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Weighted spaces of vector-valued functions and the ε -product

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Abstract

We introduce a new class $\mathcal{FV}(\Omega, E)$ of weighted spaces of functions on a set Ω with values in a locally convex Hausdorff space E which covers many classical spaces of vector-valued functions like continuous, smooth, holomorphic or harmonic functions. Then we exploit the construction of $\mathcal{FV}(\Omega, E)$ to derive sufficient conditions such that $\mathcal{FV}(\Omega, E)$ can be linearised, i.e. that $\mathcal{FV}(\Omega, E)$ is topologically isomorphic to the ε-product $\mathcal{FV}(\Omega)\varepsilon E$ where $\mathcal{FV}(\Omega) := \mathcal{FV}(\Omega, \mathbb{K})$ and $\mathbb{K}$ is the scalar field of E.

Keywords Vector-valued functions · ε-product · Linearisation · Weight · Semi-Montel space

Mathematics Subject Classification 46E40 · 46E10 · 46E15

1 Introduction

This work is dedicated to a classical topic, namely, the linearisation of weighted spaces of vector-valued functions. The setting we are interested in is the following. Let $\mathcal{FV}(\Omega)$ be a locally convex Hausdorff space of functions from a non-empty set Ω to a field $\mathbb{K}$ whose topology is generated by a family $\mathcal{V}$ of weight functions and E be a locally convex Hausdorff space. The ε-product of $\mathcal{FV}(\Omega)$ and E is defined as the space of linear continuous operators

$$\mathcal{FV}(\Omega)\varepsilon E := L_e(\mathcal{FV}(\Omega)'_\kappa, E)$$

equipped with the topology of uniform convergence on equicontinuous subsets of the dual $\mathcal{FV}(\Omega)'$ which itself is equipped with the topology of uniform convergence

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on absolutely convex compact subsets of $\mathcal{FV}(\Omega)$. Suppose that the point-evaluation functionals δ_x, $x \in \Omega$, belong to $\mathcal{FV}(\Omega)'$ and that there is a locally convex Hausdorff space $\mathcal{FV}(\Omega, E)$ of E-valued functions on Ω such that the map

$$S : \mathcal{FV}(\Omega)\varepsilon E \to \mathcal{FV}(\Omega, E),\ u \longmapsto [x \mapsto u(\delta_x)],$$

is well-defined. The main question we want to answer reads as follows. When is $\mathcal{FV}(\Omega)\varepsilon E$ a linearisation of $\mathcal{FV}(\Omega, E)$, i.e. when is S a topological isomorphism?

In [1–3] Bierstedt treats the space $\mathcal{CV}(\Omega, E)$ of continuous functions on a completely regular Hausdorff space Ω weighted with a Nachbin-family $\mathcal{V}$ and its topological subspace $\mathcal{CV}_0(\Omega, E)$ of functions that vanish at infinity in the weighted topology. He derives sufficient conditions on Ω, $\mathcal{V}$ and E such that the answer to our question is affirmative, i.e. S is a topological isomorphism. Schwartz answers this question for several weighted spaces of k-times continuously partially differentiable on $\Omega = \mathbb{R}^d$ like the Schwartz space in [31, 32] for quasi-complete E with regard to vector-valued distributions. Grothendieck treats the question in [15], mainly for nuclear $\mathcal{FV}(\Omega)$ and complete E. In [19–21] Komatsu gives a positive answer for ultradifferentiable functions of Beurling or Roumieu type and sequentially complete E with regard to vector-valued ultradistributions. For the space of k-times continuously partially differentiable functions on open subsets Ω of infinite dimensional spaces equipped with the topology of uniform convergence of all partial derivatives up to order k on compact subsets of Ω sufficient conditions for an affirmative answer are deduced by Meise in [27]. For holomorphic functions on open subsets of infinite dimensional spaces a positive answer is given in [9] by Dineen. Bonet, Frerick and Jordá show in [6] that S is a topological isomorphism for certain closed subsheafs of the sheaf $\mathcal{C}^\infty(\Omega, E)$ of smooth functions on an open subset $\Omega \subset \mathbb{R}^d$ with the topology of uniform convergence of all partial derivatives on compact subsets of Ω and locally complete E which, in particular, covers the spaces of harmonic and holomorphic functions.

In [6, 13, 14] linearisation is used by Bonet, Frerick, Jordá and Wengenroth to derive results on extensions of vector-valued functions and weak-strong principles. Another application of linearisation is within the field of partial differential equations. Let $P(\partial)$ be an elliptic linear partial differential operator with constant coefficients and $\mathcal{C}^\infty(\Omega) := \mathcal{C}^\infty(\Omega, \mathbb{K})$. Then

$$P(\partial) : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$$

is surjective by [16, Corollary 10.6.8, p. 43] and [16, Corollary 10.8.2, p. 51]. Due to [18, Satz 10.24, p. 255], the nuclearity of $\mathcal{C}^\infty(\Omega)$ and the topological isomorphism $\mathcal{C}^\infty(\Omega, E) \cong \mathcal{C}^\infty(\Omega)\varepsilon E$ for locally complete E, we immediately get the surjectivity of

$$P(\partial)^E : \mathcal{C}^\infty(\Omega, E) \to \mathcal{C}^\infty(\Omega, E)$$

for Fréchet spaces E where $P(\partial)^E$ is the version of $P(\partial)$ for E-valued functions. Thanks to the splitting theory of Vogt for Fréchet spaces and of Bonet and Domański for PLS-spaces we even have that $P(\partial)^E$ for $d > 1$ is surjective if $E := F_b'$ where F is a Fréchet space satisfying the condition (DN) by [34, Theorem 2.6, p. 174] or if E is an ultrabornological PLS-space having the property (PA) by [10, Corollary 3.9,

p. 1112] since ker$P(\partial)$ has the property (Ω) by [34, Proposition 2.5 (b), p. 173]. For examples of such PLS-spaces see [10, Corollary 4.8, p. 1116] and for more details on the properties (DN), (Ω) and (PA) see [5, 28].

