, Eq. (24), p. 577]).

Examples of Fréchet spaces with (DN) are the spaces of rapidly decreasing sequences $s(\mathbb{N}^{d})$, $s(\mathbb{N}_{0}^{d})$ and $s(\mathbb{Z}^{d})$, the space $\mathcal{C}^{\infty}([a,b])$ of all $\mathcal{C}^{\infty}$-smooth functions on (a,b) such that all derivatives can be continuously extended to the boundary and the space of smooth functions $\mathcal{C}_{2\pi}^{\infty}(\mathbb{R}^{d})$ that are 2π-periodic in each variable. Examples of ultrabornological PLS-space with (PA) are Fréchet–Schwartz spaces, the space of tempered distributions $\mathcal{S}(\mathbb{R}^{d})_{b}'$, the space of distributions $\mathcal{D}(\Omega)_{b}'$ and ultradistributions of Beurling type $\mathcal{D}_{(\omega)}(\Omega)_{b}'$ on an open set $\Omega\subset\mathbb{R}^{d}$. These and many more examples may be found in [26], [54, Corollary 4.8, p. 1116] and [112, Example 3, p. 7].

5.1.1. Proposition. *a) Let Y be a Fréchet space and X a semi-reflexive lcHs. Then $L_{b}(X_{b}',Y_{b}')\cong L_{b}(Y,(X_{b}')_{b}')$ via taking adjoints.*

b) Let E be an lcHs and X a Montel space. Then $L_{b}(X_{b}',E)\cong X\varepsilon E$ where the isomorphism is the identity map.

Proof. a) We consider the map

$$^{t}(\cdot)\colon L_b(X_b', Y_b') \to L_b(Y, (X_b')_b'),\ u \mapsto {}^{t}u,$$

defined by ${}^{t}u(y)(x') := u(x')(y)$ for $y \in Y$ and $x' \in X'$. First, we prove that ${}^{t}(\cdot)$ is well-defined. Let $u \in L(X_b', Y_b')$ and $y \in Y$. Since $u \in L(X_b', Y_b')$ and $\{y\}$ is bounded in Y, there are a bounded set $B \subset X$ and $C > 0$ such that

$$|{}^{t}u(y)(x')| = |u(x')(y)| \leq C \sup_{x \in B} |x'(x)|$$

for all $x' \in X'$, implying ${}^{t}u(y) \in (X_b')'$.

Let us denote by $(\|\cdot\|_{Y,n})_{n\in\mathbb{N}}$ the (directed) system of seminorms generating the metrisable locally convex topology of Y. The canonical embedding $J\colon Y \to (Y_b')_b'$ is an isomorphism between Y and $J(Y)$ by [131, Corollary 25.10, p. 298] because Y is a Fréchet space. For a bounded set $M \subset X_b'$ we note that

$$\sup_{x' \in M} |{}^{t}u(y)(x')| = \sup_{x' \in M} |u(x')(y)| = \sup_{x' \in M} |\langle J(y), u(x')\rangle|.$$

The next step is to prove that $u(M)$ is bounded in Y_b'. Let $N \subset Y$ be bounded. Since $u \in L(X_b', Y_b')$, there are again a bounded set $B \subset X$ and a constant $C > 0$ such that

$$\sup_{x' \in M} \sup_{y \in N} |u(x')(y)| \leq C \sup_{x' \in M} \sup_{x \in B} |x'(x)| < \infty$$

where the last estimate follows from the boundedness of $M \subset X_b'$. Hence $u(M)$ is bounded in Y_b'. By the remark about the canonical embedding there are $n \in \mathbb{N}$ and $C_0 > 0$ such that

$$\sup_{x' \in M} |{}^{t}u(y)(x')| = \sup_{y' \in u(M)} |\langle J(y), y'\rangle| \leq C_0 \|y\|_{Y,n},$$

so ${}^{t}u \in L(Y, (X_b')_b')$ and the map ${}^{t}(\cdot)$ is well-defined.

Let us turn to injectivity. Let $u, v \in L(X_b', Y_b')$ with ${}^{t}u = {}^{t}v$. This is equivalent to

$$u(x')(y) = {}^{t}u(y)(x') = {}^{t}v(y)(x') = v(x')(y)$$

for all $y \in Y$ and $x' \in X'$. This implies $u(x') = v(x')$ for all $x' \in X'$, hence $u = v$.

Next, we turn to surjectivity. We consider the map

$$^{\mathrm{t}}(\cdot)\colon L_b(Y, (X_b')_b') \to L_b(X_b', Y_b'),\ u \mapsto {}^{\mathrm{t}}u,$$

defined by ${}^{\mathrm{t}}u(x')(y) := u(y)(x')$ for $x' \in X'$ and $y \in Y$. We show that this map is well-defined. Let $u \in L_b(Y, (X_b')_b')$ and $x' \in X'$. Since $u \in L_b(Y, (X_b')_b')$ and $\{x'\}$ is bounded in X', there are $n \in \mathbb{N}$ and $C > 0$ such that

$$|{}^{\mathrm{t}}u(x')(y)| = |u(y)(x')| \leq C\|y\|_{Y,n}$$

for all $y \in Y$, yielding ${}^{\mathrm{t}}u(x') \in Y'$. Let $B \subset Y$ be bounded. The semi-reflexivity of X implies that for every $u(y)$, $y \in B$, there is a unique $x_{u(y)} \in X$ such that $u(y)(x') = x'(x_{u(y)})$ for all $x' \in X'$. Then we get

$$\sup_{y \in B} |{}^{\mathrm{t}}u(x')(y)| = \sup_{y \in B} |u(y)(x')| = \sup_{y \in B} |x'(x_{u(y)})|.$$

We claim that $D := \{x_{u(y)} \mid y \in B\}$ is a bounded set in X. Let $N \subset X'$ be finite. Then the set $M := \{{}^{\mathrm{t}}u(x') \mid x' \in N\} \subset Y'$ is finite. We have

$$\sup_{y \in B} \sup_{x' \in N} |x'(x_{u(y)})| = \sup_{y \in B} \sup_{x' \in N} |{}^{\mathrm{t}}u(x')(y)| = \sup_{y \in B} \sup_{y' \in M} |y'(y)| < \infty$$

where the last estimate follows from the fact that the bounded set B is weakly bounded. Thus D is weakly bounded and by [131, Mackey's theorem 23.15, p. 268] bounded in X. Therefore it follows from

$$\sup_{y \in B} |{}^{\mathrm{t}}u(x')(y)| = \sup_{y \in B} |x'(x_{u(y)})| = \sup_{x \in D} |x'(x)|$$

for all $x' \in X'$ that ${}^{\mathrm{t}}u \in L(X_b', Y_b')$, which means that ${}^{\mathrm{t}}(\cdot)$ is well-defined. Let $u \in L(Y, (X_b')_b')$. Then we have ${}^{\mathrm{t}}u \in L_b(X_b', Y_b')$. In addition, for all $y \in Y$ and all $x' \in X'$

$$ {}^{t}({}^{\mathrm{t}}u)(y)(x') = {}^{\mathrm{t}}u(x')(y) = u(y)(x') $$

is valid and so ${}^{t}({}^{\mathrm{t}}u)(y) = u(y)$ for all $y \in Y$, proving the surjectivity.

The last step is to prove the continuity of ${}^{t}(\cdot)$ and its inverse. Let $M \subset Y$ and $B \subset X_b'$ be bounded sets. Then

$$ \begin{aligned} \sup_{y \in M} \sup_{x' \in B} |{}^{t}u(y)(x')| &= \sup_{y \in M} \sup_{x' \in B} |u(x')(y)| = \sup_{x' \in B} \sup_{y \in M} |u(x')(y)| \\ &= \sup_{x' \in B} \sup_{y \in M} |{}^{\mathrm{t}}({}^{t}u)(x')(y)| \end{aligned} $$

holds for all $u \in L(X_b', Y_b')$. Therefore, ${}^{t}(\cdot)$ and its inverse are continuous.

b) Let $T \in L(X_b', E)$. For $\alpha \in \mathfrak{A}$ there are a bounded set $B \subset X$ and $C > 0$ such that

$$ p_\alpha(T(x')) \leq C \sup_{x \in B} |x'(x)| \leq C \sup_{x \in \overline{\mathrm{acx}}(B)} |x'(x)| $$

for every $x' \in X'$ where $\overline{\mathrm{acx}}(B)$ is the closure of the absolutely convex hull of B. The set $\overline{\mathrm{acx}}(B)$ is absolutely convex and compact by [89, 6.2.1 Proposition, p. 103] and [89, 6.7.1 Proposition, p. 112] since B is bounded in the Montel space X. Hence we gain $T \in L(X_\kappa', E)$.

Let $M \subset X'$ be equicontinuous. Due to [89, 8.5.1 Theorem (a), p. 156] M is bounded in X_b'. Therefore,

$$ \mathrm{id}\colon L_b(X_b', E) \to L_e(X_\kappa', E) = X \varepsilon E $$

is continuous.

Let $T \in L(X_\kappa', E)$. For $\alpha \in \mathfrak{A}$ there are an absolutely convex compact set $B \subset X$ and $C > 0$ such that

$$ p_\alpha(T(x')) \leq C \sup_{x \in B} |x'(x)| $$

for every $x' \in X'$. Since the compact set B is bounded, we get $T \in L(X_b', E)$.

Let M be a bounded set in X_b'. Then M is equicontinuous by virtue of [171, Theorem 33.2, p. 349], as X, being a Montel space, is barrelled. Thus

$$ \mathrm{id}\colon L_e(X_\kappa', E) \to L_b(X_b', E) $$

is continuous. □

For part e) of the next theorem we need that our map $S_{\mathcal{F}_2(\Omega_2)}\colon \mathcal{F}_2(\Omega_2)\varepsilon E \to \mathcal{F}_2(\Omega_2, E)$ is an isomorphism and that its inverse is given as in Theorem 3.2.4, i.e. that

$$ R^t\colon \mathcal{F}_2(\Omega_2, E) \to \mathcal{F}_2(\Omega_2)\varepsilon E,\ f \mapsto \mathcal{J}^{-1} \circ R_f^t, $$

is the inverse of $S_{\mathcal{F}_2(\Omega_2)}$ where $R_f^t(f')(e') = f'(e' \circ f)$ for $f' \in \mathcal{F}_2(\Omega_2)'$ and $e' \in E'$, and $\mathcal{J}\colon E \to E'^\star$ is the canonical injection in the algebraic dual $E'^\star$ of E'.

5.1.2. Theorem. *Let E be an lcHs, $\mathcal{F}_1(\Omega_1)$ and $\mathcal{F}_1(\Omega_1, E)$ as well as $\mathcal{F}_2(\Omega_2)$ and $\mathcal{F}_2(\Omega_2, E)$ be ε-into-compatible. Let $(T^E, T^{\mathbb{K}})$ be a consistent family for $(\mathcal{F}_1, E)$ such that $T^{\mathbb{K}}\colon \mathcal{F}_1(\Omega_1) \to \mathcal{F}_2(\Omega_2)$ is continuous and $T^E\colon \mathcal{F}_1(\Omega_1, E) \to \mathcal{F}_2(\Omega_2, E)$. Then the following holds:*

a) $T^E \circ S_{\mathcal{F}_1(\Omega_1)} = S_{\mathcal{F}_2(\Omega_2)} \circ (T^{\mathbb{K}} \varepsilon \,\mathrm{id}_E)$.

b) If $S_{\mathcal{F}_1(\Omega_1)}$ is surjective and $T^{\mathbb{K}}$ is injective, then T^E is injective, continuous and

$$ T^E = S_{\mathcal{F}_2(\Omega_2)} \circ (T^{\mathbb{K}} \varepsilon \,\mathrm{id}_E) \circ S_{\mathcal{F}_1(\Omega_1)}^{-1}. $$

If in addition $S_{\mathcal{F}_2(\Omega_2)}$ is surjective and $T^{\mathbb{K}}$ an isomorphism, then T^E is an isomorphism with inverse

$$(T^E)^{-1} = S_{\mathcal{F}_1(\Omega_1)} \circ ((T^{\mathbb{K}})^{-1} \varepsilon \operatorname{id}_E) \circ S^{-1}_{\mathcal{F}_2(\Omega_2)}.$$

c) *If $S_{\mathcal{F}_2(\Omega_2)}$ and $T^{\mathbb{K}}\varepsilon \operatorname{id}_E$ are surjective, then T^E is surjective.*

d) *If $S_{\mathcal{F}_2(\Omega_2)}$ and $T^{\mathbb{K}}$ are surjective, $\mathcal{F}_1(\Omega_1)$, $\mathcal{F}_2(\Omega_2)$ and E are Fréchet spaces and*
   (i) *$\mathcal{F}_1(\Omega_1)$ and $\mathcal{F}_2(\Omega_2)$ are nuclear, or*
   (ii) *E is nuclear,*
   then T^E is surjective.

e) *If $S_{\mathcal{F}_2(\Omega_2)}$ is surjective with inverse R^t, $T^{\mathbb{K}}$ is surjective, $\mathcal{F}_1(\Omega_1)$ and $\mathcal{F}_2(\Omega_2)$ are Fréchet spaces, $\ker T^{\mathbb{K}}$ is nuclear and has (Ω), and*
   (i) *$\mathcal{F}_1(\Omega_1)$ and $\mathcal{F}_2(\Omega_2)$ are Montel spaces, $E = F_b'$ where F is a Fréchet space satisfying (DN), or*
   (ii) *$\mathcal{F}_1(\Omega_1)$ and $\mathcal{F}_2(\Omega_2)$ are Schwartz spaces, E is an ultrabornological PLS-space satisfying (PA),*
   then T^E is surjective.

Proof. a) Let $u \in \mathcal{F}_1(\Omega_1)\varepsilon E$. Then

$$\begin{aligned}(T^E \circ S_{\mathcal{F}_1(\Omega_1)})(u)(x) &= u(\delta_x \circ T^{\mathbb{K}}) = (u \circ (T^{\mathbb{K}})^t)(\delta_x) = (T^{\mathbb{K}}\varepsilon \operatorname{id}_E)(u)(\delta_x)\\ &= S_{\mathcal{F}_2(\Omega_2)}\big((T^{\mathbb{K}}\varepsilon \operatorname{id}_E)(u)\big)(x), \quad x \in \Omega_2,\end{aligned}$$

as $(T^E, T^{\mathbb{K}})$ is consistent for $(\mathcal{F}_1, E)$, which proves part a).

b) If $S_{\mathcal{F}_1(\Omega_1)}$ is surjective, then $S_{\mathcal{F}_1(\Omega_1)}$ is an isomorphism, because it is an isomorphism into, and we have

$$T^E = S_{\mathcal{F}_2(\Omega_2)} \circ (T^{\mathbb{K}}\varepsilon \operatorname{id}_E) \circ S^{-1}_{\mathcal{F}_1(\Omega_1)}$$

by part a). If $T^{\mathbb{K}}$ is injective, then $T^{\mathbb{K}}\varepsilon \operatorname{id}_E$ is also injective by [159, Chap. I, §1, Proposition 1, p. 20] and thus T^E by the formula above as well since $S_{\mathcal{F}_1(\Omega_1)}$ is an isomorphism and $S_{\mathcal{F}_2(\Omega_2)}$ an isomorphism into. If $S_{\mathcal{F}_2(\Omega_2)}$ is surjective and $T^{\mathbb{K}}$ an isomorphism, then $S_{\mathcal{F}_2(\Omega_2)}$ and $T^{\mathbb{K}}\varepsilon \operatorname{id}_E$ are isomorphisms, the latter by [159, Chap. I, §1, Proposition 1, p. 20] and its inverse is $(T^{\mathbb{K}})^{-1}\varepsilon \operatorname{id}_E$. The rest of part b) follows from the formula for T^E above.

c) Let $f \in \mathcal{F}_2(\Omega_2, E)$. Then there is $g \in \mathcal{F}_1(\Omega_1)\varepsilon E$ such that $(S_{\mathcal{F}_2(\Omega_2)} \circ (T^{\mathbb{K}}\varepsilon \operatorname{id}_E))(g) = f$. Hence we obtain $h := S_{\mathcal{F}_1(\Omega_1)}(g) \in \mathcal{F}_1(\Omega_1, E)$ and $T^E(h) = f$ by part a).

d) For $n = 1, 2$ the continuous linear injection (see (3))

$$\Theta_n \colon \mathcal{F}_n(\Omega_n) \otimes_\pi E \to \mathcal{F}_n(\Omega_n)\varepsilon E, \ \sum_{j=1}^{k} f_j \otimes e_j \longmapsto \Big[y \mapsto \sum_{j=1}^{k} y(f_j) e_j\Big],$$

from the tensor product $\mathcal{F}_n(\Omega_n) \otimes_\pi E$ with the projective topology extends to a continuous linear map $\widehat{\Theta}_n \colon \mathcal{F}_n(\Omega_n)\widehat{\otimes}_\pi E \to \mathcal{F}_n(\Omega_n)\varepsilon E$ on the completion $\mathcal{F}_n(\Omega_n)\widehat{\otimes}_\pi E$ of $\mathcal{F}_n(\Omega_n) \otimes_\pi E$. The map $\widehat{\Theta}_n$ is also a topological isomorphism since $\mathcal{F}_n(\Omega_n)$ is nuclear for $n = 1, 2$ in case (i) resp. E is nuclear in case (ii). Furthermore, $T^{\mathbb{K}} \otimes_\pi \operatorname{id}_E \colon \mathcal{F}_1(\Omega_1) \otimes_\pi E \to \mathcal{F}_2(\Omega_2) \otimes_\pi E$ is defined by the relation $\Theta_2 \circ (T^{\mathbb{K}} \otimes_\pi \operatorname{id}_E) = (T^{\mathbb{K}}\varepsilon \operatorname{id}_E) \circ \Theta_1$. We denote by $T^{\mathbb{K}} \widehat{\otimes}_\pi \operatorname{id}_E$ the continuous linear extension of $T^{\mathbb{K}} \otimes_\pi \operatorname{id}_E$ to the completion $\mathcal{F}_1(\Omega_1)\widehat{\otimes}_\pi E$. Moreover, $\mathcal{F}_n(\Omega_n)$ for $n = 1, 2$ and E are Fréchet spaces, $T^{\mathbb{K}}$ and id_E are linear, continuous and surjective, so $T^{\mathbb{K}} \widehat{\otimes}_\pi \operatorname{id}_E$ is surjective by [94, 10.24 Satz, p. 255]. We observe that

$$T^{\mathbb{K}}\varepsilon \operatorname{id}_E = \widehat{\Theta}_2 \circ (T^{\mathbb{K}} \widehat{\otimes}_\pi \operatorname{id}_E) \circ \widehat{\Theta}_1^{-1}$$

and deduce that $T^{\mathbb{K}}\varepsilon\,\mathrm{id}_E$ is surjective. Now, we apply part c), which proves part d).

e) Throughout this proof we use the notation $X'' := (X_b')_b'$ for a locally convex Hausdorff space X and $T := T^{\mathbb{K}}$. The space $\mathcal{F}_1(\Omega_1)$ is a Fréchet space and so its closed subspace $\ker T$ as well. Further, $\mathcal{F}_n(\Omega_n)$ is a Montel space for $n = 1, 2$ and $\ker T$ nuclear, thus they are reflexive. The sequence

$$0 \to \ker T \overset{i}{\to} \mathcal{F}_1(\Omega_1) \overset{T}{\to} \mathcal{F}_2(\Omega_2) \to 0, \tag{40}$$

where i means the inclusion, is a topologically exact sequence of Fréchet spaces because T is surjective by assumption. Let us denote by $J_0\colon \ker T \to (\ker T)''$ and $J_n\colon \mathcal{F}_n(\Omega_n) \to \mathcal{F}_n(\Omega_n)''$ for $n = 1, 2$ the canonical embeddings which are topological isomorphisms since $\ker T$ and $\mathcal{F}_n(\Omega_n)$ are reflexive for $n = 1, 2$. Then the exactness of (40) implies that

$$0 \to (\ker T)'' \overset{i_0}{\to} \mathcal{F}_1(\Omega_1)'' \overset{T_1}{\to} \mathcal{F}_2(\Omega_2)'' \to 0, \tag{41}$$

where $i_0 := J_0 \circ i \circ J_0^{-1}$ and $T_1 := J_2 \circ T \circ J_1^{-1}$, is an exact topological sequence. This exact sequence is topological because the (strong) bidual of a Fréchet space is again a Fréchet space by [131, Corollary 25.10, p. 298].

(i) Let $E = F_b'$ where F is a Fréchet space with (DN). Then $\operatorname{Ext}^1(F, (\ker T)'') = 0$ by [174, 5.1 Theorem, p. 186] since $\ker T$ is nuclear and satisfies (Ω) and therefore $(\ker T)''$ as well. Combined with the exactness of (41) this implies that the sequence

$$0 \to L(F, (\ker T)'') \overset{i_0^*}{\to} L(F, \mathcal{F}_1(\Omega_1)'') \overset{T_1^*}{\to} L(F, \mathcal{F}_2(\Omega_2)'') \to 0$$

is exact by [137, Proposition 2.1, p. 13–14] where $i_0^*(B) := i_0 \circ B$ and $T_1^*(D) := T_1 \circ D$ for $B \in L(F, (\ker T)'')$ and $D \in L(F, \mathcal{F}_1(\Omega_1)'')$. In particular, we obtain that

$$T_1^*\colon L(F, \mathcal{F}_1(\Omega_1)'') \to L(F, \mathcal{F}_2(\Omega_2)'') \tag{42}$$

is surjective. Via $E = F_b'$ and Proposition 5.1.1 ($X = \mathcal{F}_n(\Omega_n)$ and $Y = F$) we have the isomorphisms into

$$\psi_n := S_{\mathcal{F}_n(\Omega_n)} \circ {}^{\mathrm{t}}(\cdot)\colon L(F, \mathcal{F}_n(\Omega_n)'') \to \mathcal{F}_n(\Omega_n, E),$$
$$\psi_n(u) = \big(S_{\mathcal{F}_n(\Omega_n)} \circ {}^{\mathrm{t}}(\cdot)\big)(u) = \big[x \mapsto {}^{\mathrm{t}}u(\delta_x)\big],$$

for $n = 1, 2$ and the inverse

$$\psi_2^{-1}(f) = (S \circ {}^{\mathrm{t}}(\cdot))^{-1}(f) = ({}^{t}(\cdot) \circ S_{\mathcal{F}_2(\Omega_2)}^{-1})(f) = {}^{t}(\mathcal{J}^{-1} \circ R_f^t), \quad f \in \mathcal{F}_2(\Omega_2, E).$$

Let $g \in \mathcal{F}_2(\Omega_2, E)$. Then $\psi_2^{-1}(g) \in L(F, \mathcal{F}_2(\Omega_2)'')$ and by the surjectivity of (42) there is $u \in L(F, \mathcal{F}_1(\Omega_1)'')$ such that $T_1^* u = \psi_2^{-1}(g)$. So we get $\psi_1(u) \in \mathcal{F}_1(\Omega_1, E)$. Next, we show that $T^E \psi_1(u) = g$ is valid. Let $y \in F$ and $x \in \Omega_2$. Then

$$T^E(\psi_1(u))(x) = {}^{\mathrm{t}}u(\delta_x \circ T)$$

by consistency and

$$\begin{aligned}
T^E(\psi_1(u))(x)(y) &= {}^{\mathrm{t}}u(\delta_x \circ T)(y) = u(y)(\delta_x \circ T) = \langle \delta_x \circ T, J_1^{-1}(u(y))\rangle \\
&= \langle \delta_x, T J_1^{-1}(u(y))\rangle = \langle [J_2 \circ T \circ J_1^{-1}](u(y)), \delta_x\rangle = \langle (T_1 \circ u)(y), \delta_x\rangle \\
&= \langle (T_1^* u)(y), \delta_x\rangle = \psi_2^{-1}(g)(y)(\delta_x) = {}^{t}(\mathcal{J}^{-1} \circ R_g^t)(y)(\delta_x) \\
&= (\mathcal{J}^{-1} \circ R_g^t)(\delta_x)(y) = \mathcal{J}^{-1}(\mathcal{J}(g(x))(y) = g(x)(y).
\end{aligned}$$

Thus $T^E(\psi_1(u))(x) = g(x)$ for every $x \in \Omega_2$, which proves the surjectivity.

(ii) Let E be an ultrabornological PLS-space satisfying (PA). Since the nuclear Fréchet space $\ker T$ is also a Schwartz space, its strong dual $(\ker T)_b'$ is a DFS-space. By [26, Theorem 4.1, p. 577] we obtain $\operatorname{Ext}^1_{PLS}((\ker T)_b', E) = 0$ as the bidual $(\ker T)''$ satisfies (Ω), E is a PLS-space satisfying (PA) and condition (c) in the theorem is fulfilled because $(\ker T)_b'$ is the strong dual of a nuclear Fréchet space. Moreover, we have $\operatorname{Proj}^1 E=0$ due to [180, Corollary 3.3.10, p. 46] because E is an ultrabornological PLS-space. Then the exactness of the sequence (41), [26, Theorem 3.4, p. 567] and [26, Lemma 3.3, p. 567] (in the lemma the same condition (c) as in [26, Theorem 4.1, p. 577] is fulfilled and we choose $H=(\ker T)''$, $F=\mathcal{F}_1(\Omega_1)''$ and $G=\mathcal{F}_2(\Omega_2)''$), imply that the sequence

$$0\to L(E_b',(\ker T)'')\overset{i_0^*}{\to} L(E_b',\mathcal{F}_1(\Omega_1)'')\overset{T_1^*}{\to} L(E_b',\mathcal{F}_2(\Omega_2)'')\to 0$$

is exact. The maps i_0^* and T_1^* are defined as in part (i). Especially, we get that

$$T_1^*\colon L(E_b',\mathcal{F}_1(\Omega_1)'')\to L(E_b',\mathcal{F}_2(\Omega_2)'') \tag{43}$$

is surjective.

By [54, Remark 4.4, p. 1114] we have $L_b(\mathcal{F}_n(\Omega_n)_b',E'')\cong L_b(E_b',\mathcal{F}_n(\Omega_n)'')$ for $n=1,2$ via taking adjoints since $\mathcal{F}_n(\Omega_n)$, being a Fréchet–Schwartz space, is a PLS-space and hence its strong dual an LFS-space, which is regular by [180, Corollary 6.7, 10. ⇔ 11., p. 114], and E is an ultrabornological PLS-space, in particular, reflexive by [53, Theorem 3.2, p. 58]. In addition, the map

$$P\colon L_b(\mathcal{F}_n(\Omega_n)_b',E'')\to L_b(\mathcal{F}_n(\Omega_n)_b',E),$$

defined by $P(u)(y):=\mathcal{J}^{-1}(u(y))$ for $u\in L(\mathcal{F}_n(\Omega_n)_b',E'')$ and $y\in\mathcal{F}_n(\Omega_n)'$, is an isomorphism because E is reflexive. Due to Proposition 5.1.1 b) with $X=\mathcal{F}_n(\Omega_n)$ we obtain the isomorphisms into

$$\begin{aligned}\psi_n &:= S\circ\mathcal{J}^{-1}\circ{}^{\mathrm{t}}(\cdot)\colon L_b(E_b',\mathcal{F}_n(\Omega_n)'')\to\mathcal{F}_n(\Omega_n,E),\\ \psi_n(u)&=[S_{\mathcal{F}_n(\Omega_n)}\circ\mathcal{J}^{-1}\circ{}^{\mathrm{t}}(\cdot)](u)=\big[x\mapsto\mathcal{J}^{-1}({}^{\mathrm{t}}u(\delta_x))\big],\end{aligned}$$

for $n=1,2$ and the inverse given by

$$\begin{aligned}\psi_2^{-1}(f)&=(S_{\mathcal{F}_2(\Omega_2)}\circ\mathcal{J}^{-1}\circ{}^{\mathrm{t}}(\cdot))^{-1}(f)=[{}^t(\cdot)\circ\mathcal{J}\circ S_{\mathcal{F}_2(\Omega_2)}^{-1}](f)={}^t(\mathcal{J}\circ\mathcal{J}^{-1}\circ R_f^t)\\ &={}^t(R_f^t)\end{aligned}$$

for $f\in\mathcal{F}_2(\Omega_2,E)$.

Let $g\in\mathcal{F}_2(\Omega_2,E)$. Then $\psi_2^{-1}(g)\in L(E_b',\mathcal{F}_2(\Omega_2)'')$ and by the surjectivity of (43) there exists $u\in L(E_b',\mathcal{F}_1(\Omega_1)'')$ such that $T_1^*u=\psi_2^{-1}(g)$. So we have $\psi_1(u)\in\mathcal{F}_1(\Omega_1,E)$. The last step is to show that $T^E\psi_1(u)=g$. As in part (i) we gain for every $x\in\Omega_2$

$$T^E(\psi_1(u))(x)=\mathcal{J}^{-1}({}^{\mathrm{t}}u(\delta_x\circ T))$$

by consistency and for every $y\in E'$

$$\begin{aligned}{}^{\mathrm{t}}u(\delta_x\circ T)(y)&=u(y)(\delta_x\circ T)=(T_1^*u)(y)(\delta_x)=\psi_2^{-1}(g)(y)(\delta_x)={}^t(R_g^t)(y)(\delta_x)\\ &=\delta_x(y\circ g)=y(g(x))=\mathcal{J}(g(x))(y).\end{aligned}$$

Thus we have ${}^{\mathrm{t}}u(\delta_x\circ T)=\mathcal{J}(g(x))$ and therefore $T^E(\psi_1(u))(x)=g(x)$ for all $x\in\Omega_2$. □

Theorem 5.1.2 d) and e) are generalisations of [116, Corollary 4.3, p. 2689] and [112, Theorem 5, p. 7–8] where $T^{\mathbb{C}}$ is the Cauchy–Riemann operator $\overline{\partial}$ on certain weighted spaces $\mathcal{CV}^\infty(\Omega)$ of smooth functions. Our next result is the well-known application of tensor product theory and splitting theory to linear partial differential operators we already mentioned in the introduction.

5.1.3. Corollary. *Let E be a locally complete lcHs, $\Omega_1\subset\mathbb{R}^d$ open and $P(\partial)^{\mathbb{K}}$ be a linear partial differential operator with $\mathcal{C}^\infty$-smooth coefficients. Then the following holds:*

a) $P(\partial)^E=S_{\mathcal{C}^\infty(\Omega_1)}\circ(P(\partial)^{\mathbb{K}}\varepsilon\,\mathrm{id}_E)\circ S_{\mathcal{C}^\infty(\Omega_1)}^{-1}$.

b) If $\mathbb{K}=\mathbb{C}$, $P(D):=P(D)^{\mathbb{C}}:=P(-\mathrm{i}\partial)^{\mathbb{C}}$ has constant coefficients and is

(i) elliptic, or
(ii) hypoelliptic and Ω_1 convex,
and
(iii) E is a Fréchet space, or
(iv) $E = F_b'$ where F is a Fréchet space satisfying (DN), $d \geq 2$, or
(v) E is an ultrabornological PLS-space satisfying (PA), $d \geq 2$,
then $P(D)^E: \mathcal{C}^\infty(\Omega_1, E) \to \mathcal{C}^\infty(\Omega_1, E)$ is surjective.

PROOF. Part a) follows from Theorem 5.1.2 a), Example 3.2.8 a) and the consistency of $(P(\partial)^E, P(\partial)^{\mathbb{K}})$ because (7) holds for $u \in \mathcal{CW}^\infty(\Omega_1)\varepsilon E$ as well.

Let us turn to part b). The inverse of $S_{\mathcal{C}^\infty(\Omega_1)}$ is given by R^t by Example 3.2.8 a). The map $P(D) = P(D)^{\mathbb{C}}: \mathcal{C}^\infty(\Omega_1) \to \mathcal{C}^\infty(\Omega_1)$ is surjective by [86, Corollary 10.6.8, p. 43] and [86, Theorem 10.6.2, p. 41] in case (ii) resp. by [86, Corollary 10.8.2, p. 51] in case (i). The space $\mathcal{CW}^\infty(\Omega_1)$, i.e. $\mathcal{C}^\infty(\Omega_1)$ with its usual topology (see Example 3.1.9 b)), is a nuclear Fréchet space and thus its closed subspace $\ker P(D)^{\mathbb{C}}$ as well. In case (i) $\ker P(D)^{\mathbb{C}}$ has (Ω) due to [173, Proposition 2.5 (b), p. 173] and in case (ii) due to [140, 4.5 Corollary (a), p. 202]. Hence the surjectivity of $P(D)^E$ follows from Theorem 5.1.2 d)+e). □

Recently, it was shown in [47, Theorem 4.2, p. 13] that in the case that Ω_1 is convex, $\ker P(D)^{\mathbb{C}}$ has property (Ω) for any $P(D)$ with constant coefficients (so without the assumption of $P(D)$ being hypoelliptic). Hence we may replace the assumption of $P(D)$ being hypoelliptic in (ii) by the assumption that $P(D)^{\mathbb{C}}: \mathcal{C}^\infty(\Omega_1) \to \mathcal{C}^\infty(\Omega_1)$ is surjective. Even more recently, a necessary and sufficient condition for the surjectivity of $P(D)^{\mathbb{C}}$ and $\ker P(D)^{\mathbb{C}}$ having property (Ω) for $P(D)$ with constant coefficients and general open $\Omega_1 \subset \mathbb{R}^d$ was derived in [48, Theorem 1.1. (a), p. 3] using shifted fundamental solutions.

Even though Corollary 5.1.3 b) is known, it is often proved without using tensor products or splitting theory (see e.g. [94, Theorem 10.10, p. 240]) or it is phrased as the surjectivity of $P(D)^{\mathbb{C}} \widehat{\otimes}_\pi \mathrm{id}_E$ (see e.g. [171, Eq. (52.4), p. 541]) and the proof of the relation

$$P(D)^E = S_{\mathcal{C}^\infty(\Omega_1)} \circ (\widehat{\Theta}_1 \circ (P(D)^{\mathbb{C}} \widehat{\otimes}_\pi \mathrm{id}_E) \circ \widehat{\Theta}_1^{-1}) \circ S_{\mathcal{C}^\infty(\Omega_1)}^{-1}$$

for Fréchet spaces E is omitted (see e.g. [171, p. 545–546]), or only the surjectivity of $T_1^* = P(D)_1^*$ in part e) of Theorem 5.1.2 is actually shown and it is only stated but not proved that this implies the surjectivity of $P(D)^E$ (see e.g. the statement of surjectivity of $P(D)^E$ in [173, p. 168] for elliptic $P(D)$ and $E = F_b'$ for a Fréchet space F with (DN) and that it is 'only' shown that $P(D)_1^*$ is surjective by [173, Proposition 2.5 (b), p. 173] and [173, Theorem 2.4 (b), p. 173] where the symbol $P(D)^*$ is used instead of $P(D)_1^*$ in [173, p. 172] since the isomorphism $J_1 = J_2$ is omitted). So, apart from being the probably most classical application of tensor products or splitting theory, that is the reason why we still included Corollary 5.1.3.

Let us give another application of Theorem 5.1.2 d) and e), namely, a vector-valued Borel–Ritt theorem.

5.1.4. THEOREM. *Let E be an lcHs and $(x_n)_{n\in\mathbb{N}_0}$ a sequence in E. If*
(i) E is a Fréchet space, or
(ii) $E = F_b'$ where F is a Fréchet space satisfying (DN), or
(iii) E is an ultrabornological PLS-space satisfying (PA),
then there is $f \in \mathcal{C}_{2\pi}^\infty(\mathbb{R}, E)$ such that $(\partial^n)^E f(0) = x_n$ for all $n \in \mathbb{N}_0$.

PROOF. By the Borel–Ritt theorem [94, Satz 9.12, p. 206] the map

$$T^{\mathbb{K}}: \mathcal{C}_{2\pi}^\infty(\mathbb{R}) \to \mathbb{K}^{\mathbb{N}_0},\ T^{\mathbb{K}}(f) := \big((\partial^n)^{\mathbb{K}} f(0)\big)_{n\in\mathbb{N}_0},$$

is surjective and obviously linear and continuous as well. Now, we define the map $T^E:\mathcal{C}^\infty_{2\pi}(\mathbb{R},E)\to E^{\mathbb{N}_0}$ by replacing $\mathbb{K}$ by E in the definition of $T^{\mathbb{K}}$. Due to Example 4.2.1 $\mathbb{K}^{\mathbb{N}_0}$ and $E^{\mathbb{N}_0}$ are ε-compatible and the inverse of $S_{\mathbb{K}^{\mathbb{N}_0}}$ is given by R^t. In addition, $\mathcal{C}^\infty_{2\pi}(\mathbb{R})$ and $\mathcal{C}^\infty_{2\pi}(\mathbb{R},E)$ are ε-compatible by Example 4.2.27 as in all three cases E is complete. We observe that $(T^E,T^{\mathbb{K}})$ is consistent by Proposition 3.1.11 c). The spaces $\mathbb{K}^{\mathbb{N}_0}$ and $\mathcal{C}^\infty_{2\pi}(\mathbb{R})$ are nuclear Fréchet spaces. The first by [171, Theorem 51.1, p. 526] and the second because it is a subspace of the nuclear space $\mathcal{C}^\infty(\mathbb{R})$ by [131, Examples 28.9 (1), p. 349–350] and [131, Proposition 28.6, p. 347]. Hence in case (i) our statement follows from Theorem 5.1.2 d). Moreover, $\ker T^{\mathbb{K}}$ is nuclear since $\mathcal{C}^\infty_{2\pi}(\mathbb{R})$ is nuclear. By the proof of [131, Lemma 31.3, p. 392–393] $\ker T^{\mathbb{K}}$ is isomorphic to $s(\mathbb{N}_0)$. The space $s(\mathbb{N}_0)$ has (Ω) by [131, Lemma 29.11 (3), p. 368] and thus $\ker T^{\mathbb{K}}$ as well because (Ω) is a linear topological invariant by [131, Lemma 29.11 (1), p. 368]. Therefore our statement in case (ii) and (iii) follows from Theorem 5.1.2 e). □

We close this section with an application of Theorem 5.1.2 b) to the Fourier transformation on the Beurling–Björck spaces $\mathcal{S}_\mu(\mathbb{R}^d,E)$ from Example 4.2.26.

5.1.5. Theorem. *Let E be a locally complete lcHs over $\mathbb{C}$ and $\mu:\mathbb{R}^d\to[0,\infty)$ continuous such that $\mu(x)=\mu(-x)$ for all $x\in\mathbb{R}^d$ and condition (γ) is fulfilled.*

(i) If E has metric ccp, or
(ii) if $\mu\in\mathcal{C}^1(\mathbb{R}^d)$ and there are $k\in\mathbb{N}_0$, $C>0$ such that $|\partial^{e_n}\mu(x)|\leq C\mathrm{e}^{k\mu(x)}$ for all $x\in\mathbb{R}^d$ and $1\leq n\leq d$,

then $\mathfrak{F}^E:\mathcal{S}_\mu(\mathbb{R}^d,E)\to\mathcal{S}_\mu(\mathbb{R}^d,E)$ is an isomorphism with $\mathfrak{F}^E=S\circ(\mathfrak{F}^{\mathbb{C}}\varepsilon\,\mathrm{id}_E)\circ S^{-1}$.

Proof. Due to Example 4.2.26 $\mathcal{S}_\mu(\mathbb{R}^d)$ and $\mathcal{S}_\mu(\mathbb{R}^d,E)$ are ε-compatible. The Fourier transformation $\mathfrak{F}^{\mathbb{C}}:\mathcal{S}_\mu(\mathbb{R}^d)\to\mathcal{S}_\mu(\mathbb{R}^d)$ is a well-defined isomorphism by the definition of $\mathcal{S}_\mu(\mathbb{R}^d)$ and since $(\mathfrak{F}^{\mathbb{C}}\circ\mathfrak{F}^{\mathbb{C}})(f)(x)=f(-x)$ for all $f\in\mathcal{S}_\mu(\mathbb{R}^d)$ as well as $\mu(x)=\mu(-x)$ for all $x\in\mathbb{R}^d$. Due to (26) with $\beta=0$ we have that $(\mathfrak{F}^E,\mathfrak{F}^{\mathbb{C}})$ is a consistent family for $(\mathcal{S}_\mu,E)$ and thus it follows from Theorem 5.1.2 b) that $\mathfrak{F}^E$ is an isomorphism and $\mathfrak{F}^E=S\circ(\mathfrak{F}^{\mathbb{C}}\varepsilon\,\mathrm{id}_E)\circ S^{-1}$, which completes the proof. □

5.2. Extension of vector-valued functions

We study the problem of extending vector-valued functions via the existence of weak extensions in this section. The precise description of this problem reads as follows. Let E be a locally convex Hausdorff space over the field $\mathbb{K}$ of real or complex numbers and $\mathcal{F}(\Omega):=\mathcal{F}(\Omega,\mathbb{K})$ a locally convex Hausdorff space of $\mathbb{K}$-valued functions on a set Ω. Suppose that the point evaluations δ_x belong to the dual $\mathcal{F}(\Omega)'$ for every $x\in\Omega$ and that there is a locally convex Hausdorff space $\mathcal{F}(\Omega,E)$ of E-valued functions on Ω such that the map

$$S:\mathcal{F}(\Omega)\varepsilon E\to\mathcal{F}(\Omega,E),\ u\longmapsto[x\mapsto u(\delta_x)], \tag{44}$$

is an isomorphism into, i.e. $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ are ε-into-compatible. Thus $\mathcal{F}(\Omega)\varepsilon E$ is a linearisation of a subspace of $\mathcal{F}(\Omega,E)$. Linearisations that are based on the Dixmier–Ng theorem were used by Bonet, Domański and Lindström in [28, Lemma 10, p. 243] resp. Laitila and Tylli in [121, Lemma 5.2, p. 14] to describe the space of weakly holomorphic resp. harmonic functions on the unit disc $\Omega=\mathbb{D}\subset\mathbb{C}$ with values in a (complex) Banach space E (see also [118]).

5.2.1. Question. Let Λ be a subset of Ω and G a linear subspace of E'. Let $f:\Lambda\to E$ be such that for every $e'\in G$, the function $e'\circ f:\Lambda\to\mathbb{K}$ has an extension in $\mathcal{F}(\Omega)$. When is there an extension $F\in\mathcal{F}(\Omega,E)$ of f, i.e. $F_{|\Lambda}=f$?

An affirmative answer for $\Lambda = \Omega$ and $G = E'$ is called a *weak-strong principle*. For weighted continuous functions on a completely regular Hausdorff space Ω with values in a semi-Montel or Schwartz space E a weak-strong principle is given by Bierstedt in [17, 2.10 Lemma, p. 140]. Weak-strong principles for holomorphic functions on open subsets $\Omega \subset \mathbb{C}$ were shown by Dunford in [59, Theorem 76, p. 354] for Banach spaces E and by Grothendieck in [82, Théorème 1, p. 37–38] for quasi-complete E. For a wider class of function spaces weak-strong principles are due to Grothendieck, mainly, in the case that $\mathcal{F}(\Omega)$ is nuclear and E complete (see [83, Chap. II, §3, n°3, Théorème 13, p. 80]), which covers the case that $\mathcal{F}(\Omega)$ is the space $\mathcal{C}^\infty(\Omega)$ of smooth functions on an open set $\Omega \subset \mathbb{R}^d$ (with its usual topology).

Gramsch [77] analysed the weak-strong principles of Grothendieck and realised that they can be used to extend functions if Λ is a set of uniqueness, i.e. from $f \in \mathcal{F}(\Omega)$ and $f(x) = 0$ for all $x \in \Lambda$ follows that $f = 0$, and $\mathcal{F}(\Omega)$ a semi-Montel space, E complete and $G = E'$ (see [77, 0.1, p. 217]). An extension result for holomorphic functions where $G = E'$ and E is sequentially complete was shown by Bogdanowicz in [25, Corollary 3, p. 665].

Grosse-Erdmann proved for holomorphic functions on $\Lambda = \Omega$ in [79, 5.2 Theorem, p. 35] that it is sufficient to test locally bounded functions f with values in a locally complete space E with functionals from a weak*-dense subspace G of E'. Arendt and Nikolski [7, 8] shortened his proof in the case that E is a Fréchet space (see [7, Theorem 3.1, p. 787] and [7, Remark 3.3, p. 787]). Arendt gave an affirmative answer in [6, Theorem 5.4, p. 74] for harmonic functions on an open subset $\Lambda = \Omega \subset \mathbb{R}^d$ where the range space E is a Banach space and G a weak*-dense subspace of E'.

In [77] Gramsch also derived extension results for a large class of Fréchet–Montel spaces $\mathcal{F}(\Omega)$ in the case that Λ is a special set of uniqueness, E sequentially complete and G strongly dense in E' (see [77, 3.3 Satz, p. 228–229]). He applied it to the space of holomorphic functions and Grosse-Erdmann [81] expanded this result to the case of E being B_r-complete and G only a weak*-dense subspace of E' (see [81, Theorem 2, p. 401] and [81, Remark 2 (a), p. 406]). In a series of papers [30, 69, 70, 92, 93] these results were generalised and improved by Bonet, Frerick, Jordá and Wengenroth who used (44) to obtain extensions for vector-valued functions via extensions of linear operators. In [92, 93] this was done by Jordá for holomorphic functions on a domain (i.e. open and connected) $\Omega \subset \mathbb{C}$ and weighted holomorphic functions on a domain Ω in a Banach space. In [30] this was done by Bonet, Frerick and Jordá for closed subsheaves $\mathcal{F}(\Omega)$ of the sheaf of smooth functions $\mathcal{C}^\infty(\Omega)$ on a domain $\Omega \subset \mathbb{R}^d$. Their results implied some consequences on the work of Bierstedt and Holtmanns [18] as well. Further, in [69] this was done by Frerick and Jordá for closed subsheaves $\mathcal{F}(\Omega)$ of smooth functions on a domain $\Omega \subset \mathbb{R}^d$ which are closed in the sheaf $\mathcal{C}(\Omega)$ of continuous functions and in [70] by the first two authors and Wengenroth in the case that $\mathcal{F}(\Omega)$ is the space of bounded functions in the kernel of a hypoelliptic linear partial differential operator, in particular, the spaces of bounded holomorphic or harmonic functions.

In this section we present a unified approach to the extension problem for a large class of function spaces. The spaces we treat are usually of the kind that $\mathcal{F}(\Omega)$ belongs to the class of semi-Montel spaces, Fréchet–Schwartz spaces or Banach spaces. Even quite general weighted spaces $\mathcal{F}(\Omega)$ are treated, at least, if E is a semi-Montel space. Our approach is based on three ideas. First, it is based on the representation of (a subspace of) $\mathcal{F}(\Omega, E)$ as a space of continuous linear operators via the map S from (44). We note that almost all our examples of such spaces $\mathcal{F}(\Omega, E)$ are actually of the form of a general weighted space $\mathcal{FV}(\Omega, E)$ from Definition 3.1.4. Second, it is based on the idea to consider a set of uniqueness Λ

not necessarily as a subset of Ω but rather as a set of functionals acting on $\mathcal{F}(\Omega)$. In the definition of a set of uniqueness given above one may identify Λ with the set of functionals $\{\delta_x \mid x \in \Lambda\}$ and this shift of perspective allows us to consider certain sets of functionals of the form $T^{\mathbb{K}}_{m,x}$ as sets of uniqueness for $\mathcal{F}(\Omega)$ (see Definition 5.2.2). Third, the generalised concept of consistency and strength of a family of operators $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ acting on $(\mathcal{F}(\Omega,E), \mathcal{F}(\Omega))$ from Definition 4.3.1 enables us to generalise Question 5.2.1 and affirmatively answer this generalised question.

These three ideas are used to extend the mentioned results and we always have to balance the sets Λ from which we extend our functions and the subspaces $G \subset E'$ with which we test. The case of 'thin' sets Λ and 'thick' subspaces G is handled in Section 5.2.1, the converse case of 'thick' sets Λ and 'thin' subspaces G in Section 5.2.2.

5.2.1. Extension from thin sets. Using the functionals $T^{\mathbb{K}}_{m,x}$, we extend the definition of a set of uniqueness and a space of restrictions given in [30, Definition 4, 5, p. 230]. This prepares the ground for a generalisation of Question 5.2.1 using a strong, consistent family $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$.

5.2.2. DEFINITION (set of uniqueness). Let Ω be a non-empty set, $\mathcal{F}(\Omega) \subset \mathbb{K}^{\Omega}$ an lcHs, $(\omega_m)_{m\in M}$ be a family of non-empty sets and $T^{\mathbb{K}}_m : \mathcal{F}(\Omega) \to \mathbb{K}^{\omega_m}$ be linear for all $m \in M$. Then $U \subset \bigcup_{m\in M}(\{m\}\times\omega_m)$ is called a *set of uniqueness* for $(T^{\mathbb{K}}_m, \mathcal{F})_{m\in M}$ if

(i) $\forall\ (m,x) \in U :\ T^{\mathbb{K}}_{m,x} \in \mathcal{F}(\Omega)'$,

(ii) $\forall\ f \in \mathcal{F}(\Omega) :\ [\forall\ (m,x) \in U :\ T^{\mathbb{K}}_m(f)(x) = 0] \ \Rightarrow\ f = 0.$

We omit the index m in ω_m and $T^{\mathbb{K}}_m$ if M is a singleton and consider U as a subset of ω.

If U is a set of uniqueness for $(T^{\mathbb{K}}_m, \mathcal{F})_{m\in M}$, then $\operatorname{span}\{T^{\mathbb{K}}_{m,x} \mid (m,x) \in U\}$ is dense in $\mathcal{F}(\Omega)'_\sigma$ (and $\mathcal{F}(\Omega)'_\kappa$) by the bipolar theorem.

5.2.3. REMARK. Let Ω be a non-empty set and $\mathcal{F}(\Omega) \subset \mathbb{K}^{\Omega}$ an lcHs.

a) A simple set of uniqueness for $(\mathrm{id}_{\mathbb{K}^\Omega}, \mathcal{F})$ is given by $U := \Omega$ if $\delta_x \in \mathcal{F}(\Omega)'$ for all $x \in \Omega$.

b) If $\mathcal{F}(\Omega)$ has a Schauder basis $(f_n)_{n\in\mathbb{N}}$ with associated sequence of coefficient functionals $T^{\mathbb{K}} := (T^{\mathbb{K}}_n)_{n\in\mathbb{N}}$, then $U := \mathbb{N}$ is a set of uniqueness for $(T^{\mathbb{K}}, \mathcal{F})$.

An example for b) is the space of holomorphic functions on an open disc $\mathbb{D}_r(z_0) \subset \mathbb{C}$ with radius $0 < r \leq \infty$ and center $z_0 \in \mathbb{C}$. If we equip this space with the topology of compact convergence, then it has the shifted monomials $((\cdot - z_0)^n)_{n\in\mathbb{N}_0}$ as a Schauder basis with the point evaluations $(\delta_{z_0} \circ \partial^n_{\mathbb{C}})_{n\in\mathbb{N}_0}$ given by $(\delta_{z_0} \circ \partial^n_{\mathbb{C}})(f) := f^{(n)}(z_0)$ as associated sequence of coefficient functionals. We will explore further sets of uniqueness for concrete function spaces in the upcoming examples and come back to b) in Section 5.7.

5.2.4. DEFINITION (restriction space). Let $G \subset E'$ be a separating subspace and U a set of uniqueness for $(T^{\mathbb{K}}_m, \mathcal{F})_{m\in M}$. Let $\mathcal{F}_G(U,E)$ be the space of functions $f: U \to E$ such that for every $e' \in G$ there is $f_{e'} \in \mathcal{F}(\Omega)$ with $T^{\mathbb{K}}_m(f_{e'})(x) = (e' \circ f)(m,x)$ for all $(m,x) \in U$.

5.2.5. REMARK. Since U is a set of uniqueness, the functions $f_{e'}$ are unique and the map $\mathscr{R}_f : E' \to \mathcal{F}(\Omega)$, $\mathscr{R}_f(e') := f_{e'}$, is well-defined and linear. The map $\mathscr{R}_f$ resembles the map R_f defined above Lemma 3.2.1.

5.2.6. REMARK. Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-into-compatible. Consider a set of uniqueness U for $(T^{\mathbb{K}}_m, \mathcal{F})_{m\in M}$, a separating subspace $G \subset E'$ and a strong, consistent family $(T^{E}_{m},T^{\mathbb{K}}_{m})_{m\in M}$ for $(\mathcal{F},E)$. For $u\in\mathcal{F}(\Omega)\varepsilon E$ set $f:=S(u)$. Then $f\in\mathcal{F}(\Omega,E)$ by the ε-into-compatibility and we set $\widetilde{f}\colon U\to E$, $\widetilde{f}(m,x):=T^{E}_{m}(f)(x)$. It follows that

$$(e'\circ\widetilde{f})(m,x)=(e'\circ T^{E}_{m}(f))(x)=T^{\mathbb{K}}_{m}(e'\circ f)(x)$$

for all $(m,x)\in U$ and $f_{e'}:=e'\circ f\in\mathcal{F}(\Omega)$ for all $e'\in E'$ by the strength of the family. We conclude that $\widetilde{f}\in\mathcal{F}_{G}(U,E)$.

5.2.7. Remark. If U is a set of uniqueness for $(T^{\mathbb{K}}_{m},\mathcal{F})_{m\in M}$, then the existence of operators $(T^{E}_{m})_{m\in M}$ such that $(T^{E}_{m},T^{\mathbb{K}}_{m})_{m\in M}$ is a strong, consistent family for $(\mathcal{F},E)$ is often guaranteed by the Riesz–Markov–Kakutani representation theorems in Section 4.3.

