

Parallel Condensation in the Presence of Global Masters

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Abstract

In the dynamic analysis of structures condensation methods are often used to reduce the number of degrees of freedom to manageable size. Substructuring and choosing the master variables as the degrees of freedom on the interfaces of the substructures yield data structures which are well suited to be implemented on distributed memory parallel computers. This paper discusses a parallel condensation method in the presence of generalized global masters which are obtained in reanalysis or from prolongation of coarse grid approximations, e.g.

Keywords: generalized eigenvalue problem, condensation, parallel method, global masters

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1 Introduction

In the analysis of the dynamic response of structures using finite element methods very often prohibitively many degrees of freedom are needed to model the behaviour of the system sufficiently accurate. Static condensation is frequently employed to economize the computation of a selected group of eigenvalues and eigenvectors. These methods choose from the degrees of freedom a small number of master variables. Neglecting inertia terms the remaining variables (termed slaves) are eliminated leaving a much smaller problem for the master variables only.

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The condensation process requires the factorization of the restriction of the stiffness matrix to the slave degrees of freedom. This can be done efficiently by choosing the master variables appropriately. Partitioning the structure under consideration into substructures and choosing the degrees of freedom on the interfaces of the substructures as masters and the interior knots of the substructures as slaves the matrix which has to be factorized becomes block diagonal. At the same time one obtains data structures and formulae which are well suited to be implemented on distributed memory parallel computers. Taking advantage of these properties Rothe and the third author obtained a fully parallel condensation method for generalized eigenvalue problems (cf. [10]).

It has frequently been noted in the literature that the quality of the eigenvalue and eigenvector approximations produced by static condensation is satisfactory only for a very small part of the lower end of the spectrum. To improve the approximation properties of condensation Mackens and the third author [8] introduced general masters which allow to implement a priori information of the eigenmodes (such as eigenmodes of similar structures considered in reanalysis or prolongations of eigenvector approximations obtained on a coarser grid) into the condensation process.

In [7] it was shown that the parallelization concept of [10] can be generalized to the case of general masters if the support of each of them is contained in a single substructure and in [2] we generalized it to condensation methods in the presence of global masters. In this paper we report on first results obtained with an implementation of this method. In particular we demonstrate the gain of accuracy effectuated by prolongations of coarse grid eigenvector approximations as general masters.

2 Condensation

We consider the general eigenvalue problem

$$Kx = \lambda Mx \tag{1}$$

where $K \in \mathbb{R}^{(n,n)}$ and $M \in \mathbb{R}^{(n,n)}$ are symmetric and positive definite matrices which are usually the stiffness and mass matrix of a finite element model of a structure, respectively. The dimension n is supposed to be very large.

Equation (1) can be reduced to manageable size by nodal condensation introduced by Guyan [1] and Irons [3]. To this end the degrees of freedom are decomposed into masters and slaves. After reordering the equations and

unknowns problem (1) can be rewritten as

$$\begin{pmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{pmatrix} \begin{pmatrix} x_m \\ x_s \end{pmatrix} = \lambda \begin{pmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{pmatrix} \begin{pmatrix} x_m \\ x_s \end{pmatrix}. \quad (2)$$

Neglecting the inertia terms in the second equation, solving for x_s , and substituting x_s into the first equation one obtains the condensed problem

$$\tilde{K}_0 x_m = \lambda \tilde{M}_0 x_m \quad (3)$$

where

$$\begin{aligned} \tilde{K}_0 &:= K_{mm} - K_{ms} K_{ss}^{-1} K_{sm}, \\ \tilde{M}_0 &:= M_{mm} - K_{ms} K_{ss}^{-1} M_{sm} - M_{ms} K_{ss}^{-1} K_{sm} + K_{ms} K_{ss}^{-1} M_{ss} K_{ss}^{-1} K_{sm}. \end{aligned} \quad (4)$$

Nodal condensation has the disadvantage that it produces accurate results only for a small part of the lower end of the spectrum. Several attempts have been suggested to improve its accuracy (cf. [4], [5], [6], [9], [11], e.g.) most of them being very time-consuming because an iterative process is involved in the enhancement of every single eigenvalue.