Our goal is to give a unified and flexible approach to linearisation which is able to handle new examples and covers the already known examples. This new approach is used in [24] to lift series representations from scalar-valued functions to vector-valued functions. Let us outline the content of this paper. We begin with some notation and preliminaries in Sect. 2 and introduce in Sect. 3 the spaces of functions $\mathcal{FV}(\Omega, E)$ as subspaces of sections of domains of linear operators T^E on E^Ω having a certain growth given by a family of weight functions $\mathcal{V}$. These spaces cover many examples of classical spaces of functions appearing in analysis like the mentioned ones. Then we exploit the structure of our spaces to describe sufficient conditions, which we call consistency and strength, on the interplay of the pairs of operators $(T^E, T^{\mathbb{K}})$ and the map S as well as the spaces $\mathcal{FV}(\Omega)$ and E such that $S : \mathcal{FV}(\Omega)\varepsilon E \cong \mathcal{FV}(\Omega, E)$ becomes a topological isomorphism in our main Theorem 14. Looking at the pair of partial differential operators $(P(\partial)^E, P(\partial))$ considered above, these conditions allow us to express $P(\partial)^E$ as $P(\partial)^E = S\circ(P(\partial)\varepsilon\,\mathrm{id}_E)\circ S^{-1}$ where $P(\partial)\varepsilon\,\mathrm{id}_E$ is the ε-product of $P(\partial)$ and the identity id_E on E. Hence it becomes obvious that the surjectivity of $P(\partial)^E$ is equivalent to the surjectivity of $P(\partial)\varepsilon\,\mathrm{id}_E$. This is used in [23, 26] in the case of the Cauchy-Riemann operator $P(\partial) = \overline{\partial}$ on spaces of smooth functions with exponential growth.

2 Notation and preliminaries

We equip the spaces $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\mathbb{C}$ with the usual Euclidean norm $|\cdot|$. Furthermore, for a subset M of a topological space X we denote the closure of M by $\overline{M}$ and the boundary of M by ∂M. For a subset M of a vector space X we denote by $\mathrm{ch}(M)$ the circled hull, by $\mathrm{cx}(M)$ the convex hull and by $\mathrm{acx}(M)$ the absolutely convex hull of M. If X is a topological vector space, we write $\overline{\mathrm{acx}}(M)$ for the closure of $\mathrm{acx}(M)$ in X.

By E we always denote a non-trivial locally convex Hausdorff space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha\in\mathfrak{A}}$ and, in short, we write that E is an lcHs. If $E = \mathbb{K}$, then we set $(p_\alpha)_{\alpha\in\mathfrak{A}} := \{|\cdot|\}$. For details on the theory of locally convex spaces see [12, 17] or [28].

By X^Ω we denote the set of maps from a non-empty set Ω to a non-empty set X, by χ_K we mean the characteristic function of $K \subset \Omega$, by $\mathcal{C}(\Omega, X)$ the space of continuous functions from a topological space Ω to a topological space X and by $L(F, E)$ the space of continuous linear operators from F to E where F and E are locally convex Hausdorff spaces. If $E = \mathbb{K}$, we just write $F' := L(F, \mathbb{K})$ for the dual space and G° for the polar set of $G \subset F$. If F and E are (linearly) topologically isomorphic, we write $F \cong E$. We denote by $L_t(F, E)$ the space $L(F, E)$ equipped with the locally convex topology t of uniform convergence on the finite subsets of F if $t = \sigma$, on the absolutely convex, compact subsets of F if $t = \kappa$ and on the precompact (totally bounded) subsets of F if $t = \gamma$. We use the symbols $t(F', F)$ for the corresponding topology on F' and $t(F)$ for the corresponding bornology on F. The so-called *ε-product* of Schwartz is defined by

$$F\varepsilon E := L_e(F'_\kappa, E) \tag{1}$$

where $L(F'_\kappa, E)$ is equipped with the topology of uniform convergence on equicontinuous subsets of F'. This definition of the ε-product coincides with the original one by Schwartz [32, Chap. I, Sect. 1, Définition, p. 18]. It is symmetric which means that $F\varepsilon E \cong E\varepsilon F$. In the literature the definition of the ε-product is sometimes done the other way around, i.e. $E\varepsilon F$ is defined by the right-hand side of (1) but due to the symmetry these definitions are equivalent and for our purpose the given definition is more suitable. If we replace F'_κ by F'_γ, we obtain Grothendieck's definition of the ε-product and we remark that the two ε-products coincide if F is quasi-complete because then $F'_\gamma = F'_\kappa$ holds. However, we stick to Schwartz' definition. For more information on the theory of ε-products see [17, 18].

The sufficient conditions for the surjectivity of the map $S : \mathcal{FV}(\Omega)\varepsilon E \to \mathcal{FV}(\Omega, E)$ from the introduction, which we derive in the forthcoming, depend on assumptions on different types of completeness of E. For this purpose we recapitulate some definitions which are connected to completeness. We start with local completeness. For a disk $D \subset E$, i.e. a bounded, absolutely convex set, the vector space $E_D := \bigcup_{n\in\mathbb{N}} nD$ becomes a normed space if it is equipped with the gauge functional of D as a norm (see [17, p. 151]). The space E is called *locally complete* if E_D is a Banach space for every closed disk $D \subset E$ (see [17, 10.2.1 Proposition, p. 197]). Moreover, a locally convex Hausdorff space is locally complete if and only if it is convenient by [22, 2.14 Theorem, p. 20]. In particular, every complete locally convex Hausdorff space is quasi-complete, every quasi-complete space is sequentially complete and every sequentially complete space is locally complete and all these implications are strict. The first two by [17, p. 58] and the third by [29, 5.1.8 Corollary, p. 153] and [29, 5.1.12 Example, p. 154].

Now, let us recall the following definition from [36, 9-2-8 Definition, p. 134] and [35, p. 259]. A locally convex Hausdorff space is said to have the *[metric] convex compactness property* ([metric] ccp) if the closure of the absolutely convex hull of every [metrisable] compact set is compact. Sometimes this condition is phrased with the term convex hull instead of absolutely convex hull but these definitions coincide. Indeed, the first definition implies the second since every convex hull of a set $A \subset E$ is contained in its absolutely convex hull. On the other hand, we have $\operatorname{acx}(A) = \operatorname{cx}(\operatorname{ch}(A))$ by [17, 6.1.4 Proposition, p. 103] and the circled hull $\operatorname{ch}(A)$ of a [metrisable] compact set A is compact by [30, Chap. I, 5.2, p. 26] [and metrisable by [8, Chap. IX, §2.10, Proposition 17, p. 159] since $\mathbb{D} \times A$ is metrisable and $\operatorname{ch}(A) = M_E(\mathbb{D} \times A)$ where $M_E : \mathbb{K} \times E \to E$ is the continuous scalar multiplication and $\mathbb{D}$ the open unit disc] which yields the other implication.