Under the assumptions of Remark 5.2.6 the map

$$R_{U,G}\colon S(\mathcal{F}(\Omega)\varepsilon E)\to\mathcal{F}_{G}(U,E),\ f\mapsto(T^{E}_{m}(f)(x))_{(m,x)\in U},$$

is well-defined. The map $R_{U,G}$ is also linear since T^{E}_{m} is linear for all $m\in M$. Further, the strength of the defining family guarantees that $R_{U,G}$ is injective.

5.2.8. Proposition. *Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-into-compatible, $G\subset E'$ a separating subspace and U a set of uniqueness for $(T^{\mathbb{K}}_{m},\mathcal{F})_{m\in M}$. If $(T^{E}_{m},T^{\mathbb{K}}_{m})_{m\in M}$ is a strong family for $(\mathcal{F},E)$, then the map*

$$T^{E}\colon\mathcal{F}(\Omega,E)\to E^{U},\ f\mapsto(T^{E}_{m}(f)(x))_{(m,x)\in U},$$

is injective, in particular, $R_{U,G}$ is injective.

Proof. Let $f\in\mathcal{F}(\Omega,E)$ with $T^{E}(f)=0$. Then

$$0=(e'\circ T^{E}(f))(m,x)=(e'\circ T^{E}_{m}(f))(x)=T^{\mathbb{K}}_{m}(e'\circ f)(x),\quad(m,x)\in U,$$

and $e'\circ f\in\mathcal{F}(\Omega)$ for all $e'\in E'$ by the strength of the family. Since U is a set of uniqueness, we get that $e'\circ f=0$ for all $e'\in E'$, which implies $f=0$. □

5.2.9. Question. Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-into-compatible, $G\subset E'$ a separating subspace, $(T^{E}_{m},T^{\mathbb{K}}_{m})_{m\in M}$ a strong family for $(\mathcal{F},E)$ and U a set of uniqueness for $(T^{\mathbb{K}}_{m},\mathcal{F})_{m\in M}$. When is the injective restriction map

$$R_{U,G}\colon S(\mathcal{F}(\Omega)\varepsilon E)\to\mathcal{F}_{G}(U,E),\ f\mapsto(T^{E}_{m}(f)(x))_{(m,x)\in U},$$

surjective?

The Question 5.2.1 is a special case of this question if there is a set of uniqueness U for $(T^{\mathbb{K}}_{m},\mathcal{F})_{m\in M}$ with $\{T^{\mathbb{K}}_{m,x}\mid(m,x)\in U\}=\{\delta_{x}\mid x\in\Lambda\}$, $\Lambda\subset\Omega$. We observe that a positive answer to the surjectivity of $R_{\Omega,G}$ results in the following weak-strong principle.

5.2.10. Proposition. *Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-into-compatible, $G\subset E'$ a separating subspace such that $e'\circ f\in\mathcal{F}(\Omega)$ for all $e'\in G$ and $f\in\mathcal{F}(\Omega,E)$. If*

$$R_{\Omega,G}\colon S(\mathcal{F}(\Omega)\varepsilon E)\to\mathcal{F}_{G}(\Omega,E),\ f\mapsto f,$$

with the set of uniqueness Ω for $(\mathrm{id}_{\mathbb{K}^{\Omega}},\mathcal{F})$ is surjective, then

$$\mathcal{F}(\Omega)\varepsilon E\cong\mathcal{F}(\Omega,E)\quad\textit{via } S\quad\textit{and}\quad\mathcal{F}(\Omega,E)=\{f\colon\Omega\to E\mid\forall\ e'\in G:\ e'\circ f\in\mathcal{F}(\Omega)\}.$$

Proof. From the ε-into-compatibility and the surjectivity of $R_{\Omega,G}$ we obtain

$$\{f\colon\Omega\to E\mid\forall\ e'\in G:\ e'\circ f\in\mathcal{F}(\Omega)\}=\mathcal{F}_{G}(\Omega,E)=S(\mathcal{F}(\Omega)\varepsilon E)\subset\mathcal{F}(\Omega,E).$$

Further, the assumption that $e'\circ f\in\mathcal{F}(\Omega)$ for all $e'\in G$ and $f\in\mathcal{F}(\Omega,E)$, implies that $\mathcal{F}(\Omega,E)$ is a subspace of the space on the left-hand side, which proves our statement, in particular, the surjectivity of S. □

To answer Question 5.2.9 for general sets of uniqueness we have to restrict to a certain class of separating subspaces of E'.

5.2.11. DEFINITION (determine boundedness [30, p. 230]). A linear subspace $G \subset E'$ *determines boundedness* if every $\sigma(E,G)$-bounded set $B \subset E$ is already bounded in E.

In [67, p. 139] such a space G is called uniform boundedness deciding by Fernández et al. and in [134, p. 63] w^*-thick by Nygaard if E is a Banach space.

5.2.12. REMARK. a) Let E be an lcHs. Then $G := E'$ determines boundedness by [131, Mackey's theorem 23.15, p. 268].

b) Let X be a barrelled lcHs, Y an lcHs and $E := L_b(X,Y)$. For $x \in X$ and $y' \in Y'$ we set $\delta_{x,y'}: L(X,Y) \to \mathbb{K}$, $T \mapsto y'(T(x))$, and $G := \{\delta_{x,y'} \mid x \in X,\, y' \in Y'\} \subset E'$. Then the span of G determines boundedness (in E) by Mackey's theorem and the uniform boundedness principle. For Banach spaces X,Y this is already observed in [30, Remark 11, p. 233] and, if in addition $Y = \mathbb{K}$, in [7, Remark 1.4 b), p. 781].

c) Further examples and a characterisation of subspaces $G \subset E'$ that determine boundedness can be found in [7, Remark 1.4, p. 781–782], [134, Theorem 1.5, p. 63–64] and [134, Theorem 2.3, 2.4, p. 67–68] in the case that E is a Banach space.

$\mathcal{F}(\Omega)$ a semi-Montel space and E (sequentially) complete. Our next results are in need of spaces $\mathcal{F}(\Omega)$ such that closed graph theorems hold with Banach spaces as domain spaces and $\mathcal{F}(\Omega)$ as the range space. Let us formally define this class of spaces.

5.2.13. DEFINITION (BC-space [142, p. 395]). We call an lcHs F a *BC-space* if for every Banach space X and every linear map $f: X \to F$ with closed graph in $X \times F$, one has that f is continuous.

A characterisation of BC-spaces is given by Powell in [142, 6.1 Corollary, p. 400–401]. Since every Banach space is ultrabornological and barrelled, the [131, Closed graph theorem 24.31, p. 289] of de Wilde and the Pták–Kōmura–Adasch–Valdivia closed graph theorem [103, §34, 9.(7), p. 46] imply that webbed spaces and B_r-complete spaces are BC-spaces. We recall that an lcHs F is said to be *B_r-complete* if every $\sigma(F',F)$-dense $\sigma^f(F',F)$-closed linear subspace of F' equals F' where $\sigma^f(F',F)$ is the finest topology coinciding with $\sigma(F',F)$ on all equicontinuous sets in F' (see [103, §34, p. 26]). An lcHs F is called *B-complete* if every $\sigma^f(F',F)$-closed linear subspace of F' is weakly closed. In particular, B-complete spaces are B_r-complete and every B_r-complete space is complete by [103, §34, 2.(1), p. 26]. These definitions are equivalent to the original definitions of B_r- and B-completeness by Pták [143, Definition 2, 5, p. 50, 55] due to [103, §34, 2.(2), p. 26–27] and we note that they are also called *infra-Pták spaces* and *Pták spaces*, respectively. In particular, Fréchet spaces are B-complete by [89, 9.5.2 Krein–Šmulian Theorem, p. 184] but we will encounter non-Fréchet B-complete spaces as well.

The following proposition is a modification of [94, Satz 10.6, p. 237] and uses the map $\mathscr{R}_f: e' \mapsto f_{e'}$ from Remark 5.2.5.

5.2.14. PROPOSITION. *Let U be a set of uniqueness for $(T^{\mathbb{K}}_m, \mathcal{F})_{m\in M}$ and $\mathcal{F}(\Omega)$ a BC-space. Then $\mathscr{R}_f(B^\circ_\alpha)$ is bounded in $\mathcal{F}(\Omega)$ for every $f \in \mathcal{F}_{E'}(U,E)$ and $\alpha \in \mathfrak{A}$ where $B_\alpha := \{x \in E \mid p_\alpha(x) < 1\}$. In addition, if $\mathcal{F}(\Omega)$ is a semi-Montel space, then $\mathscr{R}_f(B^\circ_\alpha)$ is relatively compact in $\mathcal{F}(\Omega)$.*

PROOF. Let $f \in \mathcal{F}_{E'}(U,E)$ and $\alpha \in \mathfrak{A}$. The polar B_α° is compact in E'_σ and thus $E'_{B_\alpha^\circ}$ is a Banach space by [131, Corollary 23.14, p. 268]. We claim that the restriction of $\mathscr{R}_f$ to $E'_{B_\alpha^\circ}$ has closed graph. Indeed, let (e'_τ) be a net in $E'_{B_\alpha^\circ}$ converging to e' in $E'_{B_\alpha^\circ}$ and $\mathscr{R}_f(e'_\tau)$ converging to g in $\mathcal{F}(\Omega)$. For $(m,x) \in U$ we note that

$$\begin{aligned} T^{\mathbb{K}}_{m,x}\big(\mathscr{R}_f(e'_\tau)\big) &= T^{\mathbb{K}}_m(f_{e'_\tau})(x) = (e'_\tau \circ f)(m,x) \to (e' \circ f)(m,x) = T^{\mathbb{K}}_m(f_{e'})(x) \\ &= T^{\mathbb{K}}_m\big(\mathscr{R}_f(e')\big)(x). \end{aligned}$$

The left-hand side converges to $T^{\mathbb{K}}_{m,x}(g)$ since $T^{\mathbb{K}}_{m,x} \in \mathcal{F}(\Omega)'$ for all $(m,x) \in U$. Hence we have $T^{\mathbb{K}}_m(g)(x) = T^{\mathbb{K}}_m\big(\mathscr{R}_f(e')\big)(x)$ for all $(m,x) \in U$. From U being a set of uniqueness follows that $g = \mathscr{R}_f(e')$. Thus the restriction of $\mathscr{R}_f$ to $E'_{B_\alpha^\circ}$ has closed graph and is continuous since $\mathcal{F}(\Omega)$ is a BC-space. This yields that $\mathscr{R}_f(B_\alpha^\circ)$ is bounded as B_α° is bounded in $E'_{B_\alpha^\circ}$. If $\mathcal{F}(\Omega)$ is also a semi-Montel space, then $\mathscr{R}_f(B_\alpha^\circ)$ is even relatively compact. □

Now, we are ready to prove our first extension theorem. Its proof of surjectivity of $R_{U,E'}$ is just an adaptation of the proof of surjectivity of S given in Theorem 3.2.4. Let U be a set of uniqueness for $(T^{\mathbb{K}}_m, \mathcal{F})_{m\in M}$. For $f \in \mathcal{F}_{E'}(U,E)$ we consider the dual map

$$\mathscr{R}^t_f \colon \mathcal{F}(\Omega)' \to E'^\star,\ \mathscr{R}^t_f(y)(e') \coloneqq y(f_{e'}),$$

where $E'^\star$ is the algebraic dual of E'. Further, we recall the notation $\mathcal{J}\colon E \to E'^\star$ for the canonical injection.

5.2.15. THEOREM. *Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-into-compatible, $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ a strong, consistent family for $(\mathcal{F},E)$, $\mathcal{F}(\Omega)$ a semi-Montel BC-space and U a set of uniqueness for $(T^{\mathbb{K}}_m, \mathcal{F})_{m\in M}$. If*

(i) E is complete, or
(ii) E is sequentially complete and for every $f \in \mathcal{F}_{E'}(U,E)$ and $f' \in \mathcal{F}(\Omega)'$ there is a sequence $(f'_n)_{n\in\mathbb{N}}$ in $\mathcal{F}(\Omega)'$ converging to f' in $\mathcal{F}(\Omega)'_\kappa$ such that $\mathscr{R}^t_f(f'_n) \in \mathcal{J}(E)$ for every $n \in \mathbb{N}$,

then the restriction map $R_{U,E'}\colon S(\mathcal{F}(\Omega)\varepsilon E) \to \mathcal{F}_{E'}(U,E)$ is surjective.

PROOF. Let $f \in \mathcal{F}_{E'}(U,E)$. As in Theorem 3.2.4 we equip $\mathcal{J}(E)$ with the system of seminorms given by

$$p_{B_\alpha^\circ}(\mathcal{J}(x)) \coloneqq \sup_{e' \in B_\alpha^\circ} |\mathcal{J}(x)(e')| = p_\alpha(x), \quad x \in E, \tag{45}$$

for all $\alpha \in \mathfrak{A}$ where $B_\alpha \coloneqq \{x \in E \mid p_\alpha(x) < 1\}$. We claim $\mathscr{R}^t_f \in L(\mathcal{F}(\Omega)'_\kappa, \mathcal{J}(E))$. Indeed, we have for $y \in \mathcal{F}(\Omega)'$

$$p_{B_\alpha^\circ}\big(\mathscr{R}^t_f(y)\big) = \sup_{e' \in B_\alpha^\circ} |y(f_{e'})| = \sup_{x \in \mathscr{R}_f(B_\alpha^\circ)} |y(x)| \leq \sup_{x \in K_\alpha} |y(x)| \tag{46}$$

where $K_\alpha \coloneqq \overline{\mathscr{R}_f(B_\alpha^\circ)}$. Due to Proposition 5.2.14 the set $\mathscr{R}_f(B_\alpha^\circ)$ is absolutely convex and relatively compact, implying that K_α is absolutely convex and compact in $\mathcal{F}(\Omega)$ by [89, 6.2.1 Proposition, p. 103]. Further, we have for all $e' \in E'$ and $(m,x) \in U$

$$\mathscr{R}^t_f(T^{\mathbb{K}}_{m,x})(e') = T^{\mathbb{K}}_{m,x}(f_{e'}) = (e' \circ f)(m,x) = \mathcal{J}\big(f(m,x)\big)(e') \tag{47}$$

and thus $\mathscr{R}^t_f(T^{\mathbb{K}}_{m,x}) \in \mathcal{J}(E)$.

First, let condition (i) be satisfied, i.e. let E be complete, and $f' \in \mathcal{F}(\Omega)'$. The span of $\{T^{\mathbb{K}}_{m,x} \mid (m,x) \in U\}$ is dense in $\mathcal{F}(\Omega)'_\kappa$ since U is a set of uniqueness for

$\mathcal{F}(\Omega)$. Thus there is a net (f'_τ) converging to f' in $\mathcal{FV}(\Omega)'_\kappa$ with $\mathscr{R}^t_f(f'_\tau)\in\mathcal{J}(E)$ and

$$p_{B^\circ_\alpha}\big(\mathscr{R}^t_f(f'_\tau)-\mathscr{R}^t_f(f')\big)\underset{(46)}{\leq}\sup_{x\in K_\alpha}|(f'_\tau-f')(x)|\to 0 \tag{48}$$

for all $\alpha\in\mathfrak{A}$. We gain that $(\mathscr{R}^t_f(f'_\tau))$ is a Cauchy net in the complete space $\mathcal{J}(E)$. Hence it has a limit $g\in\mathcal{J}(E)$ which coincides with $\mathscr{R}^t_f(f')$ since

$$\begin{aligned}p_{B^\circ_\alpha}\big(g-\mathscr{R}^t_f(f')\big)&\leq p_{B^\circ_\alpha}\big(g-\mathscr{R}^t_f(f'_\tau)\big)+p_{B^\circ_\alpha}\big(\mathscr{R}^t_f(f'_\tau)-\mathscr{R}^t_f(f')\big)\\&\underset{(48)}{\leq}p_{B^\circ_\alpha}\big(g-\mathscr{R}^t_f(f'_\tau)\big)+\sup_{x\in K_\alpha}\big|(f'_\tau-f')(x)\big|\to 0\end{aligned}$$

for all $\alpha\in\mathfrak{A}$. We conclude that $\mathscr{R}^t_f(f')\in\mathcal{J}(E)$ for every $f'\in\mathcal{F}(\Omega)'$.

Second, let condition (ii) be satisfied and $f'\in\mathcal{F}(\Omega)'$. Then there is a sequence (f'_n) in $\mathcal{F}(\Omega)'$ converging to f' in $\mathcal{F}(\Omega)'_\kappa$ such that $\mathscr{R}^t_f(f'_n)\in\mathcal{J}(E)$ for every $n\in\mathbb{N}$. From (46) we derive that $(\mathscr{R}^t_f(f'_n))$ is a Cauchy sequence in the sequentially complete space $\mathcal{J}(E)$ converging to $\mathscr{R}^t_f(f')\in\mathcal{J}(E)$.

Therefore we obtain in both cases that $\mathscr{R}^t_f\in L(\mathcal{F}(\Omega)'_\kappa,\mathcal{J}(E))$. So we get for all $\alpha\in\mathfrak{A}$ and $y\in\mathcal{F}(\Omega)'$

$$p_\alpha\big((\mathcal{J}^{-1}\circ\mathscr{R}^t_f)(y)\big)\underset{(45)}{=}p_{B^\circ_\alpha}\big(\mathcal{J}((\mathcal{J}^{-1}\circ\mathscr{R}^t_f)(y))\big)=p_{B^\circ_\alpha}\big(\mathscr{R}^t_f(y)\big)\underset{(46)}{\leq}\sup_{x\in K_\alpha}|y(x)|.$$

This implies $\mathcal{J}^{-1}\circ\mathscr{R}^t_f\in L(\mathcal{F}(\Omega)'_\kappa,E)=\mathcal{F}(\Omega)\varepsilon E$ (as linear spaces). We set $F:=S(\mathcal{J}^{-1}\circ\mathscr{R}^t_f)$ and obtain from consistency that

$$T^E_m(F)(x)=T^E_mS(\mathcal{J}^{-1}\circ\mathscr{R}^t_f)(x)=\mathcal{J}^{-1}\big(\mathscr{R}^t_f(T^{\mathbb{K}}_{m,x})\big)\underset{(47)}{=}\mathcal{J}^{-1}\big(\mathcal{J}(f(m,x))\big)=f(m,x)$$

for every $(m,x)\in U$, which means $R_{U,E'}(F)=f$. □

If E is complete and U a set of uniqueness for $(T^{\mathbb{K}}_m,\mathcal{F})_{m\in M}$ with $\{T^{\mathbb{K}}_{m,x}\mid(m,x)\in U\}=\{\delta_x\mid x\in\Lambda\}$, $\Lambda\subset\Omega$, then we get [77, 0.1, p. 217] as a special case. Condition (i) and (ii) are adaptations of Condition 3.2.3 a) and c) from $\mathcal{FV}(\Omega,E)$ and R_f to $\mathcal{F}_{E'}(U,E)$ and $\mathscr{R}_f$. We also treat an adaptation of Condition 3.2.3 e) in Theorem 5.2.52. Condition 3.2.3 b) and d) may be adapted as well but we restrict to the ones we actually apply. First, we apply Theorem 5.2.15 to the space of bounded zero-solutions of a hypoelliptic linear partial differential operator equipped with the strict topology β from Proposition 4.2.24.

5.2.16. PROPOSITION. *Let $\Omega\subset\mathbb{R}^d$ be open and $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator. Then $(\mathcal{C}^\infty_{P(\partial),b}(\Omega),\beta)$ is a B-complete semi-Montel space.*

PROOF. Due to the proof of Proposition 4.2.24 we know that β coincides with the mixed topology $\gamma(\tau_c,\|\cdot\|_\infty)$. It is easy to check that the closed $\|\cdot\|_\infty$-unit ball $B_{\|\cdot\|_\infty}$ is τ_c-compact in $\mathcal{C}^\infty_{P(\partial),b}(\Omega)$. Thus [46, Section I.1, 1.13 Proposition, p. 11] yields that $(\mathcal{C}^\infty_{P(\partial),b}(\Omega),\beta)$ is a semi-Montel space. From [150, 2.9 Theorem, p. 185] it follows that the space is B-complete. □

5.2.17. COROLLARY. *Let $\Omega\subset\mathbb{R}^d$ be open, E a complete lcHs, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator, $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ a strong, consistent family for $((\mathcal{C}^\infty_{P(\partial),b}(\Omega),\beta),E)$ and U a set of uniqueness for $(T^{\mathbb{K}}_m,(\mathcal{C}^\infty_{P(\partial),b}(\Omega),\beta))_{m\in M}$. If $f\colon U\to E$ is a function such that there is $f_{e'}\in\mathcal{C}^\infty_{P(\partial),b}(\Omega)$ for each $e'\in E'$ with $T^{\mathbb{K}}_m(f_{e'})(x)=(e'\circ f)(m,x)$ for all $(m,x)\in U$, then there is a unique $F\in\mathcal{C}^\infty_{P(\partial),b}(\Omega,E)$ with $T^E_m(F)(x)=f(m,x)$ for all $(m,x)\in U$.*

PROOF. The space $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)$ is a semi-Montel BC-space by Proposition 5.2.16. Moreover, $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)$ and $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\beta)$ are ε-compatible by Proposition 4.2.24, yielding our statement by Theorem 5.2.15 (i) and Proposition 5.2.8. □

Especially, for any $m\in\mathbb{N}_0$ the family $((\partial^{\beta})^{E},(\partial^{\beta})^{\mathbb{K}})_{\beta\in\mathbb{N}_0^d,|\beta|\leq m}$ is strong and consistent for $((\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta),E)$ by the proof of Proposition 4.2.24. It is always possible to construct a strong, consistent family $(T^{E}_{m},T^{\mathbb{K}}_{m})_{m\in M}$ for $((\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta),E)$ from a given set of uniqueness $(T^{\mathbb{K}}_{m},(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta))_{m\in M}$ due to Remark 4.3.11 b) and c).

Similarly, we may apply Theorem 5.2.15 to the space $\mathcal{E}^{\{M_p\}}(\Omega,E)$ of ultradifferentiable functions of class $\{M_p\}$ of Roumieu-type from Example 3.1.9 f). $\mathcal{E}^{\{M_p\}}(\Omega)$ is a projective limit of a countable sequence of DFS-spaces by [99, Theorem 2.6, p. 44] and thus webbed because being webbed is stable under the formation of projective and inductive limits of countable sequences by [89, 5.3.3 Corollary, p. 92]. Further, if the sequence $(M_p)_{p\in\mathbb{N}_0}$ satisfies Komatsu's conditions (M.1) and (M.3)', then $\mathcal{E}^{\{M_p\}}(\Omega)$ is a Montel space by [99, Theorem 5.12, p. 65–66]. The spaces $\mathcal{E}^{\{M_p\}}(\Omega)$ and $\mathcal{E}^{\{M_p\}}(\Omega,E)$ are ε-compatible if (M.1) and (M.3)' hold and E is complete by Example 3.2.11 b). Hence Theorem 5.2.15 (i) is applicable.

5.2.18. REMARK. We remark that Remark 5.2.6 and Theorem 5.2.15 still hold if the map $S\colon\mathcal{F}(\Omega)\varepsilon E\to\mathcal{F}(\Omega,E)$ is only a linear isomorphism into, i.e. an isomorphism into of linear spaces, since the topological nature of ε-into-compatibility is not used in the proof. In particular, this means that it can be applied to the space $\mathcal{M}(\Omega,E)$ of meromorphic functions on an open, connected set $\Omega\subset\mathbb{C}$ with values in an lcHs E over $\mathbb{C}$ (see [29, p. 356]). The space $\mathcal{M}(\Omega)$ is a Montel LF-space, thus webbed by [89, 5.3.3 Corollary (b), p. 92], due to the proof of [80, Theorem 3 (a), p. 294–295] if it is equipped with the locally convex topology τ_{ML} given in [80, p. 292]. By [29, Proposition 6, p. 357] the map $S\colon\mathcal{M}(\Omega)\varepsilon E\to\mathcal{M}(\Omega,E)$ is a linear isomorphism if E is locally complete and does not contain the space $\mathbb{C}^{\mathbb{N}}$. Therefore we can apply Theorem 5.2.15 if E is complete and does not contain $\mathbb{C}^{\mathbb{N}}$. This augments [92, Theorem 12, p. 12] where E is assumed to be locally complete with suprabarrelled strong dual and $(T^{E},T^{\mathbb{C}})=(\mathrm{id}_{E^{\Omega}},\mathrm{id}_{\mathbb{C}^{\Omega}})$.

$\mathcal{F}(\Omega)$ a Fréchet–Schwartz space and E locally complete. We recall the following abstract extension result.

5.2.19. PROPOSITION ([30, Proposition 7, p. 231]). *Let E be a locally complete lcHs, Y a Fréchet–Schwartz space, $X\subset Y_b'(=Y_\kappa')$ dense and $\mathsf{A}\colon X\to E$ linear. Then the following assertions are equivalent:*

a) There is a (unique) extension $\widehat{\mathsf{A}}\in Y\varepsilon E$ of A.
b) $(\mathsf{A}^t)^{-1}(Y)$ $(=\{e'\in E'\mid e'\circ\mathsf{A}\in Y\})$ determines boundedness in E.

Next, we generalise [30, Theorem 9, p. 232] using the preceding proposition. The proof of the generalisation is simply obtained by replacing the set of uniqueness in the proof of [30, Theorem 9, p. 232] by our more general set of uniqueness.

5.2.20. THEOREM. *Let E be a locally complete lcHs, $G\subset E'$ determine boundedness and $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-into-compatible. Let $(T^{E}_{m},T^{\mathbb{K}}_{m})_{m\in M}$ be a strong, consistent family for $(\mathcal{F},E)$, $\mathcal{F}(\Omega)$ a Fréchet–Schwartz space and U a set of uniqueness for $(T^{\mathbb{K}}_{m},\mathcal{F})_{m\in M}$. Then the restriction map $R_{U,G}\colon S(\mathcal{F}(\Omega)\varepsilon E)\to\mathcal{F}_G(U,E)$ is surjective.*

PROOF. Let $f \in \mathcal{F}_G(U,E)$. We choose $X := \operatorname{span}\{T^{\mathbb{K}}_{m,x} \mid (m,x) \in U\}$ and $Y := \mathcal{F}(\Omega)$. Let $\mathsf{A}\colon X \to E$ be the linear map generated by $\mathsf{A}(T^{\mathbb{K}}_{m,x}) := f(m,x)$. The map A is well-defined since G is $\sigma(E',E)$-dense. Let $e' \in G$ and $f_{e'}$ be the unique element in $\mathcal{F}(\Omega)$ such that $T^{\mathbb{K}}_m(f_{e'})(x) = (e' \circ \mathsf{A})(T^{\mathbb{K}}_{m,x})$ for all $(m,x) \in U$. This equation allows us to consider $f_{e'}$ as a linear form on X (by setting $f_{e'}(T^{\mathbb{K}}_{m,x}) := (e' \circ \mathsf{A})(T^{\mathbb{K}}_{m,x})$), which yields $e' \circ \mathsf{A} \in \mathcal{F}(\Omega)$ for all $e' \in G$. It follows that $G \subset (\mathsf{A}^t)^{-1}(Y)$, implying that $(\mathsf{A}^t)^{-1}(Y)$ determines boundedness. Applying Proposition 5.2.19, there is an extension $\widehat{\mathsf{A}} \in \mathcal{F}(\Omega)\varepsilon E$ of A and we set $F := S(\widehat{\mathsf{A}})$. We note that

$$T^E_m(F)(x) = T^E_m S(\widehat{\mathsf{A}})(x) = \widehat{\mathsf{A}}(T^{\mathbb{K}}_{m,x}) = \mathsf{A}(T^{\mathbb{K}}_{m,x}) = f(m,x)$$

for all $(m,x) \in U$ by consistency, yielding $R_{U,G}(F) = f$. □

Let us apply the preceding theorem to our weighted spaces of continuously partially differentiable functions and its subspaces from Example 3.1.9 and Example 4.2.22.

5.2.21. COROLLARY. *Let E be a locally complete lcHs, $G \subset E'$ determine boundedness, $\mathcal{V}^\infty$ a directed family of weights which is locally bounded away from zero on an open set $\Omega \subset \mathbb{R}^d$, let $\mathcal{F}(\Omega)$ be a Fréchet–Schwartz space and $U \subset \mathbb{N}_0^d \times \Omega$ a set of uniqueness for $(\partial^\beta, \mathcal{F})_{\beta \in \mathbb{N}_0^d}$ where $\mathcal{F}$ stands for $\mathcal{CV}^\infty$, $\mathcal{CV}^\infty_0$, $\mathcal{CV}^\infty_{P(\partial)}$ or $\mathcal{CV}^\infty_{P(\partial),0}$. Then the following holds:*

a) *If $f\colon U \to E$ is a function such that there is $f_{e'} \in \mathcal{F}(\Omega)$ for each $e' \in G$ with $\partial^\beta f_{e'}(x) = (e' \circ f)(\beta,x)$ for all $(\beta,x) \in U$, then there is a unique $F \in \mathcal{F}(\Omega,E)$ with $(\partial^\beta)^E F(x) = f(\beta,x)$ for all $(\beta,x) \in U$.*
b) *If $U \subset \Omega$ and $f\colon U \to E$ is a function such that $e' \circ f$ admits an extension $f_{e'} \in \mathcal{F}(\Omega)$ for every $e' \in G$, then there is a unique extension $F \in \mathcal{F}(\Omega,E)$ of f.*
c) *$\mathcal{F}(\Omega,E) = \{f\colon \Omega \to E \mid \forall\, e' \in G:\ e' \circ f \in \mathcal{F}(\Omega)\}$.*

PROOF. The strength and consistency of $((\partial^\beta)^E, \partial^\beta)_{\beta \in \mathbb{N}_0^d}$ for $(\mathcal{F},E)$ and the ε-compatibility of $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ follow from Example 3.2.7 e)+f) and Example 4.2.22 c)+f). This implies that part a) and its special case part b) hold by Theorem 5.2.20 and Proposition 5.2.8. Part c) follows from part b) and Proposition 5.2.10 since $U := \Omega$ is a set of uniqueness for $(\operatorname{id}_{\mathbb{K}^\Omega}, \mathcal{F})$. □

5.2.22. REMARK. Let $\mathcal{V}^\infty$ be a directed family of weights which is locally bounded away from zero on an open set $\Omega \subset \mathbb{R}^d$.

a) Then any dense set $U \subset \Omega$ is a set of uniqueness for $(\operatorname{id}_{\mathbb{K}^\Omega}, \mathcal{F})$ with $\mathcal{F} = \mathcal{CV}^\infty$, $\mathcal{CV}^\infty_0$, $\mathcal{CV}^\infty_{P(\partial)}$ or $\mathcal{CV}^\infty_{P(\partial),0}$ due to continuity.
b) Let Ω be connected and $x_0 \in \Omega$. Then $U := \{(e_n,x) \mid 1 \leq n \leq d,\ x \in \Omega\} \cup \{(0,x_0)\}$ is a set of uniqueness for $(\partial^\beta, \mathcal{F})_{\beta \in \mathbb{N}_0}$ by the mean value theorem with $\mathcal{F}$ from a).
c) Let $\mathbb{K} := \mathbb{R}$, $d := 1$, $\Omega := (a,b) \subset \mathbb{R}$, $g\colon (a,b) \to \mathbb{N}$ and $x_0 \in (a,b)$. Then $U := \{(g(x),x) \mid x \in (a,b)\} \cup \{(n,x_0) \mid n \in \mathbb{N}_0\}$ is a set of uniqueness for $(\partial^\beta, \mathcal{F})_{\beta \in \mathbb{N}_0}$ with $\mathcal{F}$ from a). Indeed, if $f \in \mathcal{F}(\Omega)$ and $0 = \partial^{g(x)} f(x)$ for all $x \in (a,b)$, then f is a polynomial by [56, Chap. 11, Theorem, p. 53]. If, in addition, $0 = \partial^n f(x_0)$ for all $n \in \mathbb{N}_0$, then the polynomial f must vanish on the whole interval Ω.
d) Let $\Omega \subset \mathbb{C}$ be connected. Then any set $U \subset \Omega$ with an accumulation point in Ω is a set of uniqueness for $(\operatorname{id}_{\mathbb{C}^\Omega}, \mathcal{CV}^\infty_{\overline{\partial}})$ by the identity theorem for holomorphic functions.

e) Let $\Omega \subset \mathbb{C}$ be connected and $z_0 \in \Omega$. Then $U := \{(n, z_0) \mid n \in \mathbb{N}_0\}$ is a set of uniqueness for $(\partial_{\mathbb{C}}^n, \mathcal{CV}_{\overline{\partial}}^\infty)_{n\in\mathbb{N}_0}$ by local power series expansion and the identity theorem.
f) Let $\Omega \subset \mathbb{R}^d$ be connected. Then any non-empty open set $U \subset \Omega$ is a set of uniqueness for $(\mathrm{id}_{\mathbb{K}^\Omega}, \mathcal{CV}_\Delta^\infty)$ by the identity theorem for harmonic functions (see e.g. [85, Theorem 5, p. 218]).
g) Further examples of sets of uniqueness for $(\mathrm{id}_{\mathbb{K}^\Omega}, \mathcal{CV}_\Delta^\infty)$ are given in [98].

In part e) a special case of Remark 5.2.3 b) is used, namely, that $\mathcal{CW}_{\overline{\partial}}^\infty(\mathbb{D}_r(z_0))$ has a Schauder basis with associated coefficient functionals $(\delta_{z_0} \circ \partial_{\mathbb{C}}^n)_{n\in\mathbb{N}_0}$ where $0 < r \leq \infty$ is such that $\mathbb{D}_r(z_0) \subset \Omega$. In order to obtain some sets of uniqueness which are more sensible w.r.t. the family of weights $\mathcal{V}^\infty$, we turn to entire and harmonic functions fulfilling some growth conditions. For a family $\mathcal{V} := (\nu_j)_{j\in\mathbb{N}}$ of continuous weights on $\mathbb{R}^d$ set $\mathcal{V}^\infty := (\nu_{j,m})_{j\in\mathbb{N}, m\in\mathbb{N}_0}$ where $\nu_{j,m}\colon \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq m\} \times \mathbb{R}^d \to [0,\infty)$, $\nu_{j,m}(\beta, x) := \nu_j(x)$. We know that $\mathcal{CV}_{P(\partial)}(\mathbb{R}^d, E) = \mathcal{CV}_{P(\partial)}^\infty(\mathbb{R}^d, E)$ as locally convex spaces and that $\mathcal{CV}_{P(\partial)}(\mathbb{R}^d)$ is a nuclear Fréchet space for $P(\partial) = \overline{\partial}$ or $P(\partial) = \Delta$ by Proposition 4.2.19 if E is a locally complete lcHs and $\mathcal{V}$ fulfils Condition 4.2.18. In particular, Condition 4.2.18 is fulfilled if $\nu_j(x) := \exp(-(\tau + \frac{1}{j})|x|)$, $x \in \mathbb{R}^d$, for all $j \in \mathbb{N}$ and some $0 \leq \tau < \infty$ and thus we can apply Corollary 5.2.21 to the spaces $A_{\overline{\partial}}^\tau(\mathbb{C}, E) = \mathcal{CV}_{\overline{\partial}}(\mathbb{C}, E)$ of entire and $A_\Delta^\tau(\mathbb{R}^d, E) = \mathcal{CV}_\Delta(\mathbb{R}^d, E)$ of harmonic functions of exponential type τ by Remark 4.2.20. Hence we may complement our list in Remark 5.2.22 by some more examples for spaces of functions of exponential type $0 \leq \tau < \infty$.

5.2.23. Remark. The following sets $U \subset \mathbb{C}$ are sets of uniqueness for $(\mathrm{id}_{\mathbb{C}^\mathbb{C}}, A_{\overline{\partial}}^\tau)$.
a) If $\tau < \pi$, then $U := \mathbb{N}_0$ is a set of uniqueness by [21, 9.2.1 Carlson's theorem, p. 153].
b) Let $\delta > 0$ and $(\lambda_n)_{n\in\mathbb{N}} \subset (0,\infty)$ such that $\lambda_{n+1} - \lambda_n > \delta$ for all $n \in \mathbb{N}$. Then $U := (\lambda_n)_{n\in\mathbb{N}}$ is a set of uniqueness if $\limsup_{r\to\infty} r^{-2\tau/\pi}\psi(r) = \infty$ where $\psi(r) := \exp(\sum_{\lambda_n < r} \lambda_n^{-1})$, $r > 0$, by [21, 9.5.1 Fuchs's theorem, p. 157–158].

The following sets U are sets of uniqueness for $(\partial_{\mathbb{C}}^n, A_{\overline{\partial}}^\tau)_{n\in\mathbb{N}_0}$.
c) Let $(\lambda_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$ with $|\lambda_n| < 1$ for all $n \in \mathbb{N}_0$. If $\tau < \ln(2)$, then $U := \{(n, \lambda_n) \mid n \in \mathbb{N}_0\}$ is a set of uniqueness by [21, 9.11.1 Theorem, p. 172]. If $\tau < \ln(2 + \sqrt{3})$, then $U := \{(2n+1, 0) \mid n \in \mathbb{N}_0\} \cup \{(2n, \lambda_n) \mid n \in \mathbb{N}_0\}$ is a set of uniqueness by [21, 9.11.3 Theorem, p. 173].
d) Let $(\lambda_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$ with $\limsup_{n\to\infty} n^{-1}\sum_{k=1}^n |\lambda_k| \leq 1$. If $\tau < \mathrm{e}^{-1}$, then $U := \{(n, \lambda_n) \mid n \in \mathbb{N}_0\}$ is a set of uniqueness by [21, 9.11.4 Theorem, p. 173].

The following sets $U \subset \mathbb{R}^d$ are sets of uniqueness for $(\mathrm{id}_{\mathbb{R}^{\mathbb{R}^d}}, A_\Delta^\tau)$.
e) Let $d := 2$. If there is $k \in \mathbb{N}$ with $\tau < \pi/k$, then $U := \mathbb{Z} \cup (\mathbb{Z} + \mathrm{i}k)$ is a set of uniqueness by [22, Theorem 1, p. 425].
f) Let $d := 2$. If $\tau < \pi$ and $\theta \notin \pi\mathbb{Q}$, then $U := \mathbb{Z} \cup (\mathrm{e}^{\mathrm{i}\theta}\mathbb{Z})$ is a set of uniqueness by [22, Theorem 2, p. 426].
g) If $\tau < \pi$, then $U := \{0,1\} \times \mathbb{Z}^{d-1}$ is a set of uniqueness by [145, Corollary 1.8, p. 312].
h) If $\tau < \pi$ and $a \in \mathbb{R}$ with $|a| \leq \sqrt{1/(d-1)}$, then $U := \mathbb{Z}^{d-1} \times \{0, a\}$ is a set of uniqueness by [185, Theorem A, p. 335].
i) Further examples of sets of uniqueness can be found in [10].

The following sets U are sets of uniqueness for $((\partial^\beta)^\mathbb{R}, A_\Delta^\tau)_{\beta\in\mathbb{N}_0^d}$.
j) If $\tau < \pi$, then $U := \{(\beta, (x, 0)) \mid \beta \in \{0, e_d\},\ x \in \mathbb{Z}^{d-1}\}$ is a set of uniqueness by [185, Theorem B, p. 335]. Further examples can be found in [10].

We need the following weak-strong principle in our last section for the space $\mathcal{E}_0(E)$ of E-valued infinitely continuously partially differentiable functions on $(0,1)$ such that all derivatives can be continuously extended to the boundary and vanish at 1.

5.2.24. Corollary. *Let E be a locally complete lcHs and $G \subset E'$ determine boundedness. Then* $\mathcal{E}_0(E) = \{f\colon(0,1)\to E \mid \forall\, e' \in G:\ e'\circ f \in \mathcal{E}_0\}$.

Proof. By Example 4.2.29 $\mathcal{E}_0$ is a Fréchet–Schwartz space and $\mathcal{E}_0$ and $\mathcal{E}_0(E)$ are ε-compatible. We derive our statement from Theorem 5.2.20 and Proposition 5.2.10 with $(T^E,T^{\mathbb{K}}) := (\mathrm{id}_{E^U},\mathrm{id}_{\mathbb{K}^U})$ and $U := (0,1)$. □

$\mathcal{F}\nu(\Omega)$ a Banach space and E locally complete. In this subsection we consider function spaces $\mathcal{F}(\Omega,E)$ with a certain structure, namely, spaces $\mathcal{FV}(\Omega,E)$ from Definition 3.1.4 where the family of weights $\mathcal{V} = (\nu_{j,m})_{j\in J,m\in M}$ only consists of one weight function, i.e. the sets J and M can be chosen as singletons. So for two non-empty sets Ω and ω, a weight $\nu\colon\omega\to(0,\infty)$, a linear operator $T^E\colon E^\Omega \supset \operatorname{dom} T^E \to E^\omega$ and a linear subspace $\mathrm{AP}(\Omega,E)$ of E^Ω we consider the space

$$\mathcal{F}\nu(\Omega,E) = \big\{f \in F(\Omega,E) \mid \forall\, \alpha \in \mathfrak{A}:\ |f|_\alpha < \infty\big\}$$

where

$$F(\Omega,E) = \mathrm{AP}(\Omega,E) \cap \operatorname{dom} T^E$$

and

$$|f|_\alpha = |f|_{\mathcal{F}\nu(\Omega),\alpha} = \sup_{x\in\omega} p_\alpha\big(T^E(f)(x)\big)\nu(x).$$

For instance, if $\Omega := \omega$, $T^E := \mathrm{id}_{E^\Omega}$ and $\nu := 1$ on Ω, then $\mathcal{F}\nu(\Omega,E)$ is the linear subspace of $F(\Omega,E)$ consisting of bounded functions. We use the methods developed in [70, 93] where, in particular, the special case that $\mathcal{F}\nu(\Omega)$ is the space of bounded smooth functions on an open set $\Omega\subset\mathbb{R}^d$ in the kernel of a hypoelliptic linear partial differential operator resp. a weighted space of holomorphic functions on an open subset Ω of a Banach space is treated. The lack of compact subsets of an infinite dimensional Banach space $\mathcal{F}\nu(\Omega)$ is compensated in [70, 93] by equipping $F(\Omega)$ with a locally convex Hausdorff topology such that the closed unit ball of $\mathcal{F}\nu(\Omega)$ is compact in $F(\Omega)$. Among others, the space $F(\Omega,E) := (\mathcal{O}(\Omega,E),\tau_c)$ of holomorphic functions on an open set $\Omega\subset\mathbb{C}$ with values in a locally complete space E equipped with topology τ_c of compact convergence is used in [70] and the space $\mathcal{F}\nu(\Omega,E) := H^\infty(\Omega,E)$ of E-valued bounded holomorphic functions on Ω.

5.2.25. Proposition. *Let $F(\Omega)$ and $F(\Omega,E)$ be ε-into-compatible, $(T^E,T^{\mathbb{K}})$ a consistent family for (F,E) and a generator for $(\mathcal{F}\nu,E)$ and the map $i\colon\mathcal{F}\nu(\Omega)\to F(\Omega)$, $f\mapsto f$, continuous. We set*

$$\mathcal{F}_\varepsilon\nu(\Omega,E) := S(\{u\in F(\Omega)\varepsilon E \mid u(B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}) \text{ is bounded in } E\})$$

where $B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'} := \{y'\in F(\Omega)' \mid \forall\, f\in B_{\mathcal{F}\nu(\Omega)}:\ |y'(f)|\leq 1\}$ *and $B_{\mathcal{F}\nu(\Omega)}$ is the closed unit ball of $\mathcal{F}\nu(\Omega)$. Then the following holds:*

a) *$\mathcal{F}\nu(\Omega)$ is a* dom-*space.*
b) *Let $u\in F(\Omega)\varepsilon E$. Then*

$$\sup_{y'\in B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}} p_\alpha(u(y')) = |S(u)|_{\mathcal{F}\nu(\Omega),\alpha},\quad \alpha\in\mathfrak{A}.$$

In particular,

$$\mathcal{F}_\varepsilon\nu(\Omega,E) = S(\{u\in F(\Omega)\varepsilon E \mid \forall\,\alpha\in\mathfrak{A}:\ |S(u)|_{\mathcal{F}\nu(\Omega),\alpha}<\infty\}).$$

c) *$S(\mathcal{F}\nu(\Omega)\varepsilon E) \subset \mathcal{F}_{\varepsilon}\nu(\Omega, E) \subset \mathcal{F}\nu(\Omega, E)$ as linear spaces. If $F(\Omega)$ and $F(\Omega, E)$ are even ε-compatible, then $\mathcal{F}_{\varepsilon}\nu(\Omega, E) = \mathcal{F}\nu(\Omega, E)$.*

d) *If $\mathcal{F}\nu(\Omega, E)$ is Hausdorff, then*
   (i) *$(T^E, T^{\mathbb{K}})$ is a consistent generator for $(\mathcal{F}\nu, E)$.*
   (ii) *$\mathcal{F}\nu(\Omega)$ and $\mathcal{F}\nu(\Omega, E)$ are ε-into-compatible.*
   (iii) *$(T^E, T^{\mathbb{K}})$ is a strong generator for $(\mathcal{F}\nu, E)$ if it is a strong family for (F, E).*

PROOF. Part a) follows from the continuity of the map i and the ε-into-compatibility of $F(\Omega)$ and $F(\Omega, E)$. Let us turn to part b). As in Lemma 3.1.8 it follows from the bipolar theorem that

$$B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'} = \overline{\operatorname{acx}}\{T_x^{\mathbb{K}}(\cdot)\nu(x) \mid x \in \omega\},$$

where $\overline{\operatorname{acx}}$ denotes the closure w.r.t. $\kappa(F(\Omega)', \mathcal{F}\nu(\Omega))$ of the absolutely convex hull acx of the set $D := \{T_x^{\mathbb{K}}(\cdot)\nu(x) \mid x \in \omega\}$ on the right-hand side, and that

$$\begin{aligned}\sup_{y' \in B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}} p_{\alpha}(u(y')) &= \sup_{y' \in \operatorname{acx}(D)} p_{\alpha}(u(y')) = \sup_{y' \in D} p_{\alpha}(u(y')) = \sup_{x \in \omega} p_{\alpha}\big(u(T_x^{\mathbb{K}})\big)\nu(x) \\ &= \sup_{x \in \omega} p_{\alpha}\big(T^E(S(u))(x)\big)\nu(x) = |S(u)|_{\mathcal{F}\nu(\Omega), \alpha}\end{aligned}$$

by consistency, which proves part b).

Let us address part c). The continuity of the map i implies the continuity of the inclusion $\mathcal{F}\nu(\Omega)\varepsilon E \hookrightarrow F(\Omega)\varepsilon E$ and thus we obtain $u_{|F(\Omega)'} \in F(\Omega)\varepsilon E$ for every $u \in \mathcal{F}\nu(\Omega)\varepsilon E$. If $u \in \mathcal{F}\nu(\Omega)\varepsilon E$ and $\alpha \in \mathfrak{A}$, then there are $C_0, C_1 > 0$ and an absolutely convex compact set $K \subset \mathcal{F}\nu(\Omega)$ such that $K \subset C_1 B_{\mathcal{F}\nu(\Omega)}$ and

$$\sup_{y' \in B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}} p_{\alpha}(u(y')) \leq C_0 \sup_{y' \in B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}} \sup_{f \in K} |y'(f)| \leq C_0 C_1,$$

which implies $S(\mathcal{F}\nu(\Omega)\varepsilon E) \subset \mathcal{F}_{\varepsilon}\nu(\Omega, E)$. If $f := S(u) \in \mathcal{F}_{\varepsilon}\nu(\Omega, E)$ and $\alpha \in \mathfrak{A}$, then $S(u) \in F(\Omega, E)$ and

$$|f|_{\mathcal{F}\nu(\Omega), \alpha} = \sup_{x \in \omega} p_{\alpha}(u(T_x^{\mathbb{K}})\nu(x)) < \infty$$

by consistency, yielding $\mathcal{F}_{\varepsilon}\nu(\Omega, E) \subset \mathcal{F}\nu(\Omega, E)$. If $F(\Omega)$ and $F(\Omega, E)$ are even ε-compatible, then $S(F(\Omega)\varepsilon E) = F(\Omega, E)$, which yields $\mathcal{F}_{\varepsilon}\nu(\Omega, E) = \mathcal{F}\nu(\Omega, E)$ by part b).

Let us turn to part d). By part a) $\mathcal{F}\nu(\Omega)$ is a dom-space. Since $\mathcal{F}\nu(\Omega, E)$ is Hausdorff, it is also a dom-space due to Remark 3.1.6 c). We have $u_{|F(\Omega)'} \in F(\Omega)\varepsilon E$ for every $u \in \mathcal{F}\nu(\Omega)\varepsilon E$ and

$$S_{\mathcal{F}\nu(\Omega)}(u)(x) = u(\delta_x) = u_{|F(\Omega)'}(\delta_x) = S_{F(\Omega)}(u_{|F(\Omega)'})(x), \quad x \in \Omega.$$

In combination with $S(F(\Omega)\varepsilon E) \subset F(\Omega, E)$ and the consistency of $(T^E, T^{\mathbb{K}})$ for (F, E) this yields that $(T^E, T^{\mathbb{K}})$ is a consistent generator for $(\mathcal{F}\nu, E)$. Thus part (i) holds and implies part (ii) by Theorem 3.1.12. If $(T^E, T^{\mathbb{K}})$ is in addition a strong family for (F, E), then the inclusion $\mathcal{F}\nu(\Omega, E) \subset F(\Omega, E)$ implies that $e' \circ f \in F(\Omega, E)$ and $T^{\mathbb{K}}(e' \circ f)(x) = (e' \circ T^E(f))(x)$ for all $e' \in E'$, $f \in \mathcal{F}\nu(\Omega, E)$ and $x \in \omega$. It follows that $(T^E, T^{\mathbb{K}})$ is a strong generator for $(\mathcal{F}\nu, E)$. □

The canonical situation in part c) is that $\mathcal{F}_{\varepsilon}\nu(\Omega, E)$ and $\mathcal{F}\nu(\Omega, E)$ coincide as linear spaces for locally complete E as we will encounter in the forthcoming examples, e.g. if $\mathcal{F}\nu(\Omega, E) := H^{\infty}(\Omega, E)$ and $F(\Omega, E) := (\mathcal{O}(\Omega, E), \tau_c)$ for an open set $\Omega \subset \mathbb{C}$. That all three spaces in part c) coincide is usually only guaranteed by Corollary 3.2.5 (iii) if E is a semi-Montel space. Therefore the 'mingle-mangle' space $\mathcal{F}_{\varepsilon}\nu(\Omega, E)$ is a good replacement for $S(\mathcal{F}\nu(\Omega)\varepsilon E)$ for our purpose.

5.2.26. Remark. Let $(T^E, T^{\mathbb{K}})$ be a strong, consistent family for (F, E) and a generator for $(\mathcal{F}\nu, E)$. Let $F(\Omega)$ and $F(\Omega, E)$ be ε-into-compatible and the inclusion $\mathcal{F}\nu(\Omega) \hookrightarrow F(\Omega)$ continuous. Consider a set of uniqueness U for $(T^{\mathbb{K}}, \mathcal{F}\nu)$ and a separating subspace $G \subset E'$. For $u \in F(\Omega)\varepsilon E$ such that $u(B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'})$ is bounded in E, i.e. $S(u) \in \mathcal{F}_\varepsilon\nu(\Omega, E)$, we set $f := S(u)$. Then $f \in F(\Omega, E)$ by the ε-into-compatibility and we define $\widetilde{f}: U \to E$, $\widetilde{f}(x) := T^E(f)(x)$. This yields

$$(e' \circ \widetilde{f})(x) = (e' \circ T^E(f))(x) = T^{\mathbb{K}}(e' \circ f)(x) \tag{49}$$

for all $x \in U$ and $f_{e'} := e' \circ f \in F(\Omega)$ for each $e' \in E'$ by the strength of the family. Moreover, $T_x^{\mathbb{K}}(\cdot)\nu(x) \in B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}$ for every $x \in \omega$, which implies that for every $e' \in E'$ there are $\alpha \in \mathfrak{A}$ and $C > 0$ such that

$$|f_{e'}|_{\mathcal{F}\nu(\Omega)} = \sup_{x \in \omega} \left| e'\big(u(T_x^{\mathbb{K}}(\cdot)\nu(x)\big)\right| \le C \sup_{y' \in B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}} p_\alpha(u(y')) < \infty$$

by strength and consistency. Hence $f_{e'} \in \mathcal{F}\nu(\Omega)$ for every $e' \in E'$ and $\widetilde{f} \in \mathcal{F}\nu_G(U, E)$.

Under the assumptions of Remark 5.2.26 the map

$$R_{U,G}: \mathcal{F}_\varepsilon\nu(\Omega, E) \to \mathcal{F}\nu_G(U, E),\ f \mapsto (T^E(f)(x))_{x \in U}, \tag{50}$$

is well-defined and linear. In addition, we derive from (49) that $R_{U,G}$ is injective since U is a set of uniqueness and $G \subset E'$ separating. The replacement of Question 5.2.9 reads as follows.

5.2.27. Question. Let the assumptions of Remark 5.2.26 be fulfilled. When is the injective restriction map

$$R_{U,G}: \mathcal{F}_\varepsilon\nu(\Omega, E) \to \mathcal{F}\nu_G(U, E),\ f \mapsto (T^E(f)(x))_{x \in U},$$

surjective?

Due to Proposition 5.2.25 c) the Question 5.2.1 is a special case of this question if $\Lambda \subset \Omega =: \omega$ and $U := \Lambda$ is a set of uniqueness for $(\mathrm{id}_{\mathbb{K}^\Omega}, \mathcal{F}\nu)$. We recall the following extension result for continuous linear operators.