In [8] it was shown that condensation can be interpreted as one step of simultaneous inverse iteration with ansatz vectors e_1, \dots, e_m where e_j denotes the j -th unit vector. Hence, one can introduce preinformation about the modes of the structure into the condensation process using a basis transformation.

The idea of splitting a vector x into a master part x_m and a slave part x_s can be generalized by splitting x into two parts belonging to orthogonal subspaces. Let z_1, \dots, z_m be a basis of the space of master vectors, and let y_{m+1}, \dots, y_n a basis of the space of slave vectors. If we define $Z := (z_1, \dots, z_m) \in \mathbb{R}^{(n,m)}$ and $Y := (y_{m+1}, \dots, y_n) \in \mathbb{R}^{(n,n-m)}$ then x has the unique representation

$$x = Zx_m + Yx_s.$$

If we insert this into the original problem (1) and premultiply by $(Z, Y)^T$ we obtain the eigenvalue problem

$$\begin{pmatrix} K_{zz} & K_{zy} \\ K_{yz} & K_{yy} \end{pmatrix} \begin{pmatrix} x_m \\ x_s \end{pmatrix} = \lambda \begin{pmatrix} M_{zz} & M_{zy} \\ M_{yz} & M_{yy} \end{pmatrix} \begin{pmatrix} x_m \\ x_s \end{pmatrix} \quad (5)$$

where

$$L_{zz} := Z^T LZ, \quad L_{zy} := Z^T LY =: L_{yz}^T, \quad L_{yy} := Y^T LY, \quad L \in \{K, M\}. \quad (6)$$

Therefore, the stiffness and the mass matrix have been decomposed with respect to the spaces Z and Y .

In principle equation (5) could be employed to reduce the eigenvalue problem (1) using $\{z_1, \dots, z_m\}$ as master degrees of freedom in the same way as in nodal condensation. However, since in practice only the small set of masters is available, but the large set of slave vectors $\{y_{m+1}, \dots, y_n\}$ is definitely not the matrices $K_{zy}, K_{yy}, M_{zy}, M_{yy}$ are usually not at hand. Hence, the straightforward transfer of (3) and (4) to perform the reduction in the presence of general masters does not apply. In [8] it has been shown how to generate the condensed problem corresponding to the decomposition (5) with the basis z_1, \dots, z_m only.

Theorem 1 *Let $Z = (z_1, \dots, z_m) \in \mathbb{R}^{(n,m)}$ be linearly independent master vectors, let $Y \in \mathbb{R}^{(n,n-m)}$ such that $Z^T V Y = O$ for a symmetric and positive definite metric matrix $V \in \mathbb{R}^{(n,n)}$, and set $X := VZ$.*

Then the condensed problem with general masters z_1, \dots, z_m is the projected eigenvalue problem

$$P^T K P x_m = \lambda P^T M P x_m \quad (7)$$

where

$$P = K^{-1} X (X^T K^{-1} X)^{-1} X^T Z. \quad (8)$$

Since $(X^T K^{-1} X)^{-1}$ and $X^T Z$ are nonsingular matrices (7) and (8) demonstrate that condensation can be interpreted as one step of simultaneous inverse iteration. This observation indicates that the approximation quality of the condensation process can be enhanced substantially if additionally to interface masters we choose general masters z_j and the metric matrix V such that z_j are approximate eigenvectors of problem (1) and V approximates the mass matrix M .

An alternative method for the calculation of the projection matrix contained in Theorem 2 was proved in [8], too. This result served as a starting point for the development of the parallel implementation introduced in [2].