In particular, every locally convex Hausdorff space with ccp has obviously metric ccp, every quasi-complete locally convex Hausdorff space has ccp by [36, 9-2-10 Example, p. 134], every sequentially complete locally convex Hausdorff space has metric ccp by [4, A.1.7 Proposition (ii), p. 364] and every locally convex Hausdorff space with metric cpp is locally complete by [35, Remark 4.1, p. 267]. All these implications are strict. The second by [36, 9-2-10 Example, p. 134] and the others by [35, Remark 4.1, p. 267]. For more details on the [metric] convex compactness property and local completeness see [7, 35]. In addition, we remark that every semi-Montel space is semi-reflexive by [17, 11.5.1 Proposition, p. 230] and every semi-reflexive locally convex Hausdorff space is quasi-complete by [30, Chap. IV, 5.5, Corollary 1, p. 144] and these implications are strict as well. Summarizing, we have the following diagram of strict implications:

$$
\begin{tikzcd}
 & \text{semi-Montel} \arrow[r, Rightarrow] & \text{semi-reflexive} \arrow[d, Rightarrow] & & \\
\text{complete} \arrow[rr, Rightarrow] & & \text{quasi-complete} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{sequentially complete} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{locally complete} \\
 & & \text{ccp} \arrow[r, Rightarrow] & \text{metric ccp} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

Since weighted spaces of continuously partially differentiable vector-valued functions will serve as our standard examples, we recall the definition of the spaces $\mathcal{C}^k(\Omega, E)$. A function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ to an lcHs E is called continuously partially differentiable (f is $\mathcal{C}^1$) if for the nth unit vector $e_n \in \mathbb{R}^d$ the limit

$$(\partial^{e_n})^E f(x) := \lim_{\substack{h \to 0 \\ h \in \mathbb{R}, h \neq 0}} \frac{f(x + he_n) - f(x)}{h}$$

exists in E for every $x \in \Omega$ and $(\partial^{e_n})^E f$ is continuous on Ω ($(\partial^{e_n})^E f$ is $\mathcal{C}^0$) for every $1 \leq n \leq d$. For $k \in \mathbb{N}$ a function f is said to be k-times continuously partially differentiable (f is $\mathcal{C}^k$) if f is $\mathcal{C}^1$ and all its first partial derivatives are $\mathcal{C}^{k-1}$. A function f is called infinitely continuously partially differentiable (f is $\mathcal{C}^\infty$) if f is $\mathcal{C}^k$ for every $k \in \mathbb{N}$. For $k \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ the functions $f : \Omega \to E$ which are $\mathcal{C}^k$ form a linear space which is denoted by $\mathcal{C}^k(\Omega, E)$. For $\beta \in \mathbb{N}_0^d$ with $|\beta| := \sum_{n=1}^d \beta_n \leq k$ and a function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^d$ to an lcHs E we set $(\partial^{\beta_n})^E f := f$ if $\beta_n = 0$, and

$$(\partial^{\beta_n})^E f(x) := \underbrace{(\partial^{e_n})^E \cdots (\partial^{e_n})^E}_{\beta_n\text{-times}} f(x)$$

if $\beta_n \neq 0$ and the right-hand side exists in E for every $x \in \Omega$. Further, we define

$$(\partial^\beta)^E f(x) =: \big((\partial^{\beta_1})^E \cdots (\partial^{\beta_d})^E\big) f(x)$$

if the right-hand side exists in E for every $x \in \Omega$.

3 The ε-product for weighted function spaces

In this section we introduce the weighted space $\mathcal{FV}(\Omega, E)$ of E-valued functions on Ω as a subspace of sections of domains in E^Ω of linear operators T^E_m equipped with a generalised version of a weighted graph topology. This space is the role model for many function spaces and an example for these operators are the partial derivative operators. Then we treat the question whether we can identify $\mathcal{FV}(\Omega, E)$ with $\mathcal{FV}(\Omega)\varepsilon E$ topologically. This is deeply connected with the interplay of the pair of operators $(T^E_m, T^{\mathbb{K}}_m)$ with the map S from the introduction (see Definition 6). In our main theorem we give sufficient conditions such that $\mathcal{FV}(\Omega, E) \cong \mathcal{FV}(\Omega)\varepsilon E$ holds (see Theorem 14). We start with the well-known example $\mathcal{C}^k(\Omega, E)$ of k-times continuously partially differentiable E-valued functions to motivate our definition of $\mathcal{FV}(\Omega, E)$.

Example 1 Let $k \in \mathbb{N}_\infty$ and $\Omega \subset \mathbb{R}^d$ open. Consider the space $\mathcal{C}(\Omega, E)$ of continuous functions $f : \Omega \to E$ with the compact-open topology, i.e. the topology given by the seminorms

$$\|f\|_{K,\alpha} := \sup_{x\in K} p_\alpha(f(x)), \quad f \in \mathcal{C}(\Omega, E),$$

for compact $K \subset \Omega$ and $\alpha \in \mathfrak{A}$. The usual topology on the space $\mathcal{C}^k(\Omega, E)$ of k-times continuously partially differentiable functions is the graph topology generated by the partial derivative operators $(\partial^\beta)^E : \mathcal{C}^k(\Omega, E) \to \mathcal{C}(\Omega, E)$ for $\beta \in \mathbb{N}_0^d$, $|\beta| \leq k$, i.e. the topology given by the seminorms

$$\|f\|_{K,\beta,\alpha} := \max(\|f\|_{K,\alpha}, \|(\partial^\beta)^E f\|_{K,\alpha}), \quad f \in \mathcal{C}^k(\Omega, E),$$

for compact $K \subset \Omega$, $\beta \in \mathbb{N}_0^d$, $|\beta| \leq k$, and $\alpha \in \mathfrak{A}$. The same topology is induced by the directed systems of seminorms given by

$$|f|_{K,m,\alpha} := \sup_{\beta\in\mathbb{N}_0^d, |\beta|\leq m} \|f\|_{K,\beta,\alpha} = \sup_{\substack{x \in K \\ \beta \in \mathbb{N}_0^d, |\beta| \leq m}} p_\alpha\big((\partial^\beta)^E f(x)\big), \quad f \in \mathcal{C}^k(\Omega, E),$$

for compact $K \subset \Omega$, $m \in \mathbb{N}_0$, $m \leq k$, and $\alpha \in \mathfrak{A}$ and may also be seen as a weighted topology induced by the family (χ_K) of characteristic functions of the compact sets $K \subset \Omega$ by writing

$$|f|_{K,m,\alpha} = \sup_{\substack{x \in \Omega \\ \beta \in \mathbb{N}_0^d, |\beta| \leq m}} p_\alpha\big((\partial^\beta)^E f(x)\big)\chi_K(x), \quad f \in \mathcal{C}^k(\Omega, E).$$

This topology is inherited by linear subspaces of functions having additional properties like being holomorphic or harmonic.