5.2.28. Proposition ([70, Proposition 2.1, p. 691]). *Let E be a locally complete lcHs, $G \subset E'$ determine boundedness, Z a Banach space whose closed unit ball B_Z is a compact subset of an lcHs Y and $X \subset Y'$ be a $\sigma(Y', Z)$-dense subspace. If $\mathsf{A}: X \to E$ is a $\sigma(X, Z)$-$\sigma(E, G)$-continuous linear map, then there exists a (unique) extension $\widehat{\mathsf{A}} \in Y\varepsilon E$ of A such that $\widehat{\mathsf{A}}(B_Z^{\circ Y'})$ is bounded in E where $B_Z^{\circ Y'} := \{y' \in Y' \mid \forall\, z \in B_Z : |y'(z)| \le 1\}$.*

Now, we are able to generalise [70, Theorem 2.2, p. 691] and [93, Theorem 10, p. 5].

5.2.29. Theorem. *Let E be a locally complete lcHs, $G \subset E'$ determine boundedness and $F(\Omega)$ and $F(\Omega, E)$ be ε-into-compatible. Let $(T^E, T^{\mathbb{K}})$ be a generator for $(\mathcal{F}\nu, E)$ and a strong, consistent family for (F, E), $\mathcal{F}\nu(\Omega)$ a Banach space whose closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is a compact subset of $F(\Omega)$ and U a set of uniqueness for $(T^{\mathbb{K}}, \mathcal{F}\nu)$. Then the restriction map*

$$R_{U,G}: \mathcal{F}_\varepsilon\nu(\Omega, E) \to \mathcal{F}\nu_G(U, E)$$

is surjective.

Proof. Let $f \in \mathcal{F}\nu_G(U, E)$. We set $X := \operatorname{span}\{T_x^{\mathbb{K}} \mid x \in U\}$, $Y := F(\Omega)$ and $Z := \mathcal{F}\nu(\Omega)$. The consistency of $(T^E, T^{\mathbb{K}})$ for (F, E) yields that $X \subset Y'$. From U being a set of uniqueness of Z follows that X is $\sigma(Z', Z)$-dense. Since B_Z is a compact subset of Y, it follows that Z is a linear subspace of Y and the inclusion

$Z \hookrightarrow Y$ is continuous, which yields $y'_{|Z} \in Z'$ for every $y' \in Y'$. Thus X is $\sigma(Y', Z)$-dense. Let $\mathsf{A}: X \to E$ be the linear map determined by $\mathsf{A}(T_x^{\mathbb{K}}) := f(x)$. The map A is well-defined since G is $\sigma(E', E)$-dense. Due to

$$e'(\mathsf{A}(T_x^{\mathbb{K}})) = (e' \circ f)(x) = T_x^{\mathbb{K}}(f_{e'})$$

for every $e' \in G$ and $x \in U$ we have that A is $\sigma(X, Z)$-$\sigma(E, G)$-continuous. We apply Proposition 5.2.28 and gain an extension $\widehat{\mathsf{A}} \in Y \varepsilon E$ of A such that $\widehat{\mathsf{A}}(B_Z^{\circ Y'})$ is bounded in E. We set $\widetilde{F} := S(\widehat{\mathsf{A}}) \in \mathcal{F}_\varepsilon \nu(\Omega, E)$ and get for all $x \in U$ that

$$T^E(\widetilde{F})(x) = T^E S(\widehat{\mathsf{A}})(x) = \widehat{\mathsf{A}}(T_x^{\mathbb{K}}) = f(x)$$

by consistency for (F, E), implying $R_{U,G}(\widetilde{F}) = f$. □

Let $\Omega \subset \mathbb{R}^d$ be open, E an lcHs and $P(\partial)^E: \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\Omega, E)$ a linear partial differential operator which is hypoelliptic if $E = \mathbb{K}$. We consider the weighted space $\mathcal{CV}_{P(\partial)}(\Omega, E)$ of zero solutions from Proposition 4.2.14 where the familiy of weights $\mathcal{V}$ only consists of one continuous weight $\nu: \Omega \to (0, \infty)$, i.e. the space

$$\mathcal{C}\nu_{P(\partial)}(\Omega, E) = \{f \in \mathcal{C}^\infty_{P(\partial)}(\Omega, E) \mid \forall\ \alpha \in \mathfrak{A}: \ |f|_{\nu,\alpha} := \sup_{x \in \Omega} p_\alpha(f(x))\nu(x) < \infty\}.$$

5.2.30. Corollary. *Let E be a locally complete lcHs, $G \subset E'$ determine boundedness, $\Omega \subset \mathbb{R}^d$ open, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator, $\nu: \Omega \to (0, \infty)$ continuous and U a set of uniqueness for $(\mathrm{id}_{\mathbb{K}^\Omega}, \mathcal{C}\nu_{P(\partial)})$. If $f: U \to E$ is a function such that $e' \circ f$ admits an extension $f_{e'} \in \mathcal{C}\nu_{P(\partial)}(\Omega)$ for every $e' \in G$, then there exists a unique extension $F \in \mathcal{C}\nu_{P(\partial)}(\Omega, E)$ of f.*

Proof. We choose $F(\Omega) := (\mathcal{C}^\infty_{P(\partial)}(\Omega), \tau_c)$ and $F(\Omega, E) := (\mathcal{C}^\infty_{P(\partial)}(\Omega, E), \tau_c)$. Then we have $\mathcal{F}\nu(\Omega) = \mathcal{C}\nu_{P(\partial)}(\Omega)$ and $\mathcal{F}\nu(\Omega, E) = \mathcal{C}\nu_{P(\partial)}(\Omega, E)$ with the generator $(T^E, T^{\mathbb{K}}) := (\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ for $(\mathcal{F}\nu, E)$. We note that $F(\Omega)$ and $F(\Omega, E)$ are ε-compatible and $(T^E, T^{\mathbb{K}})$ is a strong, consistent family for (F, E) by Proposition 4.2.17. We observe that $\mathcal{F}\nu(\Omega)$ is a Banach space by Proposition 4.2.14 and for every compact $K \subset \Omega$ we have

$$\sup_{x \in K} |f(x)| \underset{(22)}{\le} \sup_{z \in K} \nu(z)^{-1} |f|_\nu \le \sup_{z \in K} \nu(z)^{-1}, \quad f \in B_{\mathcal{F}\nu(\Omega)},$$

yielding that $B_{\mathcal{F}\nu(\Omega)}$ is bounded in $F(\Omega)$. The space $F(\Omega) = (\mathcal{C}^\infty_{P(\partial)}(\Omega), \tau_c)$ is a Fréchet–Schwartz space, thus a Montel space, and it is easy to check that $B_{\mathcal{F}\nu(\Omega)}$ is τ_c-closed. Hence the bounded and τ_c-closed set $B_{\mathcal{F}\nu(\Omega)}$ is compact in $F(\Omega)$. Finally, we remark that the ε-compatibility of $F(\Omega)$ and $F(\Omega, E)$ in combination with the consistency of $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ for (F, E) gives $\mathcal{F}_\varepsilon \nu(\Omega, E) = \mathcal{F}\nu(\Omega, E)$ as linear spaces by Proposition 5.2.25 c). From Theorem 5.2.29 follows our statement. □

If $\Omega = \mathbb{D} \subset \mathbb{C}$ is the open unit disc, $P(\partial) = \overline{\partial}$ the Cauchy–Riemann operator and $\nu = 1$ on $\mathbb{D}$, then $\mathcal{C}\nu_{P(\partial)}(\Omega, E) = H^\infty(\mathbb{D}, E)$ and a sequence $U := (z_n)_{n \in \mathbb{N}} \subset \mathbb{D}$ of distinct elements is a set of uniqueness for $(\mathrm{id}_{\mathbb{C}^\mathbb{D}}, H^\infty)$ if and only if it satisfies the Blaschke condition $\sum_{n \in \mathbb{N}} (1 - |z_n|) = \infty$ (see e.g. [149, 15.23 Theorem, p. 303]).

For a continuous function $\nu: \mathbb{D} \to (0, \infty)$ and a complex lcHs E we define the *Bloch type spaces*

$$\mathcal{B}\nu(\mathbb{D}, E) := \{f \in \mathcal{O}(\mathbb{D}, E) \mid \forall\ \alpha \in \mathfrak{A}: \ |f|_{\nu,\alpha} < \infty\}$$

with

$$|f|_{\nu,\alpha} := \max\big(p_\alpha(f(0)), \sup_{z \in \mathbb{D}} p_\alpha((\partial^1_{\mathbb{C}})^E f(z))\nu(z)\big).$$

If $E = \mathbb{C}$, we write $f'(z) := (\partial^1_{\mathbb{C}})^{\mathbb{C}} f(z)$ for $z \in \mathbb{D}$ and $f \in \mathcal{O}(\mathbb{D})$.

5.2.31. Proposition. *If $\nu: \mathbb{D} \to (0, \infty)$ is continuous, then $\mathcal{B}\nu(\mathbb{D})$ is a Banach space.*

Proof. Let $f \in \mathcal{B}\nu(\mathbb{D})$. From the estimates

$$|f(z)| \leq |f(0)| + \Big|\int_0^z f'(\zeta)\mathrm{d}\zeta\Big| \leq |f(0)| + \frac{|z|}{\min_{\xi\in[0,z]}\nu(\xi)} \sup_{\zeta\in[0,z]} |f'(\zeta)|\nu(\zeta)$$
$$\leq 2\max\Big(1, \frac{|z|}{\min_{\xi\in[0,z]}\nu(\xi)}\Big)|f|_\nu$$

for every $z \in \mathbb{D}$ and

$$\max_{|z|\leq r} |f(z)| \leq 2\max\Big(1, \frac{r}{\min_{|z|\leq r}\nu(z)}\Big)|f|_\nu \tag{51}$$

for all $0 < r < 1$ and $f \in \mathcal{B}\nu(\mathbb{D})$ it follows that $\mathcal{B}\nu(\mathbb{D})$ is a Banach space by using the completeness of $(\mathcal{O}(\mathbb{D}), \tau_c)$ analogously to the proof of Proposition 4.2.14. □

5.2.32. Proposition. *Let $\Omega \subset \mathbb{C}$ be open and E a locally complete lcHs over $\mathbb{C}$. Then $((\partial^n_{\mathbb{C}})^E, (\partial^n_{\mathbb{C}})^{\mathbb{C}})_{n\in\mathbb{N}_0}$ is a strong, consistent family for $((\mathcal{O}(\Omega), \tau_c), E)$.*

Proof. We recall from (5) that the real and complex derivatives are related by

$$(\partial^\beta)^E f(z) = \mathrm{i}^{\beta_2}(\partial^{|\beta|}_{\mathbb{C}})^E f(z), \quad z \in \Omega, \tag{52}$$

for every $f \in \mathcal{O}(\Omega, E)$ and $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$. Further, the Fréchet space $(\mathcal{O}(\Omega), \tau_c)$ is barrelled. Due to Proposition 3.1.11 c) and (52) we have for all $u \in (\mathcal{O}(\Omega), \tau_c)\varepsilon E$

$$(\partial^n_{\mathbb{C}})^E S(u)(z) = u(\delta_z \circ (\partial^n_{\mathbb{C}})^{\mathbb{C}}), \quad n \in \mathbb{N}_0,\ z \in \Omega,$$

which means that $((\partial^n_{\mathbb{C}})^E, (\partial^n_{\mathbb{C}})^{\mathbb{C}})_{n\in\mathbb{N}_0}$ is consistent.

Moreover, we have

$$(\partial^n_{\mathbb{C}})^{\mathbb{C}}(e' \circ f)(z) = e'\big((\partial^n_{\mathbb{C}})^E f(z)\big), \quad n \in \mathbb{N}_0,\ z \in \Omega,$$

for all $e' \in E'$ and $f \in \mathcal{O}(\Omega, E)$, implying the strength of $((\partial^n_{\mathbb{C}})^E, (\partial^n_{\mathbb{C}})^{\mathbb{C}})_{n\in\mathbb{N}_0}$. □

Let E be an lcHs and $\nu\colon \mathbb{D} \to (0,\infty)$ be continuous. We set $\omega := \{0\} \cup \{(1,z) \mid z \in \mathbb{D}\}$, define the operator $T^E\colon \mathcal{O}(\mathbb{D}, E) \to E^\omega$ by

$$T^E(f)(0) := f(0) \quad \text{and} \quad T^E(f)(1,z) := (\partial^1_{\mathbb{C}})^E f(z),\ z \in \mathbb{D},$$

and the weight $\nu_*\colon \omega \to (0,\infty)$ by

$$\nu_*(0) := 1 \quad \text{and} \quad \nu_*(1,z) := \nu(z),\ z \in \mathbb{D}.$$

Then we have for every $\alpha \in \mathfrak{A}$ that

$$|f|_{\nu,\alpha} = \sup_{x\in\omega} p_\alpha\big(T^E(f)(x)\big)\nu_*(x), \quad f \in \mathcal{B}\nu(\mathbb{D}, E),$$

and with $F(\mathbb{D}, E) := \mathcal{O}(\mathbb{D}, E)$ we observe that $\mathcal{F}\nu_*(\mathbb{D}, E) = \mathcal{B}\nu(\mathbb{D}, E)$ with generator $(T^E, T^{\mathbb{C}})$.

5.2.33. Corollary. *Let E be a locally complete lcHs, $G \subset E'$ determine boundedness, $\nu\colon \mathbb{D} \to (0,\infty)$ continuous and $U_* \subset \mathbb{D}$ have an accumulation point in $\mathbb{D}$. If $f\colon \{0\} \cup (\{1\} \times U_*) \to E$ is a function such that there is $f_{e'} \in \mathcal{B}\nu(\mathbb{D})$ for each $e' \in G$ with $f_{e'}(0) = e'(f(0))$ and $f'_{e'}(z) = e'(f(1,z))$ for all $z \in U_*$, then there exists a unique $F \in \mathcal{B}\nu(\mathbb{D}, E)$ with $F(0) = f(0)$ and $(\partial^1_{\mathbb{C}})^E F(z) = f(1,z)$ for all $z \in U_*$.*

Proof. We take $F(\mathbb{D}) := (\mathcal{O}(\mathbb{D}), \tau_c)$ and $F(\mathbb{D}, E) := (\mathcal{O}(\mathbb{D}, E), \tau_c)$. Then we have $\mathcal{F}\nu_*(\mathbb{D}) = \mathcal{B}\nu(\mathbb{D})$ and $\mathcal{F}\nu_*(\Omega, E) = \mathcal{B}\nu(\mathbb{D}, E)$ with the weight ν_* and generator $(T^E, T^{\mathbb{C}})$ for $(\mathcal{F}\nu_*, E)$ described above. The spaces $F(\mathbb{D})$ and $F(\mathbb{D}, E)$ are ε-compatible by Proposition 4.2.17 in combination with (23), and the generator is a strong, consistent family for (F, E) by Proposition 5.2.32. Due to Proposition 5.2.31 $\mathcal{F}\nu_*(\mathbb{D}) = \mathcal{B}\nu(\mathbb{D})$ is a Banach space and we deduce from (51) that $B_{\mathcal{F}\nu_*(\mathbb{D})}$ is compact in the Montel space $(\mathcal{O}(\mathbb{D}), \tau_c)$. We note that the ε-compatibility of $F(\Omega)$

and $F(\Omega,E)$ in combination with the consistency of $(T^E,T^{\mathbb{C}})$ for (F,E) gives $\mathcal{F}_\varepsilon\nu_*(\mathbb{D},E)=\mathcal{F}\nu_*(\mathbb{D},E)$ as linear spaces by Proposition 5.2.25 c). In addition, $U:=\{0\}\cup\{(1,z)\mid z\in U_*\}$ is a set of uniqueness for $(T^{\mathbb{C}},\mathcal{F}\nu_*)$ by the identity theorem, proving our statement by Theorem 5.2.29. □

E **a Fréchet space.** In this section we restrict to the case that E is a Fréchet space and $G\subset E'$ is generated by a sequence that *fixes the topology* in E.

5.2.34. DEFINITION ([30, Definition 12, p. 8]). Let Y be a Fréchet space. An increasing sequence $(B_n)_{n\in\mathbb{N}}$ of bounded subsets of Y_b' *fixes the topology* in Y if $(B_n^\circ)_{n\in\mathbb{N}}$ is a fundamental system of zero neighbourhoods of Y.

5.2.35. REMARK. Let Y be a Banach space. If $B\subset Y_b'$ is bounded, i.e. bounded w.r.t. the operator norm, such that B fixes the topology in Y, i.e. B° is bounded in Y, then B is called an *almost norming* subset. Examples of almost norming subspaces are given in [7, Remark 1.2, p. 780–781]. For instance, the set of point evaluations $B:=\{\delta_{1/n}\mid n\in\mathbb{N}\}$ is almost norming for the $Y:=H^\infty(\mathbb{D}):=\mathcal{C}^\infty_{\overline{\partial},b}(\mathbb{D})$.

5.2.36. DEFINITION (sb-restriction space). Let E be a Fréchet space, (B_n) fix the topology in E and $G:=\operatorname{span}(\bigcup_{n\in\mathbb{N}}B_n)$. Let $\mathcal{FV}(\Omega)$ be a dom-space, U a set of uniqueness for $(T^{\mathbb{K}}_m,\mathcal{FV})_{m\in M}$ and set

$$\mathcal{FV}_G(U,E)_{sb}:=\{f\in\mathcal{FV}_G(U,E)\mid \forall\, n\in\mathbb{N}:\ \{f_{e'}\mid e'\in B_n\}\text{ is bounded in }\mathcal{FV}(\Omega)\}.$$

Let E be a Fréchet space, (B_n) fix the topology in E, $G:=\operatorname{span}(\bigcup_{n\in\mathbb{N}}B_n)$, $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ be a strong, consistent generator for $(\mathcal{FV},E)$ and U a set of uniqueness for $(T^{\mathbb{K}}_m,\mathcal{FV})_{m\in M}$. For $u\in\mathcal{FV}(\Omega)\varepsilon E$ we have $R_{U,G}(f)\in\mathcal{FV}_G(U,E)$ with $f:=S(u)$ by Remark 5.2.6 and for $j\in J$ and $m\in M$

$$\sup_{e'\in B_n}|f_{e'}|_{j,m}=\sup_{e'\in B_n}\sup_{x\in\omega_m}|e'(T^E_m(f)(x)\nu_{j,m}(x))|=\sup_{e'\in B_n}\sup_{y\in N_{j,m}(f)}|e'(y)|$$

with $N_{j,m}(f):=\{T^E_m(f)(x)\nu_{j,m}(x)\mid x\in\omega_m\}$. This set is bounded in E since

$$\sup_{y\in N_{j,m}(f)}p_\alpha(f)=|f|_{j,m,\alpha}<\infty$$

for all $\alpha\in\mathfrak{A}$, implying $\sup_{e'\in B_n}|f_{e'}|_{j,m}<\infty$ and $R_{U,G}(f)\in\mathcal{FV}_G(U,E)_{sb}$. Hence the injective linear map

$$R_{U,G}\colon S(\mathcal{FV}(\Omega)\varepsilon E)\to\mathcal{FV}_G(U,E)_{sb},\ f\mapsto(T^E_m(f)(x))_{(m,x)\in U},$$

is well-defined.

5.2.37. QUESTION. Let E be a Fréchet space, (B_n) fix the topology in E and $G:=\operatorname{span}(\bigcup_{n\in\mathbb{N}}B_n)$. Let $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ be a strong, consistent generator for $(\mathcal{FV},E)$ and U a set of uniqueness for $(T^{\mathbb{K}}_m,\mathcal{FV})_{m\in M}$. When is the injective restriction map

$$R_{U,G}\colon S(\mathcal{FV}(\Omega)\varepsilon E)\to\mathcal{FV}_G(U,E)_{sb},\ f\mapsto(T^E_m(f)(x))_{(m,x)\in U},$$

surjective?

5.2.38. REMARK. Let E be a Fréchet space with increasing system of seminorms $(p_{\alpha_n})_{n\in\mathbb{N}}$, $B_n:=B_{\alpha_n}^\circ$ where $B_{\alpha_n}:=\{x\in E\mid p_{\alpha_n}(x)<1\}$, $(T^E_m,T^{\mathbb{K}}_m)_{m\in M}$ a strong, consistent generator for $(\mathcal{FV},E)$ and U a set of uniqueness for $(T^{\mathbb{K}}_m,\mathcal{FV})_{m\in M}$. If $\mathcal{FV}(\Omega)$ is a BC-space, then $\mathcal{FV}_{E'}(U,E)_{sb}=\mathcal{FV}_{E'}(U,E)$ by Proposition 5.2.14. Hence Theorem 5.2.15 (i) answers Question 5.2.37 in this case.

Let us turn to the case where G need not coincide with E'.

$\mathcal{FV}(\Omega)$ a Fréchet–Schwartz space and E a Fréchet space. We recall the following result.

5.2.39. PROPOSITION ([69, Lemma 9, p. 504]). *Let E be a Fréchet space, (B_n) fix the topology in E, Y a Fréchet–Schwartz space and $X \subset Y_b'(=Y_\kappa')$ a dense subspace. Set $G := \operatorname{span}(\bigcup_{n\in\mathbb{N}} B_n)$ and let $\mathsf{A}\colon X \to E$ be a linear map which is $\sigma(X,Y)$-$\sigma(E,G)$-continuous and satisfies that $\mathsf{A}^t(B_n)$ is bounded in Y for each $n \in \mathbb{N}$. Then A has a (unique) extension $\widehat{\mathsf{A}} \in Y\varepsilon E$.*

Next, we improve [69, Theorem 1 ii), p. 501].

5.2.40. THEOREM. *Let E be a Fréchet space, (B_n) fix the topology in E and $G := \operatorname{span}(\bigcup_{n\in\mathbb{N}} B_n)$, $(T_m^E, T_m^{\mathbb{K}})_{m\in M}$ a strong, consistent generator for $(\mathcal{FV}, E)$, $\mathcal{FV}(\Omega)$ a Fréchet–Schwartz space and U a set of uniqueness for $(T_m^{\mathbb{K}}, \mathcal{FV})_{m\in M}$. Then the restriction map $R_{U,G}\colon S(\mathcal{FV}(\Omega)\varepsilon E) \to \mathcal{FV}_G(U,E)_{sb}$ is surjective.*

PROOF. Let $f \in \mathcal{FV}_G(U,E)_{sb}$. We set $X := \operatorname{span}\{T_{m,x}^{\mathbb{K}} \mid (m,x) \in U\}$ and $Y := \mathcal{FV}(\Omega)$. Let $\mathsf{A}\colon X \to E$ be the linear map determined by $\mathsf{A}(T_{m,x}^{\mathbb{K}}) := f(m,x)$ which is well-defined since G is $\sigma(E',E)$-dense. From

$$e'(\mathsf{A}(T_{m,x}^{\mathbb{K}})) = (e' \circ f)(m,x) = T_{m,x}^{\mathbb{K}}(f_{e'})$$

for every $e' \in G$ and $(m,x) \in U$ it follows that A is $\sigma(X,Y)$-$\sigma(E,G)$-continuous and

$$\sup_{e'\in B_n} |\mathsf{A}^t(e')|_{j,k} = \sup_{e'\in B_n} |f_{e'}|_{j,k} < \infty$$

for all $j \in J$, $k \in M$ and $n \in \mathbb{N}$. Due to Proposition 5.2.39 there is an extension $\widehat{\mathsf{A}} \in \mathcal{FV}(\Omega)\varepsilon E$ of A. We set $F := S(\widehat{\mathsf{A}})$ and get for all $(m,x) \in U$ that

$$T_m^E(F)(x) = T_m^E S(\widehat{\mathsf{A}})(x) = \widehat{\mathsf{A}}(T_{m,x}^{\mathbb{K}}) = f(m,x)$$

by consistency, which means $R_{U,G}(F) = f$. □

5.2.41. COROLLARY. *Let E be a Fréchet space, (B_n) fix the topology in E and $G := \operatorname{span}(\bigcup_{n\in\mathbb{N}} B_n)$. Let $\mathcal{V} := (\nu_j)_{j\in\mathbb{N}}$ be an increasing family of weights which is locally bounded away from zero on an open set $\Omega \subset \mathbb{R}^d$, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator, $\mathcal{CV}_{P(\partial)}(\Omega)$ a Schwartz space and $U \subset \Omega$ a set of uniqueness for $(\mathrm{id}_{\mathbb{K}^\Omega}, \mathcal{CV}_{P(\partial)})$. If $f\colon U \to E$ is a function such that $e' \circ f$ admits an extension $f_{e'} \in \mathcal{CV}_{P(\partial)}(\Omega)$ for each $e' \in G$ and $\{f_{e'} \mid e' \in B_n\}$ is bounded in $\mathcal{CV}_{P(\partial)}(\Omega)$ for each $n \in \mathbb{N}$, then there is a unique extension $F \in \mathcal{CV}_{P(\partial)}(\Omega, E)$ of f.*

PROOF. $\mathcal{CV}_{P(\partial)}(\Omega)$ is a Fréchet–Schwartz space and $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ a strong, consistent generator for $(\mathcal{CV}_{P(\partial)}, E)$ by Proposition 4.2.14 and the proof of Example 4.2.16 b). Now, Theorem 5.2.40 and Proposition 5.2.8 prove our statement. □

We already mentioned examples of families of weights $\mathcal{V}$ such that $\mathcal{CV}_{P(\partial)}(\mathbb{R}^d)$ is a nuclear Fréchet space and sets of uniqueness for $(\mathrm{id}_{\mathbb{K}^{\mathbb{R}^d}}, \mathcal{CV}_{P(\partial)})$ in Remark 4.2.20 and Remark 5.2.23 and if $P(\partial) = \overline{\partial}$ or $P(\partial) = \Delta$. Further sets of uniqueness are given in Remark 5.2.66. If E is a Banach space, then an almost norming set fixes the topology and examples can be found via Remark 5.2.35.

$\mathcal{F}\nu(\Omega)$ a Banach space and E a Fréchet space. Let E be a Fréchet space, (B_n) fix the topology in E and recall the assumptions of Remark 5.2.26. Let $(T^E, T^{\mathbb{K}})$ be a strong, consistent family for (F,E) and a generator for $(\mathcal{F}\nu, E)$. Let $F(\Omega)$ and $F(\Omega,E)$ be ε-into-compatible and the inclusion $\mathcal{F}\nu(\Omega) \hookrightarrow F(\Omega)$ continuous. Consider a set of uniqueness U for $(T^{\mathbb{K}}, \mathcal{F}\nu)$ and $G := \operatorname{span}(\bigcup_{n\in\mathbb{N}} B_n) \subset$

E'. For $u \in F(\Omega)\varepsilon E$ such that $u(B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'})$ is bounded in E we have $R_{U,G}(f) \in \mathcal{F}\nu_G(U,E)$ with $f := S(u) \in \mathcal{F}_\varepsilon\nu(\Omega,E)$ by (50). We note that

$$\sup_{e' \in B_n} |f_{e'}|_{\mathcal{F}\nu(\Omega)} = \sup_{e' \in B_n} \sup_{x \in \omega} |e'(T^E(f)(x)\nu(x))| = \sup_{e' \in B_n} \sup_{y \in N_\omega(f)} |e'(y)|$$

with the bounded set $N_\omega(f) := \{T^E(f)(x)\nu(x) \mid x \in \omega\} \subset E$, implying $R_{U,G}(f) \in \mathcal{F}\mathcal{V}_G(U,E)_{sb}$. Thus the injective linear map

$$R_{U,G}\colon \mathcal{F}_\varepsilon\nu(\Omega,E) \to \mathcal{F}\nu_G(U,E)_{sb},\ f \mapsto (T^E(f)(x))_{x\in U},$$

is well-defined.

5.2.42. Question. Let the assumptions of Remark 5.2.26 be fulfilled, E be a Fréchet space, (B_n) fix the topology in E and $G := \operatorname{span}(\bigcup_{n\in\mathbb{N}} B_n)$. When is the injective restriction map

$$R_{U,G}\colon \mathcal{F}_\varepsilon\nu(\Omega,E) \to \mathcal{F}\nu_G(U,E)_{sb},\ f \mapsto (T^E(f)(x))_{x\in U},$$

surjective?

Now, we can generalise [70, Corollary 2.4, p. 692] and [93, Theorem 11, p. 5].

5.2.43. Corollary. *Let E be a Fréchet space, (B_n) fix the topology in E, set $G := \operatorname{span}(\bigcup_{n\in\mathbb{N}} B_n)$ and let $F(\Omega)$ and $F(\Omega,E)$ be ε-into-compatible. Let $(T^E, T^{\mathbb{K}})$ be a generator for $(\mathcal{F}\nu, E)$ and a strong, consistent family for (F,E), $\mathcal{F}\nu(\Omega)$ a Banach space whose closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is a compact subset of $F(\Omega)$ and U a set of uniqueness for $(T^{\mathbb{K}}, \mathcal{F}\nu)$. Then the restriction map*

$$R_{U,G}\colon \mathcal{F}_\varepsilon\nu(\Omega,E) \to \mathcal{F}\nu_G(U,E)_{sb}$$

is surjective.

Proof. Let $f \in \mathcal{F}\nu_G(U,E)_{sb}$. Then $\{f_{e'} \mid e' \in B_n\}$ is bounded in $\mathcal{F}\nu(\Omega)$ for each $n \in \mathbb{N}$. We deduce for each $n \in \mathbb{N}$, $(a_k)_{k\in\mathbb{N}} \in \ell^1$ and $(e'_k)_{k\in\mathbb{N}} \subset B_n$ that $(\sum_{k\in\mathbb{N}} a_k e'_k) \circ f$ admits the extension $\sum_{k\in\mathbb{N}} a_k f_{e'_k}$ in $\mathcal{F}\nu(\Omega)$. Due to [69, Proposition 7, p. 503] the LB-space $E'((B_n)_{n\in\mathbb{N}}) := \varinjlim_{n\in\mathbb{N}} E'(B_n)$, where

$$E'(B_n) := \{\sum_{k\in\mathbb{N}} a_k e'_k \mid (a_k)_{k\in\mathbb{N}} \in \ell^1,\ (e'_k)_{k\in\mathbb{N}} \subset B_n\}$$

is endowed with its Banach space topology for $n \in \mathbb{N}$, determines boundedness in E. Hence we conclude that $f \in \mathcal{F}\nu_{E'((B_n)_{n\in\mathbb{N}})}(U,E)$, which yields that there is $u \in F(\Omega)\varepsilon E$ with bounded $u(B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}) \subset E$ such that $R_{U,G}(S(u)) = f$ by Theorem 5.2.29. □

As an application we directly obtain the following two corollaries of Corollary 5.2.43 since its assumptions are fulfilled by the proof of Corollary 5.2.30 and Corollary 5.2.33, respectively.

5.2.44. Corollary. *Let E be a Fréchet space, (B_n) fix the topology in E and $G := \operatorname{span}(\bigcup_{n\in\mathbb{N}} B_n)$, $\Omega \subset \mathbb{R}^d$ open, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator, $\nu\colon \Omega \to (0,\infty)$ continuous and U a set of uniqueness for $(\mathrm{id}_{\mathbb{K}^\Omega}, \mathcal{C}\nu_{P(\partial)})$. If $f\colon U \to E$ is a function such that $e' \circ f$ admits an extension $f_{e'} \in \mathcal{C}\nu_{P(\partial)}(\Omega)$ for each $e' \in G$ and $\{f_{e'} \mid e' \in B_n\}$ is bounded in $\mathcal{C}\nu_{P(\partial)}(\Omega)$ for each $n \in \mathbb{N}$, then there exists a unique extension $F \in \mathcal{C}\nu_{P(\partial)}(\Omega,E)$ of f.*

5.2.45. Corollary. *Let E be a Fréchet space, (B_n) fix the topology in E and $G := \operatorname{span}(\bigcup_{n\in\mathbb{N}} B_n)$, $\nu\colon \mathbb{D} \to (0,\infty)$ continuous and $U_* \subset \mathbb{D}$ have an accumulation point in $\mathbb{D}$. If $f\colon \{0\} \cup (\{1\} \times U_*) \to E$ is a function such that there is $f_{e'} \in \mathcal{B}\nu(\mathbb{D})$ for each $e' \in G$ with $f_{e'}(0) = e'(f(0))$ and $f'_{e'}(z) = e'(f(1,z))$ for all $z \in U_*$ and*

$\{f_{e'} \mid e' \in B_n\}$ is bounded in $\mathcal{B}\nu(\mathbb{D})$ for each $n \in \mathbb{N}$, then there exists a unique $F \in \mathcal{B}\nu(\mathbb{D}, E)$ with $F(0) = f(0)$ and $(\partial^1_{\mathbb{C}})^E F(z) = f(1, z)$ for all $z \in U_*$.

5.2.2. Extension from thick sets. In order to obtain an affirmative answer to Question 5.2.9 for general separating subspaces of E' we have to restrict to the spaces $\mathcal{FV}(\Omega)$ from Definition 3.1.4 and a certain class of sets of uniqueness.

5.2.46. DEFINITION (fix the topology). Let $\mathcal{FV}(\Omega)$ be a dom-space. We say that $U \subset \bigcup_{m \in M}(\{m\} \times \omega_m)$ *fixes the topology* in $\mathcal{FV}(\Omega)$ if for every $j \in J$ and $m \in M$ there are $i \in J$, $k \in M$ and $C > 0$ such that

$$|f|_{j,m} \leq C \sup_{\substack{x \in \omega_k \\ (k,x) \in U}} |T^{\mathbb{K}}_k(f)(x)| \nu_{i,k}(x), \quad f \in \mathcal{FV}(\Omega).$$

In particular, U is a set of uniqueness if it fixes the topology. The present definition of fixing the topology is a generalisation of [30, Definition 13, p. 234]. Sets that fix the topology appear under several different notions. Rubel and Shields call them dominating in [148, 4.10 Definition, p. 254] in the context of bounded holomorphic functions. In the context of the space of holomorphic functions with the topology of compact convergence studied by Grosse-Erdmann [81, p. 401] they are said to determine locally uniform convergence. Ehrenpreis [61, p. 3,4,13] (cf. [156, Definition 3.2, p. 166]) refers to them as sufficient sets when he considers inductive limits of weighted spaces of entire resp. holomorphic functions, including the case of Banach spaces. In the case of Banach spaces sufficient sets coincide with weakly sufficient sets defined by Schneider [156, Definition 2.1, p. 163] (see e.g. [102, §7, 1), p. 547]) and these notions are extended beyond spaces of holomorphic functions by Korobeĭnik [102, p. 531]. Seip [162, p. 93] uses the term sampling sets in the context of weighted Banach spaces of holomorphic functions whereas Beurling uses the term balayage in [14, p. 341] and [14, Definition, p. 343]. Leibowitz [122, Exercise 4.1.4, p. 53], Stout [170, 7.1 Definition, p. 36] and Globevnik [76, p. 291–292] call them boundaries in the context of subalgebras of the algebra $\mathcal{C}(\Omega, \mathbb{C})$ of complex-valued continuous functions on a compact Hausdorff space Ω with sup-norm. Fixing the topology is also connected to the notion of frames used by Bonet et al. in [31]. Let us set

$$\ell\mathcal{V}(U, E) := \{f : U \to E \mid \forall\, j \in J, m \in M, \alpha \in \mathfrak{A} : \ \|f\|_{j,m,\alpha} < \infty\} \tag{53}$$

with

$$\|f\|_{j,m,\alpha} := \sup_{\substack{x \in \omega_m \\ (m,x) \in U}} p_\alpha(f(m,x)) \nu_{j,m}(x)$$

for an lcHs E and a set U which fixes the topology in $\mathcal{FV}(\Omega)$. If M is a singleton, $\omega_m = \Omega = U$, then $\ell\mathcal{V}(U, E)$ coincides with the space defined right above Example 4.2.2. If U is countable, then the inclusion $\ell\mathcal{V}(U) \hookrightarrow \mathbb{K}^U$ continuous where $\mathbb{K}^U$ is equipped with the topology of pointwise convergence and $\ell\mathcal{V}(U)$ contains the space of sequences (on U) with compact support as a linear subspace, then $(T^{\mathbb{K}}_{k,x})_{(k,x) \in U}$ is an $\ell\mathcal{V}(U)$-frame in the sense of [31, Definition 2.1, p. 3].

5.2.47. DEFINITION (*lb*-restriction space). Let $\mathcal{FV}(\Omega)$ be a dom-space, U fix the topology in $\mathcal{FV}(\Omega)$ and $G \subset E'$ a separating subspace. We set

$$N_{U,i,k}(f) := \{f(k,x)\nu_{i,k}(x) \mid x \in \omega_k,\ (k,x) \in U\}$$

for $i \in J$, $k \in M$ and $f \in \mathcal{FV}_G(U, E)$ and

$$\begin{aligned}\mathcal{FV}_G(U, E)_{lb} :=& \{f \in \mathcal{FV}_G(U, E) \mid \forall\, i \in J,\ k \in M :\ N_{U,i,k}(f) \text{ bounded in } E\} \\ =& \mathcal{FV}_G(U, E) \cap \ell\mathcal{V}(U, E).\end{aligned}$$

Consider a set U which fixes the topology in $\mathcal{FV}(\Omega)$, a separating subspace $G \subset E'$ and a strong, consistent family $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ for $(\mathcal{FV}, E)$. For $u \in \mathcal{FV}(\Omega)\varepsilon E$ set $f := S(u) \in \mathcal{FV}(\Omega, E)$ by Theorem 3.1.12. Then we have $R_{U,G}(f) \in \mathcal{FV}_G(U, E)$ with $f := S(u)$ by Remark 5.2.6 and for $i \in J$ and $k \in M$

$$\sup_{y\in N_{U,i,k}(R_{U,G}(f))} p_\alpha(y) = \sup_{\substack{x\in\omega_k\\(k,x)\in U}} p_\alpha(T^E_k(f)(x))\nu_{i,k}(x) \leq |f|_{i,k,\alpha} < \infty$$

for all $\alpha \in \mathfrak{A}$, implying the boundedness of $N_{U,i,k}(R_{U,G}(f))$ in E. Thus $R_{U,G}(f) \in \mathcal{FV}_G(U, E)_{lb}$ and the injective linear map

$$R_{U,G}\colon S(\mathcal{FV}(\Omega)\varepsilon E) \to \mathcal{FV}_G(U, E)_{lb},\ f \mapsto (T^E_m(f)(x))_{(m,x)\in U},$$

is well-defined.

5.2.48. QUESTION. Let $G \subset E'$ be a separating subspace, $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ a strong, consistent generator for $(\mathcal{FV}, E)$ and U fix the topology in $\mathcal{FV}(\Omega)$. When is the injective restriction map

$$R_{U,G}\colon S(\mathcal{FV}(\Omega)\varepsilon E) \to \mathcal{FV}_G(U, E)_{lb},\ f \mapsto (T^E_m(f)(x))_{(m,x)\in U},$$

surjective?

If $G \subset E'$ determines boundedness and U fixes the topology in $\mathcal{FV}(\Omega)$, then the preceding question and Question 5.2.9 coincide.

5.2.49. REMARK. Let $G \subset E'$ determine boundedness, $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ a strong, consistent generator for $(\mathcal{FV}, E)$ and U fix the topology in $\mathcal{FV}(\Omega)$. Then

$$\mathcal{FV}_G(U, E)_{lb} = \mathcal{FV}_G(U, E).$$

PROOF. We only need to show that the inclusion '$\supset$' holds. Let $f \in \mathcal{FV}_G(U, E)$. Then there is $f_{e'} \in \mathcal{FV}(\Omega)$ with $T^{\mathbb{K}}_m(f_{e'})(x) = (e' \circ f)(m, x)$ for all $(m, x) \in U$ and

$$\sup_{y\in N_{U,i,k}(f)} |e'(y)| = \sup_{\substack{x\in\omega_k\\(k,x)\in U}} |(e' \circ f)(k, x)|\nu_{i,k}(x) \leq |f_{e'}|_{i,k} < \infty$$

for each $e' \in G$, $i \in J$ and $k \in M$. Since $G \subset E'$ determines boundedness, this means that $N_{U,i,k}(f)$ is bounded in E and hence $f \in \mathcal{FV}_G(U, E)_{lb}$. □

$\mathcal{FV}(\Omega)$ arbitrary and E a semi-Montel space.

5.2.50. DEFINITION (generalised Schwartz space). We call an lcHs E a *generalised Schwartz space* if every bounded set in E is already precompact.

In particular, semi-Montel spaces and Schwartz spaces are generalised Schwartz spaces by [89, 10.4.3 Corollary, p. 202]. Conversely, a generalised Schwartz space is a Schwartz space if it is quasi-normable by [89, 10.7.3 Corollary, p. 215]. Moreover, looking at the proof of Lemma 3.2.2 b), we see that this lemma not only holds for semi-Montel or Schwartz spaces but for all generalised Schwartz spaces.

5.2.51. PROPOSITION. *Let E be an lcHs, $\mathcal{FV}(\Omega)$ a* dom-*space and U fix the topology in $\mathcal{FV}(\Omega)$. Then $\mathscr{R}_f \in L(E'_b, \mathcal{FV}(\Omega))$ and $\mathscr{R}_f(B^\circ_\alpha)$ is bounded in $\mathcal{FV}(\Omega)$ for every $f \in \mathcal{FV}_{E'}(U, E)_{lb}$ and $\alpha \in \mathfrak{A}$ where $B_\alpha := \{x \in E \mid p_\alpha(x) < 1\}$ and $\mathscr{R}_f$ is the map from Remark 5.2.5. In addition, if E is a generalised Schwartz space, then $\mathscr{R}_f \in L(E'_\gamma, \mathcal{FV}(\Omega))$ and $\mathscr{R}_f(B^\circ_\alpha)$ is relatively compact in $\mathcal{FV}(\Omega)$.*

PROOF. Let $f \in \mathcal{FV}_{E'}(U, E)_{lb}$, $j \in J$ and $m \in M$. Then there are $i \in J$, $k \in M$ and $C > 0$ such that for every $e' \in E'$

$$|\mathscr{R}_f(e')|_{j,m} = |f_{e'}|_{j,m} \leq C \sup_{\substack{x\in\omega_k\\(k,x)\in U}} |T^{\mathbb{K}}_k(f_{e'})(x)|\nu_{i,k}(x)$$

$$= C \sup_{\substack{x\in\omega_k \\ (k,x)\in U}} |(e'\circ f)(k,x)|\nu_{i,k}(x) = C \sup_{y\in N_{U,i,k}(f)} |e'(y)|,$$

which proves the first part because $N_{U,i,k}(f)$ is bounded in E. Let us consider the second part. The bounded set $N_{U,i,k}(f)$ is already precompact in E because E is a generalised Schwartz space. Therefore we have $\mathscr{R}_f \in L(E'_\gamma, \mathcal{FV}(\Omega))$. The polar B_α° is relatively compact in E'_γ for every $\alpha \in \mathfrak{A}$ by the Alaoğlu–Bourbaki theorem and thus $\mathscr{R}_f(B_\alpha^\circ)$ in $\mathcal{FV}(\Omega)$ as well. □

5.2.52. THEOREM. *Let E be a semi-Montel space, $(T_m^E, T_m^{\mathbb{K}})_{m\in M}$ a strong, consistent generator for $(\mathcal{FV}, E)$ and U fix the topology in $\mathcal{FV}(\Omega)$. Then the restriction map $R_{U,E'}\colon S(\mathcal{FV}(\Omega)\varepsilon E) \to \mathcal{FV}_{E'}(U,E)_{lb}$ is surjective.*

PROOF. Let $f \in \mathcal{FV}_{E'}(\Omega, E)_{lb}$ and $e' \in E'$. For every $f' \in \mathcal{FV}(\Omega)'$ there are $j \in J$, $m \in M$ and $C_0 > 0$ with

$$|\mathscr{R}_f^t(f')(e')| = |f'(f_{e'})| \le C_0 |f_{e'}|_{j,m}.$$

By the proof of Proposition 5.2.51 there are $i \in J$, $k \in M$ and $C > 0$ such that

$$|\mathscr{R}_f^t(f')(e')| \le C_0 C \sup_{y\in N_{U,i,k}(f)} |e'(y)| \le C_0 C \sup_{y\in \overline{\operatorname{acx}}(N_{U,i,k}(f))} |e'(y)|.$$

The set $\overline{\operatorname{acx}}(N_{U,i,k}(f))$ is absolutely convex and compact by [89, 6.2.1 Proposition, p. 103] and [89, 6.7.1 Proposition, p. 112] because E is a semi-Montel space. Therefore $\mathscr{R}_f^t(f') \in (E'_\kappa)' = \mathcal{J}(E)$ by the Mackey–Arens theorem. As in Theorem 5.2.15 we obtain $\mathcal{J}^{-1}\circ\mathscr{R}_f^t \in \mathcal{FV}(\Omega)\varepsilon E$ by (45), (46) and Proposition 5.2.51. Setting $F := S(\mathcal{J}^{-1}\circ\mathscr{R}_f^t)$, we conclude $T_m^E(F)(x) = f(m,x)$ for all $(m,x) \in U$ by (47) and so $R_{U,E'}(F) = f$. □

5.2.53. REMARK. Let E be a Fréchet space with increasing system of seminorms $(p_{\alpha_n})_{n\in\mathbb{N}}$, $B_n := B_{\alpha_n}^\circ$ where $B_{\alpha_n} := \{x \in E \mid p_{\alpha_n}(x) < 1\}$, $(T_m^E, T_m^{\mathbb{K}})_{m\in M}$ a strong, consistent generator for $(\mathcal{FV}, E)$ and U a set of uniqueness for $(T_m^{\mathbb{K}}, \mathcal{FV})_{m\in M}$. If U fixes the topology of $\mathcal{FV}(\Omega)$, then $\mathcal{FV}_{E'}(U,E)_{sb} = \mathcal{FV}_{E'}(U,E)$ by Remark 5.2.49 and Proposition 5.2.51. Hence Theorem 5.2.52 answers Question 5.2.37 if E is a Fréchet–Montel space.

Our first application of Theorem 5.2.52 concerns the space $\mathcal{C}_{bu}(\Omega, E)$ of bounded uniformly continuous functions from a metric space Ω to an lcHs E from Example 4.2.7.

5.2.54. COROLLARY. *Let Ω be a metric space, $U \subset \Omega$ a dense subset and E a semi-Montel space. If $f\colon U \to E$ is a function such that $e'\circ f$ admits an extension $f_{e'} \in \mathcal{C}_{bu}(\Omega)$ for each $e' \in E'$, then there is a unique extension $F \in \mathcal{C}_{bu}(\Omega, E)$ of f. In particular,*

$$\mathcal{C}_{bu}(\Omega, E) = \{f\colon \Omega \to E \mid \forall\ e' \in E' :\ e'\circ f \in \mathcal{C}_{bu}(\Omega)\}.$$

PROOF. $(\mathrm{id}_{E^\Omega}, \mathrm{id}_{\mathbb{K}^\Omega})$ is a strong, consistent generator for $(\mathcal{C}_{bu}, E)$ and we have $\mathcal{C}_{bu}(\Omega)\varepsilon E \cong \mathcal{C}_{bu}(\Omega, E)$ via S by Example 4.2.7. Due to Theorem 5.2.52, Proposition 5.2.8 and Remark 5.2.49 with $G = E'$ the extension F exists and is unique because the dense set $U \subset \Omega$ fixes the topology in $\mathcal{C}_{bu}(\Omega)$. The rest follows from Proposition 5.2.10. □

Next, we consider the space $\mathcal{A}(\overline{\Omega}, E)$ of continuous functions from $\overline{\Omega}$ to an lcHs E over $\mathbb{C}$ which are holomorphic on an open and bounded set $\Omega \subset \mathbb{C}$ from Example 4.2.13.

5.2.55. Corollary. *Let $\Omega \subset \mathbb{C}$ be open and bounded, $U \subset \overline{\Omega}$ fix the topology in $\mathcal{A}(\overline{\Omega})$ and E a semi-Montel space over $\mathbb{C}$. If $f\colon U \to E$ is a function such that $e' \circ f$ admits an extension $f_{e'} \in \mathcal{A}(\overline{\Omega})$ for each $e' \in E'$, then there is a unique extension $F \in \mathcal{A}(\overline{\Omega}, E)$ of f. In particular,*

$$\mathcal{A}(\overline{\Omega}, E) = \{f\colon \overline{\Omega} \to E \mid \forall\, e' \in E' : \, e' \circ f \in \mathcal{A}(\overline{\Omega})\}.$$

Proof. $(\mathrm{id}_{E^{\overline{\Omega}}}, \mathrm{id}_{\mathbb{C}^{\overline{\Omega}}})$ is a strong, consistent generator for $(\mathcal{A}, E)$ and $\mathcal{A}(\overline{\Omega})\varepsilon E \cong \mathcal{A}(\overline{\Omega}, E)$ via S by Example 4.2.13. Due to Theorem 5.2.52, Proposition 5.2.8 and Remark 5.2.49 with $G = E'$ the extension F exists and is unique. The remaining part follows from Proposition 5.2.10. □

If $\Omega \subset \mathbb{C}$ is connected, then the boundary $\partial\Omega$ of Ω fixes the topology in $\mathcal{A}(\overline{\Omega})$ by the maximum principle. If $\Omega = \mathbb{D}$, then $\partial\mathbb{D}$ is the intersection of all sets that fix the topology in $\mathcal{A}(\overline{\mathbb{D}})$ by [170, 7.7 Example, p. 39].

If E is a generalised Schwartz space which is not a semi-Montel space, we do not know whether the extension results in Corollary 5.2.54 and Corollary 5.2.55 hold but we still have a weak-strong principle due to the following observation which is based on [87, Chap. 3, §9, Proposition 2, p. 231] with $\sigma(E, E')$ replaced by $\sigma(E, G)$.

5.2.56. Proposition. *If*

(i) E is a semi-Montel space and $G \subset E'$ a separating subspace, or
(ii) E is a generalised Schwartz space and $G \subset \widehat{E}'$ a separating subspace, i.e. separates the points of the completion $\widehat{E}$,

then the initial topology of E and the topology $\sigma(E, G)$ coincide on the bounded sets of E.

Proof. (i) Let $B \subset E$ be a bounded set. If E is a semi-Montel space, then the closure $\overline{B}$ is compact in E. The topology induced by $\sigma(E, G)$ on $\overline{B}$ is Hausdorff and weaker than the initial topology induced by E. Thus the two topologies coincide on $\overline{B}$ and so on B by the remarks above [87, Chap. 3, §9, Proposition 2, p. 231].

(ii) Let $B \subset E$ be a bounded set. If E is a generalised Schwartz space, then B is precompact in E and relatively compact in the completion $\widehat{E}$ by [89, 3.5.1 Theorem, p. 64]. Hence the closure $\overline{B}$ is compact in $\widehat{E}$. The topology induced by $\sigma(\widehat{E}, G)$ on $\overline{B}$ is Hausdorff and weaker than the initial topology induced by $\widehat{E}$, implying that the two topologies coincide on $\overline{B}$ as in part (i). This yields that $\sigma(E, G)$ and the initial topology of E coincide on B because $\sigma(E, G) = \sigma(\widehat{E}, G)$ on B and the initial topologies of E and $\widehat{E}$ coincide on B as well. □

Concerning (ii), we note that a separating subspace $G \subset E'$ of E need not separate the points of $\widehat{E}$ by [79, 5.4 Example, p. 36] (even though $E' = \widehat{E}'$ by [89, 3.4.2 Theorem, p. 61–62]). Next, we apply Proposition 5.2.56 to the space $\mathcal{A}(\overline{\Omega}, E)$.

5.2.57. Remark. Let E be an lcHs over $\mathbb{C}$ and $\Omega \subset \mathbb{C}$ open and bounded. If

(i) E is a semi-Montel space and $G \subset E'$ determines boundedness, or
(ii) E is a generalised Schwartz space and $G \subset \widehat{E}'$ a separating subspace which determines boundedness in E,

then

$$\mathcal{A}(\overline{\Omega}, E) = \{f\colon \overline{\Omega} \to E \mid \forall\, e' \in G : \, e' \circ f \in \mathcal{A}(\overline{\Omega})\}.$$

Indeed, let us denote the right-hand side by $\mathcal{A}(\overline{\Omega}, E)_\sigma$ and set $E_\sigma := (E, \sigma(E, G))$. Then $\mathcal{A}(\overline{\Omega}, E)_\sigma = \mathcal{A}(\overline{\Omega}, E_\sigma)$ and $f(\overline{\Omega})$ is bounded for every $f \in \mathcal{A}(\overline{\Omega}, E)_\sigma$ as G determines boundedness in E. The initial topology of E and $\sigma(E, G)$ coincide on the bounded range $f(\overline{\Omega})$ of $f \in \mathcal{A}(\overline{\Omega}, E)_\sigma$ by Proposition 5.2.56. Hence we deduce that

$$\mathcal{A}(\overline{\Omega}, E)_\sigma = \mathcal{A}(\overline{\Omega}, E_\sigma) = \mathcal{A}(\overline{\Omega}, E).$$

In this way Bierstedt proves his weak-strong principles for weighted continuous functions in [17, 2.10 Lemma, p. 140] with $G = E' = \widehat{E}'$.

$\mathcal{FV}(\Omega)$ a Fréchet–Schwartz space and E locally complete.

5.2.58. Definition (chain-structured). Let $\mathcal{FV}(\Omega)$ be a dom-space. We say that $U \subset \bigcup_{m\in\mathbb{N}}(\{m\}\times\omega_m)$ is *chain-structured* if

(i) $(k,x)\in U \ \Rightarrow\ \forall\, m\in\mathbb{N},\, m\geq k:\ (m,x)\in U$,
(ii) $\forall\,(k,x)\in U,\, m\in\mathbb{N},\, m\geq k,\, f\in\mathcal{FV}(\Omega):\ T^{\mathbb{K}}_k(f)(x) = T^{\mathbb{K}}_m(f)(x)$.