Theorem 2 *Let V, Z and X be defined as in Theorem 1, and let $Z^T V Z = I$.*

Then the projection matrix of the condensed problem (7), (8) can be calculated from

$$\begin{pmatrix} K & -X \\ -X^T & O_{m,m} \end{pmatrix} \begin{pmatrix} P \\ S \end{pmatrix} = \begin{pmatrix} O_{n,m} \\ -I_m \end{pmatrix}. \quad (9)$$

Moreover, the condensed stiffness matrix is given by

$$P^T K P = S. \quad (10)$$

3 Parallel Condensation

For nodal condensation the following strategy yields a coarse grained parallel algorithm developed in [10]. Suppose that the structure under consideration has been decomposed into r substructures and let the masters be chosen as interface degrees of freedom. Assume that the substructures connect to each other through the master variables only. If the slave variables are numbered appropriately, then the stiffness matrix is given by

$$K = \begin{pmatrix} K_{mm} & K_{ms1} & K_{ms2} & \dots & K_{msr} \\ K_{sm1} & K_{ss1} & 0 & \dots & 0 \\ K_{sm2} & 0 & K_{ss2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{smr} & 0 & 0 & \dots & K_{ssr} \end{pmatrix}, \quad (11)$$

and the mass matrix M has the same block form.

It is easily seen that in this case the reduced matrices in (4) are given by

$$\tilde{K}_0 = K_{mm} - \sum_{j=1}^r K_{mmj} := K_{mm} - \sum_{j=1}^r K_{msj} K_{ssj}^{-1} K_{smj}$$

and

$$\tilde{M}_0 = M_{mm} - \sum_{j=1}^r M_{mmj},$$

where

$$M_{mmj} := K_{msj} K_{ssj}^{-1} M_{smj} + M_{msj} K_{ssj}^{-1} K_{smj} - K_{msj} K_{ssj}^{-1} M_{ssj} K_{ssj}^{-1} K_{smj}.$$

Hence they can be computed completely in parallel.

If general masters z_1, \dots, z_g are added to the interface masters then the block structure of K in (11) is perturbed. Taking advantage of Theorem 2 we developed the following parallelization concept:

Let $Z_g := (z_1, \dots, z_g)$ and denote by Z_{gj} the restriction of Z_g to the j -th substructure. Let $V := \text{diag}(I_m, V_1, \dots, V_r)$ where m is the number of interface masters and the dimension of V_j corresponds to the number of slaves in the j -th substructure.

To each substructure we assign a (slave-) process named ' S_j ', and we attach to the master information one further process called ' Ma '. The slave processes are connected directly or indirectly to the master process.

We assume that the matrices K_{mm} , M_{mm} , and the substructuring are known to the master process and that for every $j = 1, \dots, r$ the j -th slave processes owns the matrices K_{ssj} , M_{ssj} , K_{smj} , M_{smj} , V_j and Z_{gj} .

1. For every $j = 1, \dots, r$ process ' S_j ' does:
 - $X_j := V_j Z_{gj}$
 - factorize $K_{ssj} =: L_j L_j^T$
 - solve $L_j Y_j = X_j$ for Y_j
 - solve $L_j U_j = K_{smj}$ for U_j
 - $T_j := Y_j^T Y_j$
 - $W_j := Y_j^T U_j$
2. Determine
 - $T := \sum_{j=1}^r T_j$ and $W := \sum_{j=1}^r W_j$
 - by a fan in process such that at the end ' Ma ' owns the matrices T and W
3. Process ' Ma ' does
 - solve $T S_{10} = W$ for S_{10}
 - solve $T S_{11} = I$ for S_{11}
4. ' Ma ' broadcasts S_{10} and S_{11} to all ' S_j ', $j = 1, \dots, r$
5. For every $j = 1, \dots, r$ process ' S_j ' does:
 - solve $L_j^T P_{j0} = Y_j S_{10} - U_j$ for P_{j0}
 - $A_j := K_{msj} P_{j0}$
 - isend A_j to ' Ma ' (nonblocking communication)
 - solve $L_j^T P_{j1} = Y_j S_{11}$ for P_{j1}
 - $B_j := P_{j0}^T M_{ssj}$
 - $C_j := (M_{msj} + B_j) P_{j1}$
 - $E_j := B_j P_{j0}$
 - $F_j := M_{msj} P_{j0}$
 - $G_j := P_{j1}^T M_{ssj} P_{j1}$
 - $H_j := E_j + F_j + F_j^T$
6. ' Ma ' computes $S_{00} := K_{mm} + \sum_{j=1}^r A_j$
7. Determine
 - $MZ_{1,1} := M_{mm} + \sum_{j=1}^r H_j$
 - $MZ_{1,2} := \sum_{j=1}^r C_j$
 - $MZ_{2,2} := \sum_{j=1}^r G_j$
 - by a fan in process such that ' Ma ' owns $MZ_{1,1}$, $MZ_{1,2}$ and $MZ_{2,2}$
8. ' Ma ' determines desired eigenvalues $\lambda_1, \dots, \lambda_k$ and corresponding eigenvectors u_1, \dots, u_k of the reduced eigenvalue problem