We turn to the weight functions which we use to define a kind of weighted graph topology.

Definition 2 (*Weight function)* Let J be a non-empty set and $(\omega_m)_{m\in M}$ a family of non-empty sets. We call $\mathcal{V} := (v_{j,m})_{j\in J,m\in M}$ a family of *weight functions* on $(\omega_m)_{m\in M}$ if it fulfils $v_{j,m} : \omega_m \to [0,\infty)$ for all $j \in J$, $m \in M$ and

$$\forall\, m \in M,\, x \in \omega_m\ \exists\, j \in J :\ 0 < v_{j,m}(x). \tag{2}$$

From the structure of Example 1 we arrive at the following definition of the weighted spaces of vector-valued functions we want to consider.

Definition 3 Let Ω be a non-empty set, $\mathcal{V} := (v_{j,m})_{j\in J,m\in M}$ a family of weight functions on $(\omega_m)_{m\in M}$ and $T^E_m : E^\Omega \supset \operatorname{dom} T^E_m \to E^{\omega_m}$ a linear map for every $m \in M$. Let $\mathrm{AP}(\Omega, E)$ be a linear subspace of E^Ω and define the space of intersections

$$\mathcal{F}(\Omega, E) := \mathrm{AP}(\Omega, E) \cap \left(\bigcap_{m\in M} \operatorname{dom} T^E_m \right)$$

as well as

$$\mathcal{FV}(\Omega, E) := \left\{ f \in \mathcal{F}(\Omega, E) \mid \forall\, j \in J,\, m \in M,\, \alpha \in \mathfrak{A} :\ |f|_{j,m,\alpha} < \infty \right\}$$

where

$$|f|_{j,m,\alpha} := \sup_{x\in\omega_m} p_\alpha\left(T^E_m(f)(x)\right) v_{j,m}(x) = \sup_{e\in N_{j,m}(f)} p_\alpha(e)$$

with

$$N_{j,m}(f) := \{T^E_m(f)(x) v_{j,m}(x) \mid x \in \omega_m\}.$$

Further, we write $\mathcal{F}(\Omega) := \mathcal{F}(\Omega, \mathbb{K})$ and $\mathcal{FV}(\Omega) := \mathcal{FV}(\Omega, \mathbb{K})$. If we want to emphasise dependencies, we write $M(E)$ instead of M and $\mathrm{AP}_{\mathcal{FV}}(\Omega, E)$ instead of $\mathrm{AP}(\Omega, E)$.

The space $\mathrm{AP}(\Omega, E)$ is a placeholder where we collect *additional properties* (AP) of our functions not being reflected by the operators T^E_m which we integrated in the topology. However, these additional properties might come from being in the domain or kernel of additional operators, e.g. harmonicity means being in the kernel of the Laplacian. But often $\mathrm{AP}(\Omega, E)$ can be chosen as E^Ω or $\mathcal{C}^0(\Omega, E)$. The space $\mathcal{FV}(\Omega, E)$ is locally convex but need not be Hausdorff. Since it is easier to work with Hausdorff spaces and a directed family of seminorms plus the point evaluation functionals $\delta_x : \mathcal{FV}(\Omega) \to \mathbb{K}$, $f \mapsto f(x)$, for $x \in \Omega$ and their continuity play a big role, we introduce the following definition.

Definition 4 (dom-*space and* $T^E_{m,x}$) We call $\mathcal{FV}(\Omega, E)$ a dom-*space* if it is a locally convex Hausdorff space, the system of seminorms $(|f|_{j,m,\alpha})_{j\in J,m\in M,\alpha\in\mathfrak{A}}$ is directed and, in addition, $\delta_x \in \mathcal{FV}(\Omega)'$ for every $x \in \Omega$ if $E = \mathbb{K}$. We define the point evaluation of T^E_m by $T^E_{m,x} :\ \operatorname{dom} T^E_m \to E$, $T^E_{m,x}(f) := T^E_m(f)(x)$, for $m \in M$ and $x \in \omega_m$.

Remark 5

a) It is easy to see that $\mathcal{FV}(\Omega, E)$ is Hausdorff if there is $m \in M$ such that $\omega_m = \Omega$ and $T^E_m = \mathrm{id}_{E^\Omega}$ since E is Hausdorff.
b) If $E = \mathbb{K}$, then $T^{\mathbb{K}}_{m,x} \in \mathcal{FV}(\Omega)'$ for every $m \in M$ and $x \in \omega_m$. Indeed, for $m \in M$ and $x \in \omega_m$ there exists $j \in J$ such that $\nu_{j,m}(x) > 0$ by (2), implying for every $f \in \mathcal{FV}(\Omega)$ that

$$|T^{\mathbb{K}}_{m,x}(f)| = \frac{1}{\nu_{j,m}(x)} |T^{\mathbb{K}}_m(f)(x)| \nu_{j,m}(x) \leq \frac{1}{\nu_{j,m}(x)} |f|_{j,m}.$$

In particular, this implies $\delta_x \in \mathcal{FV}(\Omega)'$ for all $x \in \Omega$ if there is $m \in M$ such that $\omega_m = \Omega$ and $T^{\mathbb{K}}_m = \mathrm{id}_{\mathbb{K}^\Omega}$.
c) The system of seminorms $(|f|_{j,m,\alpha})_{j \in J, m \in M, \alpha \in \mathfrak{A}}$ is directed if the family of weight functions $\mathcal{V}$ is *directed*, i.e.

$$\begin{aligned}&\forall\, j_1, j_2 \in J,\ m_1, m_2 \in M\ \exists\, j_3 \in J,\ m_3 \in M,\ C > 0\ \forall\, i \in \{1,2\} :\\ &(\omega_{m_1} \cup \omega_{m_2}) \subset \omega_{m_3} \quad \text{and} \quad \nu_{j_i,m_i} \leq C\nu_{j_3,m_3},\end{aligned}$$

since the system $(p_\alpha)_{\alpha \in \mathfrak{A}}$ of E is already directed.