5.2.59. Remark. Let $\Omega\subset\mathbb{R}^d$ be open and $\mathcal{V}^\infty$ a directed family of weights. Concerning the operators $(T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ of $\mathcal{CV}^\infty(\Omega)$ from Example 3.1.9 a) where $\omega_m := \{\beta\in\mathbb{N}_0^d \mid |\beta|\leq m\}\times\Omega$ resp. $\omega_m := \mathbb{N}_0^d\times\Omega$, we have for all $k\in\mathbb{N}_0$ and $f\in\mathcal{CV}^\infty(\Omega)$ that

$$T^{\mathbb{K}}_k(f)(\beta,x) = \partial^\beta f(x) = T^{\mathbb{K}}_m(f)(\beta,x),\quad \beta\in\mathbb{N}_0^d,\ |\beta|\leq k,\ x\in\Omega,$$

for all $m\in\mathbb{N}_0$, $m\geq k$. Hence condition (ii) of Definition 5.2.58 is fulfilled for any $U\subset\bigcup_{m\in\mathbb{N}_0}(\{m\}\times\omega_m)$ in this case. Condition (i) says that once a 'link' (k,β,x) belongs to U for some order k, then the 'link' (m,β,x) belongs to U for any higher order m as well.

5.2.60. Definition (diagonally dominated, increasing). We say that a family $\mathcal{V} := (\nu_{j,m})_{j,m\in\mathbb{N}}$ of weights on Ω is *diagonally dominated and increasing* if $\omega_m\subset\omega_{m+1}$ for all $m\in\mathbb{N}$ and $\nu_{j,m}\leq\nu_{\max(j,m),\max(j,m)}$ on $\omega_{\min(j,m)}$ for all $j,m\in\mathbb{N}$ as well as $\nu_{j,j}\leq\nu_{j+1,j+1}$ on ω_j for all $j\in\mathbb{N}$.

5.2.61. Remark. Let $\mathcal{FV}(\Omega)$ be a dom-space, $U\subset\bigcup_{m\in\mathbb{N}}(\{m\}\times\omega_m)$ chain-structured, $G\subset E'$ a separating subspace and $\mathcal{V}$ diagonally dominated and increasing.

a) If U fixes the topology in $\mathcal{FV}(\Omega)$, then
$$\mathcal{FV}_G(U,E)_{lb} = \{f\in\mathcal{FV}_G(U,E) \mid \forall\, i\in\mathbb{N}:\ N_{U,i}(f) \text{ bounded in } E\}$$
with $N_{U,i}(f) := N_{U,i,i}(f)$.
b) Let $\mathcal{FV}(\Omega)$ be a Fréchet space. We set $U_m := \{(m,x)\in U \mid x\in\omega_m\}$ and $B_j := \bigcup_{m=1}^{j}\{T^{\mathbb{K}}_{m,x}(\cdot)\nu_{m,m}(x) \mid (m,x)\in U_m\}\subset\mathcal{FV}(\Omega)'$ for $j\in\mathbb{N}$. Then U fixes the topology in $\mathcal{FV}(\Omega)$ in the sense of Definition 5.2.46 if and only if the sequence $(B_j)_{j\in\mathbb{N}}$ fixes the topology in $\mathcal{FV}(\Omega)$ in the sense of Definition 5.2.34.

Proof. Let us begin with a). We only need to show that the inclusion '$\supset$' holds. Let f be an element of the right-hand side and $i,k\in\mathbb{N}$. We set $m:=\max(i,k)$ and observe that for $(k,x)\in U$ we have $(m,x)\in U$ by (i) and

$$(e'\circ f)(k,x) = T^{\mathbb{K}}_k(f_{e'})(x) \underset{\text{(ii)}}{=} T^{\mathbb{K}}_m(f_{e'})(x) = (e'\circ f)(m,x)$$

for each $e'\in G$ with (i) and (ii) from the definition of U being chain-structured. Since G is separating, it follows that $f(k,x) = f(m,x)$. Hence we get for all $\alpha\in\mathfrak{A}$

$$\begin{aligned}\sup_{y\in N_{U,i,k}(f)} p_\alpha(y) &= \sup_{\substack{x\in\omega_k\\(k,x)\in U}} p_\alpha(f(k,x))\nu_{i,k}(x) \underset{\text{(i)}}{\leq} \sup_{\substack{x\in\omega_m\\(m,x)\in U}} p_\alpha(f(k,x))\nu_{m,m}(x)\\ &= \sup_{\substack{x\in\omega_m\\(m,x)\in U}} p_\alpha(f(m,x))\nu_{m,m}(x) < \infty\end{aligned}$$

using that $\omega_k\subset\omega_m$ and $\mathcal{V}$ is diagonally dominated.

Let us turn to part b). '$\Rightarrow$': Let $j\in\mathbb{N}$ and $A\subset\mathcal{FV}(\Omega)$ be bounded. Then

$$\sup_{y\in B_j}\sup_{f\in A}|y(f)| = \sup_{\substack{1\leq m\leq j\\(m,x)\in U_m}}\sup_{f\in A}|T^{\mathbb{K}}_m(f)(x)|\nu_{m,m}(x) \leq \sup_{f\in A}\sup_{1\leq m\leq j}|f|_{m,m} < \infty$$

since A is bounded, implying that B_j is bounded in $\mathcal{FV}(\Omega)_b'$. Further, (B_j) is increasing by definition. Additionally, for all $j\in\mathbb{N}$

$$\begin{aligned}B_j^\circ &= \bigcap_{m=1}^{j}\{f\in\mathcal{FV}(\Omega)\mid \sup_{\substack{x\in\omega_m\\(m,x)\in U}}|T_m^{\mathbb{K}}(f)(x)|\nu_{m,m}(x)\leq 1\}\\ &= \{f\in\mathcal{FV}(\Omega)\mid \sup_{\substack{x\in\omega_j\\(j,x)\in U}}|T_j^{\mathbb{K}}(f)(x)|\nu_{j,j}(x)\leq 1\}\end{aligned}$$

because U is chain-structured and $\mathcal{V}$ increasing. Thus (B_j°) is a fundamental system of zero neighbourhoods of $\mathcal{FV}(\Omega)$ if U fixes the topology.
'$\Leftarrow$': Let $j,m\in\mathbb{N}$. Then there are $i\in\mathbb{N}$ and $\varepsilon>0$ such that

$$\varepsilon B_i^\circ\subset\{f\in\mathcal{FV}(\Omega)\mid |f|_{j,m}\leq 1\}=:D_{j,m}$$

which follows from fixing the topology in the sense of Definition 5.2.34. Let $f\in D_{j,m}$ and set

$$|f|_{U_i}:=\sup_{(i,x)\in U_i}|T_i^{\mathbb{K}}(f)(x)|\nu_{i,i}(x).$$

If $|f|_{U_i}=0$, then $tf\in\varepsilon B_i^\circ$ for all $t>0$ and hence $t|f|_{j,m}=|tf|_{j,m}\leq 1$ for all $t>0$, which yields $|f|_{j,m}=0=|f|_{U_i}$. If $|f|_{U_i}\neq 0$, then $\frac{f}{|f|_{U_i}}\in B_i^\circ$ and thus $\varepsilon\frac{f}{|f|_{U_i}}\in D_{j,m}$, implying

$$|f|_{j,m}=\frac{1}{\varepsilon}|f|_{U_i}\Big|\varepsilon\frac{f}{|f|_{U_i}}\Big|_{j,m}\leq\frac{1}{\varepsilon}|f|_{U_i}.$$

The inequality $|f|_{j,m}\leq\frac{1}{\varepsilon}|f|_{U_i}$ still holds if $|f|_{U_i}=0$. □

5.2.62. THEOREM ([30, Theorem 16, p. 236]). *Let Y be a Fréchet–Schwartz space, $(B_j)_{j\in\mathbb{N}}$ fix the topology in Y and $\mathsf{A}\colon X:=\operatorname{span}(\bigcup_{j\in\mathbb{N}}B_j)\to E$ be a linear map which is bounded on each B_j. If*

a) $(\mathsf{A}^t)^{-1}(Y)$ is dense in E_b' and E locally complete, or
b) $(\mathsf{A}^t)^{-1}(Y)$ is dense in E_σ' and E is B_r-complete,

then A has a (unique) extension $\widehat{\mathsf{A}}\in Y\varepsilon E$.

Now, we generalise [30, Theorem 17, p. 237].

5.2.63. THEOREM. *Let E be an lcHs and $G\subset E'$ a separating subspace. Let $(T_m^E,T_m^{\mathbb{K}})_{m\in M}$ be a strong, consistent generator for $(\mathcal{FV},E)$, $\mathcal{FV}(\Omega)$ a Fréchet–Schwartz space, $\mathcal{V}$ diagonally dominated and increasing and U be chain-structured and fix the topology in $\mathcal{FV}(\Omega)$. If*

a) G is dense in E_b' and E locally complete, or
b) E is B_r-complete,

then the restriction map $R_{U,G}\colon S(\mathcal{FV}(\Omega)\varepsilon E)\to\mathcal{FV}_G(U,E)_{lb}$ is surjective.

PROOF. Let $f\in\mathcal{FV}_G(U,E)_{lb}$. We set $X:=\operatorname{span}(\bigcup_{j\in\mathbb{N}}B_j)$ with B_j from Remark 5.2.61 b) and $Y:=\mathcal{FV}(\Omega)$. Let $\mathsf{A}\colon X\to E$ be the linear map determined by

$$\mathsf{A}(T_{m,x}^{\mathbb{K}}(\cdot)\nu_{m,m}(x)):=f(m,x)\nu_{m,m}(x)$$

for $1\leq m\leq j$ and $(m,x)\in U_m$ with U_m from Remark 5.2.61 b). The map A is well-defined since G is $\sigma(E',E)$-dense, and bounded on each B_j because $\mathsf{A}(B_j)=\bigcup_{m=1}^{j}N_{U,m}(f)$. Let $e'\in G$ and $f_{e'}$ be the unique element in $\mathcal{FV}(\Omega)$ such that $T_m^{\mathbb{K}}(f_{e'})(x)=(e'\circ f)(m,x)$ for all $(m,x)\in U$, which implies $T_m^{\mathbb{K}}(f_{e'})(x)\nu_{m,m}(x)=(e'\circ\mathsf{A})(T_{m,x}^{\mathbb{K}}(\cdot)\nu_{m,m}(x))$ for all $(m,x)\in U_m$. This equation allows us to consider $f_{e'}$ as a linear form on X (by $f_{e'}(T_{m,x}^{\mathbb{K}}(\cdot)\nu_{m,m}(x)):=(e'\circ\mathsf{A})(T_{m,x}^{\mathbb{K}}(\cdot)\nu_{m,m}(x))$), which yields $e'\circ\mathsf{A}\in\mathcal{FV}(\Omega)$ for all $e'\in G$. It follows that $G\subset(\mathsf{A}^t)^{-1}(Y)$. Noting that G is $\sigma(E',E)$-dense, we apply Theorem 5.2.62 and obtain an extension $\widehat{\mathsf{A}}\in\mathcal{FV}(\Omega)\varepsilon E$

of A. We set $F := S(\widehat{\mathsf{A}})$ and observe that for all $(m,x) \in U$ there is $j \in \mathbb{N}$, $j \geq m$, such that $(j,x) \in U_j$ and $\nu_{j,j}(x) > 0$ by (6) and because U is chain-structured and $\mathcal{V}$ diagonally dominated and increasing. Due to the proof of Remark 5.2.61 a) we have $f(j,x) = f(m,x)$ and thus

$$\begin{aligned} T_m^E(F)(x) &= T_m^E S(\widehat{\mathsf{A}})(x) = \widehat{\mathsf{A}}(T_{m,x}^{\mathbb{K}}) = \frac{1}{\nu_{j,j}(x)}\widehat{\mathsf{A}}(T_{m,x}^{\mathbb{K}}(\cdot)\nu_{j,j}(x)) \\ &= \frac{1}{\nu_{j,j}(x)}\widehat{\mathsf{A}}(T_{j,x}^{\mathbb{K}}(\cdot)\nu_{j,j}(x)) = f(j,x) = f(m,x) \end{aligned}$$

by consistency, yielding $R_{U,G}(F) = f$. □

In particular, condition a) is fulfilled if E is semi-reflexive. Indeed, if E is semi-reflexive, then E is quasi-complete by [153, Chap. IV, 5.5, Corollary 1, p. 144] and $\overline{G}^{b(E',E)} = \overline{G}^{\tau(E',E)} = E'$ by [89, 11.4.1 Proposition, p. 227] and the bipolar theorem. For instance, condition b) is satisfied if E is a Fréchet space or $E = (\mathcal{C}^{\infty}_{\overline{\partial},b}(\mathbb{D}),\beta)$ which is a B_r-complete space by Proposition 5.2.16 and is not a Fréchet space by Remark 4.2.23.

As stated, our preceding theorem generalises [30, Theorem 17, p. 237] where $\mathcal{FV}(\Omega)$ is a closed subspace of $\mathcal{CW}^{\infty}(\Omega)$ for open, connected $\Omega \subset \mathbb{R}^d$. A characterisation of sets that fix the topology in the space $\mathcal{CW}^{\infty}_{\overline{\partial}}(\Omega)$ of holomorphic functions on an open, connected set $\Omega \subset \mathbb{C}$ is given in [30, Remark 14, p. 235]. The characterisation given in [30, Remark 14 (b), p. 235] is still valid and applied in [30, Corollary 18, p. 238] for closed subspaces of $\mathcal{CW}^{\infty}_{P(\partial)}(\Omega)$ where $P(\partial)^{\mathbb{K}}$ is a hypoelliptic linear partial differential operator which satisfies the maximum principle, namely, that $U \subset \Omega$ fixes the topology if and only if there is a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of relatively compact, open subsets of Ω with $\bigcup_{n\in\mathbb{N}}\Omega_n = \Omega$ such that $\partial\Omega_n \subset \overline{U \cap \Omega_{n+1}}$ for all $n \in \mathbb{N}$. Among the hypoelliptic operators $P(\partial)^{\mathbb{K}}$ satisfying the maximum principle are the Cauchy–Riemann operator $\overline{\partial}$ and the Laplacian Δ. Further examples can be found in [74, Corollary 3.2, p. 33]. The statement of [30, Corollary 18, p. 238] for the space of holomorphic functions is itself a generalisation of [81, Theorem 2, p. 401] with [81, Remark 2 (a), p. 406] where E is B_r-complete and of [92, Theorem 6, p. 10] where E is semi-reflexive. The case that G is dense in E_b' and E is sequentially complete is covered by [77, 3.3 Satz, p. 228–229], not only for spaces of holomorphic functions, but for several classes of function spaces.

Let us turn to other families of weights than $\mathcal{W}^{\infty}$. Due to Proposition 4.2.19 we already know that $U := \{0\} \times \mathbb{C}$ fixes the topology in $\mathcal{CV}^{\infty}_{\overline{\partial}}(\mathbb{C}) = \mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ and $U := \{0\} \times \mathbb{R}^d$ in $\mathcal{CV}^{\infty}_{\Delta}(\mathbb{R}^d) = \mathcal{CV}_{\Delta}(\mathbb{R}^d)$ if $\mathcal{V} := (\nu_j)_{j\in\mathbb{N}}$ fulfils Condition 4.2.18 and $\mathcal{V}^{\infty} := (\nu_{j,m})_{j\in\mathbb{N},m\in\mathbb{N}_0}$ where $\nu_{j,m}\colon \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq m\} \times \mathbb{R}^d \to [0,\infty)$, $\nu_{j,m}(\beta,x) := \nu_j(x)$.

5.2.64. Corollary. *Let E be an lcHs, $G \subset E'$ a separating subspace, $\mathcal{V} := (\nu_j)_{j\in\mathbb{N}}$ an increasing family of weights which is locally bounded away from zero on an open set $\Omega \subset \mathbb{R}^d$, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator, $\mathcal{CV}_{P(\partial)}(\Omega)$ a Schwartz space and $U \subset \Omega$ fix the topology of $\mathcal{CV}_{P(\partial)}(\Omega)$. If*

a) G is dense in E_b' and E locally complete, or
b) E is B_r-complete,

and $f\colon U \to E$ is a function in $\ell\mathcal{V}(U)$ such that $e' \circ f$ admits an extension $f_{e'} \in \mathcal{CV}_{P(\partial)}(\Omega)$ for each $e' \in G$, then there is a unique extension $F \in \mathcal{CV}_{P(\partial)}(\Omega, E)$ of f.

Proof. The existence of F follows from Proposition 4.2.14, Example 4.2.16 b) and Theorem 5.2.63 with $(T_m^E, T_m^{\mathbb{K}})_{m\in M} := (\mathrm{id}_{E^{\Omega}}, \mathrm{id}_{\mathbb{K}^{\Omega}})$. The uniqueness of F is a result of Proposition 5.2.8. □

We have the following sufficient conditions on a family of weights $\mathcal{V}$ which guarantee the existence of a countable set $U \subset \mathbb{C}$ that fixes the topology of $\mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ due to Abanin and Varziev [2].

5.2.65. PROPOSITION. *Let $\mathcal{V} := (\nu_j)_{j\in\mathbb{N}}$ where $\nu_j(z) := \exp(a_j\mu(z)-\varphi(z))$, $z \in \mathbb{C}$, with some continuous, subharmonic function $\mu\colon \mathbb{C} \to [0,\infty)$, a continuous function $\varphi\colon \mathbb{C} \to \mathbb{R}$ and a strictly increasing, positive sequence $(a_j)_{j\in\mathbb{N}}$ with $a := \lim_{j\to\infty} a_j \in (0,\infty]$. Let there be*

(i) $s \geq 0$ and $C > 0$ such that $|\varphi(z)-\varphi(\zeta)| \leq C$ and $|\mu(z)-\mu(\zeta)| \leq C$ for all $z,\zeta \in \mathbb{C}$ with $|z-\zeta| \leq (1+|z|)^{-s}$,

(ii) $\max(\varphi(z),\mu(z)) \leq |z|^q + C_0$ for some $q, C_0 > 0$ and

(iii) $\ln(|z|) = O(\mu(z))$ as $|z| \to \infty$ if $a = \infty$, or $\ln(|z|) = o(\mu(z))$ as $|z| \to \infty$ if $0 < a < \infty$.

Let $(\lambda_k)_{k\in\mathbb{N}}$ be the sequence of simple zeros of a function $L \in \mathcal{C}\widetilde{\mathcal{V}}_{\overline{\partial}}(\mathbb{C})$ having no other zeros where $\widetilde{\mathcal{V}} := (\nu_j^2/\nu_{m_j})_{j\in\mathbb{N}}$ for some sequence $(m_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$. Suppose that there are $j_0 \in \mathbb{N}$ and a sequence of circles $\{z \in \mathbb{C} \mid |z| = R_m\}$ with $R_m \nearrow \infty$ such that

$$|L(z)|\nu_{j_0}(z) \geq C_m, \quad m \in \mathbb{N},\, z \in \mathbb{C},\, |z| = R_m,$$

for some $C_m \nearrow \infty$ and

$$|L'(\lambda_k)|\nu_{j_0}(\lambda_k) \geq 1 \quad \text{for all sufficiently large } k \in \mathbb{N}.$$

Then $\mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ is a nuclear Fréchet space for all $a \in (0,\infty]$ and $U := (\lambda_k)_{k\in\mathbb{N}}$ fixes the topology of $\mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ if $a = \infty$. If μ is a radial function, i.e. $\mu(z) = \mu(|z|)$, $z \in \mathbb{C}$, with $\mu(2z) \sim \mu(z)$ as $|z| \to \infty$, then U fixes the topology of $\mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ for all $a \in (0,\infty]$.

PROOF. First, we check that Condition 4.2.18 is satisfied, which implies that $\mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ is a nuclear Fréchet space by Proposition 4.2.19. We set $k := \max(s,2)$ and observe that (i) is also fulfilled with k instead of s. Let $z \in \mathbb{C}$ and $\|\zeta\|_\infty, \|\eta\|_\infty \leq (1/\sqrt{2})(1+|z|)^{-k} =: r(z)$. From $|\cdot| \leq \sqrt{2}\|\cdot\|_\infty$ and (i) it follows

$$|\mu(z+\zeta)-\mu(z+\eta)| \leq |\mu(z+\zeta)-\mu(z)| + |\mu(z)-\mu(z+\eta)| \leq 2C$$

and thus $\mu(z+\zeta) \leq 2C+\mu(z+\eta)$. In the same way we obtain $-\varphi(z+\zeta) \leq 2C-\varphi(z+\eta)$. Hence we have

$$a_j\mu(z+\zeta)-\varphi(z+\zeta) \leq 2C(a_j+1) + a_j\mu(z+\eta)-\varphi(z+\eta)$$

for $j \in \mathbb{N}$, implying

$$\nu_j(z+\zeta) \leq \mathrm{e}^{2C(a_j+1)}\nu_j(z+\eta),$$

which means that $(\alpha.1)$ of Condition 4.2.18 holds. By (iii) there are $\varepsilon > 0$ and $R > 0$ such that $\ln(|z|) \leq \varepsilon\mu(z)$ for all $z \in \mathbb{C}$ with $|z| \geq R$ if $a = \infty$. This yields for all $|z| \geq \max(2,R)$ that

$$a_j\mu(z) + k\ln(1+|z|) \leq a_j\mu(z) + k\ln(|z|^2) = a_j\mu(z) + 2k\ln(|z|) \leq a_j\mu(z) + 2k\varepsilon\mu(z).$$

Since $a = \infty$, there is $n \in \mathbb{N}$ such that $a_n \geq a_j + 2k\varepsilon$, resulting in

$$a_j\mu(z) + k\ln(1+|z|) \leq a_n\mu(z)$$

for all $|z| \geq \max(2,R)$. Therefore we derive

$$a_j\mu(z) + k\ln(1+|z|) \leq a_n\mu(z) + k\ln(1+\max(2,R)) \tag{54}$$

for all $z \in \mathbb{C}$, which means that $(\alpha.2)$ and $(\alpha.3)$ hold with $\psi_j(z) := r(z)$. If $0 < a < \infty$, for every $\varepsilon > 0$ there is $R > 0$ such that $\ln(|z|) \leq \varepsilon\mu(z)$ for all $z \in \mathbb{C}$ with $|z| \geq R$ by (iii). Thus we may choose $\varepsilon > 0$ such that $a_{j+1}-a_j \geq 2k\varepsilon > 0$ because (a_j) is strictly increasing. We deduce that (54) with $n := j+1$ holds in this case as well and $(\alpha.2)$ and $(\alpha.3)$, too.

Observing that the condition that $U = (\lambda_k)_{k\in\mathbb{N}}$ is the sequence of simple zeros of a function $L \in \mathcal{C}\widetilde{\mathcal{V}}_{\overline{\partial}}(\mathbb{C})$ means that $L \in \mathscr{L}(\Phi^a_{\varphi,\mu};U)$ and (i) that φ and μ vary slowly w.r.t. $r(z) := (1+|z|)^{-s}$ in the notation of [2, Definition, p. 579, 584] and [2, p. 585], respectively, the statement that U fixes the topology is a consequence of [2, Theorem 2, p. 585–586]. □

5.2.66. REMARK. a) Let $D \subset \mathbb{C}$ be convex, bounded and open with $0 \in D$. Let $\varphi(z) := H_D(z) := \sup_{\zeta \in D} \operatorname{Re}(z\zeta)$, $z \in \mathbb{C}$, be the supporting function of D, $\mu(z) := \ln(1+|z|)$, $z \in \mathbb{C}$, and $a_j := j$, $j \in \mathbb{N}$. Then φ and μ fulfil the conditions of Proposition 5.2.65 with $a = \infty$ by [2, p. 586] and the existence of an entire function L which fulfils the conditions of Proposition 5.2.65 is guaranteed by [3, Theorem 1.6, p. 1537]. Thus there is a countable set $U := (\lambda_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ which fixes the topology in $A_D^{-\infty} := \mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ with $\mathcal{V} := (\exp(a_j\mu - \varphi))_{j \in \mathbb{N}}$.

b) An explicit construction of a set $U := (\lambda_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ which fixes the topology in $A_D^{-\infty}$ is given in [1, Algorithm 3.2, p. 3629]. This construction does not rely on the entire function L. In particular (see [31, p. 15]), for $D := \mathbb{D}$ we have $\varphi(z) = |z|$, for each $k \in \mathbb{N}$ we may take $l_k \in \mathbb{N}$, $l_k > 2\pi k^2$, and set $\lambda_{k,j} := kr_{k,j}$, $1 \leq j \leq l_k$, where $r_{k,j}$ denote the l_k-roots of unity. Ordering $\lambda_{k,j}$ in a sequence of one index appropriately, we obtain a sequence which fixes the topology of $A_{\mathbb{D}}^{-\infty}$.

c) Let $\mu\colon \mathbb{C} \to [0,\infty)$ be a continuous, subharmonic, radial function which increases with $|z|$ and satisfies
   (i) $\sup_{\zeta \in \mathbb{C}, \|\zeta\|_\infty \leq r(z)} \mu(z+\zeta) \leq \inf_{\zeta \in \mathbb{C}, \|\zeta\|_\infty \leq r(z)} \mu(z+\zeta) + C$ for some continuous function $r\colon \mathbb{C} \to (0,1]$ and $C > 0$,
   (ii) $\ln(1+|z|^2) = o(\mu(|z|))$ as $|z| \to \infty$,
   (iii) $\mu(2|z|) = O(\mu(|z|))$ as $|z| \to \infty$.
   Then $\mathcal{V} := (\exp(-(1/j)\mu))_{j \in \mathbb{N}}$ fulfils Condition 4.2.18 where $(\alpha.1)$ follows from (i) and $(\alpha.2)$, $(\alpha.3)$ as in the proof of Proposition 5.2.65. Thus $\mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ is a nuclear Fréchet space by Proposition 4.2.19. If $\mu(|z|) = o(|z|^2)$ as $|z| \to \infty$ or $\mu(|z|) = |z|^2$, $z \in \mathbb{C}$, then $U := \{\alpha n + \mathrm{i}\beta m \mid n, m \in \mathbb{Z}\}$ fixes the topology in the space $A_\mu^0 := \mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ for any $\alpha, \beta > 0$ by [31, Corollary 4.6, p. 20] and [31, Proposition 4.7, p. 20], respectively.

d) For instance, the conditions on μ in c) are fulfilled for $\mu(z) := |z|^\gamma$, $z \in \mathbb{C}$, with $0 < \gamma \leq 2$ by [130, 1.5 Examples (3), p. 205]. If $\gamma = 1$, then $A_\mu^0 = A_{\overline{\partial}}^0(\mathbb{C})$ is the space of entire functions of exponential type zero (see Remark 4.2.20).

e) More general characterisations of countable sets that fix the topology of $\mathcal{CV}_{\overline{\partial}}(\mathbb{C})$ can be found in [2, Theorem 1, p. 580] and [31, Theorem 4.5, p. 17].

The spaces A_μ^0 from c) are known as *Hörmander algebras* and the space $A_D^{-\infty}$ considered in a) is isomorphic to the strong dual of the *Korenblum space* $A^{-\infty}(D)$ via Laplace transform by [132, Proposition 4, p. 580].

$\mathcal{F}\nu(\Omega)$ a Banach space and E locally complete. For a dom-space $\mathcal{F}\nu(\Omega)$, a set U that fixes the topology in $\mathcal{F}\nu(\Omega)$ and a separating subspace $G \subset E'$ we have

$$\begin{aligned}\mathcal{F}\nu_G(U,E)_{lb} &= \{f \in \mathcal{F}\nu_G(U,E) \mid N_U(f) \text{ bounded in } E\}\\ &= \mathcal{F}\nu_G(U,E) \cap \ell\nu(U,E)\end{aligned}$$

where $N_U(f) := \{f(x)\nu(x) \mid x \in U\}$. Let us recall the assumptions of Remark 5.2.26 but now U fixes the topology. Let $(T^E, T^{\mathbb{K}})$ be a strong, consistent family for (F,E) and a generator for $(\mathcal{F}\nu, E)$. Let $F(\Omega)$ and $F(\Omega,E)$ be ε-into-compatible and the inclusion $\mathcal{F}\nu(\Omega) \hookrightarrow F(\Omega)$ continuous. Consider a set U which fixes the topology in $\mathcal{F}\nu(\Omega)$ and a separating subspace $G \subset E'$. For $u \in F(\Omega)\varepsilon E$ such that $u(B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'})$ is bounded in E we have $R_{U,G}(f) \in \mathcal{F}\nu_G(U,E)$ with $f := S(u) \in \mathcal{F}_\varepsilon\nu(\Omega,E)$ by (50). Further, $T_x^{\mathbb{K}}(\cdot)\nu(x) \in B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}$ for every $x \in \omega$, which implies that

$$\sup_{x\in U} p_\alpha(R_{U,G}(f)(x))\nu(x) = \sup_{x\in U} p_\alpha\big(u(T_x^{\mathbb{K}}(\cdot)\nu(x))\big) \leq \sup_{y'\in B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}} p_\alpha(u(y')) < \infty$$

for all $\alpha \in \mathfrak{A}$ by consistency. Hence $R_{U,G}(f) \in \mathcal{F}\nu_G(U,E)_{lb}$. Therefore the injective linear map

$$R_{U,G}\colon \mathcal{F}_\varepsilon\nu(\Omega,E) \to \mathcal{F}\nu_G(U,E)_{lb},\ f \mapsto (T^E(f)(x))_{x\in U},$$

is well-defined and the question we want to answer is:

5.2.67. Question. Let the assumptions of Remark 5.2.26 be fulfilled and U fix the topology in $\mathcal{F}\nu(\Omega)$. When is the injective restriction map

$$R_{U,G}\colon \mathcal{F}_\varepsilon\nu(\Omega,E) \to \mathcal{F}\nu_G(U,E)_{lb},\ f \mapsto (T^E(f)(x))_{x\in U},$$

surjective?

5.2.68. Proposition ([70, Proposition 3.1, p. 692]). *Let E be a locally complete lcHs, $G \subset E'$ a separating subspace and Z a Banach space whose closed unit ball B_Z is a compact subset of an lcHs Y. Let $B_1 \subset B_Z^{\circ Y'}$ such that $B_1^{\circ Z} := \{z \in Z \mid \forall\, y' \in B_1 : |y'(z)| \leq 1\}$ is bounded in Z. If $\mathsf{A}\colon X := \operatorname{span} B_1 \to E$ is a linear map which is bounded on B_1 such that there is a $\sigma(E',E)$-dense subspace $G \subset E'$ with $e' \circ \mathsf{A} \in Z$ for all $e' \in G$, then there exists a (unique) extension $\widehat{\mathsf{A}} \in Y\varepsilon E$ of A such that $\widehat{\mathsf{A}}(B_Z^{\circ Y'})$ is bounded in E.*

The following theorem is a generalisation of [70, Theorem 3.2, p. 693] and [93, Theorem 12, p. 5].

5.2.69. Theorem. *Let E be a locally complete lcHs, $G \subset E'$ a separating subspace and $F(\Omega)$ and $F(\Omega,E)$ be ε-into-compatible. Let $(T^E,T^{\mathbb{K}})$ be a generator for $(\mathcal{F}\nu,E)$ and a strong, consistent family for (F,E), $\mathcal{F}\nu(\Omega)$ a Banach space whose closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is a compact subset of $F(\Omega)$ and U fix the topology in $\mathcal{F}\nu(\Omega)$. Then the restriction map*

$$R_{U,G}\colon \mathcal{F}_\varepsilon\nu(\Omega,E) \to \mathcal{F}\nu_G(U,E)_{lb}$$

is surjective.

Proof. Let $f \in \mathcal{F}\nu_G(U,E)_{lb}$. We set $B_1 := \{T_x^{\mathbb{K}}(\cdot)\nu(x) \mid x \in U\}$, $X := \operatorname{span} B_1$, $Y := F(\Omega)$ and $Z := \mathcal{F}\nu(\Omega)$. We have $B_1 \subset Y'$ since $(T^E,T^{\mathbb{K}})$ is a consistent family for (F,E). If $f \in B_Z$, then

$$|T_x^{\mathbb{K}}(f)\nu(x)| \leq |f|_{\mathcal{F}\nu(\Omega)} \leq 1$$

for all $x \in U$ and thus $B_1 \subset B_Z^{\circ Y'}$. Further on, there is $C > 0$ such that for all $f \in B_1^{\circ Z}$

$$|f|_{\mathcal{F}\nu(\Omega)} \leq C \sup_{x\in U} |T_x^{\mathbb{K}}(f)|\nu(x) \leq C$$

as U fixes the topology in Z, implying the boundedness of $B_1^{\circ Z}$ in Z. Let $\mathsf{A}\colon X \to E$ be the linear map determined by

$$\mathsf{A}(T_x^{\mathbb{K}}(\cdot)\nu(x)) := f(x)\nu(x).$$

The map A is well-defined since G is $\sigma(E',E)$-dense, and bounded on B_1 because $\mathsf{A}(B_1) = N_U(f)$. Let $e' \in G$ and $f_{e'}$ be the unique element in $\mathcal{F}\nu(\Omega)$ such that $T^{\mathbb{K}}(f_{e'})(x) = (e' \circ f)(x)$ for all $x \in U$, which implies $T^{\mathbb{K}}(f_{e'})(x)\nu(x) = (e'\circ\mathsf{A})(T_x^{\mathbb{K}}(\cdot)\nu(x))$. Again, this equation allows us to consider $f_{e'}$ as a linear form on X (by setting $f_{e'}(T_x^{\mathbb{K}}(\cdot)\nu(x)) := (e'\circ\mathsf{A})(T_x^{\mathbb{K}}(\cdot)\nu(x))$), which yields $e'\circ\mathsf{A} \in \mathcal{F}\nu(\Omega) = Z$ for all $e' \in G$. Hence we can apply Proposition 5.2.68 and obtain an extension

$\widehat{\mathsf{A}} \in Y\varepsilon E$ of A such that $\widehat{\mathsf{A}}(B_Z^{\circ Y'})$ is bounded in E. We set $\widetilde{F} := S(\widehat{\mathsf{A}}) \in \mathcal{F}_\varepsilon\nu(\Omega, E)$ and get for all $x \in U$ that

$$T^E(\widetilde{F})(x) = T^E S(\widehat{\mathsf{A}})(x) = \widehat{\mathsf{A}}(T_x^{\mathbb{K}}) = \frac{1}{\nu(x)}\mathsf{A}(T_x^{\mathbb{K}}(\cdot)\nu(x)) = f(x)$$

by consistency for (F, E), yielding $R_{U,G}(\widetilde{F}) = f$. □

5.2.70. Corollary. *Let E be a locally complete lcHs, $G \subset E'$ a separating subspace, $\Omega \subset \mathbb{R}^d$ open, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator, $\nu\colon \Omega \to (0, \infty)$ continuous and U fix the topology in $\mathcal{C}\nu_{P(\partial)}(\Omega)$. If $f\colon U \to E$ is a function in $\ell\nu(U, E)$ such that $e' \circ f$ admits an extension $f_{e'} \in \mathcal{C}\nu_{P(\partial)}(\Omega)$ for every $e' \in G$, then there exists a unique extension $F \in \mathcal{C}\nu_{P(\partial)}(\Omega, E)$ of f.*

Proof. Observing that $f \in \mathcal{F}\nu_G(U, E)_{lb}$ with $\mathcal{F}\nu(\Omega) = \mathcal{C}\nu_{P(\partial)}(\Omega)$, our statement follows directly from Theorem 5.2.69 whose conditions are fulfilled by the proof of Corollary 5.2.30. □

Sets that fix the topology in $\mathcal{C}\nu_{P(\partial)}(\Omega)$ for different weights ν are well-studied if $P(\partial) = \overline{\partial}$ is the Cauchy–Riemann operator. If $\Omega \subset \mathbb{C}$ is open, $P(\partial) = \overline{\partial}$ and $\nu = 1$, then $\mathcal{C}\nu_{P(\partial)}(\Omega) = H^\infty(\Omega)$ is the space of bounded holomorphic functions on Ω. Brown, Shields and Zeller characterise the countable discrete sets $U := (z_n)_{n\in\mathbb{N}} \subset \Omega$ that fix the topology in $H^\infty(\Omega)$ with $C = 1$ and equality in Definition 5.2.46 for Jordan domains Ω in [36, Theorem 3, p. 167]. In particular, they prove for $\Omega = \mathbb{D}$ that a discrete $U = (z_n)_{n\in\mathbb{N}}$ fixes the topology in $H^\infty(\mathbb{D})$ if and only if almost every boundary point is a non-tangential limit of a sequence in U. Bonsall obtains the same characterisation for bounded harmonic functions, i.e. $P(\partial) = \Delta$ and $\nu = 1$, on $\Omega = \mathbb{D}$ by [32, Theorem 2, p. 473]. An example of such a set $U = (z_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ is constructed in [36, Remark 6, p. 172]. Probably the first example of a countable discrete set $U \subset \mathbb{D}$ that fixes the topology in $H^\infty(\mathbb{D})$ is given by Wolff in [183, p. 1327] (cf. [81, Theorem (Wolff), p. 402]). In [148, 4.14 Theorem, p. 255] Rubel and Shields give a charaterisation of sets $U \subset \Omega$ that fix the topology in $H^\infty(\Omega)$ by means of bounded complex measures where $\Omega \subset \mathbb{C}$ is open and connected. The existence of a countable U fixing the topology in $H^\infty(\Omega)$ is shown in [148, 4.15 Proposition, p. 256]. In the case of several complex variables the existence of such a countable U is treated by Sibony in [167, Remarques 4 b), p. 209] and by Massaneda and Thomas in [128, Theorem 2, p. 838].

If $\Omega = \mathbb{C}$ and $P(\partial) = \overline{\partial}$, then $\mathcal{C}\nu_{P(\partial)}(\Omega) =: F_\nu^\infty(\mathbb{C})$ is a generalised L^∞-version of the Bargmann–Fock space. In the case that $\nu(z) = \exp(-\alpha|z|^2/2)$, $z \in \mathbb{C}$, for some $\alpha > 0$, Seip and Wallstén show in [162, Theorem 2.3, p. 93] that a countable discrete set $U \subset \mathbb{C}$ fixes the topology in $F_\nu^\infty(\mathbb{C})$ if and only if U contains a uniformly discrete subset U' with lower uniform density $D^-(U') > \alpha/\pi$ (the proof of sufficiency is given in [165] and the result was announced in [161, Theorem 1.3, p. 324]). A generalisation of this result using lower angular densities is given by Lyubarskiĭ and Seip in [124, Theorem 2.2, p. 162] to weights of the form $\nu(z) = \exp(-\phi(\arg z)|z|^2/2)$, $z \in \mathbb{C}$, with a 2π-periodic 2-trigonometrically convex function ϕ such that $\phi \in \mathcal{C}^2([0, 2\pi])$ and $\phi(\theta) + (1/4)\phi''(\theta) > 0$ for all $\theta \in [0, 2\pi]$. An extension of the results in [162] to weights of the form $\nu(z) = \exp(-\phi(z))$, $z \in \mathbb{C}$, with a subharmonic function ϕ such that $\Delta\phi(z) \sim 1$ is given in [135, Theorem 1, p. 249] by Ortega-Cerdà and Seip. Here, $f(x) \sim g(x)$ for two functions $f, g\colon \Omega \to \mathbb{R}$ means that there are $C_1, C_2 > 0$ such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ for all $x \in \Omega$. Marco, Massaneda and Ortega-Cerdà describe sets that fix the topology in $F_\nu^\infty(\mathbb{C})$ with $\nu(z) = \exp(-\phi(z))$, $z \in \mathbb{C}$, for some subharmonic function ϕ whose Laplacian $\Delta\phi$ is a doubling measure (see [126, Definition 5, p. 868]), e.g. $\phi(z) = |z|^\beta$ for some $\beta > 0$, in [126, Theorem A, p. 865]. The case of several complex variables is handled by Ortega-Cerdà, Schuster and Varolin in [136, Theorem 2, p. 81].

If $\Omega = \mathbb{D}$ and $P(\partial) = \overline{\partial}$, then $\mathcal{C}\nu_{P(\partial)}(\Omega) =: A_\nu^\infty(\mathbb{D})$ is a generalised L^∞-version of the weighted Bergman space (and of $H^\infty(\mathbb{D})$). For $\nu(z) = (1-|z|^2)^n$, $z \in \mathbb{D}$, for some $n \in \mathbb{N}$, Seip proves that a countable discrete set $U \subset \mathbb{D}$ fixes the topology in $A_\nu^\infty(\mathbb{D})$ if and only if U contains a uniformly discrete subset U' with lower uniform density $D^-(U') > n$ by [163, Theorem 1.1, p. 23], and gives a typical example in [163, p. 23]. Later on, this is extended by Seip in [164, Theorem 2, p. 718] to weights $\nu(z) = \exp(-\phi(z))$, $z \in \mathbb{D}$, with a subharmonic function ϕ such that $\Delta\phi(z) \sim (1-|z|^2)^{-2}$, e.g. $\phi(z) = -\beta\ln(1-|z|^2)$, $z \in \mathbb{D}$, for some $\beta > 0$. Domański and Lindström give necessary resp. sufficient conditions for fixing the topology in $A_\nu^\infty(\mathbb{D})$ in the case that ν is an essential weight on $\mathbb{D}$, i.e. there is $C > 0$ with $\nu(z) \leq \widetilde{\nu}(z) \leq C\nu(z)$ for each $z \in \mathbb{D}$ where $\widetilde{\nu}(z) := (\sup\{|f(z)| \mid f \in B_{A_\nu^\infty(\mathbb{D})}\})^{-1}$ is the associated weight. In [55, Theorem 29, p. 260] they describe necessary resp. sufficient conditions for fixing the topology if the upper index U_ν is finite (see [55, p. 242]), and necessary and sufficient conditions in [55, Corollary 31, p. 261] if $0 < L_\nu = U_\nu < \infty$ holds where L_ν is the lower index (see [55, p. 243]), which for example can be applied to $\nu(z) = (1-|z|^2)^n(\ln(\frac{\mathrm{e}}{1-|z|}))^\beta$, $z \in \mathbb{D}$, for some $n > 0$ and $\beta \in \mathbb{R}$. The case of simply connected open $\Omega \subset \mathbb{C}$ is considered in [55, Corollary 32, p. 261–262].

Borichev, Dhuez and Kellay treat $A_\nu^\infty(\mathbb{D})$ and $F_\nu^\infty(\mathbb{C})$ simultaneously. Let $\Omega_R := \mathbb{D}$, if $R = 1$, and $\Omega_R := \mathbb{C}$ if $R = \infty$. They take $\nu(z) = \exp(-\phi(z))$, $z \in \Omega_R$, where $\phi\colon [0,R) \to [0,\infty)$ is an increasing function such that $\phi(0) = 0$, $\lim_{r\to R}\phi(r) = \infty$, ϕ is extended to Ω_R by $\phi(z) := \phi(|z|)$, $\phi \in \mathcal{C}^2(\Omega_R)$, and, in addition $\Delta\phi(z) \geq 1$ if $R = \infty$ (see [33, p. 564–565]). Then they set $\rho\colon [0,R) \to \mathbb{R}$, $\rho(r) := [\Delta\phi(r)]^{-1/2}$, and suppose that ρ decreases to 0 near R, $\rho'(r) \to 0$, $r \to R$, and either $(I_\mathbb{D})$ the function $r \mapsto \rho(r)(1-r)^{-C}$ increases for some $C \in \mathbb{R}$ and for r close to 1, resp. $(I_\mathbb{C})$ the function $r \mapsto \rho(r)r^C$ increases for some $C \in \mathbb{R}$ and for large r, or (II_{Ω_R}) that $\rho'(r)\ln(1/\rho(r)) \to 0$, $r \to R$ (see [33, p. 567–569]). Typical examples for $(I_\mathbb{D})$ are

$$\phi(r) = \ln(\ln(\tfrac{1}{1-r}))\ln(\tfrac{1}{1-r}) \quad \text{or} \quad \phi(r) = \tfrac{1}{1-r},$$

a typical example for $(II_\mathbb{D})$ is $\phi(r) = \exp(\frac{1}{1-r})$, for $(I_\mathbb{C})$

$$\phi(r) = r^2\ln(\ln(r)) \quad \text{or} \quad \phi(r) = r^p, \text{ for some } p > 2,$$

and a typical example for $(II_\mathbb{C})$ is $\phi(r) = \exp(r)$. Sets that fix the topology in $A_\nu^\infty(\mathbb{D})$ are described by densities in [33, Theorem 2.1, p. 568] and sets that fix the topology in $F_\nu^\infty(\mathbb{C})$ in [33, Theorem 2.5, p. 569].

Wolf uses sets that fix the topology in $A_\nu^\infty(\mathbb{D})$ for the characterisation of weighted composition operators on $A_\nu^\infty(\mathbb{D})$ with closed range in [182, Theorem 1, p. 36] for bounded ν.

5.2.71. Corollary. *Let E be a locally complete lcHs, $G \subset E'$ a separating subspace, $\nu\colon \mathbb{D} \to (0,\infty)$ continuous and $U := \{0\} \cup (\{1\} \times U_*)$ fix the topology in $\mathcal{B}\nu(\mathbb{D})$ with $U_* \subset \mathbb{D}$. If $f\colon U \to E$ is a function in $\ell\nu_*(U,E)$ such that there is $f_{e'} \in \mathcal{B}\nu(\mathbb{D})$ for each $e' \in G$ with $f_{e'}(0) = e'(f(0))$ and $f_{e'}'(z) = e'(f(1,z))$ for all $z \in U_*$, then there exists a unique $F \in \mathcal{B}\nu(\mathbb{D},E)$ with $F(0) = f(0)$ and $(\partial_\mathbb{C}^1)^E F(z) = f(1,z)$ for all $z \in U_*$.*

Proof. As in Corollary 5.2.70 but with $\mathcal{F}\nu_*(\mathbb{D}) = \mathcal{B}\nu(\mathbb{D})$ and Corollary 5.2.33 instead of Corollary 5.2.30. □

Sets that fix the topology in $\mathcal{B}\nu(\mathbb{D})$ play an important role in the characterisation of composition operators on $\mathcal{B}\nu(\mathbb{D})$ with closed range. Chen and Gauthier give a characterisation in [42] for weights of the form $\nu(z) = (1-|z|^2)^\alpha$, $z \in \mathbb{D}$, for some $\alpha \geq 1$. We recall the following definitions which are needed to phrase this characterisation. For a continuous function $\nu\colon \mathbb{D} \to (0,\infty)$ and a non-constant holomorphic function $\phi\colon \mathbb{D} \to \mathbb{D}$ we set

$$\tau_\phi^\nu(z) := \frac{\nu(z)|\phi'(z)|}{\nu(\phi(z))},\ z \in \mathbb{D}, \quad \text{and} \quad \Omega_\varepsilon^\nu := \{z \in \mathbb{D} \mid \tau_\phi^\nu(z) \geq \varepsilon\},\ \varepsilon > 0,$$

and define the *pseudohyperbolic distance*

$$\rho(z,w) := \left|\frac{z-w}{1-\overline{z}w}\right|,\ z,w \in \mathbb{D}.$$

For $0 < r < 1$ a set $B \subset \mathbb{D}$ is called a *pseudo r-net* if for every $w \in \mathbb{D}$ there is $z \in B$ with $\rho(z,w) \leq r$ (see [42, p. 195]). A set $U_\star \subset \mathbb{D}$ is a sampling set for $\mathcal{B}\nu(\mathbb{D})$ with ν as above in the sense of [42, p. 198] if and only if $\{0\} \cup (\{1\} \times U_\star)$ fixes the topology in $\mathcal{B}\nu(\mathbb{D})$ (see the definitions above Corollary 5.2.33).

5.2.72. Theorem ([42, Theorem 3.1, p. 199, Theorem 4.3, p. 202]). *Let $\phi\colon \mathbb{D} \to \mathbb{D}$ be a non-constant holomorphic function and $\nu(z) = (1-|z|^2)^\alpha$, $z \in \mathbb{D}$, for some $\alpha \geq 1$. Then the following statements are equivalent.*

- *(i) The composition operator $C_\phi\colon \mathcal{B}\nu(\mathbb{D}) \to \mathcal{B}\nu(\mathbb{D})$, $C_\phi(f) := f \circ \phi$, is bounded below (i.e. has closed range).*
- *(ii) There is $\varepsilon > 0$ such that $\{0\} \cup (\{1\} \times \phi(\Omega_\varepsilon^\nu))$ fixes the topology in $\mathcal{B}\nu(\mathbb{D})$.*
- *(iii) There are $\varepsilon > 0$ and $0 < r < 1$ such that $\phi(\Omega_\varepsilon^\nu)$ is a pseudo r-net.*

This theorem has some predecessors. The implications (i)⇒(iii) and (iii), $r < 1/4$ ⇒(i) for $\alpha = 1$ are due to Ghatage, Yan and Zheng by [72, Proposition 1, p. 2040] and [72, Theorem 2, p. 2043]. This was improved by Chen to (i)⇔(iii) for $\alpha = 1$ by removing the restriction $r < 1/4$ in [41, Theorem 1, p. 840]. The proof of the equivalence (i)⇔(ii) given in [73, Theorem 1, p. 1372] for $\alpha = 1$ is due to Ghatage, Zheng and Zorboska. A non-trivial example of a sampling set for $\alpha = 1$ can be found in [73, Example 2, p. 1376] (cf. [42, p. 203]). In the case of several complex variables a characterisation corresponding to Theorem 5.2.72 is given by Chen in [41, Theorem 2, p. 844] and Deng, Jiang and Ouyang in [50, Theorem 1–3, p. 1031–1032, 1034] where Ω is the unit ball of $\mathbb{C}^d$. Giménez, Malavé and Ramos-Fernández extend Theorem 5.2.72 by [75, Theorem 3, p. 112] and [75, Corollary 1, p. 113] to more general weights of the form $\nu(z) = \mu(1-|z|^2)$ with some continuous function $\mu\colon (0,1] \to (0,\infty)$ such that $\mu(r) \to 0$, $r \to 0+$, which can be extended to a holomorphic function μ_0 on $\mathbb{D}_1(1)$ without zeros in $\mathbb{D}_1(1)$ and fulfilling $\mu(1-|1-z|) \leq C|\mu_0(z)|$ for all $z \in \mathbb{D}_1(1)$ and some $C > 0$ (see [75, p. 109]). Examples of such functions μ are $\mu_1(r) := r^\alpha$, $\alpha > 0$, $\mu_2 := r\ln(2/r)$ and $\mu_3(r) := r^\beta \ln(1-r)$, $\beta > 1$, for $r \in (0,1]$ (see [75, p. 110]) and with $\nu(z) = \mu_1(1-|z|^2) = (1-|z|^2)^\alpha$, $z \in \mathbb{D}$, one gets Theorem 5.2.72 back for $\alpha \geq 1$. For $0 < \alpha < 1$ and $\nu(z) = \mu_1(1-|z|^2)$, $z \in \mathbb{D}$, the equivalence (i)⇔(ii) is given in [184, Proposition 4.4, p. 14] of Yoneda as well and a sufficient condition implying (ii) in [184, Corollary 4.5, p. 15]. Ramos-Fernández generalises the results given in [75] to bounded essential weights ν on $\mathbb{D}$ by [144, Theorem 4.3, p. 85] and [144, Remark 4.2, p. 84]. In [141, Theorem 2.4, p. 3106] Pirasteh, Eghbali and Sanatpour use sets that fix the topology in $\mathcal{B}\nu(\mathbb{D})$ for radial essential ν to characterise Li–Stević integral-type operators on $\mathcal{B}\nu(\mathbb{D})$ with closed range instead of composition operators. The composition operator on the harmonic variant of the Bloch type space $\mathcal{B}\nu(\mathbb{D})$ with $\nu(z) = (1-|z|^2)^\alpha$, $z \in \mathbb{D}$, for some $\alpha > 0$ is considered by Esmaeili, Estaremi and Ebadian, who give a corresponding result in [64, Theorem 2.8, p. 542].

5.3. Weak-strong principles for differentiability of finite order

This section is dedicated to $\mathcal{C}^k$-weak-strong principles for differentiable functions. So the question is:

5.3.1. Question. Let E be an lcHs, $G \subset E'$ a separating subspace, $\Omega \subset \mathbb{R}^d$ open and $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f\colon \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^k(\Omega)$ for each $e' \in G$, does $f \in \mathcal{C}^k(\Omega, E)$ hold?

An affirmative answer to the preceding question is called a $\mathcal{C}^k$-weak-strong principle. It is a result of Bierstedt [17, 2.10 Lemma, p. 140] that for $k = 0$ the $\mathcal{C}^0$-weak-strong principle holds if $\Omega \subset \mathbb{R}^d$ is open (or more general a Hausdorff $k_{\mathbb{R}}$-space), $G = E'$ and E is such that every bounded set is already precompact in E, i.e. E is a generalised Schwartz space (see Definition 5.2.50 and Remark 5.2.57). For instance, the last condition is fulfilled if E is a semi-Montel or Schwartz space. The $\mathcal{C}^0$-weak-strong principle does not hold for general E by [94, Beispiel, p. 232].