$$\begin{pmatrix} S_{00} & S_{10}^T \\ S_{10} & S_{11} \end{pmatrix} u = \lambda \begin{pmatrix} MZ_{1,1} & MZ_{1,2} \\ MZ_{1,2}^T & MZ_{2,2} \end{pmatrix} u$$

9. 'Ma' broadcasts $\lambda_1, \dots, \lambda_k$ and u_1, \dots, u_k to all ' S_j ', $j = 1, \dots, r$
10. For $j = 1, \dots, r$ process ' S_j ' determines the j -th slaveportion $u_{i,j}$ of the i -th eigenvector

$$(K_{ssj} - \lambda_i M_{ssj})u_{i,j} = (K_{smj} - \lambda_i M_{smj})u_i$$

4 Numerical Experiments

We implemented the algorithm of the last section in FORTRAN 90 using LAPACK 3 and BLAS routines for the linear algebra and MPI 1.05 for message passing. Depending on the dimension of the condensed eigenvalue problem in step 8. we solved it with the LAPACK routine `dspgvx` or the SCALAPACK routine `pdsygvx`.

We tested the program on a heterogeneous workstation cluster consisting of one HP C3000, one HP J2240, and five HP 9000, 712/100 connected by fast-ethernet and on an HP N-Class parallel computer with 8 and 16 HP-PA 8500/440MHz processors organized as one and two clusters, respectively. The user can only assign each process to one of the clusters whereas the local scheduling is organized by the operating system. Moreover the computer is run in a multi-user environment. Hence we are not able to report on the load balancing of the processors.

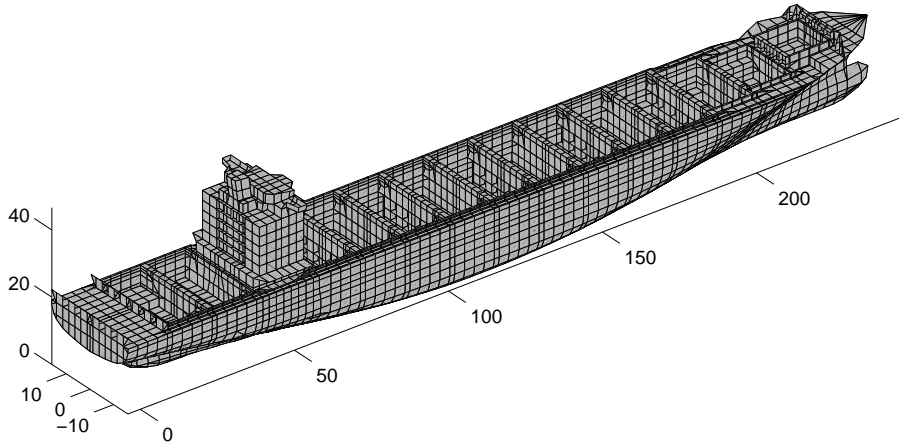
Our test example was a finite element model of a container ship with 35262 degrees of freedom given in Figure 1. We subdivided it into 6 and 9 substructures, respectively, each of them consisting of between 3000 and 8000 unknowns (cf. Figure 2), and obtained a reduced problem of dimension 2013 and 4479, respectively. This model is still very coarse for practical purposes, and work is in progress to study more realistic problems.