For the lcHs E over $\mathbb{K}$ we want to define a natural E-valued version of a dom -space $\mathcal{FV}(\Omega) = \mathcal{FV}(\Omega, \mathbb{K})$. The natural E-valued version of $\mathcal{FV}(\Omega)$ should be a dom -space $\mathcal{FV}(\Omega, E)$ such that there is a canonical relation between the families $(T^{\mathbb{K}}_m)$ and (T^E_m). This canonical relation will be explained in terms of their interplay with the map

$$S : \mathcal{FV}(\Omega)\varepsilon E \to E^\Omega,\ u \longmapsto [x \mapsto u(\delta_x)].$$

Further, the elements of our E-valued version $\mathcal{FV}(\Omega, E)$ of $\mathcal{FV}(\Omega)$ should be compatible with a weak definition in the sense that $e' \circ f \in \mathcal{FV}(\Omega)$ should hold for every $e' \in E'$ and $f \in \mathcal{FV}(\Omega, E)$.

Definition 6 (*Generator, consistent, strong*) Let $\mathcal{FV}(\Omega)$ and $\mathcal{FV}(\Omega, E)$ be dom-spaces such that $M := M(\mathbb{K}) = M(E)$.

a) We call $(T^E_m, T^{\mathbb{K}}_m)_{m \in M}$ a *generator* for $(\mathcal{FV}(\Omega), E)$, in short, $(\mathcal{FV}, E)$.
b) We call $(T^E_m, T^{\mathbb{K}}_m)_{m \in M}$ *consistent* if we have for all $u \in \mathcal{FV}(\Omega)\varepsilon E$ that $S(u) \in \mathcal{F}(\Omega, E)$ and

$$\forall\, m \in M,\ x \in \omega_m :\ \big(T^E_m S(u)\big)(x) = u(T^{\mathbb{K}}_{m,x}).$$

c) We call $(T^E_m, T^{\mathbb{K}}_m)_{m \in M}$ *strong* if we have for all $e' \in E'$, $f \in \mathcal{FV}(\Omega, E)$ that $e' \circ f \in \mathcal{F}(\Omega)$ and

$$\forall\, m \in M,\ x \in \omega_m :\ T^{\mathbb{K}}_m(e' \circ f)(x) = \big(e' \circ T^E_m(f)\big)(x).$$

More precisely, $T^{\mathbb{K}}_{m,x}$ in b) means the restriction of $T^{\mathbb{K}}_{m,x}$ to $\mathcal{FV}(\Omega)$ and the term $u(T^{\mathbb{K}}_{m,x})$ is well-defined by Remark 5 b). Consistency will guarantee that the map $S : \mathcal{FV}(\Omega)\varepsilon E \to \mathcal{FV}(\Omega, E)$ is a well-defined topological isomorphism into, i.e. to its range, and strength will help us to prove its surjectivity under some additional assumptions on $\mathcal{FV}(\Omega)$ and E. Let us come to a lemma which describes the topology of $\mathcal{FV}(\Omega)\varepsilon E$ in terms of the operators $T^{\mathbb{K}}_m$ with $m \in M$. It was the motivation for the definition of consistency and allows us to consider $\mathcal{FV}(\Omega)\varepsilon E$ as a topological subspace of $\mathcal{FV}(\Omega, E)$ via S, assuming consistency.

Lemma 7 *Let $\mathcal{FV}(\Omega)$ be a* dom-*space. Then the topology of $\mathcal{FV}(\Omega)\varepsilon E$ is given by the system of seminorms defined by*

$$\|u\|_{j,m,\alpha} := \sup_{x\in\omega_m} p_\alpha\Big(u(T^{\mathbb{K}}_{m,x})\Big)\nu_{j,m}(x), \quad u \in \mathcal{FV}(\Omega)\varepsilon E,$$

for $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$.

Proof We set $D_{j,m} := \{T^{\mathbb{K}}_{m,x}(\cdot)\nu_{j,m}(x) \mid x \in \omega_m\}$ and $B_{j,m} := \{f \in \mathcal{FV}(\Omega) \mid |f|_{j,m} \le 1\}$ for every $j \in J$ and $m \in M$. We claim that $\operatorname{acx}(D_{j,m})$ is dense in the polar $B^\circ_{j,m}$ with respect to $\kappa(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$. The observation

$$\begin{aligned} D^\circ_{j,m} &= \{T^{\mathbb{K}}_{m,x}(\cdot)\nu_{j,m}(x) \mid x \in \omega_m\}^\circ \\ &= \{f \in \mathcal{FV}(\Omega) \mid \forall x \in \omega_m : |T^{\mathbb{K}}_m(f)(x)|\nu_{j,m}(x) \le 1\} \\ &= \{f \in \mathcal{FV}(\Omega) \mid |f|_{j,m} \le 1\} = B_{j,m} \end{aligned}$$

yields

$$\overline{\operatorname{acx}}(D_{j,m})^{\kappa(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))} = (D_{j,m})^{\circ\circ} = B^\circ_{j,m}$$

by the bipolar theorem. By [17, 8.4, p. 152, 8.5, p. 156-157] the system of seminorms defined by

$$q_{j,m,\alpha}(u) := \sup_{y\in B^\circ_{j,m}} p_\alpha(u(y)), \quad u \in \mathcal{FV}(\Omega)\varepsilon E,$$

for $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$ gives the topology on $\mathcal{FV}(\Omega)\varepsilon E$ (here it is used that the system of seminorms $(|\cdot|_{j,m})$ of $\mathcal{FV}(\Omega)$ is directed). As every $u \in \mathcal{FV}(\Omega)\varepsilon E$ is continuous on $B^\circ_{j,m}$, we may replace $B^\circ_{j,m}$ by a $\kappa(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$-dense subset. Therefore we obtain

$$q_{j,m,\alpha}(u) = \sup\left\{p_\alpha(u(y)) \mid y \in \operatorname{acx}(D_{j,m})\right\}.$$