Grothendieck sketches in a footnote [82, p. 39] (cf. [84, Chap. 3, Sect. 8, Corollary 1, p. 134]) the proof that for $k < \infty$ a weakly-$\mathcal{C}^{k+1}$ function $f\colon \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ with values in a quasi-complete lcHs E is already $\mathcal{C}^k$, i.e. that from $e' \circ f \in \mathcal{C}^{k+1}(\Omega)$ for all $e' \in E'$ it follows $f \in \mathcal{C}^k(\Omega, E)$. A detailed proof of this statement is given by Schwartz in [158], simultaneously weakening the condition from quasi-completeness of E to sequential completeness and from weakly-$\mathcal{C}^{k+1}$ to weakly-$\mathcal{C}^{k,1}_{loc}$.

5.3.2. Theorem ([158, Appendice, Lemme II, Remarques 1^0), p. 146–147]). *Let E be a sequentially complete lcHs, $\Omega \subset \mathbb{R}^d$ open and $k \in \mathbb{N}_0$.*

a) If $f\colon \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^{k,1}_{loc}(\Omega)$ for all $e' \in E'$, then $f \in \mathcal{C}^k(\Omega, E)$.
b) If $f\colon \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^{k+1}(\Omega)$ for all $e' \in E'$, then $f \in \mathcal{C}^k(\Omega, E)$.

Here $\mathcal{C}^{k,1}_{loc}(\Omega)$ denotes the space of functions in $\mathcal{C}^k(\Omega)$ whose partial derivatives of order k are locally Lipschitz continuous. Part b) clearly implies a $\mathcal{C}^\infty$-weak-strong principle for open $\Omega \subset \mathbb{R}^d$, $G = E'$ and sequentially complete E. This can be generalised to locally complete E. Waelbroeck has shown in [177, Proposition 2, p. 411] and [176, Definition 1, p. 393] that the $\mathcal{C}^\infty$-weak-strong principle holds if Ω is a manifold, $G = E'$ and E is locally complete. It is a result of Bonet, Frerick and Jordá that the $\mathcal{C}^\infty$-weak-strong principle still holds by [30, Theorem 9, p. 232] if $\Omega \subset \mathbb{R}^d$ is open, $G \subset E'$ determines boundedness and E is locally complete. Due to [104, 2.14 Theorem, p. 20] of Kriegl and Michor an lcHs E is locally complete if and only if the $\mathcal{C}^\infty$-weak-strong principle holds for $\Omega = \mathbb{R}$ and $G = E'$.

One of the goals of this section is to improve Theorem 5.3.2. We recall the following definition from Example 4.2.28. For $k \in \mathbb{N}_0$ the space of k-times continuously partially differentiable E-valued functions on an open set $\Omega \subset \mathbb{R}^d$ whose partial derivatives up to order k are continuously extendable to the boundary of Ω is

$$\mathcal{C}^k(\overline{\Omega}, E) = \{f \in \mathcal{C}^k(\Omega, E) \mid (\partial^\beta)^E f \text{ cont. extendable on } \overline{\Omega} \text{ for all } \beta \in \mathbb{N}_0^d,\ |\beta| \leq k\}$$

equipped with the system of seminorms given by

$$|f|_{\mathcal{C}^k(\overline{\Omega}),\alpha} = \sup_{\substack{x \in \Omega \\ \beta \in \mathbb{N}_0^d, |\beta| \leq k}} p_\alpha((\partial^\beta)^E f(x)), \quad f \in \mathcal{C}^k(\overline{\Omega}, E),\ \alpha \in \mathfrak{A}.$$

The space of functions in $\mathcal{C}^k(\overline{\Omega}, E)$ such that all its k-th partial derivatives are γ-Hölder continuous with $0 < \gamma \leq 1$ is given by

$$\mathcal{C}^{k,\gamma}(\overline{\Omega}, E) := \left\{f \in \mathcal{C}^k(\overline{\Omega}, E) \mid \forall\ \alpha \in \mathfrak{A} : |f|_{\mathcal{C}^{k,\gamma}(\overline{\Omega}),\alpha} < \infty\right\}$$

where

$$|f|_{\mathcal{C}^{k,\gamma}(\overline{\Omega}),\alpha} := \max\Big(|f|_{\mathcal{C}^k(\overline{\Omega}),\alpha}, \sup_{\beta \in \mathbb{N}_0^d, |\beta| = k} |(\partial^\beta)^E f|_{\mathcal{C}^{0,\gamma}(\Omega),\alpha}\Big)$$

with

$$|f|_{\mathcal{C}^{0,\gamma}(\Omega),\alpha} := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{p_\alpha(f(x)-f(y))}{|x-y|^\gamma}.$$

We set $\mathcal{C}^{k,\gamma}(\overline{\Omega}) := \mathcal{C}^{k,\gamma}(\overline{\Omega},\mathbb{K})$ and

$$\omega_1 := \{\beta\in\mathbb{N}_0^d \mid |\beta|\leq k\}\times\Omega \quad\text{and}\quad \omega_2 := \{\beta\in\mathbb{N}_0^d \mid |\beta|=k\}\times(\Omega^2\smallsetminus\{(x,x)\mid x\in\Omega\})$$

as well as $\omega := \omega_1\cup\omega_2$. We define the operator $T^E\colon\mathcal{C}^k(\Omega,E)\to E^\omega$ by

$$\begin{aligned} T^E(f)(\beta,x) &:= (\partial^\beta)^E(f)(x) &&, (\beta,x)\in\omega_1,\\ T^E(f)(\beta,(x,y)) &:= (\partial^\beta)^E(f)(x)-(\partial^\beta)^E(f)(y) &&, (\beta,(x,y))\in\omega_2. \end{aligned}$$

and the weight $\nu\colon\omega\to(0,\infty)$ by

$$\nu(\beta,x) := 1,\ (\beta,x)\in\omega_1, \quad\text{and}\quad \nu(\beta,(x,y)) := \frac{1}{|x-y|^\gamma},\ (\beta,(x,y))\in\omega_2.$$

By setting $F(\Omega,E) := \mathcal{C}^k(\overline{\Omega},E)$ and observing that

$$|f|_{\mathcal{C}^{k,\gamma}(\overline{\Omega}),\alpha} = \sup_{x\in\omega} p_\alpha(T^E(f)(x))\nu(x), \quad f\in\mathcal{C}^{k,\gamma}(\overline{\Omega},E),\ \alpha\in\mathfrak{A},$$

we have $\mathcal{F}\nu(\Omega,E) = \mathcal{C}^{k,\gamma}(\overline{\Omega},E)$ with generator $(T^E,T^{\mathbb{K}})$.

5.3.3. Corollary. *Let E be a locally complete lcHs, $G\subset E'$ determine boundedness, $\Omega\subset\mathbb{R}^d$ open and bounded, $k\in\mathbb{N}_0$ and $0<\gamma\leq 1$. In the case $k\geq 1$, assume additionally that Ω has Lipschitz boundary. If $f\colon\Omega\to E$ is such that $e'\circ f\in\mathcal{C}^{k,\gamma}(\overline{\Omega})$ for all $e'\in G$, then $f\in\mathcal{C}^{k,\gamma}(\overline{\Omega},E)$.*

Proof. We take $F(\Omega) := \mathcal{C}^k(\overline{\Omega})$ and $F(\Omega,E) := \mathcal{C}^k(\overline{\Omega},E)$ and have $\mathcal{F}\nu(\Omega) = \mathcal{C}^{k,\gamma}(\overline{\Omega})$ and $\mathcal{F}\nu(\Omega,E) = \mathcal{C}^{k,\gamma}(\overline{\Omega},E)$ with the weight ν and generator $(T^E,T^{\mathbb{K}})$ for $(\mathcal{F}\nu,E)$ described above. Due to the proof of Example 4.2.28 and Theorem 3.1.12 the spaces $F(\Omega)$ and $F(\Omega,E)$ are ε-into-compatible for any lcHs E (the condition that E has metric ccp in Example 4.2.28 is only needed for ε-compatibility). Another consequence of Example 4.2.28 is that

$$T^E(S(u))(\beta,x) = (\partial^\beta)^E(S(u))(x) = u(\delta_x\circ(\partial^\beta)^{\mathbb{K}}) = u(T^{\mathbb{K}}_{\beta,x}), \quad (\beta,x)\in\omega_1,$$

holds for all $u\in F(\Omega)\varepsilon E$, implying

$$\begin{aligned} T^E(S(u))(\beta,(x,y)) &= T^E(S(u))(\beta,x) - T^E(S(u))(\beta,y) = u(T^{\mathbb{K}}_{\beta,x}) - u(T^{\mathbb{K}}_{\beta,y})\\ &= u(T^{\mathbb{K}}_{\beta,(x,y)}), \quad (\beta,(x,y))\in\omega_2. \end{aligned}$$

Thus $(T^E,T^{\mathbb{K}})$ is a consistent family for (F,E) and its strength is easily seen. In addition, $\mathcal{F}\nu(\Omega) = \mathcal{C}^{k,\gamma}(\overline{\Omega})$ is a Banach space by [57, Theorem 9.8, p. 110] (cf. [4, 1.7 Hölderstetige Funktionen, p. 46]) whose closed unit ball is compact in $F(\Omega) = \mathcal{C}^k(\overline{\Omega})$ by [4, 8.6 Einbettungssatz in Hölder-Räumen, p. 338]. Moreover, the ε-into-compatibility of $F(\Omega)$ and $F(\Omega,E)$ in combination with the consistency of $(T^E,T^{\mathbb{K}})$ for (F,E) implies $\mathcal{F}_\varepsilon\nu(\Omega,E)\subset\mathcal{F}\nu(\Omega,E)$ as linear spaces by Proposition 5.2.25 c). Hence our statement follows from Theorem 5.2.29 with the set of uniqueness $U := \{0\}\times\Omega$ for $(T^{\mathbb{K}},\mathcal{F}\nu)$. □

5.3.4. Remark. We point out that Corollary 5.3.3 corrects our result [117, Corollary 5.3, p. 16] by adding the missing assumption that Ω should additionally have Lipschitz boundary in the case $k\geq 1$. This is needed to deduce that the closed unit ball of $\mathcal{C}^{k,\gamma}(\overline{\Omega})$ is compact in $\mathcal{C}^k(\overline{\Omega})$ by [4, 8.6 Einbettungssatz in Hölder-Räumen, p. 338] (in the notation of [74] Ω having Lipschitz boundary means that it is a $\mathcal{C}^{0,1}$ domain, see [74, Lemma 6.36, p. 136] and the comments below and above this lemma). This additional assumption is missing in [57, Theorem 14.32, p. 232], which is our main reference in [117] for the compact embedding, but it is needed due to [4, U8.1 Gegenbeispiel zu Einbettungssätzen, p. 365] (cf. [74, p. 53]). However, this only affects the result [117, Corollary 6.3, p. 21–22] where we have to add this missing assumption as well (see Corollary 5.4.4 for this). The other results of [117] derived from [117, Corollary 5.3, p. 16] are not affected by this missing assumption since they are all a consequence of [117, Corollary 5.4, p. 17] and [117, Corollary 6.4, p. 22], whose proofs can be adjusted without additional assumptions (see Corollary 5.3.5 and Corollary 5.4.5 for this).

Next, we use the preceding corollary to generalise the theorem of Grothendieck and Schwartz on weakly $\mathcal{C}^{k+1}$-functions. For $k \in \mathbb{N}_0$ and $0 < \gamma \leq 1$ we define the space of k-times continuously partially differentiable E-valued functions with locally γ-Hölder continuous partial derivatives of k-th order on an open set $\Omega \subset \mathbb{R}^d$ by

$$\mathcal{C}^{k,\gamma}_{loc}(\Omega, E) := \{f \in \mathcal{C}^k(\Omega, E) \mid \forall\ K \subset \Omega \text{ compact}, \alpha \in \mathfrak{A} : |f|_{K,\alpha} < \infty\}$$

where

$$|f|_{K,\alpha} := \max\Big(|f|_{\mathcal{C}^k(K),\alpha}, \sup_{\beta \in \mathbb{N}_0^d, |\beta|=k} |(\partial^\beta)^E f|_{\mathcal{C}^{0,\gamma}(K),\alpha}\Big)$$

with

$$|f|_{\mathcal{C}^k(K),\alpha} := \sup_{\substack{x \in K \\ \beta \in \mathbb{N}_0^d, |\beta| \leq k}} p_\alpha((\partial^\beta)^E f(x))$$

and

$$|f|_{\mathcal{C}^{0,\gamma}(K),\alpha} := \sup_{\substack{x,y \in K \\ x \neq y}} \frac{p_\alpha(f(x) - f(y))}{|x-y|^\gamma}.$$

Further, we set $\mathcal{C}^{k,\gamma}_{loc}(\Omega) := \mathcal{C}^{k,\gamma}_{loc}(\Omega, \mathbb{K})$. Using Corollary 5.3.3, we are able to improve Theorem 5.3.2 to the following form.

5.3.5. Corollary. *Let E be a locally complete lcHs, $G \subset E'$ determine boundedness, $\Omega \subset \mathbb{R}^d$ open, $k \in \mathbb{N}_0$ and $0 < \gamma \leq 1$.*

a) *If $f : \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega)$ for all $e' \in G$, then $f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$.*
b) *If $f : \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^{k+1}(\Omega)$ for all $e' \in G$, then $f \in \mathcal{C}^{k,1}_{loc}(\Omega, E)$.*

Proof. Let us start with a). Let $f : \Omega \to E$ be such that $e' \circ f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega)$ for all $e' \in G$. Let $(\Omega_n)_{n \in \mathbb{N}}$ be an exhaustion of Ω with open, relatively compact sets $\Omega_n \subset \Omega$ with Lipschitz boundaries $\partial\Omega_n$ (e.g. choose each Ω_n as the interior of a finite union of closed axis-parallel cubes, see the proof of [168, Theorem 1.4, p. 7] for the construction) that satisfies $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$. Then the restriction of $e' \circ f$ to Ω_n is an element of $\mathcal{C}^{k,\gamma}(\overline{\Omega}_n)$ for every $e' \in G$ and $n \in \mathbb{N}$. Due to Corollary 5.3.3 we obtain that $f \in \mathcal{C}^{k,\gamma}(\overline{\Omega}_n, E)$ for every $n \in \mathbb{N}$. Thus $f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$ since differentiability is a local property and for each compact $K \subset \Omega$ there is $n \in \mathbb{N}$ such that $K \subset \Omega_n$.

Let us turn to b), i.e. let $f : \Omega \to E$ be such that $e' \circ f \in \mathcal{C}^{k+1}(\Omega)$ for all $e' \in G$. Since $\Omega \subset \mathbb{R}^d$ is open, for every $x \in \Omega$ there is $\varepsilon_x > 0$ such that $\overline{\mathbb{B}_{\varepsilon_x}(x)} \subset \Omega$. For all $e' \in G$, $\beta \in \mathbb{N}_0^d$ with $|\beta| = k$ and $w, y \in \overline{\mathbb{B}_{\varepsilon_x}(x)}$, $w \neq y$, it holds that

$$\frac{|(\partial^\beta)^{\mathbb{K}}(e' \circ f)(w) - (\partial^\beta)^{\mathbb{K}}(e' \circ f)(y)|}{|w-y|} \leq C_d \max_{1 \leq n \leq d} \max_{z \in \overline{\mathbb{B}_{\varepsilon_x}(x)}} |(\partial^{\beta+e_n})^{\mathbb{K}}(e' \circ f)(z)|$$

by the mean value theorem applied to the real and imaginary part where $C_d := \sqrt{d}$ if $\mathbb{K} = \mathbb{R}$, and $C_d := 2\sqrt{d}$ if $\mathbb{K} = \mathbb{C}$. Thus $e' \circ f \in \mathcal{C}^{k,1}_{loc}(\Omega)$ for all $e' \in G$ since for each compact set $K \subset \Omega$ there are $m \in \mathbb{N}$ and $x_i \in \Omega$, $1 \leq i \leq m$, such that $K \subset \bigcup_{i=1}^m \mathbb{B}_{\varepsilon_{x_i}}(x_i)$. It follows from part a) that $f \in \mathcal{C}^{k,1}_{loc}(\Omega, E)$. □

If $\Omega = \mathbb{R}$, $\gamma = 1$ and $G = E'$, then part a) of Corollary 5.3.5 is already known by [104, 2.3 Corollary, p. 15]. A 'full' $\mathcal{C}^k$-weak-strong principle for $k < \infty$, i.e. the conditions of part b) imply $f \in \mathcal{C}^{k+1}(\Omega, E)$, does not hold in general (see [104, p. 11–12]) but it holds if we restrict the class of admissible lcHs E.

5.3.6. Theorem. *Let E be a semi-Montel space, $G \subset E'$ determine boundedness, $\Omega \subset \mathbb{R}^d$ open and $k \in \mathbb{N}$. If $f\colon \Omega \to E$ is such that $e' \circ f \in \mathcal{C}^k(\Omega)$ for all $e' \in G$, then $f \in \mathcal{C}^k(\Omega, E)$.*

Proof. Let $f\colon \Omega \to E$ be such that $e' \circ f \in \mathcal{C}^k(\Omega)$ for all $e' \in G$. Due to Corollary 5.3.5 b) we already know that $f \in \mathcal{C}^{k-1,1}_{loc}(\Omega, E)$ since semi-Montel spaces are quasi-complete and thus locally complete. Now, let $x \in \Omega$, $\varepsilon_x > 0$ such that $\overline{\mathbb{B}_{\varepsilon_x}(x)} \subset \Omega$, $\beta \in \mathbb{N}_0^d$ with $|\beta| = k - 1$ and $n \in \mathbb{N}$ with $1 \leq n \leq d$. The set

$$B := \Big\{ \frac{(\partial^\beta f)^E(x + he_n) - (\partial^\beta f)^E f(x)}{h} \mid h \in \mathbb{R},\, 0 < h \leq \varepsilon_x \Big\}$$

is bounded in E because $f \in \mathcal{C}^{k-1,1}_{loc}(\Omega, E)$. As E is a semi-Montel space, the closure $\overline{B}$ is compact in E. Let $(h_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $0 < h_m \leq \varepsilon_x$ for all $m \in \mathbb{N}$. From the compactness of $\overline{B}$ we deduce that there is a subnet $(h_{m_\iota})_{\iota\in I}$ of $(h_m)_{m\in\mathbb{N}}$ and $y_x \in \overline{B}$ with

$$y_x = \lim_{\iota \in I} \frac{(\partial^\beta f)^E(x + h_{m_\iota} e_n) - (\partial^\beta f)^E f(x)}{h_{m_\iota}} =: \lim_{\iota\in I} y_\iota .$$

Further, we note that the limit

$$(\partial^{\beta+e_n})^{\mathbb{K}}(e' \circ f)(x) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}, h\neq 0}} \frac{\partial^\beta (e' \circ f)(x + he_n) - \partial^\beta (e' \circ f)(x)}{h} \tag{55}$$

exists for all $e' \in G$ and that $(e'(y_\iota))_{\iota\in I}$ is a subnet of the net of difference quotients on the right-hand side of (55) as $(\partial^\beta)^{\mathbb{K}}(e' \circ f) = e' \circ (\partial^\beta)^E f$. Therefore

$$\begin{aligned}(\partial^{\beta+e_n})^{\mathbb{K}}(e' \circ f)(x) &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}, h\neq 0}} e'\Big(\frac{(\partial^\beta)^E f(x + he_n) - (\partial^\beta)^E f(x)}{h} \Big) \\ &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}, 0<h\leq\varepsilon_x}} e'\Big(\frac{(\partial^\beta)^E f(x + he_n) - (\partial^\beta)^E f(x)}{h} \Big) = \lim_{\iota\in I} e'(y_\iota) \\ &= e'(y_x) \end{aligned} \tag{56}$$

for all $e' \in G$. By Proposition 5.2.56 (i) the topology $\sigma(E, G)$ and the initial topology of E coincide on $\overline{B}$. Combining this fact with (56), we deduce that

$$(\partial^{\beta+e_n})^E f(x) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}, h\neq 0}} \frac{(\partial^\beta)^E f(x + he_n) - (\partial^\beta)^E f(x)}{h} = y_x .$$

In addition, $e' \circ (\partial^{\beta+e_n})^E f = (\partial^{\beta+e_n})^{\mathbb{K}}(e' \circ f)$ is continuous on $\overline{\mathbb{B}_{\varepsilon_x}(x)}$ for all $e' \in G$, meaning that the restriction of $(\partial^{\beta+e_n})^E f$ on $\overline{\mathbb{B}_{\varepsilon_x}(x)}$ to $(E, \sigma(E, G))$ is continuous, and the range $(\partial^{\beta+e_n})^E f(\overline{\mathbb{B}_{\varepsilon_x}(x)})$ is bounded in E. As before we use that $\sigma(E, G)$ and the initial topology of E coincide on $(\partial^{\beta+e_n})^E f(\overline{\mathbb{B}_{\varepsilon_x}(x)})$, which implies that the restriction of $(\partial^{\beta+e_n})^E f$ on $\overline{\mathbb{B}_{\varepsilon_x}(x)}$ is continuous w.r.t. the initial topology of E. Since continuity is a local property and $x \in \Omega$ is arbitrary, we conclude that $(\partial^{\beta+e_n})^E f$ is continuous on Ω. □

In the special case that $\Omega = \mathbb{R}$, $G = E'$ and E is a Montel space, i.e. a barrelled semi-Montel space, a different proof of the preceding weak-strong principle can be found in the proof of [40, Lemma 4, p. 15]. This proof uses the Banach–Steinhaus theorem and needs the barrelledness of the Montel space E_b'. Our weak-strong principle Theorem 5.3.6 does not need the barrelledness of E, hence can be applied to $E = (\mathcal{C}^{\infty}_{\overline{\partial},b}(\mathbb{D}),\beta)$ which is a non-barrelled semi-Montel space by Remark 4.2.23 and Proposition 5.2.16.

Besides the 'full' $\mathcal{C}^k$-weak-strong principle for $k<\infty$ and semi-Montel E, part b) of Corollary 5.3.5 also suggests an 'almost' $\mathcal{C}^k$-weak-strong principle in terms of [66, 3.1.6 Rademacher's theorem, p. 216], which we prepare next.

5.3.7. DEFINITION (generalised Gelfand space). We say that an lcHs E is a *generalised Gelfand space* if every Lipschitz continuous map $f\colon[0,1]\to E$ is differentiable almost everywhere w.r.t to the one-dimensional Lebesgue measure.

If E is a real Fréchet space ($\mathbb{K}=\mathbb{R}$), then this definition coincides with the definition of a Fréchet–Gelfand space given in [125, Definition 2.2, p. 17]. In particular, every real nuclear Fréchet lattice (see [78, Theorem 6, Corollary, p. 375, 378]) and more general every real Fréchet–Montel space is a generalised Gelfand space by [125, Theorem 2.9, p. 18]. If E is a Banach space, then this definition coincides with the definition of a Gelfand space given in [51, Definition 4.3.1, p. 106–107] by [9, Proposition 1.2.4, p. 18]. A Banach space is a Gelfand space if and only if it has the Radon–Nikodým property by [51, Theorem 4.3.2, p. 107]. Thus separable duals of Banach spaces, reflexive Banach spaces and $\ell^1(\Gamma)$ for any set Γ are generalised Gelfand spaces by [51, Theorem 3.3.1 (Dunford-Pettis), p. 79], [51, Corollary 3.3.4 (Phillips), p. 82] and [51, Corollary 3.3.8, p. 83]. The Banach spaces c_0, ℓ^∞, $\mathcal{C}([0,1])$, $\mathcal{L}^1([0,1])$ and $\mathcal{L}^\infty([0,1])$ do not have the Radon–Nikodým property and hence are not generalised Gelfand spaces by [9, Proposition 1.2.9, p. 20], [9, Example 1.2.8, p. 20] and [9, Proposition 1.2.10, p. 21].

5.3.8. COROLLARY. *Let E be a locally complete generalised Gelfand space, $G\subset E'$ determine boundedness, $\Omega\subset\mathbb{R}$ open and $k\in\mathbb{N}$. If $f\colon\Omega\to E$ is such that $e'\circ f\in\mathcal{C}^k(\Omega)$ for all $e'\in G$, then $f\in\mathcal{C}^{k-1,1}_{loc}(\Omega,E)$ and the derivative $(\partial^k)^E f(x)$ exists for Lebesgue almost all $x\in\Omega$.*

PROOF. The first part follows from Corollary 5.3.5 b). Now, let $[a,b]\subset\Omega$ be a bounded interval. We set $F\colon[0,1]\to E$, $F(x):=(\partial^{k-1})^E f(a+x(b-a))$. Then F is Lipschitz continuous as $f\in\mathcal{C}^{k-1,1}_{loc}(\Omega,E)$. This yields that F is differentiable on $[0,1]$ almost everywhere because E is a generalised Gelfand space, implying that $(\partial^{k-1})^E f$ is differentiable on $[a,b]$ almost everywhere. Since the open set $\Omega\subset\mathbb{R}$ can be written as a countable union of disjoint open intervals I_n, $n\in\mathbb{N}$, and each I_n is a countable union of closed bounded intervals $[a_m,b_m]$, $m\in\mathbb{N}$, our statement follows from the fact that the countable union of null sets is a null set. □

To the best of our knowledge there are still some open problems for continuously partially differentiable functions of finite order.

5.3.9. QUESTION.
(i) Are there other spaces than semi-Montel spaces E for which the 'full' $\mathcal{C}^k$-weak-strong principle Theorem 5.3.6 with $k<\infty$ is true? For instance, if $k=0$, then it is still true if E is a generalised Schwartz space by [17, 2.10 Lemma, p. 140]. Does this hold for $0<k<\infty$ as well?
(ii) Does the 'almost' $\mathcal{C}^k$-weak-strong principle Corollary 5.3.8 also hold for $d>1$?
(iii) For every $\varepsilon>0$ does there exist a function $g\in\mathcal{C}^k(\mathbb{R},E)$ such that $\lambda(\{x\in\Omega\mid f(x)\neq g(x)\})<\varepsilon$ in Corollary 5.3.8 where λ is the one-dimensional Lebesgue measure. In the case that $E=\mathbb{R}^n$ this is true by [66, Theorem 3.1.15, p. 227].

(iv) Is there a 'Radon–Nikodým type' characterisation of generalised Gelfand spaces as in the Banach case?

5.4. Vector-valued Blaschke theorems

In this section we prove several convergence theorems for Banach-valued functions in the spirit of Blaschke's convergence theorem [38, Theorem 7.4, p. 219] as it is done in [7, Theorem 2.4, p. 786] and [7, Corollary 2.5, p. 786–787] for bounded holomorphic functions and more general in [70, Corollary 4.2, p. 695] for bounded functions in the kernel of a hypoelliptic linear partial differential operator. *Blaschke's convergence theorem* says that if $(z_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ is a sequence of distinct elements with $\sum_{n\in\mathbb{N}}(1-|z_n|)=\infty$ and if $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $H^\infty(\mathbb{D})$ such that $(f_k(z_n))_k$ converges in $\mathbb{C}$ for each $n\in\mathbb{N}$, then there is $f\in H^\infty(\mathbb{D})$ such that $(f_k)_k$ converges uniformly to f on the compact subsets of $\mathbb{D}$, i.e. w.r.t. to τ_c.

5.4.1. Proposition ([70, Proposition 4.1, p. 695]). *Let $(E,\|\cdot\|)$ be a Banach space, Z a Banach space whose closed unit ball B_Z is a compact subset of an lcHs Y and let $(\mathsf{A}_\iota)_{\iota\in I}$ be a net in $Y\varepsilon E$ such that*

$$\sup_{\iota\in I}\{\|\mathsf{A}_\iota(y)\| \mid y\in B_Z^{\circ Y'}\}<\infty.$$

Assume further that there exists a $\sigma(Y',Z)$-dense subspace $X\subset Y'$ such that $\lim_\iota \mathsf{A}_\iota(x)$ exists for each $x\in X$. Then there is $\mathsf{A}\in Y\varepsilon E$ with $\mathsf{A}(B_Z^{\circ Y'})$ bounded and $\lim_\iota \mathsf{A}_\iota=\mathsf{A}$ uniformly on the equicontinuous subsets of Y', i.e. for all equicontinuous $B\subset Y'$ and $\varepsilon>0$ there exists $\varsigma\in I$ such that

$$\sup_{y\in B}\|\mathsf{A}_\iota(y)-\mathsf{A}(y)\|<\varepsilon$$

for each $\iota\geq\varsigma$.

Next, we generalise [70, Corollary 4.2, p. 695].

5.4.2. Corollary. *Let $(E,\|\cdot\|)$ be a Banach space and $F(\Omega)$ and $F(\Omega,E)$ be ε-into-compatible. Let $(T^E,T^{\mathbb{K}})$ be a generator for $(\mathcal{F}\nu,E)$ and a strong, consistent family for (F,E), $\mathcal{F}\nu(\Omega)$ a Banach space whose closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is a compact subset of $F(\Omega)$ and U a set of uniqueness for $(T^{\mathbb{K}},\mathcal{F}\nu)$.*

If $(f_\iota)_{\iota\in I}\subset\mathcal{F}_\varepsilon\nu(\Omega,E)$ is a bounded net in $\mathcal{F}\nu(\Omega,E)$ such that $\lim_\iota T^E(f_\iota)(x)$ exists for all $x\in U$, then there is $f\in\mathcal{F}_\varepsilon\nu(\Omega,E)$ such that $(f_\iota)_{\iota\in I}$ converges to f in $F(\Omega,E)$.

Proof. We set $X:=\operatorname{span}\{T_x^{\mathbb{K}}\mid x\in U\}$, $Y:=F(\Omega)$ and $Z:=\mathcal{F}\nu(\Omega)$. As in the proof of Theorem 5.2.29 we observe that X is $\sigma(Y',Z)$-dense in Y'. From $(f_\iota)_{\iota\in I}\subset\mathcal{F}_\varepsilon\nu(\Omega,E)$ follows that there are $\mathsf{A}_\iota\in F(\Omega)\varepsilon E$ with $S(\mathsf{A}_\iota)=f_\iota$ for all $\iota\in I$. Since $(f_\iota)_{\iota\in I}$ is a bounded net in $\mathcal{F}\nu(\Omega,E)$, we note that

$$\begin{aligned}\sup_{\iota\in I}\sup_{x\in\omega}\|\mathsf{A}_\iota(T_x^{\mathbb{K}}(\cdot)\nu(x))\| &= \sup_{\iota\in I}\sup_{x\in\omega}\|T^E S(\mathsf{A}_\iota)(x)\|\nu(x)=\sup_{\iota\in I}\sup_{x\in\omega}\|T^E f_\iota(x)\|\nu(x)\\ &=\sup_{\iota\in I}|f_\iota|_{\mathcal{F}\nu(\Omega,E)}<\infty\end{aligned}$$

by consistency. Further, $\lim_\iota S(\mathsf{A}_\iota)(T_x^{\mathbb{K}})=\lim_\iota T^E(f_\iota)(x)$ exists for each $x\in U$, implying the existence of $\lim_\iota S(\mathsf{A}_\iota)(x)$ for each $x\in X$ by linearity. We apply Proposition 5.4.1 and obtain $f:=S(\mathsf{A})\in\mathcal{F}_\varepsilon\nu(\Omega,E)$ such that $(\mathsf{A}_\iota)_{\iota\in I}$ converges to A in $F(\Omega)\varepsilon E$. From $F(\Omega)$ and $F(\Omega,E)$ being ε-into-compatible it follows that $(f_\iota)_{\iota\in I}$ converges to f in $F(\Omega,E)$. □

First, we apply the preceding corollary to the space $\mathcal{C}_z^{[\gamma]}(\Omega, E)$ of γ-Hölder continuous functions on Ω that vanish at a fixed point $z \in \Omega$ from Example 4.2.9 a). We recall that for a metric space (Ω, d), $z \in \Omega$, an lcHs E and $0 < \gamma \leq 1$ we have

$$\mathcal{C}_z^{[\gamma]}(\Omega, E) = \{f \in E^\Omega \mid f(z) = 0 \text{ and } \forall\ \alpha \in \mathfrak{A}: |f|_{\mathcal{C}^{0,\gamma}(\Omega),\alpha} < \infty\}.$$

Further, we set $\omega := \Omega^2 \smallsetminus \{(x,x) \mid x \in \Omega\}$, $F(\Omega, E) := \{f \in \mathcal{C}(\Omega, E) \mid f(z) = 0\}$ and $T^E: F(\Omega, E) \to E^\omega$, $T^E(f)(x,y) := f(x) - f(y)$, and

$$\nu: \omega \to [0, \infty),\ \nu(x,y) := \frac{1}{\mathrm{d}(x,y)^\gamma}.$$

Then we have for every $\alpha \in \mathfrak{A}$ that

$$|f|_{\mathcal{C}^{0,\gamma}(\Omega),\alpha} = \sup_{x \in \omega} p_\alpha\big(T^E(f)(x)\big)\nu(x), \quad f \in \mathcal{C}_z^{[\gamma]}(\Omega, E),$$

and observe that $\mathcal{F}\nu(\Omega, E) = \mathcal{C}_z^{[\gamma]}(\Omega, E)$ with generator $(T^E, T^{\mathbb{K}})$.

5.4.3. Corollary. *Let E be a Banach space, (Ω, d) a metric space, $z \in \Omega$ and $0 < \gamma \leq 1$. If $(f_\iota)_{\iota \in I}$ is a bounded net in $\mathcal{C}_z^{[\gamma]}(\Omega, E)$ such that* $\lim_\iota f_\iota(x)$ *exists for all x in a dense subset $U \subset \Omega$, then there is $f \in \mathcal{C}_z^{[\gamma]}(\Omega, E)$ such that $(f_\iota)_{\iota \in I}$ converges to f in $\mathcal{C}(\Omega, E)$ uniformly on compact subsets of Ω.*

Proof. We choose $F(\Omega) := \{f \in \mathcal{C}(\Omega) \mid f(z) = 0\}$ and $F(\Omega, E) := \{f \in \mathcal{C}(\Omega, E) \mid f(z) = 0\}$. Then we have $\mathcal{F}\nu(\Omega) = \mathcal{C}_z^{[\gamma]}(\Omega)$ and $\mathcal{F}\nu(\Omega, E) = \mathcal{C}_z^{[\gamma]}(\Omega, E)$ with the weight ν and generator $(T^E, T^{\mathbb{K}})$ for $(\mathcal{F}\nu, E)$ described above. Due to [17, 3.1 Bemerkung, p. 141] the spaces $F(\Omega)$ and $F(\Omega, E)$, equipped with the topology τ_c of compact convergence, are ε-compatible. Obviously, $(T^E, T^{\mathbb{K}})$ is a strong, consistent family for $(\mathcal{F}, E)$. In addition, $\mathcal{F}\nu(\Omega) = \mathcal{C}_z^{[\gamma]}(\Omega)$ is a Banach space by [179, Proposition 1.6.2, p. 20]. For all f from the closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ of $\mathcal{F}\nu(\Omega)$ we have

$$|f(x) - f(y)| \leq \mathrm{d}(x,y)^\gamma, \quad x, y \in \Omega,$$

and

$$|f(x)| = |f(x) - f(z)| \leq \mathrm{d}(x,z)^\gamma, \quad x \in \Omega.$$

It follows that $B_{\mathcal{F}\nu(\Omega)}$ is (uniformly) equicontinuous and $\{f(x) \mid f \in B_{\mathcal{F}\nu(\Omega)}\}$ is bounded in $\mathbb{K}$ for all $x \in \Omega$. Ascoli's theorem (see e.g. [133, Theorem 47.1, p. 290]) implies the compactness of $B_{\mathcal{F}\nu(\Omega)}$ in $F(\Omega)$ (see also [118, 3.7 Theorem (a), p. 10]). Furthermore, the ε-compatibility of $F(\Omega)$ and $F(\Omega, E)$ in combination with the consistency of $(T^E, T^{\mathbb{K}})$ for (F, E) gives $\mathcal{F}_\varepsilon\nu(\Omega, E) = \mathcal{F}\nu(\Omega, E)$ as linear spaces by Proposition 5.2.25 c). We note that $\lim_\iota f_\iota(x) = \lim_\iota T^E(f_\iota)(x, z)$ for all x in U, proving our claim by Corollary 5.4.2. □

The space $\mathcal{C}_z^{[\gamma]}(\Omega)$ is named $\mathrm{Lip}_0(\Omega^\gamma)$ in [179] (see [179, Definition 1.6.1 (b), p. 19] and [179, Definition 1.1.2, p. 2]). Corollary 5.4.3 generalises [179, Proposition 2.1.7, p. 38] (in combination with [179, Proposition 1.2.4, p. 5]) where Ω is compact, $U = \Omega$ and $E = \mathbb{K}$.

5.4.4. Corollary. *Let E be a Banach space, $\Omega \subset \mathbb{R}^d$ open and bounded, $k \in \mathbb{N}_0$ and $0 < \gamma \leq 1$. In the case $k \geq 1$, assume additionally that Ω has Lipschitz boundary. If $(f_\iota)_{\iota \in I}$ is a bounded net in $\mathcal{C}^{k,\gamma}(\overline{\Omega}, E)$ such that*

- *(i)* $\lim_\iota f_\iota(x)$ *exists for all x in a dense subset $U \subset \Omega$, or if*
- *(ii)* $\lim_\iota (\partial^{e_n})^E f_\iota(x)$ *exists for all $1 \leq n \leq d$ and x in a dense subset $U \subset \Omega$, Ω is connected and there is $x_0 \in \overline{\Omega}$ such that* $\lim_\iota f_\iota(x_0)$ *exists and $k \geq 1$,*

then there is $f \in \mathcal{C}^{k,\gamma}(\overline{\Omega}, E)$ such that $(f_\iota)_{\iota \in I}$ converges to f in $\mathcal{C}^k(\overline{\Omega}, E)$.

PROOF. As in Corollary 5.3.3 we take $F(\Omega) := \mathcal{C}^k(\overline{\Omega})$ and $F(\Omega, E) := \mathcal{C}^k(\overline{\Omega}, E)$ as well as $\mathcal{F}\nu(\Omega) := \mathcal{C}^{k,\gamma}(\overline{\Omega})$ and $\mathcal{F}\nu(\Omega, E) := \mathcal{C}^{k,\gamma}(\overline{\Omega}, E)$ with the weight ν and generator $(T^E, T^{\mathbb{K}})$ for $(\mathcal{F}\nu, E)$ described above of Corollary 5.3.3. By the proof of Corollary 5.3.3 all conditions of Corollary 5.4.2 are satisfied, which implies our statement. □

We recall that $\mathcal{CW}^k(\Omega, E)$ is the space $\mathcal{C}^k(\Omega, E)$ equipped with its usual topology for an open set $\Omega \subset \mathbb{R}^d$, $k \in \mathbb{N}_\infty \cup \{0\}$ and an lcHs E (see Example 3.1.9 b) for $k \in \mathbb{N}_\infty$ and the definition above Proposition 3.1.11 for $k = 0$).

5.4.5. COROLLARY. *Let E be a Banach space, $\Omega \subset \mathbb{R}^d$ open, $k \in \mathbb{N}_0$ and $0 < \gamma \leq 1$. If $(f_\iota)_{\iota \in I}$ is a bounded net in $\mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$ such that*

(i) $\lim_\iota f_\iota(x)$ exists for all x in a dense subset $U \subset \Omega$, or if

(ii) $\lim_\iota (\partial^{e_n})^E f_\iota(x)$ exists for all $1 \leq n \leq d$ and x in a dense subset $U \subset \Omega$, Ω is connected and there is $x_0 \in \Omega$ such that $\lim_\iota f_\iota(x_0)$ exists and $k \geq 1$,

then there is $f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$ such that $(f_\iota)_{\iota \in I}$ converges to f in $\mathcal{CW}^k(\Omega, E)$.

PROOF. Let $(\Omega_n)_{n \in \mathbb{N}}$ be an exhaustion of Ω with open, relatively compact sets $\Omega_n \subset \Omega$ such that Ω_n has Lipschitz boundary, $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$ and, in addition, $x_0 \in \Omega_1$ and Ω_n is connected for each $n \in \mathbb{N}$ in case (ii) (see the proof of Corollary 5.3.5). The restriction of $(f_\iota)_{\iota \in I}$ to Ω_n is a bounded net in $\mathcal{C}^{k,\gamma}(\overline{\Omega}_n, E)$ for each $n \in \mathbb{N}$. By Corollary 5.4.4 there is $F_n \in \mathcal{C}^{k,\gamma}(\overline{\Omega}_n, E)$ for each $n \in \mathbb{N}$ such that the restriction of $(f_\iota)_{\iota \in I}$ to Ω_n converges to F_n in $\mathcal{C}^k(\overline{\Omega}_n, E)$ since $U \cap \Omega_n$ is dense in Ω_n due to Ω_n being open and x_0 being an element of the connected set Ω_n in case (ii). The limits F_{n+1} and F_n coincide on Ω_n for each $n \in \mathbb{N}$. Thus the definition $f := F_n$ on Ω_n for each $n \in \mathbb{N}$ gives a well-defined function $f \in \mathcal{C}^{k,\gamma}_{loc}(\Omega, E)$, which is a limit of $(f_\iota)_{\iota \in I}$ in $\mathcal{CW}^k(\Omega, E)$. □

5.4.6. COROLLARY. *Let E be a Banach space, $\Omega \subset \mathbb{R}^d$ open and $k \in \mathbb{N}_0$. If $(f_\iota)_{\iota \in I}$ is a bounded net in $\mathcal{C}^{k+1}(\Omega, E)$ such that*

(i) $\lim_\iota f_\iota(x)$ exists for all x in a dense subset $U \subset \Omega$, or if

(ii) $\lim_\iota (\partial^{e_n})^E f_\iota(x)$ exists for all $1 \leq n \leq d$ and x in a dense subset $U \subset \Omega$, Ω is connected and there is $x_0 \in \Omega$ such that $\lim_\iota f_\iota(x_0)$ exists,

then there is $f \in \mathcal{C}^{k,1}_{loc}(\Omega, E)$ such that $(f_\iota)_{\iota \in I}$ converges to f in $\mathcal{CW}^k(\Omega, E)$.

PROOF. By Corollary 5.3.5 b) $(f_\iota)_{\iota \in I}$ is a bounded net in $\mathcal{C}^{k,1}_{loc}(\Omega, E)$. Hence our statement is a consequence of Corollary 5.4.5. □

The preceding result directly implies a $\mathcal{C}^\infty$-smooth version.

5.4.7. COROLLARY. *Let E be a Banach space and $\Omega \subset \mathbb{R}^d$ open. If $(f_\iota)_{\iota \in I}$ is a bounded net in $\mathcal{C}^\infty(\Omega, E)$ such that*

(i) $\lim_\iota f_\iota(x)$ exists for all x in a dense subset $U \subset \Omega$, or if

(ii) $\lim_\iota (\partial^{e_n})^E f_\iota(x)$ exists for all $1 \leq n \leq d$ and x in a dense subset $U \subset \Omega$, Ω is connected and there is $x_0 \in \Omega$ such that $\lim_\iota f_\iota(x_0)$ exists,

then there is $f \in \mathcal{C}^\infty(\Omega, E)$ such that $(f_\iota)_{\iota \in I}$ converges to f in $\mathcal{CW}^\infty(\Omega, E)$.

Now, we turn to weighted kernels of hypoelliptic linear partial differential operators.

5.4.8. COROLLARY. *Let E be a Banach space, $\Omega \subset \mathbb{R}^d$ open, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator, $\nu\colon \Omega \to (0, \infty)$ continuous and $U \subset \Omega$ a set of uniqueness for $(\mathrm{id}_{\mathbb{K}^\Omega}, \mathcal{C}\nu_{P(\partial)})$. If $(f_\iota)_{\iota \in I}$ is a bounded net in $(\mathcal{C}\nu_{P(\partial)}(\Omega, E), |\cdot|_\nu)$ such that $\lim_\iota f_\iota(x)$ exists for all $x \in U$, then there is $f \in \mathcal{C}\nu_{P(\partial)}(\Omega, E)$ such that $(f_\iota)_{\iota \in I}$ converges to f in $(\mathcal{C}^\infty_{P(\partial)}(\Omega, E), \tau_c)$.*

PROOF. Our statement follows from Corollary 5.4.2 since by the proof of Corollary 5.2.30 all conditions needed are fulfilled. □

For $\nu = 1$ on Ω the preceding corollary is included in [70, Corollary 4.2, p. 695] but then an even better result is available, whose proof we prepare next. We recall the definition of the space $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\beta)$ with the strict topology β from Proposition 4.2.24. For an open set $\Omega\subset\mathbb{R}^d$, an lcHs E and a linear partial differential operator $P(\partial)^E$ which is hypoelliptic if $E=\mathbb{K}$ the space of bounded zero solutions is

$$\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E)=\{f\in\mathcal{C}^{\infty}_{P(\partial)}(\Omega,E)\mid \forall\ \alpha\in\mathfrak{A}:\ \|f\|_{\infty,\alpha}=\sup_{x\in\Omega}p_{\alpha}(f(x))<\infty\}.$$

We equip this space with strict topology β induced by the seminorms

$$|f|_{\widetilde{\nu},\alpha}:=\sup_{x\in\Omega}p_{\alpha}(f(x))|\widetilde{\nu}(x)|,\quad f\in\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),$$

for $\widetilde{\nu}\in\mathcal{C}_0(\Omega)$. Now, we phrase for $\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E)=\mathcal{C}\nu_{P(\partial)}(\Omega,E)$ with $\nu=1$ on Ω the improved version of Corollary 5.4.8.

5.4.9. COROLLARY. *Let E be a Banach space, $\Omega\subset\mathbb{R}^d$ open, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator and $U\subset\Omega$ a set of uniqueness for $(\mathrm{id}_{\mathbb{K}^{\Omega}},\mathcal{C}^{\infty}_{P(\partial),b})$. If $(f_{\iota})_{\iota\in I}$ is a bounded net in $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\|\cdot\|_{\infty})$ such that $\lim_{\iota}f_{\iota}(x)$ exists for all $x\in U$, then there is $f\in\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E)$ such that $(f_{\iota})_{\iota\in I}$ converges to f in $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\beta)$.*

PROOF. We take $F(\Omega):=(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\beta)$ and $F(\Omega,E):=(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\beta)$ as well as $\mathcal{F}\nu(\Omega):=(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega),\|\cdot\|_{\infty})$ and $\mathcal{F}\nu(\Omega,E):=(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\|\cdot\|_{\infty})$ with the weight $\nu(x):=1$, $x\in\Omega$, and generator $(\mathrm{id}_{E^{\Omega}},\mathrm{id}_{\Omega^{\mathbb{K}}})$ for $(\mathcal{F}\nu,E)$. The generator is strong and consistent for (F,E) and $F(\Omega)$ and $F(\Omega,E)$ are ε-compatible by Proposition 4.2.24. The space $\mathcal{F}\nu(\Omega)$ is a Banach space as a closed subspace of the Banach space $(\mathcal{C}_b(\Omega),\|\cdot\|_{\infty})$. Its closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is τ_c-compact because $(\mathcal{C}^{\infty}_{P(\partial)}(\Omega),\tau_c)$ is a Fréchet–Schwartz space, in particular, a Montel space. Thus $B_{\mathcal{F}\nu(\Omega)}$ is $\|\cdot\|_{\infty}$-bounded and τ_c-compact, which implies that it is also β-compact by [45, Proposition 1 (viii), p. 586] and [45, Proposition 3, p. 590]. In addition, the ε-compatibility of $F(\Omega)$ and $F(\Omega,E)$ in combination with the consistency of $(\mathrm{id}_{E^{\Omega}},\mathrm{id}_{\mathbb{K}^{\Omega}})$ for (F,E) gives $\mathcal{F}_{\varepsilon}\nu(\Omega,E)=\mathcal{F}\nu(\Omega,E)$ as linear spaces by Proposition 5.2.25 c), verifying our statement by Corollary 5.4.2. □

A direct consequence of Corollary 5.4.9 is the following remark.

5.4.10. REMARK. Let E be a Banach space, $\Omega\subset\mathbb{R}^d$ open, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator and $(f_{\iota})_{\iota\in I}$ a bounded net in the space $(\mathcal{C}^{\infty}_{P(\partial),b}(\Omega,E),\|\cdot\|_{\infty})$. Then the following statements are equivalent:

(i) (f_{ι}) converges pointwise,
(ii) (f_{ι}) converges uniformly on compact subsets of Ω,
(iii) (f_{ι}) is β-convergent.

In the case of complex-valued bounded holomorphic functions of one variable, i.e. $E=\mathbb{C}$, $\Omega\subset\mathbb{C}$ is open and $P(\partial)=\overline{\partial}$ is the Cauchy–Riemann operator, convergence w.r.t. β is known as bounded convergence (see [147, p. 13–14, 16]) and the preceding remark is included in [148, 3.7 Theorem, p. 246] for connected sets Ω.

A similar improvement of Corollary 5.4.3 for the space $\mathcal{C}^{[\gamma]}_{z}(\Omega,E)$ of γ-Hölder continuous functions on a metric space (Ω,d) that vanish at a given point $z\in\Omega$ is possible, using the strict topology β on $\mathcal{C}_z^{[\gamma]}(\Omega)$ given by the seminorms

$$|f|_\nu := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\gamma}|\nu(x,y)|, \quad f\in\mathcal{C}_z^{[\gamma]}(\Omega),$$

for $\nu\in\mathcal{C}_0(\omega)$ with $\omega=\Omega^2\smallsetminus\{(x,x)\mid x\in\Omega\}$. If Ω is compact and E a Banach space, this follows as in Corollary 5.4.9 from the observation that β is the mixed topology $\gamma(|\cdot|_{\mathcal{C}^{0,\gamma}(\Omega)},\tau_c)$ by [90, Theorem 3.3, p. 645], that a set is β-compact if and only if it is $|\cdot|_{\mathcal{C}^{0,\gamma}(\Omega)}$-bounded and τ_c-compact by [90, Theorem 2.1 (6), p. 642], the ε-compatibility $(\mathcal{C}_z^{[\gamma]}(\Omega),\beta)\varepsilon E\cong(\mathcal{C}_z^{[\gamma]}(\Omega,E),\gamma\tau_\gamma)$ by [90, Theorem 4.4, p. 648] where the topology $\gamma\tau_\gamma$ is described in [90, Definition 4.1, p. 647] and coincides with β if $E=\mathbb{K}$ by [90, Proposition 4.3 (i), p. 647].

Let us turn to Bloch type spaces. The result corresponding to Corollary 5.4.8 for Bloch type spaces reads as follows.

5.4.11. Corollary. *Let E be a Banach space, $\nu\colon\mathbb{D}\to(0,\infty)$ continuous and $U_*\subset\mathbb{D}$ have an accumulation point in $\mathbb{D}$. If $(f_\iota)_{\iota\in I}$ is a bounded net in $\mathcal{B}\nu(\mathbb{D},E)$ such that $\lim_\iota f_\iota(0)$ and $\lim_\iota(\partial_{\mathbb{C}}^1)^E f_\iota(z)$ exist for all $z\in U_*$, then there is $f\in\mathcal{B}\nu(\mathbb{D},E)$ such that $(f_\iota)_{\iota\in I}$ converges to f in $(\mathcal{O}(\mathbb{D},E),\tau_c)$.*

Proof. Due to the proof of Corollary 5.2.33 all conditions needed to apply Corollary 5.4.2 are fulfilled, which proves our statement. □

5.5. Wolff type results

The following theorem gives us a Wolff type description of the dual of $F(\Omega)$ and generalises [70, Theorem 3.3, p. 693] and [70, Corollary 3.4, p. 694] whose proofs only need a bit of adaptation. *Wolff's theorem* [183, p. 1327] (cf. [81, Theorem (Wolff), p. 402]) phrased in a functional analytic way (see [70, p. 240]) says: if $\Omega\subset\mathbb{C}$ is a domain (i.e. open and connected), then for each $\mu\in\mathcal{O}(\Omega)'$ there are a sequence $(z_n)_{n\in\mathbb{N}}$ which is relatively compact in Ω and a sequence $(a_n)_{n\in\mathbb{N}}$ in ℓ^1 such that $\mu=\sum_{n=1}^\infty a_n\delta_{z_n}$.