To demonstrate the gain of accuracy effectuated by prolongations of coarse grid eigenvector approximations as general masters we consider the free vibration problem of a uniform thin clamped plate covering the rectangular region $\Omega := (0, 5) \times (0, 3)$ which are governed by the eigenvalue problem

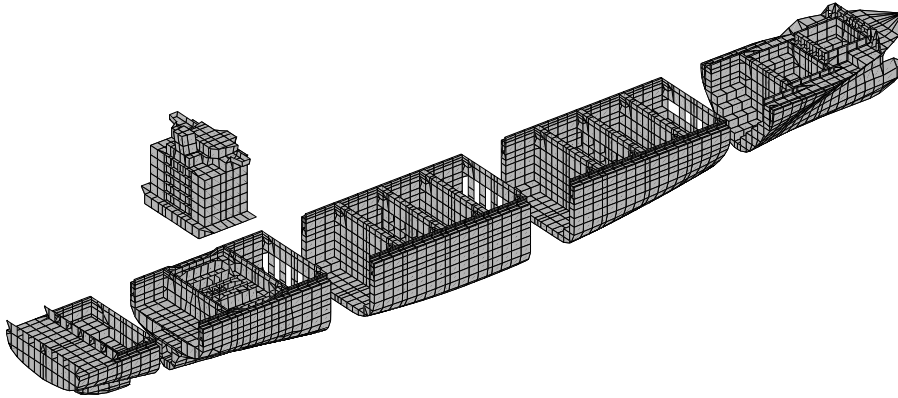
We discretized this problem by Bogner-Fox-Schmidt elements (with node variables u , u_x , u_y and u_{xy}) on a quadratic mesh of meshsize $h = 0.1$, and obtained a discrete problem of dimension $n = 5684$. Dividing Ω into 15 identical substructures each of them being a square of sidelength 1 and choosing all interface degrees of freedom as masters we obtained a reduced problem of dimension $m = 824$.

Additionally we constructed generalized masters in the following way: We discretized the plate problem with Bogner-Fox-Schmidt elements with stepsize $h = 1$ resulting in a matrix eigenvalue problem of dimension 32, and

Container ship



Substructuring of container ship



prolongated eigenvectors to the fine grid. Table 1 contains the smallest 12 eigenvalues of the problem, the relative errors of the condensed problem using nodal masters only, and the relative errors of the condensed problem in the presence of $g = 5$ and $g = 10$ general masters. Although the 32-dimensional approximate eigenvalue problem yields only very poor approximations of the eigenvalues and eigenvectors, the general masters improve the corresponding approximations of the condensed problem by at least 2 orders of magnitude (cf. Table 1).

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Table 1: Clamped plate: 0, 5 and 10 general masters

	eigenvalues	nodal cond.	5 gen. mast.	10 gen. mast.
1	8.2745284	3.04e-03	6.95e-08	4.13e-08
2	17.1453152	6.05e-03	9.36e-07	7.88e-07
3	39.9903040	1.28e-02	7.30e-06	6.94e-06
4	52.4244861	1.34e-02	1.75e-05	1.73e-05
5	71.1276841	1.77e-02	3.43e-05	3.13e-05
6	87.9305922	1.88e-02	1.88e-02	8.82e-05
7	109.7988780	2.53e-02	2.53e-02	8.40e-05
8	175.8636959	4.99e-02	4.98e-02	3.31e-04
9	179.2798277	6.40e-02	5.86e-02	2.90e-04
10	191.0277193	9.93e-02	9.66e-02	3.68e-04
11	224.8689790	1.12e-01	1.09e-01	1.09e-01
12	288.5281177	1.28e-01	1.26e-01	1.22e-01

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