For $y \in \operatorname{acx}(D_{j,m})$ there are $n \in \mathbb{N}$, $\lambda_k \in \mathbb{K}$, $x_k \in \omega_m$, $1 \le k \le n$, with $\sum_{k=1}^n |\lambda_k| \le 1$ such that $y = \sum_{k=1}^n \lambda_k T^{\mathbb{K}}_{m,x_k}(\cdot)\nu_{j,m}(x_k)$. Then we have for every $u \in \mathcal{FV}(\Omega)\varepsilon E$

$$p_\alpha(u(y)) \le \sum_{k=1}^{n} |\lambda_k| p_\alpha\Big(u(T^{\mathbb{K}}_{m,x_k})\Big) v_{j,m}(x_k) \le \|u\|_{j,m,\alpha},$$

thus $q_{j,m,\alpha}(u) \le \|u\|_{j,m,\alpha}$. On the other hand, we derive

$$q_{j,m,\alpha}(u) \ge \sup_{y \in D_{j,m}} p_\alpha(u(y)) = \sup_{x \in \omega_m} p_\alpha\Big(u(T^{\mathbb{K}}_{m,x})\Big) v_{j,m}(x) = \|u\|_{j,m,\alpha}.$$

□

Let us turn to a more general version of Example 1, namely, to weighted spaces of k-times continuously partially differentiable functions and kernels of partial differential operators in these spaces.

Example 8 Let $k \in \mathbb{N}_\infty$ and $\Omega \subset \mathbb{R}^d$ be open. We consider the cases

(i) $\omega_m := M_m \times \Omega$ with $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \le \min(m,k)\}$ for all $m \in \mathbb{N}_0$, or
(ii) $\omega_m := \mathbb{N}_0^d \times \Omega$ for all $m \in \mathbb{N}_0$ and $k = \infty$,

and let $\mathcal{V}^k := (v_{j,m})_{j \in J, m \in \mathbb{N}_0}$ be a directed family of weights on $(\omega_m)_{m \in \mathbb{N}_0}$.

a) We define the weighted space of k-times continuously partially differentiable functions with values in an lcHs E as

$$\mathcal{C}\mathcal{V}^k(\Omega, E) := \{f \in \mathcal{C}^k(\Omega, E) \mid \forall\, j \in J,\ m \in \mathbb{N}_0,\ \alpha \in \mathfrak{A} :\ |f|_{j,m,\alpha} < \infty\}$$

where

$$|f|_{j,m,\alpha} := \sup_{(\beta,x) \in \omega_m} p_\alpha\big((\partial^\beta)^E f(x)\big) v_{j,m}(\beta, x).$$

Setting $\operatorname{dom} T^E_m := \mathcal{C}^k(\Omega, E)$ and

$$T^E_m : \mathcal{C}^k(\Omega, E) \to E^{\omega_m},\ f \longmapsto [(\beta, x) \mapsto (\partial^\beta)^E f(x)],$$

as well as $\mathrm{AP}(\Omega, E) := E^\Omega$, we observe that $\mathcal{C}\mathcal{V}^k(\Omega, E)$ is a dom-space and

$$|f|_{j,m,\alpha} = \sup_{x \in \omega_m} p_\alpha\big(T^E_m f(x)\big) v_{j,m}(x).$$

b) The space $\mathcal{C}^k(\Omega, E)$ with its usual topology given in Example 1 is a special case of a)(i) with $J := \{K \subset \Omega \mid K \text{ compact}\}$, $v_{K,m}(\beta, x) := \chi_K(x)$, $(\beta, x) \in \omega_m$, for all $m \in \mathbb{N}_0$ and $K \in J$ where χ_K is the characteristic function of K. In this case we write $\mathcal{W}^k := \mathcal{V}^k$ for the family of weight functions.
c) The *Schwartz space* is defined by

$$\mathcal{S}(\mathbb{R}^d, E) := \{f \in \mathcal{C}^\infty(\mathbb{R}^d, E) \mid \forall\, m \in \mathbb{N}_0,\ \alpha \in \mathfrak{A} :\ |f|_{m,\alpha} < \infty\}$$

where

$$|f|_{m,\alpha} := \sup_{\substack{x \in \mathbb{R}^d \\ \beta \in \mathbb{N}_0^d, |\beta| \leq m}} p_\alpha\big((\partial^\beta)^E f(x)\big)(1+|x|^2)^{m/2}.$$

This is a special case of a)(i) with $k=\infty$, $\Omega=\mathbb{R}^d$, $J=\{1\}$ and $v_{1,m}(\beta,x) := (1+|x|^2)^{m/2}$, $(\beta,x)\in\omega_m$, for all $m\in\mathbb{N}_0$.

d) Let $n\in\mathbb{N}$, $\beta_i\in\mathbb{N}_0^d$ with $|\beta_i|\leq k$ and $a_i : \Omega\to\mathbb{K}$ for $1\leq i\leq n$. We set

$$P(\partial)^E : \mathcal{C}^k(\Omega,E)\to E^\Omega,\ P(\partial)^E(f)(x) := \sum_{i=1}^n a_i(x)(\partial^{\beta_i})^E(f)(x).$$

and obtain the (topological) subspace of $\mathcal{CV}^k(\Omega,E)$ given by

$$\mathcal{CV}^k_{P(\partial)}(\Omega,E) := \{f\in\mathcal{CV}^k(\Omega,E)\mid f\in\ker P(\partial)^E\}.$$

Choosing $\mathrm{AP}(\Omega,E) := \ker P(\partial)^E$, we see that this is also a dom-space by a). If $P(\partial)^E$ is the Cauchy–Riemann operator or the Laplacian, we obtain the weighted space of holomorphic resp. harmonic functions.

We note that Example 8 a)(ii) covers spaces of ultradifferentiable functions. Let us show that the generator of these spaces is strong and consistent. In order to obtain consistency for their generator we have to restrict to directed families of weights which are *locally bounded away from zero* on Ω, i.e.

$$\forall\, K\subset\Omega \text{ compact},\ m\in\mathbb{N}_0\ \exists\, j\in J\ \forall\, \beta\in\mathbb{N}_0^d,\ |\beta|\leq\min(m,k) :\ \inf_{x\in K} v_{j,m}(\beta,x) > 0.$$

This condition on $\mathcal{V}^k$ guarantees that the map $I : \mathcal{CV}^k(\Omega)\to\mathcal{CW}^k(\Omega)$, $f\mapsto f$, is continuous which is needed for consistency.