5.5.1. Theorem. *Let $F(\Omega)$ and $F(\Omega,E)$ be ε-into-compatible, $(T^E,T^{\mathbb{K}})$ be a generator for $(\mathcal{F}\nu,E)$ and a strong, consistent family for (F,E) for every Banach space E. Let $F(\Omega)$ be a nuclear Fréchet space and $\mathcal{F}\nu(\Omega)$ a Banach space whose closed unit ball $B_{\mathcal{F}\nu(\Omega)}$ is a compact subset of $F(\Omega)$ and $(x_n)_{n\in\mathbb{N}}$ fixes the topology in $\mathcal{F}\nu(\Omega)$.*

a) *Then there is $0<\lambda\in\ell^1$, i.e. $\lambda\in\ell^1$ and $\lambda_n>0$ for all $n\in\mathbb{N}$, such that for every bounded $B\subset F(\Omega)_b'$ there is $C\geq1$ with*

$$\{\mu_{|\mathcal{F}\nu(\Omega)}\mid\mu\in B\}\subset\{\sum_{n=1}^\infty a_n\nu(x_n)T_{x_n}^{\mathbb{K}}\in\mathcal{F}\nu(\Omega)'\mid a\in\ell^1,\ \forall\,n\in\mathbb{N}\colon|a_n|\leq C\lambda_n\}.$$

b) *Let $(\|\cdot\|_k)_{k\in\mathbb{N}}$ denote the system of seminorms generating the topology of $F(\Omega)$. Then there is a decreasing zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that for all $k\in\mathbb{N}$ there is $C\geq1$ with*

$$\|f\|_k\leq C\sup_{n\in\mathbb{N}}|T^{\mathbb{K}}(f)(x_n)|\nu(x_n)\varepsilon_n,\quad f\in\mathcal{F}\nu(\Omega).$$

Proof. We start with part a). Let $B_1:=\{T_{x_n}^{\mathbb{K}}(\cdot)\nu(x_n)\mid n\in\mathbb{N}\}\subset F(\Omega)'$, $X:=\operatorname{span}B_1$, $Y:=F(\Omega)$, $Z:=\mathcal{F}\nu(\Omega)$ and $E_1:=\{\sum_{n=1}^\infty a_n\nu(x_n)T_{x_n}^{\mathbb{K}}\mid a\in\ell^1\}$. From

$$|j_1(a)(f)|:=|\sum_{n=1}^\infty a_n\nu(x_n)T_{x_n}^{\mathbb{K}}(f)|\leq\sup_{n\in\mathbb{N}}|T^{\mathbb{K}}(f)(x_n)|\nu(x_n)\|a\|_{\ell^1}\leq|f|_{\mathcal{F}\nu(\Omega)}\|a\|_{\ell^1}$$

for all $f\in\mathcal{F}\nu(\Omega)$ and $a\in\ell^1$ it follows that E_1 is a linear subspace of $\mathcal{F}\nu(\Omega)'$ and the continuity of the map $j_1\colon\ell^1\to\mathcal{F}\nu(\Omega)'$ where $\mathcal{F}\nu(\Omega)'$ is equipped with the operator norm. In addition, we deduce that the linear map $j\colon \ell^1/\ker j_1 \to \mathcal{F}\nu(\Omega)'$, $j([a]) := j_1(a)$, where $[a]$ denotes the equivalence class of $a \in \ell^1$ in the quotient space $\ell^1/\ker j_1$, is continuous w.r.t. the quotient norm since

$$\|j([a])\|_{\mathcal{F}\nu(\Omega)'} \leq \inf_{b\in\ell^1,[b]=[a]} \|b\|_{\ell^1} = \|[a]\|_{\ell^1/\ker j_1}.$$

By setting $E := j(\ell^1/\ker j_1)$ and $\|j([a])\|_E := \|[a]\|_{\ell^1/\ker j_1}$, $a \in \ell^1$, and observing that $\ell^1/\ker j_1$ is a Banach space, we obtain that E is also a Banach space, which is continuously embedded in $\mathcal{F}\nu(\Omega)'$.

We denote by $\mathsf{A}\colon X \to E$ the restriction to $Z = \mathcal{F}\nu(\Omega)$ determined by

$$\mathsf{A}(T^{\mathbb{K}}_{x_n}(\cdot)\nu(x_n)) := T^{\mathbb{K}}_{x_n}(\cdot)_{|\mathcal{F}\nu(\Omega)}\nu(x_n) = j([e_n])$$

where e_n is the n-th unit sequence in ℓ^1. We consider $\mathcal{F}\nu(\Omega)$ as a subspace of E' via

$$f(j([a])) := j([a])(f) = \sum_{n=1}^{\infty} a_n \nu(x_n) T^{\mathbb{K}}(f)(x_n), \quad a \in \ell^1,$$

for $f \in \mathcal{F}\nu(\Omega)$. The space $G := \mathcal{F}\nu(\Omega)$ clearly separates the points of E, thus is $\sigma(E', E)$-dense and

$$(f \circ \mathsf{A})(T^{\mathbb{K}}_{x_n}(\cdot)\nu(x_n)) = \mathsf{A}(T^{\mathbb{K}}_{x_n}(\cdot)\nu(x_n))(f) = j([e_n])(f) = f(j([e_n]))$$

for all $n \in \mathbb{N}$. Hence we may consider $f \circ \mathsf{A}$ by identification with f as an element of $Z = \mathcal{F}\nu(\Omega)$ for all $f \in G = \mathcal{F}\nu(\Omega)$. It follows from Proposition 5.2.68 that there is a unique extension $\widehat{\mathsf{A}} \in F(\Omega)\varepsilon E$ of A such that $S(\widehat{\mathsf{A}}) \in \mathcal{F}_\varepsilon\nu(\Omega, E)$.

For each $e' \in E'$ there are $C_0, C_1 > 0$ and an absolutely convex compact set $K \subset F(\Omega)$ such that

$$|(e' \circ \widehat{\mathsf{A}})(\mu)| \leq C_0 \|\widehat{\mathsf{A}}(\mu)\|_E \leq C_0 C_1 \sup_{f\in K} |\mu(f)|$$

for all $\mu \in F(\Omega)'$, implying $e' \circ \widehat{\mathsf{A}} \in (F(\Omega)'_b)'$. Due to the reflexivity of the nuclear Fréchet space $F(\Omega)$ we obtain $e' \circ \widehat{\mathsf{A}} \in F(\Omega)$ for each $e' \in E'$. Further, for each $e' \in E'$ we have

$$\begin{aligned}\|e' \circ \widehat{\mathsf{A}}\|_{\mathcal{F}\nu(\Omega)} &= \sup_{x\in\omega} |T^{\mathbb{K}}(e' \circ \widehat{\mathsf{A}})(x)|\nu(x) = \sup_{x\in\omega} |(e' \circ \widehat{\mathsf{A}})(T^{\mathbb{K}}_x(\cdot)\nu(x))| \\ &\leq C_0 \sup_{x\in\omega} \|\widehat{\mathsf{A}}(T^{\mathbb{K}}_x(\cdot)\nu(x))\|_E < \infty\end{aligned}$$

since $\widehat{\mathsf{A}}(B^{\circ F(\Omega)'}_{\mathcal{F}\nu(\Omega)})$ is bounded in E. This yields $e' \circ \widehat{\mathsf{A}} \in \mathcal{F}\nu(\Omega)$ for each $e' \in E'$. In particular, we get that $\widehat{\mathsf{A}}$ is $\sigma(F(\Omega)', \mathcal{F}\nu(\Omega))$-$\sigma(E, E')$ continuous. The restriction $r\colon F(\Omega)' \to \mathcal{F}\nu(\Omega)'$, $r(\mu) := \mu_{|\mathcal{F}\nu(\Omega)}$, is $\sigma(F(\Omega)', \mathcal{F}\nu(\Omega))$-$\sigma(\mathcal{F}\nu(\Omega)', \mathcal{F}\nu(\Omega))$ continuous and coincides with $\widehat{\mathsf{A}}$ on the $\sigma(F(\Omega)', \mathcal{F}\nu(\Omega))$-dense subspace $X = \operatorname{span} B_1 \subset F(\Omega)'$. Therefore $\widehat{\mathsf{A}}(\mu) = r(\mu) = \mu_{|\mathcal{F}\nu(\Omega)}$ for all $\mu \in F(\Omega)'$.

Let B be an absolutely convex, closed and bounded subset of $F(\Omega)'_b$. We endow $W := \operatorname{span} B$ with the Minkowski functional of B. Due to the nuclearity of $F(\Omega)$, there are an absolutely convex, closed and bounded subset $V \subset F(\Omega)'_b$, $(w'_k)_{k\in\mathbb{N}} \subset B_{W'}$, $(\mu_k)_{k\in\mathbb{N}} \subset V$ and $0 \leq \gamma \in \ell^1$, i.e. $\gamma \in \ell^1$ and $\gamma_n \geq 0$ for all $n \in \mathbb{N}$, such that

$$\mu = \sum_{k=1}^{\infty} \gamma_k w'_k(\mu)\mu_k, \quad \mu \in B,$$

by [24, 2.9.1 Theorem, p. 134, 2.9.2 Definition, p. 135]. The boundedness of $\widehat{\mathsf{A}}(V)$ in E and the definition of E give us a bounded sequence $([\beta^{(k)}])_{k\in\mathbb{N}} \subset E$ with

$$\mu_{k|\mathcal{F}\nu(\Omega)} = \widehat{\mathsf{A}}(\mu_k) = \sum_{n=1}^{\infty} \beta^{(k)}_n \nu(x_n) T^{\mathbb{K}}_{x_n}$$

for all $k \in \mathbb{N}$. The sequence $(\beta^{(k)})_{k\in\mathbb{N}} \subset \ell^1$ is also bounded by [131, Remark 5.11, p. 36] and we set $\rho_n := \sum_{k=1}^{\infty} \gamma_k |\beta_n^{(k)}|$ for $n \in \mathbb{N}$. With $\rho := (\rho_n)_{n\in\mathbb{N}}$ we have

$$\|\rho\|_{\ell^1} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \gamma_k |\beta_n^{(k)}| \leq \sum_{n=1}^{\infty} \sup_{l\in\mathbb{N}} |\beta_n^{(l)}| \sum_{k=1}^{\infty} \gamma_k = \sup_{l\in\mathbb{N}} \|\beta^{(l)}\|_{\ell^1} \|\gamma\|_{\ell^1} < \infty,$$

which means that $\rho \in \ell^1$. For every $\mu \in B$ we set $a_n := \sum_{k=1}^{\infty} \gamma_k w_k'(\mu)\beta_n^{(k)}$, $n \in \mathbb{N}$, and conclude that $a \in \ell^1$ with $|a_n| \leq \rho_n$ for all $n \in \mathbb{N}$ and

$$\mu_{|\mathcal{F}\nu(\Omega)} = \sum_{n=1}^{\infty} a_n \nu(x_n) T_{x_n}^{\mathbb{K}}. \tag{57}$$

The strong dual $F(\Omega)_b'$ of the Fréchet–Schwartz space $F(\Omega)$ is a DFS-space and thus there is a fundamental sequence of bounded (closed, absolutely convex) sets $(B_l)_{l\in\mathbb{N}}$ in $F(\Omega)_b'$ by [131, Proposition 25.19, p. 303]. Due to our preceding results there is $\rho^{(l)} \in \ell^1$ with (57) for each $l \in \mathbb{N}$. Finally, part a) follows from choosing $0 < \lambda \in \ell^1$ such that each $\rho^{(l)}$ is componentwise smaller than a multiple of λ, i.e. we choose λ in a way that for each $l \in \mathbb{N}$ there is $C_l \geq 1$ with $\rho_n^{(l)} \leq C_l\lambda_n$ for all $n \in \mathbb{N}$ (w.l.o.g. we may assume (the worst case) that $\lim_{n\to\infty} \rho_n^{(l+1)}/\rho_n^{(l)} = \infty$ for each $l \in \mathbb{N}$. Then the construction of a suitable $0 < \lambda \in \ell^1$ is given in [97, Chap. IX, §41, 7., p. 301–302]: set $c_n^{(l)} := \rho_n^{(l)}$ for all $l, n \in \mathbb{N}$ and define $\lambda_n := c_n + \frac{1}{n^2}$ for all $n \in \mathbb{N}$ with the $(c_n) \in \ell^1$ constructed there. Then set $C_1 := 1$ and $C_l := (\max\{c_n^{(l)} \mid 1 \leq n \leq n_{l-1}\}/\min\{\lambda_n \mid 1 \leq n \leq n_{l-1}\}) + 1$, $l \geq 2$, for the sequence of indices $(n_l)_{l\in\mathbb{N}}$ from the construction of (c_n).).

Let us turn to part b). We choose $\lambda \in \ell^1$ from part a) and a decreasing zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $(\frac{\lambda_n}{\varepsilon_n})_{n\in\mathbb{N}}$ still belongs to ℓ^1 (e.g. take $\varepsilon_n := (\sum_{k=n}^{\infty} \lambda_k)^{1/2}$ for $n \in \mathbb{N}$ by [97, Chap. IX, §39, Theorem of Dini, p. 293]). For $k \in \mathbb{N}$ we set

$$\widetilde{B}_k := \{f \in F(\Omega) \mid \|f\|_k \leq 1\}$$

and note that the polar $\widetilde{B}_k^{\circ}$ is bounded in $F(\Omega)_b'$. Due to part a) there exists $C \geq 1$ such that

$$\widehat{\mathsf{A}}(\widetilde{B}_k^{\circ}) \subset \{\sum_{n=1}^{\infty} a_n \nu(x_n) T_{x_n}^{\mathbb{K}} \in \mathcal{F}\nu(\Omega)' \mid a \in \ell^1,\ \forall\ n \in \mathbb{N}:\ |a_n| \leq C\lambda_n\}.$$

By [131, Proposition 22.14, p. 256] the formula

$$\|f\|_k = \sup_{y'\in\widetilde{B}_k^{\circ}} |y'(f)|, \quad f \in F(\Omega),$$

is valid and hence

$$\begin{aligned}\|f\|_k &= \sup_{y'\in\widetilde{B}_k^{\circ}} |r(y')(f)| = \sup_{y'\in\widetilde{B}_k^{\circ}} |\widehat{\mathsf{A}}(y')(f)| \leq C \sup_{\substack{a\in\ell^1\\ |a_n|\leq\lambda_n}} \Big|\sum_{n=1}^{\infty} a_n \nu(x_n) T^{\mathbb{K}}(f)(x_n)\Big| \\ &\leq C \Big\|\Big(\frac{\lambda_n}{\varepsilon_n}\Big)_n\Big\|_{\ell^1} \sup_{n\in\mathbb{N}} |T^{\mathbb{K}}(f)(x_n)|\nu(x_n)\varepsilon_n\end{aligned}$$

for all $f \in \mathcal{F}\nu(\Omega)$. □

5.5.2. Remark. The proof of Theorem 5.5.1 shows it is not needed that the assumption that $F(\Omega)$ and $F(\Omega, E)$ are ε-into-compatible, $(T^E, T^{\mathbb{K}})$ is a generator for $(\mathcal{F}\nu, E)$ and a strong, consistent family for (F, E) is fulfilled for every Banach space E. It is sufficient that it is fulfilled for the Banach space $E := j(\ell^1/\ker j_1)$.

We recall from (53) that for a positive sequence $\nu := (\nu_n)_{n\in\mathbb{N}}$ and an lcHs E we have

$$\ell\nu(\mathbb{N}, E) = \{x = (x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}} \mid \forall\ \alpha \in \mathfrak{A}:\ \|x\|_{\alpha} = \sup_{n\in\mathbb{N}} p_{\alpha}(x_n)\nu_n < \infty\}.$$

Further, we equip the space $E^{\mathbb{N}}$ of all sequences in E from Example 4.2.1 with the topology of pointwise convergence, i.e. the topology generated by the seminorms

$$|x|_{k,\alpha} := \sup_{1\leq n\leq k} p_\alpha(x_n), \quad x = (x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}},$$

for $k \in \mathbb{N}$ and $\alpha \in \mathfrak{A}$.

5.5.3. Corollary. *Let $\nu := (\nu_n)_{n\in\mathbb{N}}$ be a positive sequence.*

a) Then there is $0 < \lambda \in \ell^1$ such that for every bounded $B \subset (\mathbb{K}^{\mathbb{N}})_b'$ there is $C \geq 1$ with

$$\{\mu_{|\ell\nu(\mathbb{N})} \mid \mu \in B\} \subset \{\sum_{n=1}^{\infty} a_n\nu_n\delta_n \in \ell\nu(\mathbb{N})' \mid a \in \ell^1,\ \forall\, n \in \mathbb{N}:\ |a_n| \leq C\lambda_n\}.$$

b) Then there is a decreasing zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$ there is $C \geq 1$ with

$$\sup_{1\leq n\leq k} |x_n| \leq C \sup_{n\in\mathbb{N}} |x_n|\nu_n\varepsilon_n, \quad x = (x_n)_{n\in\mathbb{N}} \in \ell\nu(\mathbb{N}).$$

Proof. We take $F(\mathbb{N}) := \mathbb{K}^{\mathbb{N}}$ and $F(\mathbb{N},E) := E^{\mathbb{N}}$ as well as $\mathcal{F}\nu(\mathbb{N}) := \ell\nu(\mathbb{N})$ and $\mathcal{F}\nu(\mathbb{N},E) := \ell\nu(\mathbb{N},E)$ where $(T^E,T^{\mathbb{K}}) := (\mathrm{id}_{E^{\mathbb{N}}},\mathrm{id}_{\mathbb{K}^{\mathbb{N}}})$ is the generator for $(\mathcal{F}\nu,E)$. We remark that $F(\mathbb{N})$ and $F(\mathbb{N},E)$ are ε-compatible and $(T^E,T^{\mathbb{K}})$ is a strong, consistent family for (F,E) by Example 4.2.1 for every Banach space E. Moreover, $\mathcal{F}\nu(\mathbb{N}) = \ell\nu(\mathbb{N})$ is a Banach space by [131, Lemma 27.1, p. 326] since $\ell\nu(\mathbb{N}) = \lambda^\infty(A)$ with the Köthe matrix $A := (a_{n,j})_{n,j\in\mathbb{N}}$ given by $a_{n,j} := \nu_n$ for all $n,j \in \mathbb{N}$. In addition, we have for every $k \in \mathbb{N}$

$$\sup_{1\leq n\leq k} |x_n| \leq \sup_{1\leq n\leq k} \nu_n^{-1}|x|_\nu \leq \sup_{1\leq n\leq k} \nu_n^{-1}, \quad x = (x_n)_{n\in\mathbb{N}} \in B_{\mathcal{F}\nu(\mathbb{N})},$$

which means that $B_{\mathcal{F}\nu(\mathbb{N})}$ is bounded in $F(\mathbb{N})$. The space $F(\mathbb{N}) = \mathbb{K}^{\mathbb{N}}$ is a nuclear Fréchet space and $B_{\mathcal{F}\nu(\mathbb{N})}$ is obviously closed in $\mathbb{K}^{\mathbb{N}}$. Thus the bounded and closed set $B_{\mathcal{F}\nu(\mathbb{N})}$ is compact in $F(\mathbb{N})$, implying our statement by Theorem 5.5.1. □

5.5.4. Corollary. *Let $\Omega \subset \mathbb{R}^d$ be open, $P(\partial)^{\mathbb{K}}$ a hypoelliptic linear partial differential operator, $\nu\colon\Omega \to (0,\infty)$ continuous and $(x_n)_{n\in\mathbb{N}}$ fix the topology in $\mathcal{C}\nu_{P(\partial)}(\Omega)$.*

a) Then there is $0 < \lambda \in \ell^1$ such that for every bounded $B \subset (\mathcal{C}^\infty_{P(\partial)}(\Omega),\tau_c)_b'$ there is $C \geq 1$ with

$$\{\mu_{|\mathcal{C}\nu_{P(\partial)}(\Omega)} \mid \mu \in B\} \subset \{\sum_{n=1}^{\infty} a_n\nu(x_n)\delta_{x_n} \in \mathcal{C}\nu_{P(\partial)}(\Omega)' \mid a \in \ell^1,\ \forall\, n \in \mathbb{N}:\ |a_n| \leq C\lambda_n\}.$$

b) Then there is a decreasing zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that for all compact $K \subset \Omega$ there is $C \geq 1$ with

$$\sup_{x\in K} |f(x)| \leq C \sup_{n\in\mathbb{N}} |f(x_n)|\nu(x_n)\varepsilon_n, \quad f \in \mathcal{C}\nu_{P(\partial)}(\Omega).$$

Proof. Due to the proof of Corollary 5.2.30 and the observation that the space $F(\Omega) = (\mathcal{C}^\infty_{P(\partial)}(\Omega),\tau_c)$ is a nuclear Fréchet space all conditions of Theorem 5.5.1 are fulfilled, which yields our statement. □

5.6. Series representation of vector-valued functions via Schauder decompositions

The purpose of this section is to lift series representations known from scalar-valued functions to vector-valued functions and its underlying idea was derived from the classical example of the (local) power series representation of a holomorphic function. We recall that a $\mathbb{C}$-valued function f on the open disc $\mathbb{D}_r(0)$ around zero with radius $r > 0$ belongs to the space $\mathcal{O}(\mathbb{D}_r(0))$ of holomorphic functions on $\mathbb{D}_r(0)$ if the limit

$$f^{(1)}(z) := \lim_{\substack{h\to 0\\ h\in\mathbb{C}, h\neq 0}} \frac{f(z+h)-f(z)}{h}, \quad z\in\mathbb{D}_r(0), \tag{58}$$

exists in $\mathbb{C}$. It is well-known that every $f\in\mathcal{O}(\mathbb{D}_r(0))$ can be written as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad z\in\mathbb{D}_r(0),$$

where the power series on the right-hand side converges uniformly on every compact subset of $\mathbb{D}_r(0)$ and $f^{(n)}(0)$ is the n-th complex derivative of f at 0 which is defined from (58) by the recursion $f^{(0)} := f$ and $f^{(n)} := (f^{(n-1)})^{(1)}$ for $n\in\mathbb{N}$. By [79, 2.1 Theorem and Definition, p. 17–18] and [79, 5.2 Theorem, p. 35], this series representation remains valid if f is a holomorphic function on $\mathbb{D}_r(0)$ with values in a locally complete locally convex Hausdorff space E over $\mathbb{C}$ where holomorphy means that the limit (58) exists in E and the higher complex derivatives are defined recursively as well. Analysing this example, we observe that $\mathcal{O}(\mathbb{D}_r(0))$, equipped with the topology τ_c of uniform convergence on compact subsets of $\mathbb{D}_r(0)$, is a Fréchet space, in particular, barrelled, with a Schauder basis formed by the monomials $z\mapsto z^n$. Further, the formulas for the complex derivatives of a $\mathbb{C}$-valued resp. an E-valued function f on $\mathbb{D}_r(0)$ are built up in the same way by (58) (see Chapter 2).

Our goal is to derive a mechanism which uses these observations and transfers known series representations for other spaces of scalar-valued functions to their vector-valued counterparts. Let us describe the general setting. We recall from [89, 14.2, p. 292] that a sequence (f_n) in a locally convex Hausdorff space F over a field $\mathbb{K}$ is called a *topological basis*, or simply a basis, if for every $f\in F$ there is a unique sequence of coefficients $(\lambda_n^{\mathbb{K}}(f))$ in $\mathbb{K}$ such that

$$f = \sum_{n=1}^{\infty} \lambda_n^{\mathbb{K}}(f) f_n \tag{59}$$

where the series converges in F. Due to the uniqueness of the coefficients the map $\lambda_n^{\mathbb{K}}\colon f\mapsto \lambda_n^{\mathbb{K}}(f)$ is well-defined, linear and called the *n-th coefficient functional associated to* (f_n). Further, for each $k\in\mathbb{N}$ the map

$$P_k\colon F\to F,\ P_k(f) := \sum_{n=1}^{k} \lambda_n^{\mathbb{K}}(f) f_n,$$

is a linear projection whose range is $\operatorname{span}\{f_1,\dots,f_n\}$ and it is called the *k-th expansion operator associated to* (f_n). A basis (f_n) of F is called *equicontinuous* if the expansion operators P_k form an equicontinuous sequence in the linear space $L(F,F)$ of continuous linear maps from F to F (see [89, 14.3, p. 296]). A basis (f_n) of F is called a *Schauder basis* if the coefficient functionals are continuous, i.e. $\lambda_n^{\mathbb{K}}\in F'$ for each $n\in\mathbb{N}$. In particular, this is already fulfilled if F is a Fréchet space by [131, Corollary 28.11, p. 351]. If F is barrelled, then a Schauder basis of F is already equicontinuous and F has the (bounded) approximation property by the uniform boundedness principle.

The starting point for our approach is equation (59). Let F and E be non-trivial locally convex Hausdorff spaces over a field $\mathbb{K}$ where F has an equicontinuous Schauder basis (f_n) with associated coefficient functionals $(\lambda_n^{\mathbb{K}})$. The expansion operators (P_k) form a so-called *Schauder decomposition* of F (see [27, p. 77]), i.e. they are continuous projections on F such that

(i) $P_kP_j = P_{\min(j,k)}$ for all $j,k\in\mathbb{N}$,
(ii) $P_k\neq P_j$ for $k\neq j$,

(iii) $(P_k f)$ converges to f for each $f \in F$.

This operator theoretic definition of a Schauder decomposition is equivalent to the usual definition in terms of closed subspaces of F given in [96, p. 377] (see [123, p. 219]). In our main Theorem 5.6.1 of this section we prove that $(P_k \varepsilon \operatorname{id}_E)$ is a Schauder decomposition of Schwartz' ε-product $F\varepsilon E$ and each $u \in F\varepsilon E$ has the series representation

$$u(f') = \sum_{n=1}^{\infty} u(\lambda_n^{\mathbb{K}}) f'(f_n), \quad f' \in F'.$$

Now, suppose that $F = \mathcal{F}(\Omega)$ is a space of $\mathbb{K}$-valued functions on a set Ω with a topology such that the point-evaluation functionals δ_x, $x \in \Omega$, belong to $\mathcal{F}(\Omega)'$ and that there is a locally convex Hausdorff space $\mathcal{F}(\Omega, E)$ of functions from Ω to E such that the map

$$S\colon \mathcal{F}(\Omega)\varepsilon E \to \mathcal{F}(\Omega, E),\ u \longmapsto [x \mapsto u(\delta_x)],$$

is an isomorphism, i.e. suppose that $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega, E)$ are ε-compatible. Assuming that for each $n \in \mathbb{N}$ and $u \in \mathcal{F}(\Omega)\varepsilon E$ there is $\lambda_n^E(S(u)) \in E$ with

$$\lambda_n^E(S(u)) = u(\lambda_n^{\mathbb{K}}), \tag{60}$$

i.e. $(\lambda^E, \lambda^{\mathbb{K}})$ is consistent, we obtain in Corollary 5.6.5 that $(S \circ (P_k \varepsilon \operatorname{id}_E) \circ S^{-1})_k$ is a Schauder decomposition of $\mathcal{F}(\Omega, E)$ and

$$f = \lim_{k\to\infty} (S \circ (P_k \varepsilon \operatorname{id}_E) \circ S^{-1})(f) = \sum_{n=1}^{\infty} \lambda_n^E(f) f_n, \quad f \in \mathcal{F}(\Omega, E),$$

which is the desired series representation in $\mathcal{F}(\Omega, E)$. In particular, the consistency condition (60) guarantees in the case of E-valued holomorphic functions on $\mathbb{D}_r(0)$ that the complex derivatives at 0 appear in the Schauder decomposition of $\mathcal{O}(\mathbb{D}_r(0), E)$ since $(\partial_{\mathbb{C}}^n)^E S(u)(0) = u(\delta_0 \circ (\partial_{\mathbb{C}}^n)^{\mathbb{C}})$ for all $u \in \mathcal{O}(\mathbb{D}_r(0))\varepsilon E$ and $n \in \mathbb{N}_0$ by Proposition 5.2.32 if E is locally complete. We apply our result to sequence spaces, spaces of continuously differentiable functions on a compact interval, the space of holomorphic functions, the Schwartz space and the space of smooth functions which are 2π-periodic in each variable.

As a byproduct of Theorem 5.6.1 we obtain that every element of the completion $F \widehat{\otimes}_\varepsilon E$ of the injective tensor product $F \otimes_\varepsilon E$ has a series representation as well if F is a complete space with an equicontinuous Schauder basis and E is complete. Concerning series representation in $F \widehat{\otimes}_\varepsilon E$, little seems to be known whereas for the completion $F \widehat{\otimes}_\pi E$ of the projective tensor product $F \otimes_\pi E$ of two metrisable locally convex spaces F and E it is well-known that every $f \in F \widehat{\otimes}_\pi E$ has a series representation

$$f = \sum_{n=1}^{\infty} a_n f_n \otimes e_n$$

where $(a_n) \in \ell^1$, i.e. (a_n) is absolutely summable, and (f_n) and (e_n) are null sequences in F and E, respectively (see e.g. [83, Chap. I, §2 , n°1, Théorème 1, p. 51] or [89, 15.6.4 Corollary, p. 334]). If F and E are metrisable and one of them is nuclear, then the isomorphism $F \widehat{\otimes}_\pi E \cong F \widehat{\otimes}_\varepsilon E$ holds and we trivially have a series representation of the elements of $F \widehat{\otimes}_\varepsilon E$ as well. Other conditions on the existence of series representations of the elements of $F \widehat{\otimes}_\varepsilon E$ can be found in [151, Proposition 4.25, p. 88], where F and E are Banach spaces and both of them have a Schauder basis, and in [91, Theorem 2, p. 283], where F and E are locally convex Hausdorff spaces and both of them have an equicontinuous Schauder basis.

5.6.1. Schauder decomposition. Let us start with our main theorem on Schauder decompositions of ε-products. We recall from (3) that we consider the tensor product $F \otimes E$ as a linear subspace of $F\varepsilon E$ for two locally convex Hausdorff spaces F and E by means of the linear injection

$$\Theta\colon F \otimes E \to F\varepsilon E,\ \sum_{n=1}^{k} f_n \otimes e_n \longmapsto \big[y \mapsto \sum_{n=1}^{k} y(f_n)e_n\big].$$

The next theorem is essentially due to José Bonet, improving a previous version of us which became Corollary 5.6.5.

5.6.1. THEOREM. *Let F and E be lcHs, $(f_n)_{n\in\mathbb{N}}$ an equicontinuous Schauder basis of F with associated coefficient functionals $(\lambda_n)_{n\in\mathbb{N}}$ and set $Q_n\colon F \to F$, $Q_n(f) := \lambda_n(f)f_n$ for every $n \in \mathbb{N}$. Then the following holds:*

a) The sequence $(P_k)_{k\in\mathbb{N}}$ given by $P_k := \big(\sum_{n=1}^{k} Q_n\big)\varepsilon\,\mathrm{id}_E$ is a Schauder decomposition of $F\varepsilon E$.

b) Each $u \in F\varepsilon E$ has the series representation

$$u(f') = \sum_{n=1}^{\infty} u(\lambda_n)f'(f_n), \quad f' \in F'.$$

c) $F \otimes E$ is sequentially dense in $F\varepsilon E$.

PROOF. Since (f_n) is a Schauder basis of F, the sequence $(\sum_{n=1}^{k} Q_n)$ converges to id_F in $L_\sigma(F,F)$. Thus we deduce from the equicontinuity of (f_n) that $(\sum_{n=1}^{k} Q_n)$ converges to id_F in $L_\kappa(F,F)$ by [89, Theorem 8.5.1 (b), p. 156]. For $f' \in F'$ and $f \in F$ it holds

$$\begin{aligned}(Q_n^t \circ Q_m^t)(f')(f) &= Q_m^t(f')(Q_n(f)) = Q_m^t(f')(\lambda_n(f)f_n) = f'(\lambda_m(\lambda_n(f)f_n)f_m)\\ &= \lambda_m(f_n)\lambda_n(f)f'(f_m) = \begin{cases}\lambda_n(f)f'(f_n) &, m = n,\\ 0 &, m \neq n,\end{cases}\end{aligned}$$

due to the uniqueness of the coefficient functionals (λ_n) (see [89, 14.2.1 Proposition, p. 292]) and it follows for $k, j \in \mathbb{N}$ that

$$\Big(\sum_{n=1}^{j} Q_n^t \circ \sum_{m=1}^{k} Q_m^t\Big)(f')(f) = \sum_{n=1}^{\min(j,k)} \lambda_n(f)f'(f_n) = \sum_{n=1}^{\min(j,k)} Q_n^t(f')(f).$$

This implies that

$$(P_kP_j)(u) = u \circ \sum_{n=1}^{j} Q_n^t \circ \sum_{m=1}^{k} Q_m^t = u \circ \sum_{n=1}^{\min(j,k)} Q_n^t = P_{\min(j,k)}(u)$$

for all $u \in F\varepsilon E$. If $k \neq j$, w.l.o.g. $k > j$, we choose $x \in E$, $x \neq 0$,[3] and consider $f_k \otimes x$ as an element of $F\varepsilon E$ via the map Θ. Then

$$(P_k - P_j)(f_k \otimes x) = \sum_{n=j+1}^{k} (f_k \otimes x) \circ Q_n^t = f_k \otimes x \neq 0$$

since

$$((f_k\otimes x)\circ Q_n^t)(f') = (f_k\otimes x)(\lambda_n(\cdot)f'(f_n)) = \lambda_n(f_k)f'(f_n)x = \begin{cases}(f_k \otimes x)(f') &, n = k,\\ 0 &, n \neq k.\end{cases}$$

It remains to prove that for each $u \in F\varepsilon E$

$$\lim_{k\to\infty} P_k(u) = u$$

[3]The lcHs E is non-trivial by our assumptions in Chapter 2.

in $F\varepsilon E$. Let $(q_\beta)_{\beta\in\mathfrak{B}}$ denote the system of seminorms inducing the locally convex topology of F. Let $u \in F\varepsilon E$ and $\alpha \in \mathfrak{A}$. Due to the continuity of u there are an absolutely convex compact set $K = K(u,\alpha) \subset F$ and $C_0 = C_0(u,\alpha) > 0$ such that for each $f' \in F'$ we have

$$p_\alpha\big((P_k(u) - u)(f')\big) = p_\alpha\big(u\big((\sum_{n=1}^{k} Q_n^t - \mathrm{id}_{F'})(f')\big)\big) \le C_0 \sup_{f\in K}\big|(\sum_{n=1}^{k} Q_n^t - \mathrm{id}_{F'})(f')(f)\big|$$
$$= C_0 \sup_{f\in K}\big|f'(\sum_{n=1}^{k} Q_n f - f)\big|.$$

Let V be an absolutely convex zero neighbourhood in F. As a consequence of the equicontinuity of the polar V° there are $C_1 > 0$ and $\beta \in \mathfrak{B}$ such that

$$\sup_{f'\in V^\circ} p_\alpha\big((P_k(u) - u)(f')\big) \le C_0 C_1 \sup_{f\in K} q_\beta(\sum_{n=1}^{k} Q_n f - f).$$

In combination with the convergence of $(\sum_{n=1}^{k} Q_n)$ to id_F in $L_\kappa(F,F)$ this yields the convergence of $(P_k(u))$ to u in $F\varepsilon E$ and settles part a).

Let us turn to b) and c). Since

$$P_k(u)(f') = u\big(\sum_{n=1}^{k} Q_n^t(f')\big) = \sum_{n=1}^{k} u(\lambda_n) f'(f_n)$$

for every $f' \in F'$, we note that the range of $P_k(u)$ is contained in $\mathrm{span}\{u(\lambda_n) \mid 1 \le n \le k\}$ for each $u \in F\varepsilon E$ and $k \in \mathbb{N}$. Hence $P_k(u)$ has finite rank and thus belongs to $F \otimes E$, implying the sequential density of $F \otimes E$ in $F\varepsilon E$ and the desired series representation by part a). □

The index set of the equicontinuous Schauder basis of F in Theorem 5.6.1 need not be $\mathbb{N}$ (or $\mathbb{N}_0$) but may be any other countable index set as long as the equicontinuous Schauder basis is unconditional which is, for instance, always fulfilled if F is nuclear by [89, 21.10.1 Dynin-Mitiagin Theorem, p. 510].

5.6.2. Remark. If F and E are complete, we have under the assumption of Theorem 5.6.1 that $F\widehat{\otimes}_\varepsilon E \cong F\varepsilon E$ by c) since $F\varepsilon E$ is complete by [94, Satz 10.3, p. 234] and $F\widehat{\otimes}_\varepsilon E$ is the closure of $F \otimes E$ in $F\varepsilon E$. Thus each element of $F\widehat{\otimes}_\varepsilon E$ has a series representation.

Let us apply the preceding theorem to spaces of Lebesgue integrable functions. We consider the measure space $([0,1], \mathscr{L}([0,1]), \lambda)$ of Lebesgue measurable sets and use the notation $\mathcal{L}^p[0,1]$ for the space of (equivalence classes) of Lebesgue p-integrable functions on $[0,1]$. The *Haar system* $h_n\colon [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, given by $h_1(x) := 1$ for all $x \in [0,1]$ and

$$h_{2^k+j}(x) := \begin{cases} 1 & ,(2j-2)/2^{k+1} \le x < (2j-1)/2^{k+1}, \\ -1 & ,(2j-1)/2^{k+1} \le x < 2j/2^{k+1}, \\ 0 & ,\text{else}, \end{cases}$$

for $k \in \mathbb{N}_0$ and $1 \le j \le 2^k$ forms a Schauder basis of $\mathcal{L}^p[0,1]$ for every $1 \le p < \infty$ and the associated coefficient functionals are

$$\lambda_n(f) := \int_{[0,1]} f(x) h_n(x) \mathrm{d}\lambda(x), \quad f \in \mathcal{L}^p[0,1],\ n \in \mathbb{N},$$

(see [154, Satz I, p. 317]). Because $\mathcal{L}^p[0,1]$ is Banach space and thus barrelled, its Schauder basis (h_n) is equicontinuous and we directly obtain from Theorem 5.6.1 the following corollary.

5.6.3. COROLLARY. *Let E be an lcHs and $1 \leq p < \infty$. $(\sum_{n=1}^{k} \lambda_n(\cdot) h_n \varepsilon \operatorname{id}_E)_{k\in\mathbb{N}}$ is a Schauder decomposition of $\mathcal{L}^p[0,1]\varepsilon E$ and for each $u \in \mathcal{L}^p[0,1]\varepsilon E$ it holds*

$$u(f') = \sum_{n=1}^{\infty} u(\lambda_n) f'(h_n), \quad f' \in \mathcal{L}^p[0,1]'.$$

Defining $\mathcal{L}^p([0,1],E) := \mathcal{L}^p[0,1]\varepsilon E$, we can read the corollary above as a statement on series representations in the vector-valued version of $\mathcal{L}^p[0,1]$. However, in many cases of spaces $\mathcal{F}(\Omega)$ of scalar-valued functions there is a more natural way to define the vector-valued version $\mathcal{F}(\Omega,E)$ of $\mathcal{F}(\Omega)$, namely, that $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ are ε-compatible.

5.6.4. REMARK. If $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ are ε-into-compatible, then we get by identification of isomorphic subspaces

$$\mathcal{F}(\Omega) \otimes_\varepsilon E \subset \mathcal{F}(\Omega)\varepsilon E \subset \mathcal{F}(\Omega,E)$$

and the embedding $\mathcal{F}(\Omega) \otimes E \hookrightarrow \mathcal{F}(\Omega,E)$ is given by $f \otimes e \longmapsto [x \mapsto f(x)e]$.

PROOF. The inclusions obviously hold and $\mathcal{F}(\Omega)\varepsilon E$ and $\mathcal{F}(\Omega,E)$ induce the same topology on $\mathcal{F}(\Omega) \otimes E$. Further, we have

$$f \otimes e \overset{\Theta}{\longmapsto} [y \mapsto y(f)e] \overset{S}{\longmapsto} [x \longmapsto [y \mapsto y(f)e](\delta_x)] = [x \mapsto f(x)e]. \qquad \square$$

5.6.5. COROLLARY. *Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-compatible, $(f_n)_{n\in\mathbb{N}}$ an equicontinuous Schauder basis of $\mathcal{F}(\Omega)$ with associated coefficient functionals $\lambda^{\mathbb{K}} := (\lambda_n^{\mathbb{K}})_{n\in\mathbb{N}}$. Let there be $\lambda^E\colon \mathcal{F}(\Omega,E) \to E^{\mathbb{N}}$ such that $(\lambda^E,\lambda^{\mathbb{K}})$ is a consistent family for $(\mathcal{F},E)$, and set $Q_n^E\colon \mathcal{F}(\Omega,E) \to \mathcal{F}(\Omega,E)$, $Q_n^E(f) := \lambda_n^E(f) f_n$ for every $n \in \mathbb{N}$. Then the following holds:*

a) *The sequence $(P_k^E)_{k\in\mathbb{N}}$ given by $P_k^E := \sum_{n=1}^{k} Q_n^E$ is a Schauder decomposition of $\mathcal{F}(\Omega,E)$.*
b) *Each $f \in \mathcal{F}(\Omega,E)$ has the series representation*

$$f = \sum_{n=1}^{\infty} \lambda_n^E(f) f_n.$$

c) *$\mathcal{F}(\Omega) \otimes E$ is sequentially dense in $\mathcal{F}(\Omega,E)$.*

PROOF. For each $u \in \mathcal{F}(\Omega)\varepsilon E$ and $x \in \Omega$ we note that with P_k from Theorem 5.6.1 it holds

$$\begin{aligned}(S \circ P_k)(u)(x) &= u\Big(\sum_{n=1}^{k} Q_n^t(\delta_x)\Big) = u\Big(\sum_{n=1}^{k} \lambda_n^{\mathbb{K}}(\cdot) f_n(x)\Big) = \sum_{n=1}^{k} u(\lambda_n^{\mathbb{K}}) f_n(x) \\ &= \sum_{n=1}^{k} \lambda_n^E(S(u)) f_n(x) = (P_k^E \circ S)(u)(x),\end{aligned}$$

which means that $S \circ P_k = P_k^E \circ S$. This implies part a) and b) by Theorem 5.6.1 a) since S is an isomorphism. Part c) is a direct consequence of Theorem 5.6.1 c) and the isomorphism $\mathcal{F}(\Omega)\varepsilon E \cong \mathcal{F}(\Omega,E)$. $\square$

In the preceding corollary we used the isomorphism S to obtain a Schauder decomposition. On the other hand, if S is an isomorphism into, which is often the case (see Theorem 3.1.12), we can use a Schauder decomposition of $\mathcal{F}(\Omega,E)$ to prove the surjectivity of S.

5.6.6. PROPOSITION. *Let $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ be ε-into-compatible. Let there be $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}(\Omega)$ and for every $f \in \mathcal{F}(\Omega,E)$ a sequence $(\lambda_n^E(f))_{n\in\mathbb{N}}$ in E such that*

$$f = \sum_{n=1}^{\infty} \lambda_n^E(f) f_n, \quad f \in \mathcal{F}(\Omega,E).$$

Then the following holds:

a) $\mathcal{F}(\Omega)\otimes E$ *is sequentially dense in* $\mathcal{F}(\Omega,E)$*.*
b) If $\mathcal{F}(\Omega)$ *and* E *are sequentially complete, then*

$$\mathcal{F}(\Omega,E)\cong\mathcal{F}(\Omega)\varepsilon E.$$

c) If $\mathcal{F}(\Omega)$ *and* E *are complete, then*

$$\mathcal{F}(\Omega,E)\cong\mathcal{F}(\Omega)\varepsilon E\cong\mathcal{F}(\Omega)\widehat{\otimes}_{\varepsilon}E.$$

Proof. Let $f\in\mathcal{F}(\Omega,E)$ and observe that

$$P_k^E(f):=\sum_{n=1}^{k}\lambda_n^E(f)f_n=\sum_{n=1}^{k}f_n\otimes\lambda_n^E(f)\in\mathcal{F}(\Omega)\otimes E$$

for every $k\in\mathbb{N}$ by Remark 5.6.4. Due to our assumption we have the convergence $P_k^E(f)\to f$ in $\mathcal{F}(\Omega,E)$. Thus $\mathcal{F}(\Omega)\otimes E$ is sequentially dense in $\mathcal{F}(\Omega,E)$.

Let us turn to part b). If $\mathcal{F}(\Omega)$ and E are sequentially complete, then $\mathcal{F}(\Omega)\varepsilon E$ is sequentially complete by [94, Satz 10.3, p. 234]. Since S is an isomorphism into and

$$S(\Theta(\sum_{n=q}^{k}f_n\otimes\lambda_n^E(f)))=\sum_{n=q}^{k}\lambda_n^E(f)f_n$$

for all $k,q\in\mathbb{N}$, $k>q$, we get that $(\Theta(\sum_{n=1}^{k}f_n\otimes\lambda_n^E(f))$ is a Cauchy sequence in $\mathcal{F}(\Omega)\varepsilon E$ and thus convergent. Hence we deduce that

$$S(\lim_{k\to\infty}\Theta(\sum_{n=1}^{k}f_n\otimes\lambda_n^E(f)))=\lim_{k\to\infty}\sum_{n=1}^{k}(S\circ\Theta)(f_n\otimes\lambda_n^E(f))=\sum_{n=1}^{\infty}\lambda_n^E(f)f_n=f,$$

which proves the surjectivity of S.

If $\mathcal{F}(\Omega)$ and E are complete, then $\mathcal{F}(\Omega)\widehat{\otimes}_{\varepsilon}E$ is the closure of $\mathcal{F}(\Omega)\otimes_{\varepsilon}E$ in the complete space $\mathcal{F}(\Omega)\varepsilon E$ by [94, Satz 10.3, p. 234]. As $\lim_{k\to\infty}\Theta(\sum_{n=1}^{k}f_n\otimes\lambda_n^E(f))$ is an element of the closure, we obtain part c). □

5.6.2. Examples of Schauder decompositions.

Sequence spaces. For our first application we recall the definition of some sequence spaces. For an lcHs E and a Köthe matrix $A:=(a_{k,j})_{k,j\in\mathbb{N}}$ we define the topological subspace of $\lambda^{\infty}(A,E)$ from Corollary 4.2.3 a) by

$$c_0(A,E):=\{x=(x_k)\in E^{\mathbb{N}}\mid\forall\ j\in\mathbb{N}:\ \lim_{k\to\infty}x_ka_{k,j}=0\}.$$

In particular, the space $c_0(\mathbb{N},E)$ of null-sequences in E is obtained as $c_0(\mathbb{N},E)=c_0(A,E)$ with $a_{k,j}:=1$ for all $k,j\in\mathbb{N}$. The space of convergent sequences in E is defined by

$$c(\mathbb{N},E):=\{x\in E^{\mathbb{N}}\mid x=(x_k)\text{ converges in }E\}$$

and equipped with the system of seminorms

$$|x|_{\alpha}:=\sup_{k\in\mathbb{N}}p_{\alpha}(x_k),\quad x\in c(\mathbb{N},E),$$

for $\alpha\in\mathfrak{A}$. Further, we set $c_0(A):=c_0(A,\mathbb{K})$, $c_0(\mathbb{N}):=c_0(\mathbb{N},\mathbb{K})$ and $c(\mathbb{N}):=c(\mathbb{N},\mathbb{K})$. Furthermore, we equip the space $E^{\mathbb{N}}$ with the system of seminorms given by

$$\|x\|_{l,\alpha}:=\sup_{k\in\mathbb{N}}p_{\alpha}(x_k)\chi_{\{1,\dots,l\}}(k),\quad x=(x_k)\in E^{\mathbb{N}},$$

for $l\in\mathbb{N}$ and $\alpha\in\mathfrak{A}$. For a non-empty set Ω we define for $n\in\Omega$ the n-th unit function by

$$\varphi_{n,\Omega}\colon\Omega\to\mathbb{K},\ \varphi_{n,\Omega}(k):=\begin{cases}1&,\ k=n,\\0&,\ \text{else},\end{cases}$$

and we simply write φ_n instead of $\varphi_{n,\Omega}$ if no confusion seems to be likely. Further, we set $\varphi_{\infty}\colon\mathbb{N}\to\mathbb{K}$, $\varphi_{\infty}(k):=1$, and $x_{\infty}:=\delta_{\infty}(x):=\lim_{k\to\infty}x_k$ for $x\in c(\mathbb{N},E)$. For series representations of the elements in these sequence spaces we do not need

Corollary 5.6.5 due to the subsequent proposition but we can use the representation to obtain the surjectivity of S for sequentially complete E.

5.6.7. PROPOSITION. *Let E be an lcHs and $\ell(\Omega, E)$ one of the spaces $c_0(A, E)$, $E^{\mathbb{N}}$, $s(\mathbb{N}^d, E)$, $s(\mathbb{N}_0^d, E)$ or $s(\mathbb{Z}^d, E)$.*

a) *Then $(\sum_{n\in\Omega,|n|\leq k}\delta_n\varphi_n)_{k\in\mathbb{N}}$ is a Schauder decomposition of $\ell(\Omega, E)$ and*
$$x = \sum_{n\in\Omega} x_n\varphi_n, \quad x \in \ell(\Omega, E).$$

b) *Then $(\delta_\infty\varphi_\infty + \sum_{n=1}^{k}(\delta_n - \delta_\infty)\varphi_n)_{k\in\mathbb{N}}$ is a Schauder decomposition of $c(\mathbb{N}, E)$ and*
$$x = x_\infty\varphi_\infty + \sum_{n=1}^{\infty}(x_n - x_\infty)\varphi_n, \quad x \in c(\mathbb{N}, E).$$

PROOF. Let us begin with a). First, we note that $(\varphi_n)_{n\in\Omega}$ is an unconditional equicontinuous Schauder basis of $s(\Omega)$, $\Omega = \mathbb{N}^d$, $\mathbb{N}_0^d$, $\mathbb{Z}^d$, since $s(\Omega)$ is a nuclear Fréchet space. Now, for $x = (x_n) \in \ell(\Omega, E)$ let (P_k^E) be the sequence in $\ell(\Omega, E)$ given by $P_k^E(x) := \sum_{|n|\leq k} x_n\varphi_n$. It is easy to see that P_k^E is a continuous projection on $\ell(\Omega, E)$, $P_k^E P_j^E = P_{\min(k,j)}^E$ for all $k, j \in \mathbb{N}$ and $P_k^E \neq P_j^E$ for $k \neq j$. Let $\varepsilon > 0$, $\alpha \in \mathfrak{A}$ and $j \in \mathbb{N}$. For $x \in c_0(A, E)$ there is $N_0 \in \mathbb{N}$ such that $p_\alpha(x_n a_{n,j}) < \varepsilon$ for all $n \geq N_0$. Hence we have for $x \in c_0(A, E)$
$$|x - P_k^E(x)|_{j,\alpha} = \sup_{n>k} p_\alpha(x_n)a_{n,j} \leq \sup_{n\geq N_0} p_\alpha(x_n)a_{n,j} \leq \varepsilon$$
for all $k \geq N_0$. For $x \in E^{\mathbb{N}}$ and $l \in \mathbb{N}$ we have
$$\left\|x - P_k^E(x)\right\|_{l,\alpha} = 0 < \varepsilon$$
for all $k \geq l$. For $x \in s(\Omega, E)$, $\Omega = \mathbb{N}^d$, $\mathbb{N}_0^d$, $\mathbb{Z}^d$, we notice that there is $N_1 \in \mathbb{N}$ such that for all $n \in \Omega$ with $|n| \geq N_1$ we have
$$\frac{(1 + |n|^2)^{j/2}}{(1 + |n|^2)^j} = (1 + |n|^2)^{-j/2} < \varepsilon.$$
Thus we deduce for $|n| \geq N_1$
$$p_\alpha(x_n)(1 + |n|^2)^{j/2} < \varepsilon p_\alpha(x_n)(1 + |n|^2)^j \leq \varepsilon|x|_{2j,\alpha}$$
and hence
$$|x - P_k^E(x)|_{j,\alpha} = \sup_{|n|>k} p_\alpha(x_n)(1 + |n|^2)^{j/2} \leq \sup_{|n|\geq N_1} p_\alpha(x_n)(1 + |n|^2)^{j/2} \leq \varepsilon|x|_{2j,\alpha}$$
for all $k \geq N_1$. Therefore $(P_k^E(x))$ converges to x in $\ell(\Omega, E)$ and
$$x = \lim_{k\to\infty} P_k^E(x) = \sum_{n\in\Omega} x_n\varphi_n.$$

Now, we turn to b). For $x = (x_n) \in c(\mathbb{N}, E)$ let $(\widetilde{P}_k^E(x))$ be the sequence in $c(\mathbb{N}, E)$ given by $\widetilde{P}_k^E(x) := x_\infty\varphi_\infty + \sum_{n=1}^{k}(x_n - x_\infty)\varphi_n$. Again, it is easy to see that $\widetilde{P}_k^E$ is a continuous projection on $c(\mathbb{N}, E)$, $\widetilde{P}_k^E\widetilde{P}_j^E = \widetilde{P}_{\min(k,j)}^E$ for all $k, j \in \mathbb{N}$ and $\widetilde{P}_k^E \neq \widetilde{P}_j^E$ for $k \neq j$. Let $\varepsilon > 0$ and $\alpha \in \mathfrak{A}$. Then there is $N_2 \in \mathbb{N}$ such that $p_\alpha(x_n - x_\infty) < \varepsilon$ for all $n \geq N_2$. Thus we obtain
$$|x - \widetilde{P}_k^E(x)|_\alpha = \sup_{n>k} p_\alpha(x_n - x_\infty) \leq \sup_{n\geq N_2} p_\alpha(x_n - x_\infty) \leq \varepsilon$$
for all $k \geq N_2$, implying that $(\widetilde{P}_k^E(x))$ converges to x in $c(\mathbb{N}, E)$ and
$$x = \lim_{k\to\infty} \widetilde{P}_k^E(x) = x_\infty\varphi_\infty + \sum_{n=1}^{\infty}(x_n - x_\infty)\varphi_n. \qquad \square$$

5.6.8. THEOREM. *Let E be a sequentially complete lcHs and $\ell(\Omega,E)$ one of the spaces $c_0(A,E)$, $E^{\mathbb{N}}$, $s(\mathbb{N}^d,E)$, $s(\mathbb{N}_0^d,E)$ or $s(\mathbb{Z}^d,E)$. Then*

$$(i)\ \ell(\Omega,E)\cong\ell(\Omega)\varepsilon E,\quad (ii)\ c(\mathbb{N},E)\cong c(\mathbb{N})\varepsilon E.$$

PROOF. The map $S_{\ell(\Omega)}$ is an isomorphism into by Theorem 3.1.12 and, in addition, by Proposition 4.1.9 (i) if $\ell(\Omega,E)=c_0(A,E)$. Considering $c(\mathbb{N},E)$, we observe that for $x\in c(\mathbb{N})$

$$\delta_n(x)=x_n\to x_\infty=\delta_\infty(x),$$

which implies the convergence $\delta_n\to\delta_\infty$ in $c(\mathbb{N})'_\gamma$ by the Banach–Steinhaus theorem since $c(\mathbb{N})$ is a Banach space. Hence we get

$$u(\delta_\infty)=\lim_{n\to\infty}u(\delta_n)=\lim_{n\to\infty}S(u)(n)=\delta_\infty(S(u))$$

for every $u\in c(\mathbb{N})\varepsilon E$, which implies that $S_{c(\mathbb{N})}$ is an isomorphism into by Theorem 3.1.12. From Proposition 5.6.7 and Proposition 5.6.6 we deduce our statement. □

More general, we note that Theorem 5.6.8 holds for any lcHs E if $\ell(\Omega,E)=E^{\mathbb{N}}$ by Example 4.2.1, for E with metric ccp if $\ell(\Omega,E)=c_0(A,E)$ by Example 4.2.11 (ii), and for locally complete E if $\ell(\Omega,E)=s(\Omega,E)$ with $\Omega=\mathbb{N}^d$, $\mathbb{N}_0^d$, $\mathbb{Z}^d$ by Corollary 4.2.3 b).