Proposition 9 *Let E be an lcHs, $k\in\mathbb{N}_\infty$, $\mathcal{V}^k$ be a directed family of weights which is locally bounded away from zero on an open set $\Omega\subset\mathbb{R}^d$. The generator of $(\mathcal{CV}^k,E)$ resp. $(\mathcal{CV}^k_{P(\partial)},E)$ from Example 8 is strong and consistent if $\mathcal{CV}^k(\Omega)$ resp. $\mathcal{CV}^k_{P(\partial)}(\Omega)$ is barrelled.*

Proof We recall the definitions from Example 8. We have $\omega_m := M_m\times\Omega$ with $M_m := \{\beta\in\mathbb{N}_0^d \mid |\beta|\leq\min(m,k)\}$ for all $m\in\mathbb{N}_0$ or $\omega_m := \mathbb{N}_0^d\times\Omega$ for all $m\in\mathbb{N}_0$. Further, $\mathrm{AP}_{\mathcal{CV}^k}(\Omega,E) = E^\Omega$, $\mathrm{AP}_{\mathcal{CV}^k_{P(\partial)}}(\Omega,E) = \ker P(\partial)^E$, $\operatorname{dom} T^E_m := \mathcal{C}^k(\Omega,E)$ and

$$T^E_m : \mathcal{C}^k(\Omega,E)\to E^{\omega_m},\ f\longmapsto[(\beta,x)\mapsto(\partial^\beta)^E f(x)],$$

for all $m\in\mathbb{N}_0$ and the same with $\mathbb{K}$ instead of E. The family $(T^E_m, T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ is a strong generator for $(\mathcal{CV}^k,E)$ because

$$(\partial^\beta)^{\mathbb{K}}(e'\circ f)(x) = e'\big((\partial^\beta)^E f(x)\big),\quad (\beta,x)\in\omega_m,$$

for all $e' \in E'$, $f \in \mathcal{CV}^k(\Omega, E)$ and $m \in \mathbb{N}_0$ due to the linearity and continuity of $e' \in E'$. In addition, $e' \circ f \in \ker P(\partial)^{\mathbb{K}}$ for all $e' \in E'$ and $f \in \mathcal{CV}^k_{P(\partial)}(\Omega, E)$ which implies that $(T^E_m, T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ is also a strong generator for $(\mathcal{CV}^k_{P(\partial)}, E)$.

For consistency we need to prove that

$$(\partial^\beta)^E S(u)(x) = u(\delta_x \circ (\partial^\beta)^{\mathbb{K}}), \quad (\beta, x) \in \omega_m,$$

for all $u \in \mathcal{CV}^k(\Omega)\varepsilon E$ resp. $u \in \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$. This follows from the subsequent Proposition 10 b) since $\mathcal{FV}(\Omega) = \mathcal{CV}^k(\Omega)$ resp. $\mathcal{FV}(\Omega) = \mathcal{CV}^k_{P(\partial)}(\Omega)$ is barrelled and $\mathcal{V}^k$ locally bounded away from zero on Ω. Thus $(T^E_m, T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ is a consistent generator for $(\mathcal{CV}^k, E)$. In addition, we have with $P(\partial)^E$ from Example 8 d) that

$$\begin{aligned} P(\partial)^E(S(u))(x) &= \sum_{i=1}^n a_i(x)(\partial^{\beta_i})^E(S(u))(x) = u\left(\sum_{i=1}^n a_i(x)(\delta_x \circ (\partial^{\beta_i})^{\mathbb{K}})\right) \\ &= u(\delta_x \circ P(\partial)^{\mathbb{K}}) = 0, \quad x \in \Omega, \end{aligned}$$

for every $u \in \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$, yielding that $S(u) \in \ker P(\partial)^E$ for every $u \in \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$. Therefore $(T^E_m, T^{\mathbb{K}}_m)_{m\in\mathbb{N}_0}$ is a consistent generator for $(\mathcal{CV}^k_{P(\partial)}, E)$ as well. □

Let us turn to the postponed part in the proof of consistency. We denote by $\mathcal{CW}(\Omega)$ the space of scalar-valued continuous functions on an open set $\Omega \subset \mathbb{R}^d$ with the topology of uniform convergence on compact subsets, i.e. the weighted topology given by the family of weights $\mathcal{W} := \mathcal{W}^0 := \{\chi_K \mid K \subset \Omega \text{ compact}\}$, and we set $\delta(x) := \delta_x$ for $x \in \Omega$.

Proposition 10 *Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}_\infty$ and $\mathcal{FV}(\Omega)$ a* dom-*space.*

a) If $T \in L(\mathcal{FV}(\Omega), \mathcal{CW}(\Omega))$, then $\delta \circ T \in \mathcal{C}(\Omega, \mathcal{FV}(\Omega)'_\gamma)$.

b) If the inclusion $I : \mathcal{FV}(\Omega) \to \mathcal{CW}^k(\Omega)$, $f \mapsto f$, is continuous and $\mathcal{FV}(\Omega)$ barrelled, then $S(u) \in \mathcal{C}^k(\Omega, E)$ and

$$(\partial^\beta)^E S(u)(x) = u(\delta_x \circ (\partial^\beta)^{\mathbb{K}}), \quad \beta \in \mathbb{N}_0^d,\ |\beta| \le k,\ x \in \Omega,$$

for all $u \in \mathcal{FV}(\Omega)\varepsilon E$.