Continuous and differentiable functions on a compact interval. We start with continuous functions on compact sets. Let E be an lcHs and $\Omega\subset\mathbb{R}^d$ compact. We equip the space $\mathcal{C}(\Omega,E)$ of continuous functions on Ω with values in E with the system of seminorms given by

$$|f|_\alpha:=\sup_{x\in\Omega}p_\alpha(f(x)),\quad f\in\mathcal{C}(\Omega,E),$$

for $\alpha\in\mathfrak{A}$. We want to apply our preceding results to intervals. Let $-\infty<a<b<\infty$ and $T:=(t_j)_{0\leq j\leq n}$ be a partition of the interval $[a,b]$, i.e. $a=t_0<t_1<\ldots<t_n=b$. The *hat functions* $h^T_{t_j}\colon[a,b]\to\mathbb{R}$ for the partition T are given by

$$h^T_{t_j}(x):=\begin{cases}\frac{x-t_j}{t_j-t_{j-1}} & ,t_{j-1}\leq x\leq t_j,\\ \frac{t_{j+1}-x}{t_{j+1}-t_j} & ,t_j<x\leq t_{j+1},\\ 0 & ,\text{else},\end{cases}$$

for $2\leq j\leq n-1$ and

$$h^T_a(x):=\begin{cases}\frac{t_1-x}{t_1-a} & ,a\leq x\leq t_1,\\ 0 & ,\text{else},\end{cases}\qquad h^T_b(x):=\begin{cases}\frac{x-t_{n-1}}{b-t_{n-1}} & ,t_{n-1}\leq x\leq b,\\ 0 & ,\text{else}.\end{cases}$$

Let $\mathcal{T}:=(t_n)_{n\in\mathbb{N}_0}$ be a dense sequence in $[a,b]$ with $t_0=a$, $t_1=b$ and $t_n\neq t_m$ for $n\neq m$. For $T^n:=\{t_0,\ldots,t_n\}$ there is a (unique) enumeration $\sigma\colon\{0,\ldots,n\}\to\{0,\ldots,n\}$ of T^n such that $T_n:=(t_{\sigma(j)})_{0\leq j\leq n}$ is a partition of $[a,b]$ with $T^n=\{t_{\sigma(1)},\ldots,t_{\sigma(n)}\}$. The functions $\varphi^{\mathcal{T}}_0:=h^{T_1}_{t_0}$, $\varphi^{\mathcal{T}}_1:=h^{T_1}_{t_1}$ and $\varphi^{\mathcal{T}}_n:=h^{T_n}_{t_{\sigma(j)}}$ with $j=\sigma^{-1}(n)$ for $n\geq 2$ are called *Schauder hat functions* for the sequence $\mathcal{T}$ and form a Schauder basis of $\mathcal{C}([a,b])$ with associated coefficient functionals given by $\lambda^{\mathbb{K}}_0(f):=f(t_0)$, $\lambda^{\mathbb{K}}_1(f):=f(t_1)$ and

$$\lambda^{\mathbb{K}}_{n+1}(f):=f(t_{n+1})-\sum_{k=0}^{n}\lambda^{\mathbb{K}}_k(f)\varphi^{\mathcal{T}}_k(t_{n+1}),\quad f\in\mathcal{C}([a,b]),\ n\geq 1,$$

by [166, 2.3.5 Proposition, p. 29]. Looking at the coefficient functionals, we see that the right-hand sides even make sense if $f\in\mathcal{C}([a,b],E)$ and thus we define λ^E_n on $\mathcal{C}([a,b],E)$ for $n\in\mathbb{N}_0$ accordingly.

5.6.9. Theorem. *Let E be an lcHs with metric ccp and $\mathcal{T}:=(t_n)_{n\in\mathbb{N}_0}$ a dense sequence in $[a,b]$ with $t_0=a$, $t_1=b$ and $t_n\neq t_m$ for $n\neq m$. Then $(\sum_{n=0}^{k}\lambda_k^E\varphi_n^{\mathcal{T}})_{k\in\mathbb{N}_0}$ is a Schauder decomposition of $\mathcal{C}([a,b],E)$ and*

$$f=\sum_{n=0}^{\infty}\lambda_n^E(f)\varphi_n^{\mathcal{T}},\quad f\in\mathcal{C}([a,b],E).$$

Proof. The spaces $\mathcal{C}([a,b])$ and $\mathcal{C}([a,b],E)$ are ε-compatible by Example 4.2.12 if E has metric ccp. $\mathcal{C}([a,b])$ is a Banach space and thus barrelled, implying that its Schauder basis $(\varphi_n^{\mathcal{T}})$ is equicontinuous. We note that for all $u\in\mathcal{C}([a,b])\varepsilon E$ and $x\in[a,b]$

$$\lambda_n^E(S(u))(x)=u(\delta_{t_n})=u(\lambda_n^{\mathbb{K}}),\quad n\in\{0,1\},$$

and by induction

$$\begin{aligned}\lambda_{n+1}^E(S(u))(x)&=u(\delta_{t_{n+1}})-\sum_{k=0}^{n}\lambda_k^E(S(u))\varphi_k^{\mathcal{T}}(t_{n+1})=u(\delta_{t_{n+1}})-\sum_{k=0}^{n}u(\lambda_k^{\mathbb{K}})\varphi_k^{\mathcal{T}}(t_{n+1})\\&=u(\lambda_{n+1}^{\mathbb{K}}),\quad n\geq 1.\end{aligned}$$

Thus $(\lambda^E,\lambda^{\mathbb{K}})$ is consistent, proving our claim by Corollary 5.6.5. □

If $a=0$, $b=1$ and $\mathcal{T}$ is the sequence of dyadic numbers given in [166, 2.1.1 Definitions, p. 21], then $(\varphi_n^{\mathcal{T}})$ is the so-called Faber–Schauder system. Using the Schauder basis and coefficient functionals of the space $\mathcal{C}_0(\mathbb{R})$ of continuous functions vanishing at infinity given in [166, 2.7.1, p. 41–42] and [166, 2.7.4 Corollary, p. 43] and that $S_{\mathcal{C}_0(\mathbb{R})}$ is an isomorphism by Example 4.2.11 (ii) if E has metric ccp, the corresponding result for the E-valued counterpart $\mathcal{C}_0(\mathbb{R},E)$ holds as well by a similar reasoning. Another corresponding result holds for the space $\mathcal{C}_{0,0}^{[\gamma]}([0,1],E)$, $0<\gamma<1$, of γ-Hölder continuous functions on $[0,1]$ with values in E that vanish at zero and at infinity if one uses the Schauder basis and coefficient functionals of $\mathcal{C}_{0,0}^{[\gamma]}([0,1])$ from [44, Theorem 2, p. 220] and [43, Theorem 3, p. 230]. This result is a bit weaker since Example 4.2.9 only guarantees that $S_{\mathcal{C}_{0,0}^{[\gamma]}([0,1])}$ is an isomorphism if E is quasi-complete.

Now, we turn to the spaces $\mathcal{C}^k([a,b],E)$ of continuously differentiable functions on an interval (a,b) with values in an lcHs E such that all derivatives can be continuously extended to the boundary from Example 4.2.28. We set $f^{(k)}(x):=(\partial^k)^{\mathbb{K}}f(x)$ for $x\in(a,b)$ and $f\in\mathcal{C}^k([a,b])$. From the Schauder hat functions $(\varphi_n^{\mathcal{T}})$ for a dense sequence $\mathcal{T}:=(t_n)_{n\in\mathbb{N}_0}$ in $[a,b]$ with $t_0=a$, $t_1=b$ and $t_n\neq t_m$ for $n\neq m$ and the associated coefficient functionals $\lambda_n^{\mathbb{K}}$ we can easily get a Schauder basis for the space $\mathcal{C}^k([a,b])$, $k\in\mathbb{N}$, by applying $\int_a^{(\cdot)}$ k-times to the series representation

$$f^{(k)}=\sum_{n=0}^{\infty}\lambda_n^{\mathbb{K}}(f^{(k)})\varphi_n^{\mathcal{T}},\quad f\in\mathcal{C}^k([a,b]),$$

where we identified $f^{(k)}$ with its continuous extension. The resulting Schauder basis $f_n^{\mathcal{T}}:[a,b]\to\mathbb{R}$ and associated coefficient functionals $\mu_n^{\mathbb{K}}:\mathcal{C}^k([a,b])\to\mathbb{K}$, $n\in\mathbb{N}_0$, are

$$\begin{aligned}f_n^{\mathcal{T}}(x)&=\frac{1}{n!}(x-a)^n, & \mu_n^{\mathbb{K}}(f)&=f^{(n)}(a), & 0\leq n\leq k-1,\\ f_n^{\mathcal{T}}(x)&=\int_a^x\int_a^{s_{k-1}}\cdots\int_a^{s_2}\int_a^{s_1}\varphi_{n-k}^{\mathcal{T}}\mathrm{d}s\mathrm{d}s_1\ldots\mathrm{d}s_{k-1}, & \mu_n^{\mathbb{K}}(f)&=\lambda_{n-k}^{\mathbb{K}}(f^{(k)}), & n\geq k,\end{aligned}$$

for $x\in[a,b]$ and $f\in\mathcal{C}^k([a,b])$ (see e.g. [157, p. 586–587], [166, 2.3.7, p. 29]). Again, the mapping rule for the coefficient functionals still makes sense if $f\in\mathcal{C}^k([a,b],E)$ and so we define μ_n^E on $\mathcal{C}^k([a,b],E)$ for $n\in\mathbb{N}_0$ accordingly.

5.6.10. THEOREM. *Let E be an lcHs with metric ccp, $k \in \mathbb{N}$, $\mathcal{T} := (t_n)_{n\in\mathbb{N}_0}$ a dense sequence in $[a,b]$ with $t_0 = a$, $t_1 = b$ and $t_n \neq t_m$ for $n \neq m$. Then $(\sum_{n=0}^{l} \mu_n^E f_n^{\mathcal{T}})_{l\in\mathbb{N}_0}$ is a Schauder decomposition of $\mathcal{C}^k([a,b],E)$ and*

$$f = \sum_{n=0}^{\infty} \mu_n^E(f) f_n^{\mathcal{T}}, \quad f \in \mathcal{C}^k([a,b],E).$$

PROOF. The spaces $\mathcal{C}^k([a,b])$ and $\mathcal{C}^k([a,b],E)$ are ε-compatible by Example 4.2.28 if E has metric ccp. The Banach space $\mathcal{C}^k([a,b])$ is barrelled giving the equicontinuity of its Schauder basis. Due to Proposition 3.1.11 c) we have for all $u \in \mathcal{C}^k([a,b])\varepsilon E$, $\beta \in \mathbb{N}_0$, $\beta \leq k$, and $x \in (a,b)$

$$(\partial^\beta)^E S(u)(x) = u(\delta_x \circ (\partial^\beta)^{\mathbb{K}}).$$

Further, for every sequence (x_n) in (a,b) converging to $t \in \{a,b\}$ we obtain by Proposition 4.1.7 in combination with Lemma 4.1.8 applied to $T := (\partial^\beta)^{\mathbb{K}}$

$$\lim_{n\to\infty} (\partial^\beta)^E S(u)(x_n) = u(\lim_{n\to\infty} \delta_{x_n} \circ (\partial^\beta)^{\mathbb{K}}).$$

From these observations we deduce that $\mu_n^E(S(u)) = u(\mu_n^{\mathbb{K}})$ for all $n \in \mathbb{N}_0$, i.e. $(\mu^E, \mu^{\mathbb{K}})$ is consistent. Therefore our statement is a consequence of Corollary 5.6.5. □

Holomorphic functions. In this short subsection we show how to get the result on power series expansion of holomorphic functions from the introduction. Let E be an lcHs over $\mathbb{C}$, $z_0 \in \mathbb{C}$, $r \in (0,\infty]$ and equip $\mathcal{O}(\mathbb{D}_r(z_0),E)$ with the topology τ_c of compact convergence.

5.6.11. THEOREM. *Let E be a locally complete lcHs over $\mathbb{C}$, $z_0 \in \mathbb{C}$ and $r \in (0,\infty]$. Then $(f \mapsto \sum_{n=0}^{k} \frac{(\partial_{\mathbb{C}}^n)^E f(z_0)}{n!}(\cdot - z_0)^n)_{k\in\mathbb{N}_0}$ is a Schauder decomposition of $\mathcal{O}(\mathbb{D}_r(z_0),E)$ and*

$$f = \sum_{n=0}^{\infty} \frac{(\partial_{\mathbb{C}}^n)^E f(z_0)}{n!}(\cdot - z_0)^n, \quad f \in \mathcal{O}(\mathbb{D}_r(z_0),E).$$

PROOF. The spaces $\mathcal{O}(\mathbb{D}_r(z_0))$ and $\mathcal{O}(\mathbb{D}_r(z_0),E)$ are ε-compatible by Proposition 4.2.17 and (23) (cf. [30, Theorem 9, p. 232]) if E is locally complete. Further, the Schauder basis $((\cdot - z_0)^n)$ of $\mathcal{O}(\mathbb{D}_r(z_0))$ is equicontinuous since the Fréchet space $\mathcal{O}(\mathbb{D}_r(z_0))$ is barrelled. Due to Proposition 5.2.32 we have for all $u \in \mathcal{O}(\mathbb{D}_r(z_0))\varepsilon E$

$$(\partial_{\mathbb{C}}^n)^E S(u)(z) = u(\delta_z \circ (\partial_{\mathbb{C}}^n)^{\mathbb{C}}), \quad n \in \mathbb{N}_0,\ z \in \mathbb{D}_r(z_0),$$

which yields that $(\lambda^E, \lambda^{\mathbb{C}})$ is consistent where $\lambda^E\colon \mathcal{O}(\mathbb{D}_r(z_0),E) \to E^{\mathbb{N}_0}$ is given by $\lambda_n^E(f) := \frac{(\partial_{\mathbb{C}}^n)^E f(z_0)}{n!}$ for $n \in \mathbb{N}_0$ (and analogously for E replaced by $\mathbb{C}$). Hence Corollary 5.6.5 implies our statement. □

Theorem 5.6.11 holds for holomorphic functions in several variables as well (see [113, Theorem 5.7, p. 264]).

Fourier expansions. In this subsection we turn our attention to Fourier expansions in the Schwartz space $\mathcal{S}(\mathbb{R}^d,E)$ and in the space $\mathcal{C}_{2\pi}^\infty(\mathbb{R}^d,E)$ of smooth functions that are 2π-periodic in each variable.

We recall the definition of the Hermite functions. For $n \in \mathbb{N}_0$ we set

$$h_n\colon \mathbb{R} \to \mathbb{R},\ h_n(x) := (2^n n! \sqrt{\pi})^{-1/2}\Big(x - \frac{d}{dx}\Big)^n \mathrm{e}^{-x^2/2} = (2^n n! \sqrt{\pi})^{-1/2} H_n(x) \mathrm{e}^{-x^2/2},$$

with the *Hermite polynomials* H_n of degree n which can be computed recursively by

$$H_0(x) = 1,\ H_{n+1}(x) = 2x H_n(x) - H_n'(x) \text{ and } H_n'(x) = 2n H_{n-1}(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}_0.$$

For $n=(n_k)\in\mathbb{N}_0^d$ we define the *n-th Hermite function* by

$$h_n\colon\mathbb{R}^d\to\mathbb{R},\ h_n(x):=\prod_{k=1}^d h_{n_k}(x_k),\quad\text{and}\quad H_n\colon\mathbb{R}^d\to\mathbb{R},\ H_n(x):=\prod_{k=1}^d H_{n_k}(x_k).$$

5.6.12. PROPOSITION. *Let E be a locally complete lcHs, $f\in\mathcal{S}(\mathbb{R}^d,E)$ and $n\in\mathbb{N}_0^d$. Then fh_n is Pettis-integrable on $\mathbb{R}^d$.*

PROOF. First, we set $\psi\colon\mathbb{R}^d\to\mathbb{R}$, $\psi(x):=\mathrm{e}^{-|x|^2/2}$, as well as $g\colon\mathbb{R}^d\to[0,\infty)$, $g(x):=\mathrm{e}^{|x|^2/2}$. Then $\psi\in\mathcal{L}^1(\mathbb{R}^d,\lambda)$ and $\psi g=1$. Moreover, let $u\colon\mathbb{R}^d\to E$, $u(x):=f(x)h_n(x)g(x)$, and note that

$$(\partial^{e_j})^E u(x)=(\partial^{e_j})^E f(x)h_n(x)g(x)+f(x)g(x)\partial^{e_j}h_n(x)+f(x)h_n(x)g(x)x_j$$

where

$$\begin{aligned}\partial^{e_j}h_n(x)&=(2^{n_j}n_j!\sqrt{\pi})^{-1/2}(H_{n_j}'(x_j)\mathrm{e}^{-x_j^2/2}-H_{n_j}(x_j)x_j\mathrm{e}^{-x_j^2/2})\prod_{k=1,k\neq j}^d h_{n_k}(x_k)\\&=(2^{n_j}n_j!\sqrt{\pi})^{-1/2}(2n_jH_{n_j-1}(x_j)-x_jH_{n_j}(x_j))\mathrm{e}^{-x_j^2/2}\prod_{k=1,k\neq j}^d h_{n_k}(x_k)\end{aligned}$$

for all $x=(x_k)\in\mathbb{R}^d$ and $1\leq j\leq d$. We set $C_n:=(\prod_{i=1}^d 2^{n_i}n_i!\sqrt{\pi})^{-1/2}$ and observe that

$$g(x)\partial^{e_j}h_n(x)=\mathrm{e}^{|x|^2/2}\partial^{e_j}h_n(x)=C_n(2n_jH_{n_j-1}(x_j)-x_jH_{n_j}(x_j))\prod_{k=1,k\neq j}^d H_{n_k}(x_k)$$

is a polynomial in d variables. The functions given by

$$h_n(x)g(x)=\mathrm{e}^{|x|^2/2}h_n(x)=C_nH_n(x)\quad\text{and}\quad h_n(x)g(x)x_j=C_nH_n(x)x_j$$

are polynomials in d variables as well. Thus there are $m\in\mathbb{N}$ and $C>0$ such that

$$\max\big(|h_n(x)g(x)|,|g(x)\partial^{e_j}h_n(x)|,|h_n(x)g(x)x_j|\big)\leq C(1+|x|^2)^{m/2}$$

for all $x\in\mathbb{R}^d$ and $1\leq j\leq d$, which implies

$$p_\alpha((\partial^{e_j})^E u(x))\leq C\big(p_\alpha((\partial^{e_j})^E f(x))(1+|x|^2)^{m/2}+2p_\alpha(f(x))(1+|x|^2)^{m/2}\big)$$

for all $\alpha\in\mathfrak{A}$ and hence

$$\sup_{\substack{x\in\mathbb{R}^d\\ \beta\in\mathbb{N}_0^d,|\beta|\leq 1}} p_\alpha((\partial^\beta)^E u(x))\leq 3C|f|_{\mathcal{S}(\mathbb{R}^d),m,\alpha}.$$

Therefore $u=fh_ng$ is (weakly) $\mathcal{C}_b^1$, which yields $u\in\mathcal{C}_b^{[1]}(\mathbb{R}^d,E)$ by Proposition A.1.5. Further, we set $h\colon\mathbb{R}^d\to(0,\infty)$, $h(x):=1+|x|^2$, and observe that

$$\sup_{x\in\mathbb{R}^d}p_\alpha(u(x)h(x))\leq\sup_{x\in\mathbb{R}^d}p_\alpha(f(x))|h_n(x)g(x)h(x)|\leq C|f|_{m+2,\alpha}<\infty$$

for all $\alpha\in\mathfrak{A}$. In addition, we remark that for every $\varepsilon>0$ there is $r>0$ such that $1\leq\varepsilon h(x)$ for all $x\notin\overline{\mathbb{B}_r(0)}=:K$. We deduce from Proposition A.2.7 (iii) that fh_n is Pettis-integrable on $\mathbb{R}^d$. □

Due to the previous proposition we can define the n-th Fourier coefficient of $f\in\mathcal{S}(\mathbb{R}^d,E)$ by

$$\widehat{f}(n):=\mathscr{F}_n^E(f):=\int_{\mathbb{R}^d}f(x)\overline{h_n(x)}\mathrm{d}x=\int_{\mathbb{R}^d}f(x)h_n(x)\mathrm{d}x,\quad n\in\mathbb{N}_0^d,$$

if E is locally complete. We know that the map

$$\mathscr{F}^{\mathbb{K}}\colon\mathcal{S}(\mathbb{R}^d)\to s(\mathbb{N}_0^d),\ \mathscr{F}^{\mathbb{K}}(f):=\big(\widehat{f}(n)\big)_{n\in\mathbb{N}_0^d},$$

is an isomorphism (see e.g. [94, Satz 3.7, p. 66]). We improve this result to locally complete E and derive a Schauder decomposition of $\mathcal{S}(\mathbb{R}^d, E)$ as well.

5.6.13. THEOREM. *Let E be a locally complete lcHs. Then the following holds:*

a) $(\sum_{n\in\mathbb{N}_0^d, |n|\leq k} \mathscr{F}_n^E h_n)_{k\in\mathbb{N}}$ *is a Schauder decomposition of* $\mathcal{S}(\mathbb{R}^d, E)$ *and*

$$f = \sum_{n\in\mathbb{N}_0^d} \widehat{f}(n) h_n, \quad f \in \mathcal{S}(\mathbb{R}^d, E).$$

b) The map

$$\mathscr{F}^E : \mathcal{S}(\mathbb{R}^d, E) \to s(\mathbb{N}_0^d, E),\ \mathscr{F}^E(f) := \big(\widehat{f}(n)\big)_{n\in\mathbb{N}_0^d},$$

is an isomorphism and

$$\mathscr{F}^E = S_{s(\mathbb{N}_0^d)} \circ (\mathscr{F}^{\mathbb{K}} \varepsilon \operatorname{id}_E) \circ S_{\mathcal{S}(\mathbb{R}^d)}^{-1}.$$

PROOF. Let us begin with part a). Due to Corollary 3.2.10 the spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d, E)$ are ε-compatible and the inverse of the isomorphism $S: \mathcal{S}(\mathbb{R}^d)\varepsilon E \to \mathcal{S}(\mathbb{R}^d, E)$ is given by the map $R^t: \mathcal{S}(\mathbb{R}^d, E) \to \mathcal{S}(\mathbb{R}^d)\varepsilon E$, $f \mapsto \mathcal{J}^{-1} \circ R_f^t$, according to Theorem 3.2.4. Moreover, $\mathcal{S}(\mathbb{R}^d)$ is a nuclear Fréchet space, thus barrelled, and hence its Schauder basis (h_n) is equicontinuous and unconditional. From the Pettis-integrability of fh_n by Proposition 5.6.12 and Proposition 4.3.3 with $(T_0^E, T_0^{\mathbb{K}}) := (h_n \operatorname{id}_{E^{\mathbb{R}^d}}, h_n \operatorname{id}_{\mathbb{K}^{\mathbb{R}^d}})$ we obtain that $(\mathscr{F}^E, \mathscr{F}^{\mathbb{K}})$ is consistent. Hence we conclude our statement from Corollary 5.6.5.

Let us turn to part b). First, we show that the map $\mathscr{F}^E$ is well-defined. Let $f \in \mathcal{S}(\mathbb{R}^d, E)$. Then $e' \circ f \in \mathcal{S}(\mathbb{R}^d)$ and

$$\langle e', \mathscr{F}^E(f)_n \rangle = \langle e', \widehat{f}(n) \rangle = \widehat{e' \circ f}(n) = \mathscr{F}^{\mathbb{K}}(e' \circ f)_n$$

for every $n \in \mathbb{N}_0^d$ and $e' \in E'$. Thus we have $\mathscr{F}^{\mathbb{K}}(e' \circ f) \in s(\mathbb{N}_0^d)$ for every $e' \in E'$, which implies by [131, Mackey's theorem 23.15, p. 268] that $\mathscr{F}^E(f) \in s(\mathbb{N}_0^d, E)$ and that $\mathscr{F}^E$ is well-defined. Due to Corollary 3.2.10 and Corollary 4.2.3 the maps $S_{\mathcal{S}(\mathbb{R}^d)}$ and $S_{s(\mathbb{N}_0^d)}$ are isomorphisms, which implies that $\mathscr{F}^E$ is also an isomorphism with $\mathscr{F}^E = S_{s(\mathbb{N}_0^d)} \circ (\mathscr{F}^{\mathbb{K}} \varepsilon \operatorname{id}_E) \circ S_{\mathcal{S}(\mathbb{R}^d)}^{-1}$ by Theorem 5.1.2 b). □

Our last example of this subsection is devoted to Fourier expansions in the space $\mathcal{C}_{2\pi}^\infty(\mathbb{R}^d, E)$. We recall that $\mathcal{C}_{2\pi}^\infty(\mathbb{R}^d, E)$ denotes the topological subspace of $\mathcal{CW}^\infty(\mathbb{R}^d, E)$ consisting of the functions which are 2π-periodic in each variable. Due to Lemma A.2.2 we are able to define the n-th Fourier coefficient of $f \in \mathcal{C}_{2\pi}^\infty(\mathbb{R}^d, E)$ by

$$\widehat{f}(n) := \mathfrak{F}_n^E(f) := (2\pi)^{-d} \int_{[-\pi,\pi]^d} f(x) e^{-i\langle n, x\rangle} dx, \quad n \in \mathbb{Z}^d,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^d$, if E is locally complete. We know that the map

$$\mathfrak{F}^{\mathbb{C}} : \mathcal{C}_{2\pi}^\infty(\mathbb{R}^d) \to s(\mathbb{Z}^d),\ \mathfrak{F}^{\mathbb{C}}(f) := \big(\widehat{f}(n)\big)_{n\in\mathbb{Z}^d},$$

is an isomorphism (see e.g. [94, Satz 1.7, p. 18]), which we lift to the E-valued case.

5.6.14. THEOREM. *Let E be a locally complete lcHs over $\mathbb{C}$.*

a) Then $(\sum_{n\in\mathbb{Z}^d, |n|\leq k} \mathfrak{F}_n^E e^{i\langle n, \cdot\rangle})_{k\in\mathbb{N}}$ *is a Schauder decomposition of* $\mathcal{C}_{2\pi}^\infty(\mathbb{R}^d, E)$ *and*

$$f = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n) e^{i\langle n, \cdot\rangle}, \quad f \in \mathcal{C}_{2\pi}^\infty(\mathbb{R}^d, E).$$

b) The map

$$\mathfrak{F}^E\colon \mathcal{C}^\infty_{2\pi}(\mathbb{R}^d,E)\to s(\mathbb{Z}^d,E),\ \mathfrak{F}^E(f):=\big(\widehat{f}(n)\big)_{n\in\mathbb{Z}^d},$$

is an isomorphism and

$$\mathfrak{F}^E=S_{s(\mathbb{Z}^d)}\circ(\mathfrak{F}^{\mathbb{C}}\varepsilon\operatorname{id}_E)\circ S^{-1}_{\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)}.$$

PROOF. The spaces $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)$ and $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d,E)$ are ε-compatible by Example 4.2.27.

The space $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)$ is barrelled since it is a nuclear Fréchet space and thus its Schauder basis $(\mathrm{e}^{\mathrm{i}\langle n,\cdot\rangle})$ is equicontinuous and unconditional. By Theorem 3.2.4 the inverse of $S_{\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)}$ is given by $R^t\colon \mathcal{C}^\infty_{2\pi}(\mathbb{R}^d,E)\to\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)\varepsilon E,\ f\mapsto \mathcal{J}^{-1}\circ R^t_f$. From the Pettis-integrability of $f\mathrm{e}^{-\mathrm{i}\langle n,\cdot\rangle}$ and Proposition 4.3.3 with $(T^E_0,T^{\mathbb{K}}_0):=(\mathrm{e}^{-\mathrm{i}\langle n,\cdot\rangle}\operatorname{id}_{E^{\mathbb{R}^d}},\mathrm{e}^{-\mathrm{i}\langle n,\cdot\rangle}\operatorname{id}_{\mathbb{C}^{\mathbb{R}^d}})$ we obtain that $(\mathfrak{F}^E,\mathfrak{F}^{\mathbb{C}})$ is consistent. Hence we conclude part a) from Corollary 5.6.5.

Let us turn to part b). As in Theorem 5.6.13 it follows from [131, Mackey's theorem 23.15, p. 268] that the map $\mathfrak{F}^E$ is well-defined. Due to Corollary 4.2.3 and Example 4.2.27 the maps $S_{s(\mathbb{Z}^d)}$ and $S_{\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)}$ are isomorphisms, which implies that $\mathfrak{F}^E$ is an isomorphism as well with $\mathfrak{F}^E=S_{s(\mathbb{Z}^d)}\circ(\mathscr{F}^{\mathbb{C}}\varepsilon\operatorname{id}_E)\circ S^{-1}_{\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d)}$ by Theorem 5.1.2 b). □

For quasi-complete E Theorem 5.6.14 is already known by [94, Satz 10.8, p. 239].

5.7. Representation by sequence spaces

Our last section is dedicated to the representation of weighted spaces of E-valued functions by weighted spaces of E-valued sequences if there is a counterpart of this representation in the scalar-valued case involving the coefficient functionals associated to a Schauder basis (see Remark 5.2.3 b)). We only touched upon this problem in Section 5.6 for special cases like $\mathcal{S}(\mathbb{R}^d,E)$ and $\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d,E)$ in Theorem 5.6.13 b) and Theorem 5.6.14 b). We solve this problem in a different way by an application of our extension results from Section 5.2. As an example we treat the space $\mathcal{O}(\mathbb{D}_R(0),E)$ of holomorphic functions and the multiplier space $\mathcal{O}_M(\mathbb{R},E)$ of the Schwartz space (see Corollary 5.7.3).

5.7.1. THEOREM. *Let E be a locally complete lcHs, $G\subset E'$ determine boundedness and $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ resp. $\ell(\mathbb{N})$ and $\ell(\mathbb{N},E)$ be ε-into-compatible with $e'\circ g\in\ell(\mathbb{N})$ for all $e'\in E'$ and $g\in\ell(\mathbb{N},E)$. Let $(f_n)_{n\in\mathbb{N}}$ be an equicontinuous Schauder basis of $\mathcal{F}(\Omega)$ with associated coefficient functionals $(T^{\mathbb{K}}_n)_{n\in\mathbb{N}}$ such that*

$$T^{\mathbb{K}}\colon\mathcal{F}(\Omega)\to\ell(\mathbb{N}),\ T^{\mathbb{K}}(f):=(T^{\mathbb{K}}_n(f))_{n\in\mathbb{N}},$$

is an isomorphism and let there be $T^E\colon\mathcal{F}(\Omega,E)\to E^{\mathbb{N}}$ such that $(T^E,T^{\mathbb{K}})$ is a strong, consistent family for $(\mathcal{F},E)$. If

(i) $\mathcal{F}(\Omega)$ is a Fréchet–Schwartz space, or

(ii) E is sequentially complete, $G=E'$ and $\mathcal{F}(\Omega)$ is a semi-Montel BC-space,

then the following holds:

a) $\mathcal{F}_G(\mathbb{N},E)=\ell(\mathbb{N},E)$.

b) $\ell(\mathbb{N})$ and $\ell(\mathbb{N},E)$ are ε-compatible, in particular, $\ell(\mathbb{N})\varepsilon E\cong\ell(\mathbb{N},E)$.

c) The map

$$T^E\colon\mathcal{F}(\Omega,E)\to\ell(\mathbb{N},E),\ T^E(f):=(T^E_n(f))_{n\in\mathbb{N}},$$

is a well-defined isomorphism, $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$ are ε-compatible, in particular, $\mathcal{F}(\Omega)\varepsilon E\cong\mathcal{F}(\Omega,E)$, and $T^E=S_{\ell(\mathbb{N})}\circ(T^{\mathbb{K}}\varepsilon\operatorname{id}_E)\circ S^{-1}_{\mathcal{F}(\Omega)}$.

PROOF. a)(1) First, we remark that $\mathbb{N}$ is a set of uniqueness for $(T^{\mathbb{K}},\mathcal{F})$. Let $u\in\mathcal{F}(\Omega)\varepsilon E$ and $n\in\mathbb{N}$. Then

$$\begin{aligned}R_{\mathbb{N},G}(S_{\mathcal{F}(\Omega)}(u))(n)&=(T^{E}\circ S_{\mathcal{F}(\Omega)})(u)(n)=T^{E}_{n}(S_{\mathcal{F}(\Omega)}(u))=u(T^{\mathbb{K}}_{n})=u(\delta_{n}\circ T^{\mathbb{K}})\\&=(u\circ(T^{\mathbb{K}})^{t})(\delta_{n})=(T^{\mathbb{K}}\varepsilon\operatorname{id}_{E})(u)(\delta_{n})\\&=\big(S_{\ell(\mathbb{N})}\circ(T^{\mathbb{K}}\varepsilon\operatorname{id}_{E})\big)(u)(n)\end{aligned}\tag{61}$$

by consistency and the ε-into-compatibility, yielding $\mathcal{F}_{G}(\mathbb{N},E)\subset\ell(\mathbb{N},E)$ once we have shown that $R_{\mathbb{N},G}$ is surjective, which we postpone to part b).

a)(2) Let $g\in\ell(\mathbb{N},E)$. Then $e'\circ g\in\ell(\mathbb{N})$ for all $e'\in E'$ and $g_{e'}:=(T^{\mathbb{K}})^{-1}(e'\circ g)\in\mathcal{F}(\Omega)$. We note that $T^{\mathbb{K}}_{n}(g_{e'})=(e'\circ g)(n)$ for all $n\in\mathbb{N}$, which implies $\ell(\mathbb{N},E)\subset\mathcal{F}_{G}(\mathbb{N},E)$.

b) We only need to show that $S_{\ell(\mathbb{N})}$ is surjective. Let $g\in\ell(\mathbb{N},E)$, which implies $g\in\mathcal{F}_{G}(\mathbb{N},E)$ by part a)(2).

We claim that $R_{\mathbb{N},G}$ is surjective. In case (i) this follows directly from Theorem 5.2.20. Let us turn to case (ii) and denote by $(f_{n})_{n\in\mathbb{N}}$ the equicontinuous Schauder basis of $\mathcal{F}(\Omega)$ associated to $(T^{\mathbb{K}}_{n})_{n\in\mathbb{N}}$. We check that condition (ii) of Theorem 5.2.15 is fulfilled. Let $f'\in\mathcal{F}(\Omega)'$ and set

$$f'_{k}\colon\mathcal{F}(\Omega)\to\mathbb{K},\;f'_{k}(f):=\sum_{n=1}^{k}T^{\mathbb{K}}_{n}(f)f'(f_{n}),$$

for $k\in\mathbb{N}$. Then $f'_{k}\in\mathcal{F}(\Omega)'$ for every $k\in\mathbb{N}$ and (f'_{k}) converges to f' in $\mathcal{F}(\Omega)'_{\sigma}$ since $(\sum_{n=1}^{k}T^{\mathbb{K}}_{n}(f)f_{n})$ converges to f in $\mathcal{F}(\Omega)$. From the equicontinuity of the Schauder basis we deduce that (f'_{k}) converges to f' in $\mathcal{F}(\Omega)'_{\kappa}$ by [89, 8.5.1 Theorem (b), p. 156]. Let $f\in\mathcal{F}_{E'}(\mathbb{N},E)$. For each $e'\in E'$ and $k\in\mathbb{N}$ we have

$$\mathscr{R}^{t}_{f}(f'_{k})(e')=f'_{k}(f_{e'})=\sum_{n=1}^{k}T^{\mathbb{K}}_{n}(f_{e'})f'(f_{n})=e'\Big(\sum_{n=1}^{k}f(n)f'(f_{n})\Big)$$

since $f\in\mathcal{F}_{E'}(\mathbb{N},E)$, implying $\mathscr{R}^{t}_{f}(f'_{k})\in\mathcal{J}(E)$. Hence we can apply Theorem 5.2.15 (ii) and obtain that $R_{\mathbb{N},E'}$ is surjective, finishing the proof of part a)(1).

Thus there is $u\in\mathcal{F}(\Omega)\varepsilon E$ such that $R_{\mathbb{N},E'}(S_{\mathcal{F}(\Omega)}(u))=g$ in both cases. Then $(T^{\mathbb{K}}\varepsilon\operatorname{id}_{E})(u)\in\ell(\mathbb{N})\varepsilon E$ and from (61) we derive

$$S_{\ell(\mathbb{N})}((T^{\mathbb{K}}\varepsilon\operatorname{id}_{E})(u))=R_{\mathbb{N},G}(S_{\mathcal{F}(\Omega)}(u))=g,$$

proving the surjectivity of $S_{\ell(\mathbb{N})}$.

c) First, we note that the map T^{E} is well-defined. Indeed, we have $(e'\circ T^{E})(f)=T^{\mathbb{K}}(e'\circ f)\in\ell(\mathbb{N})$ for all $f\in\mathcal{F}(\Omega,E)$ and $e'\in E'$ by the strength of the family. Part a) implies that $T^{E}(f)\in\mathcal{F}_{G}(\mathbb{N},E)=\ell(\mathbb{N},E)$ and thus the map T^{E} is well-defined and its linearity follows from the linearity of the T^{E}_{n} for $n\in\mathbb{N}$. Next, we prove that T^{E} is surjective. Let $g\in\ell(\mathbb{N},E)$. Since $T^{\mathbb{K}}\varepsilon\operatorname{id}_{E}$ is an isomorphism and $S_{\ell(\mathbb{N})}$ by part b) as well, we obtain that $u:=((T^{\mathbb{K}}\varepsilon\operatorname{id}_{E})^{-1}\circ S^{-1}_{\ell(\mathbb{N})})(g)\in\mathcal{F}(\Omega)\varepsilon E$. Therefore $S_{\mathcal{F}(\Omega)}(u)\in\mathcal{F}(\Omega,E)$ and from (61) we get

$$T^{E}(S_{\mathcal{F}(\Omega)}(u))=(T^{E}\circ S_{\mathcal{F}(\Omega)})(u)=\big(S_{\ell(\mathbb{N})}\circ(T^{\mathbb{K}}\varepsilon\operatorname{id}_{E})\big)(u)=g,$$

which means that T^{E} is surjective. The injectivity of T^{E} by Proposition 5.2.8, implies that

$$S_{\mathcal{F}(\Omega)}=(T^{E})^{-1}\circ\big(S_{\ell(\mathbb{N})}\circ(T^{\mathbb{K}}\varepsilon\operatorname{id}_{E})\big),$$

yielding the surjectivity of $S_{\mathcal{F}(\Omega)}$ and thus the ε-compatibility of $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega,E)$. Furthermore, we have $T^{E}=S_{\ell(\mathbb{N})}\circ(T^{\mathbb{K}}\varepsilon\operatorname{id}_{E})\circ S^{-1}_{\mathcal{F}(\Omega)}$, resulting in T^{E} being an isomorphism. □

We note that one should not confuse the coefficient space $\ell(\mathbb{N})$ of the Schauder series expansion of functions from $\mathcal{F}(\Omega)$ in the theorem above with the space $\ell^1 = \ell^1(\mathbb{N})$ of absolutely summable sequences. We remark again (see Theorem 5.6.1) that the index set of the equicontinuous Schauder basis of $\mathcal{F}(\Omega)$ in Theorem 5.7.1 need not be $\mathbb{N}$ (or $\mathbb{N}_0$) but may be any other countable index set as long as the equicontinuous Schauder basis is unconditional which is, for instance, always fulfilled if $\mathcal{F}(\Omega)$ is nuclear by [89, 21.10.1 Dynin-Mitiagin Theorem, p. 510].

Theorem 5.7.1 (i) gives another proof of Theorem 5.6.13 b) and Theorem 5.6.14 b). Let us demonstrate an application of the preceding theorem which relates the space of $\mathcal{O}(\mathbb{D}_R(0), E)$, $0 < R \leq \infty$, of holomorphic functions on $\mathbb{D}_R(0)$ with values in a complex locally complete lcHs E (see Theorem 5.6.11) and the Köthe space $\lambda^\infty(A_R, E)$ with Köthe matrix $A_R := (r_j^k)_{k\in\mathbb{N}_0, j\in\mathbb{N}}$ for some strictly increasing sequence $(r_j)_{j\in\mathbb{N}}$ in $(0, R)$ converging to R (see Corollary 4.2.3), using the sequence of Taylor coefficients of a holomorphic function.

5.7.2. COROLLARY. *Let E be a locally complete lcHs over $\mathbb{C}$, $0 < R \leq \infty$ and define the Köthe matrix $A_R := (r_j^k)_{k\in\mathbb{N}_0, j\in\mathbb{N}}$ for some strictly increasing sequence $(r_j)_{j\in\mathbb{N}}$ in $(0, R)$ converging to R. Then $\lambda^\infty(A_R)\varepsilon E \cong \lambda^\infty(A_R, E)$ and*

$$\lambda^E \colon \mathcal{O}(\mathbb{D}_R(0), E) \to \lambda^\infty(A_R, E),\ \lambda^E(f) := \Big(\frac{(\partial^k_{\mathbb{C}})^E f(0)}{k!}\Big)_{k\in\mathbb{N}_0},$$

is an isomorphism with $\lambda^E = S_{\lambda^\infty(A_R)} \circ (\lambda^{\mathbb{C}} \varepsilon \operatorname{id}_E) \circ S^{-1}_{\mathcal{O}(\mathbb{D}_R(0))}$.

PROOF. By Proposition 4.2.17 and (23) the spaces $\mathcal{O}(\mathbb{D}_R(0))$ and $\mathcal{O}(\mathbb{D}_R(0), E)$ are ε-compatible. Moreover, $\lambda^\infty(A_R)$ and $\lambda^\infty(A_R, E)$ are ε-compatible by Corollary 4.2.3 as $\lim_{k\to\infty}(\frac{r_j}{r_{j+1}})^k = 0$ for any $j \in \mathbb{N}$. Clearly, we have $e' \circ x \in \lambda^\infty(A_R)$ for all $e' \in E'$ and $x \in \lambda^\infty(A_R, E)$. The space $\mathcal{O}(\mathbb{D}_R(0))$ with the topology τ_c of compact convergence is a nuclear Fréchet space and thus a Fréchet–Schwartz space. In particular, this space is barrelled and its Schauder basis of monomials $(z \mapsto z^k)_{k\in\mathbb{N}_0}$ is equicontinuous. The corresponding coefficient functionals are given by $\lambda^{\mathbb{C}}_k$ and the map $\lambda^{\mathbb{C}}$ is an isomorphism by [131, Example 27.27, p. 341–342]. By the proof of Theorem 5.6.11 the family $(\lambda^E, \lambda^{\mathbb{C}})$ is consistent for $(\mathcal{O}, E)$ and its strength follows from Proposition 5.2.32. Now, we can apply Theorem 5.7.1 (i), yielding our statement. □

Let us present another application of Theorem 5.7.1 to the space $\mathcal{O}_M(\mathbb{R}^d, E)$ of multipliers for the Schwartz space from Example 3.1.9 d). For simplicity we restrict to the case $d = 1$. Fix a compactly supported test function $\varphi \in \mathcal{C}^\infty_c(\mathbb{R})$ with $\varphi(x) = 1$ for $x \in [0, \frac{1}{4}]$ and $\varphi(x) = 0$ for $x \geq \frac{1}{2}$. For $f \in \mathcal{C}^\infty(\mathbb{R}, E)$ we set

$$f_j(x) := f(x+j) - \sum_{k=0}^{\infty} a_k \varphi(-2^k(x-1)) f(-2^k(x-1+j)+1),\ x \in [0,1],\ j \in \mathbb{Z},$$

where

$$a_k := \prod_{j=0, j\neq k}^{\infty} \frac{1+2^j}{2^j - 2^k},\ k \in \mathbb{N}_0.$$

Fixing $x \in [0, 1)$, we observe that $f_j(x)$ is well-defined for each $j \in \mathbb{Z}$ since there are only finitely many summands due to the compact support of φ and $-2^k(x-1) \to \infty$ for $k \to \infty$. For $x = 1$ we have $f_j(1) = 0$ for each j and the convergence of the series in E follows from the uniform continuity of f on $[0, 1]$, $f(0) = 0$ and $\sum_{k=0}^{\infty} a_k = 1$ by the case $n = 0$ in [160, Lemma (iii), p. 625]. For each $e' \in E'$ and $j \in \mathbb{Z}$ we note that

$$e'(f_j(x)) = (e' \circ f)(x+j) - \sum_{k=0}^{\infty} a_k \varphi(-2^k(x-1))(e' \circ f)(-2^k(x-1+j)+1),\ x \in [0,1],$$

which implies that $e'\circ f_j \in \mathcal{E}_0$ by [11, Proposition 3.2, p. 15]. Using the weak-strong principle Corollary 5.2.24, we obtain that $f_j \in \mathcal{E}_0(E)$ for all $j\in\mathbb{Z}$ if E is locally complete. Setting

$$\rho\colon\mathbb{R}\to[0,1],\ \rho(x) := 1-\cos(\arctan(x)) = 1-\frac{1}{\sqrt{1+x^2}},$$

we deduce from the proof and with the notation of [12, Proposition 2.2, p. 1494] that $e'\circ f_j\circ\rho = (\Phi_2^{-1}\circ\Phi_1)(e'\circ f_j)$ is an element of the Schwartz space $\mathcal{S}(\mathbb{R})$ for each $e'\in E'$. The weak-strong principle Corollary 5.2.21 c) yields that $f_j\circ\rho\in\mathcal{S}(\mathbb{R},E)$ if E is locally complete. Hence $(f_j\circ\rho)\cdot h_{2n}$ is Pettis-integrable on $\mathbb{R}$ for every $j\in\mathbb{Z}$ and $n\in\mathbb{N}_0$ by Proposition 5.6.12 if E is locally complete where h_n is the n-th Hermite function. Therefore the Pettis-integral

$$b_{n,j}(f) := \langle f_j\circ\rho, h_{2n}\rangle_{\mathcal{L}^2} := \int_{\mathbb{R}} f_j(\rho(x))h_{2n}(x)\mathrm{d}x,\ j\in\mathbb{Z},\ n\in\mathbb{N}_0,$$

is a well-defined element of E by Proposition 5.6.12 if E is locally complete. By [12, Theorem 2.1, p. 1496–1497] (cf. [172, Theorem 3, p. 478]) the map

$$\Phi^{\mathbb{K}}\colon\mathcal{O}_M(\mathbb{R})\to s(\mathbb{N})'_b\widehat{\otimes}_\pi s(\mathbb{N}),\ \Phi^{\mathbb{K}}(f) := (b_{\sigma(n,j)}(f))_{(n,j)\in\mathbb{N}^2},$$

is an isomorphism where $\sigma\colon\mathbb{N}^2\to\mathbb{N}_0\times\mathbb{Z}$ is the enumeration given by $\sigma(n,j) := (n-1,(j-1)/2)$ if j is odd, and $\sigma(n,j) := (n-1,-j/2)$ if j is even. Here, we have to interpret $\Phi^{\mathbb{K}}(f)$ as an element of $s(\mathbb{N})'_b\widehat{\otimes}_\pi s(\mathbb{N})$ by identification of isomorphic spaces. Namely,

$$s(\mathbb{N})'_b\widehat{\otimes}_\pi s(\mathbb{N}) \cong s(\mathbb{N})\widehat{\otimes}_\pi s(\mathbb{N})'_b \cong s(\mathbb{N})\varepsilon s(\mathbb{N})'_b \cong s(\mathbb{N},s(\mathbb{N})'_b)$$

holds where the first isomorphism is due to the commutativity of $\widehat{\otimes}_\pi$, the second due to the nuclearity of $s(\mathbb{N})$ and the last due to Corollary 4.2.3 b) via $S_{s(\mathbb{N})}$. Then we interpret $\Phi^{\mathbb{K}}(f)$ as an element of $s(\mathbb{N},s(\mathbb{N})'_b)$ by means of

$$j\in\mathbb{N}\longmapsto\Big[a\in s(\mathbb{N})\mapsto\sum_{n\in\mathbb{N}}a_n b_{\sigma(n,j)}\Big]$$

(see also (62) below).

5.7.3. Corollary. *If E is a sequentially complete lcHs, then the map*

$$\Phi^{E}\colon\mathcal{O}_M(\mathbb{R},E)\to s(\mathbb{N},L_b(s(\mathbb{N}),E)),\ \Phi^{E}(f) := (b_{\sigma(n,j)}(f))_{(n,j)\in\mathbb{N}^2},$$

is an isomorphism where we interpret $\Phi^{E}(f)$ as an element of $s(\mathbb{N},L_b(s(\mathbb{N}),E))$.

Proof. The spaces $\mathcal{O}_M(\mathbb{R})$ and $\mathcal{O}_M(\mathbb{R},E)$ are ε-compatible by Corollary 3.2.10 with the inverse of $S_{\mathcal{O}_M(\mathbb{R})}$ given by the map $R^t\colon\mathcal{O}_M(\mathbb{R},E)\to\mathcal{O}_M(\mathbb{R})\varepsilon E$, $f\mapsto\mathcal{J}^{-1}\circ R^t_f$, according to Theorem 3.2.4. The barrelled nuclear space $\mathcal{O}_M(\mathbb{R})$ has the equicontinuous unconditional Schauder basis $(\psi_{\sigma(n,j)})_{(n,j)\in\mathbb{N}^2}$ with associated coefficient functionals $\delta_{n,j}\circ\Phi^{\mathbb{K}} = b_{\sigma(n,j)}$ given in [12, Proposition 3.2, p. 1499]. Next, we show that $(\Phi^E,\Phi^{\mathbb{K}})$ is a strong, consistent family for $(\mathcal{O}_M,E)$. Let $f\in\mathcal{O}_M(\mathbb{R},E)$. For each $e'\in E'$ and $(n,j)\in\mathbb{N}^2$ we have

$$\begin{aligned}\delta_{n,j}\circ\Phi^{\mathbb{K}}(e'\circ f) = b_{\sigma(n,j)}(e'\circ f) &= \int_{\mathbb{R}}(e'\circ f)_{(j-1)/2}(\rho(x))h_{2(n-1)}(x)\mathrm{d}x\\ &= \langle e',\int_{\mathbb{R}}f_{(j-1)/2}(\rho(x))h_{2(n-1)}(x)\mathrm{d}x\rangle = \langle e',\delta_{n,j}\circ\Phi^E(f)\rangle\\ &= e'(b_{\sigma(n,j)}(f))\end{aligned}$$

if j is odd since $(f_{(j-1)/2}\circ\rho)\cdot h_{2(n-1)}$ is Pettis-integrable on $\mathbb{R}$. The analogous result holds for even j as well. This implies the strength of the family. Due to

Proposition 4.3.3 with $(T_0^E, T_0^{\mathbb{K}})$ given by $T_0^E(f) := (f_j \circ \rho)h_{2n}$, $f \in \mathcal{O}_M(\mathbb{R}, E)$, and $T_0^{\mathbb{K}}(f) := (f_j \circ \rho)h_{2n}$, $f \in \mathcal{O}_M(\mathbb{R})$, the family $(\Phi^E, \Phi^{\mathbb{K}})$ is consistent.

In order to apply Theorem 5.7.1 we need spaces $\ell\mathcal{V}(\mathbb{N}^2)$ and $\ell\mathcal{V}(\mathbb{N}^2, E)$ of sequences with values in $\mathbb{K}$ and E, respectively. In addition, the space $\ell\mathcal{V}(\mathbb{N}^2)$ has to be isomorphic to $s(\mathbb{N}, s(\mathbb{N})_b')$ so that $\Phi^{\mathbb{K}}: \mathcal{O}_M(\mathbb{R}) \to s(\mathbb{N}, s(\mathbb{N})_b') \cong \ell\mathcal{V}(\mathbb{N}^2)$ becomes the isomorphism we need for Theorem 5.7.1. We set

$$\ell\mathcal{V}(\mathbb{N}^2, E) := \{x = (x_{n,j}) \in E^{\mathbb{N}^2} \mid \forall\ k \in \mathbb{N},\ B \subset s(\mathbb{N}) \text{ bounded}, \alpha \in \mathfrak{A}: \ \|x\|_{k,B,\alpha} < \infty\}$$

where

$$\|x\|_{k,B,\alpha} := \sup_{(j,a)\in\omega_B} p_\alpha(T^E(x)(j,a))\nu_{k,B}(j,a)$$

with $\omega_B := \mathbb{N} \times B$ and $\nu_{k,B}: \omega_B \to [0,\infty)$, $\nu_{k,B}(j,a) := (1+j^2)^{k/2}$, and

$$T^E(x)(j,a) := \sum_{n\in\mathbb{N}} a_n x_{n,j}.$$

We claim that the map

$$T^E: \ell\mathcal{V}(\mathbb{N}^2, E) \to s(\mathbb{N}, L_b(s(\mathbb{N}), E)),\ x \mapsto (T^E(x)(j,\cdot))_{j\in\mathbb{N}}, \tag{62}$$

is an isomorphism. We remark for each $k \in \mathbb{N}$, bounded $B \subset s(\mathbb{N})$ and $\alpha \in \mathfrak{A}$ that

$$|T^E(x)|_{s(\mathbb{N}),k,(B,\alpha)} = \sup_{j\in\mathbb{N}} \sup_{a\in B} p_\alpha(T^E(x)(j,a))(1+j^2)^{k/2} = \|x\|_{k,B,\alpha}$$

for all $x \in \ell\mathcal{V}(\mathbb{N}^2, E)$, implying that T^E is an isomorphism into. Let $y := (y_j) \in s(\mathbb{N}, L_b(s(\mathbb{N}), E))$. Then $y_j \in L_b(s(\mathbb{N}), E)$ for $j \in \mathbb{N}$ and we set $x_{n,j} := y_j(e_n)$ for $n \in \mathbb{N}$ where e_n is the n-th unit sequence in $s(\mathbb{N})$. We note that with $x := (x_{n,j})_{(n,j)\in\mathbb{N}^2}$

$$T^E(x)(j,a) = \sum_{n\in\mathbb{N}} a_n x_{n,j} = \sum_{n\in\mathbb{N}} a_n y_j(e_n) = y_j(\sum_{n\in\mathbb{N}} a_n e_n) = y_j(a)$$

holds for all $j \in \mathbb{N}$ and $a := (a_n) \in s(\mathbb{N})$ since (e_n) is a Schauder basis of $s(\mathbb{N})$ with associated coefficient functionals $a \mapsto a_n$. It follows that $x \in \ell\mathcal{V}(\mathbb{N}^2, E)$ and the surjectivity of T^E.

The next step is to prove that $\ell\mathcal{V}(\mathbb{N}^2)$ and $\ell\mathcal{V}(\mathbb{N}^2, E)$ are ε-into-compatible. Due to Theorem 3.1.12 we only need to show that $(T^E, T^{\mathbb{K}})$ is a consistent generator for $(\ell\mathcal{V}, E)$. Let $u \in \ell\mathcal{V}(\mathbb{N}^2)\varepsilon E$. Then

$$\sum_{n=1}^m a_n S_{\ell\mathcal{V}(\mathbb{N}^2)}(u)(j,n) = \sum_{n=1}^m a_n u(\delta_{j,n}) = u(\sum_{n=1}^m a_n \delta_{j,n}) \tag{63}$$

for all $m \in \mathbb{N}$ and $a := (a_n) \in s(\mathbb{N})$. Since

$$(\sum_{n=1}^m a_n \delta_{j,n})(x) = \sum_{n=1}^m a_n x_{j,n} \to T^{\mathbb{K}}(x)(j,a) = T^{\mathbb{K}}_{(j,a)}(x), \quad m \to \infty,$$

for all $x \in \ell\mathcal{V}(\mathbb{N}^2)$, we deduce that $(\sum_{n=1}^m a_n \delta_{j,n})_m$ converges to $T^{\mathbb{K}}_{(j,a)}(x)$ in $\ell\mathcal{V}(\mathbb{N}^2)_\kappa'$ by the Banach–Steinhaus theorem, which is applicable as $\ell\mathcal{V}(\mathbb{N}^2) \cong s(\mathbb{N}, s(\mathbb{N})_b') \cong \mathcal{O}_M(\mathbb{R})$ is barrelled. We conclude that

$$u(T^{\mathbb{K}}_{(j,a)}) = \lim_{m\to\infty} u(\sum_{n=1}^m a_n \delta_{j,n}) \underset{(63)}{=} \sum_{n=1}^\infty a_n S_{\ell\mathcal{V}(\mathbb{N}^2)}(u)(j,n) = T^E S_{\ell\mathcal{V}(\mathbb{N}^2)}(u)(j,a)$$

and thus the consistency of $(T^E, T^{\mathbb{K}})$ for $(\ell\mathcal{V}, E)$.

Furthermore, we clearly have $e' \circ x \in \ell\mathcal{V}(\mathbb{N}^2)$ for all $x \in \ell\mathcal{V}(\mathbb{N}^2, E)$ and the map $\Phi: \mathcal{O}_M(\mathbb{R}) \to s(\mathbb{N})_b' \widehat{\otimes}_\pi s(\mathbb{N}) \cong \ell\mathcal{V}(\mathbb{N}^2)$ is an isomorphism by [12, Theorem 2.1, p. 1496–1497] and (62). Due to [83, Chap. II, §4, n°4, Théorème 16, p. 131] the dual $\mathcal{O}_M(\mathbb{R})_b'$ is an LF-space and thus $\mathcal{O}_M(\mathbb{R}) \cong (\mathcal{O}_M(\mathbb{R})_b')_b'$ is the strong dual of an LF-space by reflexivity and therefore webbed by [94, Satz 7.25, p. 165]. Finally, we can apply Theorem 5.7.1 (ii), yielding our statement. □

5.7.4. Remark. The actual isomorphism in Corollary 5.7.3 (without the interpretation) is given by $\widetilde{\Phi}^E := T^E \circ \Phi^E$ with T^E from (62) and we have

$$\widetilde{\Phi}^E = T^E \circ \Phi^E = T^E \circ S_{\ell\mathcal{V}(\mathbb{N}^2)} \circ (\Phi^{\mathbb{K}} \varepsilon \operatorname{id}_E) \circ S_{\mathcal{O}_M(\mathbb{R})}^{-1}.$$

Furthermore, Corollary 5.7.3 is valid for locally complete E as well. Indeed, similar to Example 4.2.2 we may show that $\ell\mathcal{V}(\mathbb{N}^2, E) \cong \ell\mathcal{V}(\mathbb{N}^2)\varepsilon E$ for locally complete E. In combination with Corollary 3.2.10 and Theorem 5.1.2 b) this proves Corollary 5.7.3 for locally complete E as in Theorem 5.6.13 b).

Appendices

APPENDIX A

Compactness of closed absolutely convex hulls and Pettis-integrals

A.1. Compactness of closed absolutely convex hulls

In this section of the appendix we treat the question for which functions $f\colon\Omega\to E$, subsets $K\subset\Omega$ and lcHs E sets like $\overline{\operatorname{acx}}(\overline{f(K)})$ are compact or sets like

$$N_{j,m}(f):=\{T^E_m(f)(x)\nu_{j,m}(x)\mid x\in\omega_m\},\quad j\in J,\, m\in M,$$

for $f\in\mathcal{FV}(\Omega,E)$ are contained in an absolutely convex compact set. This is useful in connection with ε-compatibility due to Corollary 3.2.5 (iv) and also relevant in connection with the Pettis-integrability of a vector-valued function due to the Mackey–Arens theorem.

We recall that the space of càdlàg functions on a set $\Omega\subset\mathbb{R}$ with values in an lcHs E is defined by

$$D(\Omega,E):=\{f\in E^{\Omega}\mid\forall\, x\in\Omega:\ \lim_{w\to x+}f(w)=f(x)\text{ and }f(x-):=\lim_{w\to x-}f(w)\text{ exists}\}.^{4}$$

A.1.1. Proposition. *Let $\Omega\subset\mathbb{R}$, $K\subset\Omega$ be compact and E an lcHs. Then $f(K)$ is precompact for every $f\in D(\Omega,E)$. If E is quasi-complete, then $\overline{\operatorname{acx}}(\overline{f(K)})$ is compact.*

Proof. Let $f\in D(\Omega,E)$, $\alpha\in\mathfrak{A}$ and $\varepsilon>0$. We recall and define

$$\mathbb{B}_r(x)=\{w\in\mathbb{R}\mid|w-x|<r\}\quad\text{and}\quad B_{\varepsilon,\alpha}(y):=\{w\in E\mid p_\alpha(w-y)<\varepsilon\}$$

for every $x\in\Omega$, $y\in E$ and $r>0$. Let $x\in\Omega$. Then there is $r_{x-}>0$ such that $p_\alpha(f(w)-f(x-))<\varepsilon$ for all $w\in\mathbb{B}_{r_{x-}}(x)\cap(-\infty,x)\cap\Omega$ if x is an accumulation point of $(-\infty,x]\cap\Omega$. Further, there is $r_{x+}>0$ such that $p_\alpha(f(w)-f(x))<\varepsilon$ for all $w\in\mathbb{B}_{r_{x+}}(x)\cap[x,\infty)\cap\Omega$ if x is an accumulation point of $[x,\infty)\cap\Omega$. If x is an accumulation point of $(-\infty,x]\cap\Omega$ and $[x,\infty)\cap\Omega$, we choose $r_x:=\min(r_{x-},r_{x+})$. If x is an accumulation point of $(-\infty,x]\cap\Omega$ but not of $[x,\infty)\cap\Omega$, we choose $r_x:=r_{x-}$. If x is an accumulation point of $[x,\infty)\cap\Omega$ but not of $(-\infty,x]\cap\Omega$, we choose $r_x:=r_{x+}$. If x is neither an accumulation point of $(-\infty,x]\cap\Omega$ nor of $[x,\infty)\cap\Omega$, then there is $r_x>0$ such that $\mathbb{B}_{r_x}(x)\cap\Omega=\{x\}$.

Setting $V_x:=\mathbb{B}_{r_x}(x)\cap\Omega$, we note that the sets V_x are open in Ω with respect to the topology induced by $\mathbb{R}$ and $K\subset\bigcup_{x\in K}V_x$. Since K is compact, there are $n\in\mathbb{N}$ and $x_1,\dots,x_n\in K$ such that $K\subset\bigcup_{i=1}^{n}V_{x_i}$. W.l.o.g. each x_i is an accumulation point of $(-\infty,x_i]\cap\Omega$ and $[x_i,\infty)\cap\Omega$. Then we have $f(w)\in(B_{\varepsilon,\alpha}(f(x_i-))\cup B_{\varepsilon,\alpha}(f(x_i)))$ for all $w\in V_{x_i}$ and get

$$f(K)\subset\bigcup_{i=1}^{n}f(V_{x_i})\subset\bigcup_{i=1}^{n}\bigl(B_{\varepsilon,\alpha}(f(x_i-))\cup B_{\varepsilon,\alpha}(f(x_i))\bigr),$$

which means that $f(K)$ is precompact.

[4] We recall that for $x\in\Omega$ we only demand $\lim_{w\to x+}f(w)=f(x)$ if x is an accumulation point of $[x,\infty)\cap\Omega$, and the existence of the limit $\lim_{w\to x-}f(w)$ if x is an accumulation point of $(-\infty,x]\cap\Omega$.

If E is quasi-complete, then the precompact set $f(K)$ is relatively compact by [89, 3.5.3 Proposition, p. 65]. Hence $\overline{\operatorname{acx}}(\overline{f(K)})$ is compact as quasi-complete spaces have ccp. □

For $f \in D(\Omega, E)$ we define the *jump function* $\Delta_* f(x) := f(x) - f(x-)$, $x \in \Omega$, where we set $f(x-) := 0$ if x is not an accumulation point of $(-\infty, x] \cap \Omega$.

A.1.2. Proposition. *Let $\Omega \subset \mathbb{R}$, $K \subset \Omega$ be compact and E an lcHs. Then $\Delta_* f(K)$ is precompact for every $f \in D(\Omega, E)$. If E is quasi-complete, then the set $\overline{\operatorname{acx}}(\overline{\Delta_* f(K)})$ is compact.*

Proof. If K is a finite set, then $\Delta_* f(K)$ is finite, thus compact, and we are done. So let us assume that K is not finite. Let $\alpha \in \mathfrak{A}$ and $\varepsilon > 0$. We define $\Delta_{\varepsilon,\alpha} := \{x \in K \mid p_\alpha(\Delta_* f(x)) \geq \varepsilon\}$ and claim that $\Delta_{\alpha,\varepsilon}$ is a finite set. Let us assume the contrary. Then there is an infinite sequence (x_n) in $\Delta_{\varepsilon,\alpha} \subset K$. Due to the compactness of K there is a subsequence of (x_n) which converges to some $x \in K$. W.l.o.g. this subsequence is strictly increasing and we call this subsequence again (x_n). Since f has left limits (in left-accumulation points), for every $n \in \mathbb{N}$, $n \geq 2$, there is $w_n \in (x_{n-1}, x_n)$ such that $p_\alpha(f(x_n-) - f(w_n)) \leq \varepsilon/2$ (if x_n is not an accumulation point of $(-\infty, x_n] \cap \Omega$, then there is $w_n \in (x_{n-1}, x_n)$ with $w_n \notin \Omega$ and we set $f(w_n) := 0$). Hence we have

$$\begin{aligned} p_\alpha(f(x_n) - f(w_n)) &\geq p_\alpha(f(x_n) - f(x_n-)) - p_\alpha(f(x_n-) - f(w_n)) \\ &= p_\alpha(\Delta_* f(x_n)) - p_\alpha(f(x_n-) - f(w_n)) \geq \varepsilon/2 \end{aligned}$$

for all $n \geq 2$. But this is a contradiction because

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(w_n) = f(x-),$$

which proves our claim.

Next, we note that

$$\Delta_* f(K) \subset \big(B_{\varepsilon,\alpha}(0) \cup \Delta_* f(\Delta_{\varepsilon,\alpha})\big) \subset \bigcup_{z \in \{0\} \cup \Delta_* f(\Delta_{\varepsilon,\alpha})} z + B_{\varepsilon,\alpha}(0),$$

which implies that $\Delta_* f(K)$ is precompact as $\{0\} \cup \Delta_* f(\Delta_{\varepsilon,\alpha})$ is finite.

If E is quasi-complete, then the precompact set $\Delta_* f(K)$ is relatively compact by [89, 3.5.3 Proposition, p. 65]. Hence $\overline{\operatorname{acx}}(\overline{\Delta_* f(K)})$ is compact as quasi-complete spaces have ccp. □

Proposition A.1.1 and Proposition A.1.2 are known in the case that $\Omega = [0,1]$ and $E = \mathbb{K}$ (see the comments after [19, Chap. 3, Sect. 14, Lemma 1, p. 110]) since precompactness is equivalent to boundedness if $E = \mathbb{K}$.

A.1.3. Proposition. *Let Ω be a locally compact topological Hausdorff space and $f \in \mathcal{C}_0(\Omega, E)$. If*

(i) E is an lcHs with ccp, or

(ii) E is an lcHs with metric ccp and Ω second-countable,

then $\overline{\operatorname{acx}}(f(\Omega))$ is compact.

Proof. Let Ω be compact, then $f(\Omega)$ is compact in E as f is continuous. If Ω is even second-countable, then Ω is metrisable by [58, Chap. XI, 4.1 Theorem, p. 233] and thus $f(\Omega)$ as well by [34, Chap. IX, §2.10, Proposition 17, p. 159]. This yields that $\overline{\operatorname{acx}}(f(\Omega))$ is compact in both cases.

Let Ω be non-compact and Ω^* denote the one-point compactification of Ω. Since $f \in \mathcal{C}_0(\Omega, E)$, it has a unique continuous extension $\widehat{f}$ to Ω^* with $\widehat{f}(\infty) = 0$. Hence $K := \widehat{f}(\Omega^*)$ is a compact set in E as Ω^* is compact and $\widehat{f}$ continuous. If Ω is even second-countable, then Ω^* is metrisable by [58, Chap. XI, 8.6 Theorem,

p. 247] and thus K as well by [34, Chap. IX, §2.10, Proposition 17, p. 159]. This yields that $\overline{\operatorname{acx}}(K)$ is compact in both cases and thus the closed subset $\overline{\operatorname{acx}}(f(\Omega))$, too. □

We note that $\mathcal{C}_0(\Omega,E)=\mathcal{C}(\Omega,E)$ if Ω is compact. For our next proposition we define the space of bounded γ-Hölder continuous functions, $0<\gamma\leq 1$, from a metric space (Ω,d) to an lcHs E by

$$\mathcal{C}_b^{[\gamma]}(\Omega,E):=\Big\{f\in E^{\Omega}\mid \forall\alpha\in\mathfrak{A}:\ \sup_{x\in\Omega}p_\alpha(f(x))<\infty \text{ and } \sup_{\substack{x,y\in\Omega\\x\neq y}}\frac{p_\alpha(f(x)-f(y))}{\mathrm{d}(x,y)^\gamma}<\infty\Big\}.$$

A.1.4. Proposition. *Let (Ω,d) be a metric space, E a locally complete lcHs and $f\in\mathcal{C}_b^{[\gamma]}(\Omega,E)$ for some $0<\gamma\leq 1$. If there is $h\colon\Omega\to(0,\infty)$ such that fh is bounded on Ω and with $N:=\{x\in\Omega\mid f(x)=0\}$ it holds that*

$$\forall\,\varepsilon>0\ \exists\,K\subset\Omega \text{ compact}\ \forall\,x\in\Omega\setminus(K\cup N):\ 1\leq\varepsilon h(x),$$

then $\overline{\operatorname{acx}}(f(\Omega))$ is compact.

Proof. Since $f\in\mathcal{C}_b^{[\gamma]}(\Omega,E)$, the sets $f(\Omega)$ and

$$B_1:=\Big\{\frac{f(z)-f(t)}{\mathrm{d}(z,t)^\gamma}\mid z,t\in\Omega,\ z\neq t\Big\}$$

are bounded in E. Further, the range $(fh)(\Omega)$ is bounded in E by assumption. Thus $B:=\overline{\operatorname{acx}}(B_1\cup f(\Omega)\cup(fh)(\Omega))$ is a closed disk and E_B a Banach space with the norm $\|x\|_B:=\inf\{r>0\mid x\in rB\}$, $x\in E_B$, as E is locally complete. Next, we show that $f(\Omega)$ is precompact in E_B. Let V be a zero neighbourhood in E_B. Then there is $\varepsilon>0$ such that $U_\varepsilon:=\{x\in E_B\mid\|x\|_B\leq\varepsilon\}\subset V$. Moreover, there is a compact set $K\subset\Omega$ such that $1\leq\varepsilon h(x)$ for all $x\in\Omega\setminus(K\cup N)$. The map $f\colon\Omega\to E_B$ is well-defined and uniformly continuous because $\|f(z)-f(t)\|_B\leq\mathrm{d}(z,t)^\gamma$ for all $z,t\in\Omega$, which follows from $B_1\subset B$. We deduce that $f(K)$ is compact in E_B. We note that

$$f(x)=f(x)h(x)\frac{1}{h(x)},\quad x\in\Omega\setminus N,$$

which implies that $\|f(x)\|_B\leq\frac{1}{h(x)}$ as $(fh)(\Omega)\subset B$. Hence we have

$$\|f(x)\|_B\leq\frac{1}{h(x)}\leq\varepsilon,\quad x\in\Omega\setminus(K\cup N),$$

and the estimate $0=\|f(x)\|_B\leq\varepsilon$ is still valid for $x\in N$, yielding $f(\Omega\setminus K)\subset U_\varepsilon$. Since $f(K)$ is compact in E_B, it is also precompact and so there is a finite set $P\subset E_B$ such that $f(K)\subset P+V$. We derive that

$$f(\Omega)=\big(f(K)\cup f(\Omega\setminus K)\big)\subset\big((P+V)\cup U_\varepsilon\big)\subset\big((P\cup\{0\})+V\big),$$

which means that $f(\Omega)$ is precompact in E_B and thus $\overline{\operatorname{acx}}(f(\Omega))$ as well by [89, 6.7.1 Proposition, p. 112]. Therefore the set $\overline{\operatorname{acx}}(f(\Omega))$ is compact in the Banach space E_B and also compact in the weaker topology of E. □

The underlying idea of Proposition A.1.4 is taken from [29, Lemma 1, Proposition 2, p. 354].

A.1.5. Proposition. *Let $\Omega\subset\mathbb{R}^d$ be an open convex set, E an lcHs over $\mathbb{K}$ and $f\colon\Omega\to E$ weakly C_b^1, i.e. $e'\circ f\in\mathcal{C}_b^1(\Omega)$ for each $e'\in E'$. Then $f\in\mathcal{C}_b^{[1]}(\Omega,E)$.*

Proof. Let $z,t\in\Omega$, $z\neq t$. By the mean value theorem we have

$$\frac{|(e'\circ f)(z)-(e'\circ f)(t)|}{|z-t|}\leq C_d\max_{1\leq n\leq d}\sup_{x\in\Omega}|(\partial^{e_n})^{\mathbb{K}}(e'\circ f)(x)|\leq C_d|e'\circ f|_{\mathcal{C}_b^1(\Omega)}<\infty$$

for all $e' \in E'$ where $C_d := \sqrt{d}$ if $\mathbb{K} = \mathbb{R}$, and $C_d := 2\sqrt{d}$ if $\mathbb{K} = \mathbb{C}$. It follows from [131, Mackey's theorem 23.15, p. 268] that f is Lipschitz continuous and bounded as well, thus $f \in \mathcal{C}_b^{[1]}(\Omega, E)$. □

A.1.6. Proposition. *Let $\mathcal{FV}(\Omega, E)$ be a* dom*-space, let there be a set X, a family $\mathfrak{K}$ of sets and a map $\pi\colon \bigcup_{m\in M} \omega_m \to X$ such that $\bigcup_{K\in\mathfrak{K}} K \subset X$. If $f \in \mathcal{FV}(\Omega, E)$ fulfils*

$\forall\, \varepsilon > 0,\, j \in J,\, m \in M,\, \alpha \in \mathfrak{A}\ \exists\, K \in \mathfrak{K}:$

(i) $\sup_{\substack{x\in\omega_m,\\ \pi(x)\notin K}} p_\alpha\big(T_m^E(f)(x)\big)\nu_{j,m}(x) < \varepsilon,$

(ii) $N_{\pi\subset K,j,m}(f) := \{T_m^E(f)(x)\nu_{j,m}(x) \mid x \in \omega_m,\ \pi(x) \in K\}$ *is precompact in* E,

then the set $N_{j,m}(f)$ is precompact in E for every $j \in J$ and $m \in M$. If E is quasi-complete, then $\overline{\operatorname{acx}}(\overline{N_{j,m}(f)})$ is compact.

Proof. Let V be a zero neighbourhood in E. Then there are $\alpha \in \mathfrak{A}$ and $\varepsilon > 0$ such that $B_{\varepsilon,\alpha} \subset V$ where $B_{\varepsilon,\alpha} := \{x \in E \mid p_\alpha(x) < \varepsilon\}$. Let $j \in J$ and $m \in M$. Due to (i) there is $K \in \mathfrak{K}$ such that the set

$$N_{\pi\not\subset K,j,m}(f) := \{T_m^E(f)(x)\nu_{j,m}(x) \mid x \in \omega_m,\ \pi(x) \notin K\}$$

is contained in $B_{\varepsilon,\alpha}$. Further, the precompactness of $N_{\pi\subset K,j,m}(f)$ by (ii) implies that there exists a finite set $P \subset E$ such that $N_{\pi\subset K,j,m}(f) \subset P + V$. Hence we conclude

$$\begin{aligned} N_{j,m}(f) &= \big(N_{\pi\not\subset K,j,m}(f) \cup N_{\pi\subset K,j,m}(f)\big)\\ &\subset \big(B_{\varepsilon,\alpha} \cup (P+V)\big) \subset \big(V \cup (P+V)\big) = (P \cup \{0\}) + V, \end{aligned}$$

which means that $N_{j,m}(f)$ is precompact.

The second part of the statement follows from the fact that a precompact set in a quasi-complete space is relatively compact by [89, 3.5.3 Proposition, p. 65] and that quasi-complete spaces have ccp. □

The most common case is that $\mathfrak{K}$ consists of the compact subsets of Ω and π is a projection on $X := \Omega$ (see e.g. Example 4.2.11, Example 4.2.16 and Example 4.2.22).

A.2. The Pettis-integral

We start with the definition of the Pettis-integral which we use to define Fourier transformations of vector-valued functions (see Proposition 4.2.25, Theorem 5.6.13 and Theorem 5.6.14) and for Riesz–Markov–Kakutani theorems in Section 4.3.

Let Σ be a σ-algebra on a set X. A function $\mu\colon \Sigma \to \mathbb{K}$ is called $\mathbb{K}$*-valued measure* if $\mu(\varnothing) = 0$ and μ is *countably additive*, i.e. for any sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in Σ it holds that

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \mu(A_n) \in \mathbb{K}.$$

If $\mathbb{K} = \mathbb{R}$, μ is also called a *signed measure*, and if $\mathbb{K} = \mathbb{C}$ a *complex measure*. If $\mathbb{K}$ is replaced by $[0,\infty]$, we say that μ is a *positive measure*. For a $\mathbb{K}$-valued measure μ its *total variation* $|\mu|$ given by

$$|\mu|(A) := \sup\Big\{\sum_{n\in\mathbb{N}} |\mu(A_n)| \;\Big|\; A_n \in \Sigma, A_m \cap A_n = \varnothing \text{ if } m \neq n, A = \bigcup_{n\in\mathbb{N}} A_n\Big\}, \quad A \in \Sigma,$$

is a well-defined positive measure by [149, 6.2 Theorem, p. 117] and it is *finite* by [149, 6.4 Theorem, p. 118], i.e. $|\mu|(X) < \infty$. Obviously, a $\mathbb{K}$-valued measure μ is positive if and only if $|\mu| = \mu$. For a positive measure μ on X and $1 \leq p < \infty$ let

$$\mathfrak{L}^p(X,\mu) := \{f\colon X \to \mathbb{K} \text{ measurable} \mid q_p(f) := \int_X |f(x)|^p \mathrm{d}\mu(x) < \infty\}$$

and define the quotient space of p-integrable functions by $\mathcal{L}^p(X,\mu) := \mathfrak{L}^p(X,\mu)/\{f \in \mathfrak{L}^p(X,\mu) \mid q_p(f) = 0\}$, which becomes a Banach space if it is equipped with the norm $\|f\|_p := \|f\|_{\mathcal{L}^p} := q_p(F)^{1/p}$, $f = [F] \in \mathcal{L}^p(X,\mu)$. From now on we do not distinguish between equivalence classes and their representatives anymore.

For a $\mathbb{K}$-valued measure μ there is a unique $h \in \mathcal{L}^1(X,|\mu|)$ with $\mathrm{d}\mu = h\mathrm{d}|\mu|$ by the Radon–Nikodým theorem (see [149, 6.12 Theorem, p. 124]) and h can be chosen such that $|h| = 1$, i.e. has a representative with modulus equal to 1. Now, we say that $f \in \mathcal{L}^p(X,\mu)$ if $f \cdot h \in \mathcal{L}^p(X,|\mu|)$. For $f \in \mathcal{L}^1(X,\mu)$ we define the *integral of f on X w.r.t. μ* by

$$\int_X f(x)\mathrm{d}\mu(x) := \int_X f(x)h(x)\mathrm{d}|\mu|(x).$$

For a measure space (X,Σ,μ) and $f\colon X \to \mathbb{K}$ we say that f is *integrable on* $\Lambda \in \Sigma$ and write $f \in \mathcal{L}^1(\Lambda,\mu)$ if $\chi_\Lambda f \in \mathcal{L}^1(X,\mu)$. Then we set

$$\int_\Lambda f(x)\mathrm{d}\mu(x) := \int_X \chi_\Lambda(x)f(x)\mathrm{d}\mu(x).$$

A.2.1. Definition (Pettis-integral). Let (X,Σ,μ) be a measure space and E an lcHs. A function $f\colon X \to E$ is called *weakly measurable* if the function $e' \circ f\colon X \to \mathbb{K}$, $(e' \circ f)(x) := \langle e', f(x)\rangle := e'(f(x))$, is measurable for all $e' \in E'$. A weakly measurable function is said to be *weakly integrable* if $e' \circ f \in \mathcal{L}^1(X,\mu)$. A function $f\colon X \to E$ is called *Pettis-integrable* on $\Lambda \in \Sigma$ if it is weakly integrable on Λ and

$$\exists\, e_\Lambda \in E\ \forall e' \in E' : \langle e', e_\Lambda\rangle = \int_\Lambda \langle e', f(x)\rangle \mathrm{d}\mu(x).$$

In this case e_Λ is unique due to E being Hausdorff and we set the *Pettis-integral*

$$\int_\Lambda f(x)\mathrm{d}\mu(x) := e_\Lambda.$$

If we consider the measure space $(X,\mathscr{L}(X),\lambda)$ of Lebesgue measurable sets for $X \subset \mathbb{R}^d$, we just write $\mathrm{d}x := \mathrm{d}\lambda(x)$.

A.2.2. Lemma. *Let E be a locally complete lcHs, $\Omega \subset \mathbb{R}^d$ open and $f\colon \Omega \to E$. If f is weakly $\mathcal{C}^1$, i.e. $e' \circ f \in \mathcal{C}^1(\Omega)$ for every $e' \in E'$, then f is Pettis-integrable on every compact subset $K \subset \Omega$ with respect to any locally finite positive measure μ on Ω and*

$$p_\alpha\Big(\int_K f(x)\mathrm{d}\mu(x)\Big) \leq \mu(K)\sup_{x\in K} p_\alpha(f(x)), \quad \alpha \in \mathfrak{A}.$$

Proof. Let $K \subset \Omega$ be compact and (Ω,Σ,μ) a measure space with locally finite measure μ, i.e. Σ contains the Borel σ-algebra $\mathcal{B}(\Omega)$ on Ω and for every $x \in \Omega$ there is a neighbourhood $U_x \subset \Omega$ of x such that $\mu(U_x) < \infty$. Since the map $e' \circ f$ is differentiable for every $e' \in E'$, thus Borel-measurable, and $\mathcal{B}(\Omega) \subset \Sigma$, it is measurable. We deduce that $e' \circ f \in \mathcal{L}^1(K,\mu)$ for every $e' \in E'$ because locally finite measures are finite on compact sets. Therefore the map

$$I\colon E' \to \mathbb{K},\ I(e') := \int_K \langle e', f(x)\rangle \mathrm{d}\mu(x)$$

is well-defined and linear. We estimate

$$|I(e')| \leq |\mu(K)| \sup_{x \in f(K)} |e'(x)| \leq \mu(K) \sup_{x \in \overline{\mathrm{acx}}(f(K))} |e'(x)|, \quad e' \in E'.$$

Due to f being weakly $\mathcal{C}^1$ and [29, Proposition 2, p. 354] the absolutely convex set $\overline{\mathrm{acx}}(f(K))$ is compact, yielding $I \in (E'_\kappa)' \cong E$ by the theorem of Mackey–Arens, which means that there is $e_K \in E$ such that

$$\langle e', e_K \rangle = I(e') = \int_K \langle e', f(x) \rangle \mathrm{d}\mu(x), \quad e' \in E'.$$

Hence f is Pettis-integrable on K w.r.t. μ. For $\alpha \in \mathfrak{A}$ we set $B_\alpha := \{x \in E \mid p_\alpha(x) < 1\}$ and observe that

$$\begin{aligned} p_\alpha\Big(\int_K f(x)\mathrm{d}\mu(x)\Big) &= \sup_{e' \in B_\alpha^\circ} \Big|\langle e', \int_K f(x)\mathrm{d}\mu(x)\rangle\Big| = \sup_{e' \in B_\alpha^\circ} \Big|\int_K e'(f(x))\mathrm{d}\mu(x)\Big| \\ &\leq \mu(K) \sup_{e' \in B_\alpha^\circ} \sup_{x \in K} |e'(f(x))| = \mu(K) \sup_{x \in K} p_\alpha(f(x)) \end{aligned}$$

where we used [131, Proposition 22.14, p. 256] in the first and last equation to get from p_α to $\sup_{e' \in B_\alpha^\circ}$ and back. □

A.2.3. Lemma. *Let E be a sequentially complete lcHs, $\Omega \subset \mathbb{R}^d$ open, (Ω, Σ, μ) a measure space with locally finite positive measure μ and $f\colon \Omega \to E$. If f is weakly $\mathcal{C}^1$ and there are $\psi \in \mathcal{L}^1(\Omega, \mu)$ and $g\colon \Omega \to [0, \infty)$ measurable such that $\psi g \geq 1$ and fg is bounded on Ω, then f is Pettis-integrable on Ω and*

$$p_\alpha\Big(\int_\Omega f(x)\mathrm{d}\mu(x)\Big) \leq \|\psi\|_1 \sup_{x \in \Omega} p_\alpha(f(x)g(x)), \quad \alpha \in \mathfrak{A}.$$

Proof. Let $(K_n)_{n \in \mathbb{N}}$ be a compact exhaustion of Ω. Due to Lemma A.2.2 the Pettis-integral

$$e_n := \int_{K_n} f(x)\mathrm{d}\mu(x)$$

is a well-defined element of E for every $n \in \mathbb{N}$. Next, we show that (e_n) is a Cauchy sequence in E. Let $\alpha \in \mathfrak{A}$, $m \in \mathbb{N}_0$ and $k, n \in \mathbb{N}$ with $k > n$. We set $B_\alpha := \{x \in E \mid p_\alpha(x) < 1\}$ and $Q_{k,n} := K_k \setminus K_n$ and note that

$$\begin{aligned} p_\alpha(e_k - e_n) &= \sup_{e' \in B_\alpha^\circ} |e'(e_k - e_n)| = \sup_{e' \in B_\alpha^\circ} \Big|\int_{Q_{k,n}} e'(f(x))\mathrm{d}\mu(x)\Big| \\ &\leq \int_{Q_{k,n}} |\psi(x)|\mathrm{d}\mu(x) \sup_{e' \in B_\alpha^\circ} \sup_{x \in \Omega} |e'(f(x)g(x))| \\ &= \int_{Q_{k,n}} |\psi(x)|\mathrm{d}\mu(x) \sup_{x \in \Omega} p_\alpha(f(x)g(x)) \end{aligned} \tag{64}$$

where we used [131, Proposition 22.14, p. 256] to switch from p_α to $\sup_{e' \in B_\alpha^\circ}$ and back. Since $\psi \in \mathcal{L}^1(\Omega, \mu)$, we have that (e_n) is a Cauchy sequence in the sequentially complete space E. Thus $e_\Omega := \lim_{n \to \infty} e_n$ exists in E and the dominated convergence theorem implies

$$e'(e_\Omega) = \lim_{n \to \infty} e'(e_n) = \lim_{n \to \infty} \int_{K_n} e'(f(x))\mathrm{d}\mu(x) = \int_\Omega e'(f(x))\mathrm{d}\mu(x), \quad e' \in E'.$$

Hence f is Pettis-integrable on Ω with $\int_\Omega f(x)\mathrm{d}\mu(x) = e_\Omega$. As in (64) we have

$$p_\alpha(e_n) \leq \int_{K_n} |\psi(x)|\mathrm{d}\mu(x) \sup_{x \in \Omega} p_\alpha(f(x)g(x)) \leq \|\psi\|_1 \sup_{x \in \Omega} p_\alpha(f(x)g(x))$$

for every $n \in \mathbb{N}$. Letting $n \to \infty$, we derive the estimate in our statement. □

A.2.4. REMARK. Let μ be a $\mathbb{K}$-valued measure and Σ contain $\mathcal{B}(\Omega)$. Then Lemma A.2.2 is still valid with $\mu(K)$ replaced by $|\mu|(K)$ due to the definition of the integral w.r.t. a $\mathbb{K}$-valued measure and as $|\mu|(K) \leq |\mu|(\Omega) < \infty$. Thus Lemma A.2.3 holds in this case as well.

The following definition is analogous to the definition of the Pettis-integral.

A.2.5. DEFINITION (Pettis-summable). Let I be a non-empty set and E an lcHs. A family $(f_i)_{i\in I}$ in E is called *weakly summable* if $(\langle e', f_i\rangle)_{i\in I} \in \ell^1(I,\mathbb{K})$ for all $e' \in E'$. A family $(f_i)_{i\in I}$ in E is called *Pettis-summable* if it is weakly summable and

$$\exists\, e_I \in E \;\forall e' \in E' : \langle e', e_I\rangle = \sum_{i\in I}\langle e', f_i\rangle.$$

In this case e_I is unique due to E being Hausdorff and we set

$$\sum_{i\in I} f_i := e_I.$$

For the elements f of the space $D([0,1],E)$ of E-valued càdlàg functions on $[0,1]$ and their jump functions $\Delta_* f$ we have the following result.

A.2.6. PROPOSITION. *Let E be a quasi-complete lcHs, μ a $\mathbb{K}$-valued Borel measure on $[0,1]$ and $\psi \in \ell^1([0,1],\mathbb{K})$. Then $f \in D([0,1],E)$ is Pettis-integrable on $[0,1]$ and*

$$p_\alpha\Big(\int_{[0,1]} f(x)\mathrm{d}\mu(x)\Big) \leq |\mu|([0,1]) \sup_{x\in[0,1]} p_\alpha(f(x)), \quad \alpha \in \mathfrak{A},$$

and $(\Delta_* f)\psi$ is Pettis-summable on $[0,1]$ and

$$p_\alpha\Big(\sum_{x\in[0,1]} (\Delta_* f)(x)\psi(x)\Big) \leq \|\psi\|_{\ell^1} \sup_{x\in[0,1]} p_\alpha(\Delta_* f(x)), \quad \alpha \in \mathfrak{A}.$$

PROOF. By [19, Chap. 3, Sect. 14, Lemma 1, p. 110] $e' \circ f$ is Borel measurable for every $e' \in E'$ and integrable due to its boundedness on $[0,1]$. Thus the map

$$I\colon E' \to \mathbb{K},\; I(e') := \int_{[0,1]} e'(f(x))\mathrm{d}\mu(x),$$

is well-defined and linear. It follows from Proposition A.1.1 that $\overline{\operatorname{acx}}(\overline{f([0,1])})$ is absolutely convex and compact in E. In combination with the estimate

$$|I(e')| \leq |\mu|([0,1]) \sup_{x\in f([0,1])} |e'(x)| \leq |\mu|([0,1]) \sup_{x\in\overline{\operatorname{acx}}(\overline{f([0,1])})} |e'(x)|$$

for every $e' \in E'$ we deduce that $I \in (E'_\kappa)' \cong E$ by the theorem of Mackey–Arens, which implies that there is $e_{[0,1]} \in E$ such that

$$\langle e', e_{[0,1]}\rangle = I(e') = \int_{[0,1]} e'(f(x))\mathrm{d}\mu(x), \quad e' \in E'.$$

Thus f is Pettis-integrable on $[0,1]$.

Since $\psi \in \ell^1([0,1])$ and $e' \circ \Delta_* f$ bounded on $[0,1]$ for every $e' \in E'$, the map

$$I_0\colon E' \to \mathbb{K},\; I_0(e') := \sum_{x\in[0,1]} e'((\Delta_* f)(x)\psi(x)),$$

is well-defined and linear. Moreover, the set $\overline{\operatorname{acx}}(\overline{\Delta_* f([0,1])})$ is absolutely convex and compact by Proposition A.1.2. Again, the estimate

$$|I_0(e')| \leq \sum_{x\in[0,1]} |\psi(x)| \sup_{x\in\Delta_* f([0,1])} |e'(x)| \leq \|\psi\|_{\ell^1} \sup_{x\in\overline{\operatorname{acx}}(\overline{\Delta_* f([0,1])})} |e'(x)|$$

for every $e' \in E'$, implies our statement. The remaining estimates are deduced analogously to Lemma A.2.2. □

A.2.7. PROPOSITION. *Let E be an lcHs, Ω a topological Hausdorff space and (Ω,Σ,μ) a measure space. If $f\colon\Omega\to E$ is weakly integrable and there are $\psi\in\mathcal{L}^1(\Omega,\mu)$ and $g\colon\Omega\to[0,\infty)$ measurable such that $\psi g\geq 1$ and*

(i) *E has ccp, Ω is locally compact and $fg\in\mathcal{C}_0(\Omega,E)$, or*
(ii) *E has metric ccp, Ω is locally compact and second-countable, and $fg\in\mathcal{C}_0(\Omega,E)$, or*
(iii) *E is locally complete, Ω a metric space, $fg\in\mathcal{C}_b^{[\gamma]}(\Omega,E)$ for some $0<\gamma\leq 1$ and there is $h\colon\Omega\to(0,\infty)$ such that fgh is bounded on Ω and with $N:=\{x\in\Omega\mid f(x)g(x)=0\}$ it holds that*
$$\forall\, \varepsilon>0\ \exists\, K\subset\Omega \text{ compact } \forall\, x\in\Omega\smallsetminus(K\cup N):\ 1\leq\varepsilon h(x),$$

then f is Pettis-integrable on Ω and
$$p_\alpha\Big(\int_\Omega f(x)\mathrm{d}\mu(x)\Big)\leq\|\psi\|_1\sup_{x\in\Omega}p_\alpha(f(x)g(x)),\quad \alpha\in\mathfrak{A}.$$

PROOF. Since f is weakly integrable, the map
$$I\colon E'\to\mathbb{K},\ I(e'):=\int_\Omega e'(f(x))\mathrm{d}\mu(x),$$
is well-defined and linear. It follows from Proposition A.1.3 in case (i)-(ii) and from Proposition A.1.4 in case (iii) that $\overline{\operatorname{acx}}(fg(\Omega))$ is absolutely convex and compact in E. If μ is a positive measure, i.e. $[0,\infty]$-valued, we observe that
$$|I(e')|\leq\int_\Omega|\psi(x)|\mathrm{d}\mu(x)\sup_{x\in fg(\Omega)}|e'(x)|\leq\|\psi\|_1\sup_{x\in\overline{\operatorname{acx}}(fg(\Omega))}|e'(x)|$$
for every $e'\in E'$. If μ is a $\mathbb{K}$-valued measure, then the same estimate holds with μ replaced by $|\mu|$. We deduce from this estimate that $I\in(E'_\kappa)'\cong E$ by the theorem of Mackey–Arens, which implies that there is $e_\Omega\in E$ such that
$$\langle e',e_\Omega\rangle=I(e')=\int_\Omega e'(f(x))\mathrm{d}\mu(x),\quad e'\in E'.$$
Thus f is Pettis-integrable on Ω. The remaining estimate is deduced analogously to Lemma A.2.2. □

The idea how to prove Proposition A.2.7 (ii) for $\Omega=\mathbb{R}^d$ is due to an anonymous reviewer of [110] but did not make it into [110] because of page limits.

List of Symbols

Sets and systems of sets

$\mathrm{acx}(M)$		absolutely convex hull of the set M 17
	$\overline{\mathrm{acx}}(M)$	closure of the absolutely convex hull of the set M 17
$B_{\mathcal{F}\nu(\Omega)}^{\circ F(\Omega)'}$		the polar $\{y' \in F(\Omega)' \mid \forall\, f \in B_{\mathcal{F}\nu(\Omega)} : \lvert y'(f)\rvert \leq 1\}$ 84
$\mathbb{B}_r(x)$		ball $\{w \in \mathbb{R}^d \mid \lvert w - x\rvert < r\}$ around $x \in \mathbb{R}^d$ with radius $r > 0$ 17
$\mathrm{ch}(M)$		circled hull of the set M 17
$\mathrm{cx}(M)$		convex hull of the set M 17
$\mathbb{D}_r(z)$		disc $\{w \in \mathbb{C} \mid \lvert w - z\rvert < r\}$ around $z \in \mathbb{C}$ with radius $r > 0$ 17
	$\mathbb{D}$	open unit disc $\mathbb{D}_1(0)$ 19
G°		the polar set of G 17
M_m		the set $\{\beta \in \mathbb{N}_0^d \mid \lvert\beta\rvert \leq \min(m,k)\}$ for $m \in \mathbb{N}_0$ and $k \in \mathbb{N}_\infty$ 24
$\overline{M}$		closure of the set M 17
	$\overline{M}^t$	closure of the set M w.r.t. the topology t 17
	$\overline{M}^X$	closure of the set M in the topological space X 17
∂M		boundary of the set M 17
$\mathbb{N}_\infty$		the set $\mathbb{N} \cup \{\infty\}$ 19
$\sigma(E,G)$		weak topology induced on E by a separating subspace $G \subset E'$ 78
τ_c		topology of compact convergence 21

Locally convex Hausdorff spaces & spaces of continuous linear operators

$E'^{\star}$		algebraic dual of the dual E' 29
E		locally convex Hausdorff space 17
	E_D	space $\bigcup_{n\in\mathbb{N}} nD$ for a disk $D \subset E$ 18
F'		dual space of F 17
	$t(F',F)$	topology on F' where $t = b$, γ, κ, σ or τ 17
	$t(F)$	bornology on F that induces the topology $t(F',F)$ 17
$F \cong E$		locally convex Hausdorff spaces F and E are isomorphic 17
$F\varepsilon E$		space $L_e(F'_\kappa, E)$ where $L(F'_\kappa, E)$ is equipped with the topology of uniform convergence on the equicontinuous subsets of F' 17
$F \otimes E$		tensor product of F and E 18
	$F \otimes_\varepsilon E$	$F \otimes E$ equipped with the topology induced by $F\varepsilon E$ 18
	$F \widehat{\otimes}_\varepsilon E$	completion of $F \otimes_\varepsilon E$ 18
	$F \otimes_\pi E$	$F \otimes E$ equipped with the projective topology 70
	$F \widehat{\otimes}_\pi E$	completion of $F \otimes_\pi E$ 70
$L(F,E)$		space of continuous linear operators from F to E 17
$L_t(F,E)$		space $L(F,E)$ equipped with the topology t 17
	b	topology $t = b$ on $L(F,E)$ of uniform convergence on the bounded subsets of F 17

	γ	topology $t=\gamma$ on $L(F,E)$ of uniform convergence on the precompact (totally bounded) subsets of F 17
	κ	topology $t=\kappa$ on $L(F,E)$ of uniform convergence on the absolutely convex, compact subsets of F 17
	σ	topology $t=\sigma$ on $L(F,E)$ of uniform convergence on the finite subsets of F 17
	τ	topology $t=\tau$ on $L(F,E)$ of uniform convergence on the absolutely convex, $\sigma(F,F')$-compact subsets of F 17

$(p_\alpha)_{\alpha\in\mathfrak{A}}$ directed system of seminorms inducing the locally convex Hausdorff topology on E 17

Spaces of functions

$A^{\tau}_{\overline{\partial}}(\mathbb{C},E)$	space of holomorphic functions $f\colon\mathbb{C}\to E$ of exponential type τ 51
$A^{\tau}_{\Delta}(\mathbb{R}^d,E)$	space of harmonic functions $f\colon\mathbb{R}^d\to E$ of exponential type τ 51
$\mathcal{A}(\overline{\Omega},E)$	space of continuous functions $f\colon\overline{\Omega}\to E$ such that f is holomorphic on Ω 47
$\mathcal{B}\nu(\mathbb{D},E)$	Bloch type space 87
$\mathcal{C}(\Omega,X)$	space of continuous functions $f\colon\Omega\to X$ 17
$\mathcal{C}_0(\Omega,X)$	space of continuous functions $f\colon\Omega\to X$ that vanish at infinity 17
$c_0(A,E)$	space of elements (x_k) in the Köthe space $\lambda^\infty(A,E)$ such that $(x_k a_{k,j})$ converges to 0 in E for all $j\in\mathbb{N}$ 123
$c(\mathbb{N},E)$	space of convergent sequences in E 123
$\mathcal{C}_b(\Omega,E)$	space of bounded continuous functions $f\colon\Omega\to E$ 46
$\mathcal{C}^{[\gamma]}_b(\Omega,E)$	space of bounded γ-Hölder continuous functions $f\colon\Omega\to E$ 140
$\mathcal{C}^1_b(\Omega,E)$	space of continuously partially differentiable functions $f\colon\Omega\to E$ such that $(\partial^\beta)^E f$ is bounded on Ω for all $\lvert\beta\rvert\leq 1$ 20
$\mathcal{C}^\infty_{P(\partial)}(\Omega,E)$	kernel of the linear partial differential operator $P(\partial)^E$ in $\mathcal{C}^\infty(\Omega,E)$ 48
$\mathcal{C}^\infty_{P(\partial),b}(\Omega,E)$	space of bounded functions f in $\mathcal{C}^\infty_{P(\partial)}(\Omega,E)$ 53
$\mathcal{C}_u(\Omega,E)$	space of uniformly continuous functions $f\colon\Omega\to E$ 39
$\mathcal{C}_{bu}(\Omega,E)$	space of bounded uniformly continuous functions $f\colon\Omega\to E$ 45
$\mathcal{CC}(\Omega,E)$	space of Cauchy continuous functions $f\colon\Omega\to E$ 38
$\mathcal{C}^{ext}(\Omega,E)$	space of continuous functions $f\colon\Omega\to E$ which have a continuous extension to $\overline{\Omega}$ 39
$\mathcal{C}^{[\gamma]}_z(\Omega,E)$	space of γ-Hölder continuous functions $f\colon\Omega\to E$ such that $f(z)=0$ 45
$\mathcal{C}^{[\gamma]}_{z,0}(\Omega,E)$	space of functions f in $\mathcal{C}^{[\gamma]}_z(\Omega,E)$ that vanish at infinity 45
$\mathcal{C}^k(\Omega,E)$	space of k-times continuously partially differentiable functions $f\colon\Omega\to E$ 19
$\mathcal{C}^\infty_{2\pi}(\mathbb{R}^d,E)$	space of functions f in $\mathcal{C}^\infty(\mathbb{R}^d,E)$ which are 2π-periodic in each variable 58
$\mathcal{C}^k(\overline{\Omega},E)$	space of functions f in $\mathcal{C}^k(\Omega,E)$ such that all partial derivatives $(\partial^\beta)^E f$ up to order k are continuously extendable on $\overline{\Omega}$ 58
$\mathcal{C}^{k,\gamma}(\overline{\Omega},E)$	space of functions f in $\mathcal{C}^k(\overline{\Omega},E)$ such that all partial derivatives $(\partial^\beta)^E f$ of order k are γ-Hölder continuous 105
$\mathcal{C}^{k,\gamma}_{loc}(\Omega,E)$	space of functions f in $\mathcal{C}^k(\Omega,E)$ such that all partial derivatives $(\partial^\beta)^E f$ of order k are locally γ-Hölder continuous 107
$\mathcal{CV}^k(\Omega,E)$	space of functions f in $\mathcal{C}^k(\Omega,E)$ s.t. $(\beta,x)\mapsto(\partial^\beta)^E f(x)\nu_{j,m}(\beta,x)$ is bounded on ω_m for all $j\in J$ and $m\in\mathbb{N}_0$ 24

$\ell\mathcal{V}(\Omega, E)$ — space of functions f in E^{Ω} such that $f\nu$ is bounded on Ω for all $\nu \in \mathcal{V}$ 42
$\mathcal{M}(\Omega, E)$ — space of meromorphic functions $f\colon \Omega \to E$ 81
$\mathcal{O}_M(\mathbb{R}^d, E)$ — multiplier space for the Schwartz space 25
$\mathcal{O}(\Omega, E)$ — space of holomorphic functions $f\colon \Omega \to E$ 20
$s(\Omega, E)$ — space of sequences (x_k) in E such that the sequence $(x_k(1+|k|^2)^{j/2})$ is bounded in E for all $j \in \mathbb{N}$ 43
$\mathcal{S}_\mu(\mathbb{R}^d, E)$ — Beurling–Björck space 55
  (γ) — property of μ 55
$\mathcal{S}(\mathbb{R}^d, E)$ — Schwartz space 25
X^{Ω} — space of maps $f\colon \Omega \to X$ 17

Maps

χ_K — characteristic function of a set $K \subset \Omega$ 17
$\overline{\partial}^E f$ — Cauchy–Riemann operator applied to an E-valued function f 20
$\Delta_* f$ — the jump function of a càdlàg function f 139
δ_x — point evaluation functional $f \mapsto f(x)$ 21
H_n — n-th Hermite polynomial 128
h_n — n-th Hermite function 128
$\mathcal{J}$ — the canonical injection $E \to E'^{\star}$, $x \longmapsto [e' \mapsto e'(x)]$ 30
$|\mu|$ — total variation of a $\mathbb{K}$-valued measure μ 142
$(\partial^{\beta})^E f$ — β-th partial derivative of an E-valued function f 19
  $\partial^{\beta} f$ — β-th partial derivative $(\partial^{\beta})^{\mathbb{K}} f$ of a $\mathbb{K}$-valued function f 20
$(\partial^n_{\mathbb{C}})^E f$ — n-th complex derivative $(\partial^n_{\mathbb{C}})^E f$ of an E-valued function f 20
  $f^{(n)}$ — n-th complex derivative $(\partial^n_{\mathbb{C}})^{\mathbb{C}} f$ of a $\mathbb{C}$-valued function f 20
$\mathscr{R}_f$ — the map $E' \to \mathcal{F}(\Omega)$, $e' \mapsto f_{e'}$, for given $f \in \mathcal{F}_G(U, E)$ 76
R_f — the map $E' \to \mathcal{FV}(\Omega)$, $e' \mapsto e' \circ f$, for given $f \in \mathcal{FV}(\Omega, E)_\sigma$ 29
R_f^t — the map $\mathcal{FV}(\Omega)' \to E'^{\star}$, $f' \longmapsto \big[e' \mapsto f'\big(R_f(e')\big)\big]$, for given $f \in \mathcal{FV}(\Omega, E)_\sigma$ 29
S — the map $\mathcal{F}(\Omega)\varepsilon E \to \mathcal{F}(\Omega, E)$, $u \longmapsto [x \mapsto u(\delta_x)]$ 21
  $S_{\mathcal{F}(\Omega)}$ — the map S with an emphasis on the dependence on $\mathcal{F}(\Omega)$ 21
$T_1 \varepsilon T_2$ — ε-product of the continuous linear operators T_1 and T_2 18
$T^E_{m,x}$ — the map $f \mapsto T^E_m(f)(x)$ 23
Θ — the linear injection $F \otimes E \to F\varepsilon E$ 18
$\mathcal{V}$ — family of weight functions 22
  (V_∞) — vanishing at infinity condition on the family of weight functions 34

Miscellaneous

(DN) — property of a Fréchet space 67
(Ω) — property of a Fréchet space 67
(PA) — property of a PLS-space 67

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Bibliography

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