Proof

a) First, if $x \in \Omega$ and $(x_n)_{n\in\mathbb{N}}$ is a sequence in Ω converging to x, then we observe that

$$(\delta_{x_n} \circ T)(f) = T(f)(x_n) \to T(f)(x) = (\delta_x \circ T)(f)$$

for every $f \in \mathcal{FV}(\Omega)$ as $T(f)$ is continuous on Ω. Second, let $K \subset \Omega$ be compact. Then there are $j \in J$, $m \in M$ and $C > 0$ such that

$$\sup_{x\in K} |(\delta_x \circ T)(f)| = \sup_{x\in K} |T(f)(x)| \leq C|f|_{j,m}$$

for every $f \in \mathcal{FV}(\Omega)$. This means that $\{\delta_x \circ T \mid x \in K\}$ is equicontinuous in $\mathcal{FV}(\Omega)'$. The topologies $\sigma(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$ and $\gamma(\mathcal{FV}(\Omega)', \mathcal{FV}(\Omega))$ coincide on equicontinuous subsets of $\mathcal{FV}(\Omega)'$, implying that the restriction $(\delta \circ T)_{|_K} : K \to \mathcal{FV}(\Omega)'_\gamma$ is continuous by our first observation. As $\delta \circ T$ is continuous on every compact subset of the open set $\Omega \subset \mathbb{R}^d$, it follows that $\delta \circ T : \Omega \to \mathcal{FV}(\Omega)'_\gamma$ is well-defined and continuous.

b) We only prove b) for $k = 1$. The general statement follows by induction on the order of differentiation. Let $u \in \mathcal{FV}(\Omega)\varepsilon E$, $x \in \Omega$ and $1 \leq n \leq d$. Then $S(u) = u \circ \delta \in \mathcal{C}(\Omega, E)$ by part a) with $T = I$. There is $\varepsilon > 0$ such that $x + he_n \in \Omega$ for all $h \in \mathbb{R}$ with $0 < |h| < \varepsilon$. We note that $\delta \in \mathcal{C}(\Omega, \mathcal{FV}(\Omega)'_\kappa)$ by part a) with $T = I$, which implies $\frac{\delta_{x+he_n} - \delta_x}{h} \in \mathcal{FV}(\Omega)'$. For every $f \in \mathcal{FV}(\Omega)$ we have

$$\lim_{h\to 0} \frac{\delta_{x+he_n} - \delta_x}{h}(f) = \lim_{h\to 0} \frac{f(x+he_n) - f(x)}{h} = (\partial^{e_n})^{\mathbb{K}} f(x)$$

in $\mathbb{K}$ as $f \in \mathcal{C}^1(\Omega)$. Therefore $\frac{1}{h}(\delta_{x+he_n} - \delta_x)$ converges to $\delta_x \circ (\partial^{e_n})^{\mathbb{K}}$ in $\mathcal{FV}(\Omega)'_\sigma$ and thus in $\mathcal{FV}(\Omega)'_\kappa$ by the Banach-Steinhaus theorem as well due to the barrelledness of $\mathcal{FV}(\Omega)$. This yields

$$\begin{aligned} u(\delta_x \circ (\partial^{e_n})^{\mathbb{K}}) &= \lim_{h\to 0} \frac{1}{h}\big(u(\delta_{x+he_n}) - u(\delta_x)\big) = \lim_{h\to 0} \frac{1}{h}\big(S(u)(x+he_n) - S(u)(x)\big) \\ &= (\partial^{e_n})^E S(u)(x) \end{aligned}$$

for every $x \in \Omega$. Moreover, $\delta \circ (\partial^{e_n})^{\mathbb{K}} \in \mathcal{C}(\Omega, \mathcal{FV}(\Omega)'_\kappa)$ by part a) for $T = (\partial^{e_n})^{\mathbb{K}}$ and the continuity of I. Hence we have $S(u) \in \mathcal{C}^1(\Omega, E)$.

□

Part a) of the preceding lemma is just a modification of [2, 4.1 Lemma, p. 198], where $\mathcal{FV}(\Omega) = \mathcal{CV}(\Omega)$ is the Nachbin-weighted space of continuous functions and $T =$ id, and holds more general for $k_{\mathbb{R}}$-spaces Ω.

The Schwartz space from Example 8 c) can also be topologized by integral operators instead of partial derivative operators. Let $f \in \mathcal{S}(\mathbb{R}^d, E)$. If E is sequentially complete, then $f \circ h_n$ is Pettis-integrable on $\mathbb{R}^d$ for every $n = (n_k) \in \mathbb{N}_0^d$ by [24, 4.8 Proposition, p. 15] where $h_n : \mathbb{R}^d \to \mathbb{R}$, $h_n(x) := \prod_{k=1}^d h_{n_k}(x)$, is the nth Hermite function with

$$h_m : \mathbb{R} \to \mathbb{R},\ h_m(x) := (2^m m! \sqrt{\pi})^{-1/2} \left(x - \frac{d}{dx}\right)^m e^{-x^2/2}, \quad m \in \mathbb{N}_0.$$

Thus the nth Fourier coefficient of f given by the Pettis-integral

$$\mathscr{F}^E(f)(n) := \int_{\mathbb{R}^d} f(x) h_n(x) \mathrm{d}x \in E$$

is defined if E is sequentially complete.

Example 11 Let E be a sequentially complete lcHs and equip $\mathcal{S}(\mathbb{R}^d,E)$ with the topology generated by the seminorms

$$\|f\|_{j,\alpha} := \sup_{n\in\mathbb{N}_0^d} p_\alpha(\mathscr{F}^E(f)(n))(1+|n|^2)^{j/2}, \quad f\in\mathcal{S}(\mathbb{R}^d,E),$$

for $j\in\mathbb{N}$ and $\alpha\in\mathfrak{A}$. $\mathcal{S}(\mathbb{R}^d,E)$ equipped with this topology is a dom-space with $\omega:=\mathbb{N}_0^d$, $\operatorname{dom}T^E:=\mathcal{S}(\mathbb{R}^d,E)$, $T^E:=\mathscr{F}^E$, $\nu_j(n):=(1+|n|^2)^{j/2}$, $n\in\mathbb{N}_0^d$, for $j\in\mathbb{N}$ whose topology coincides with the one from Example 8 c) by [24, 4.9 Theorem, p. 16]. Further, by the same theorem the generator $(\mathscr{F}^E,\mathscr{F}^{\mathbb{K}})$ of $(\mathcal{S},E)$ is strong and consistent.

Among others, the techniques of the present paper are used in [24] to deduce the series expansion

$$f=\sum_{n\in\mathbb{N}_0^d}\mathscr{F}^E(f)(n)h_n, \quad f\in\mathcal{S}(\mathbb{R}^d,E),$$

and to show that $\mathscr{F}^E:\mathcal{S}(\mathbb{R}^d,E)\to s(\mathbb{N}_0^d,E)$ is a topological isomorphism for sequentially complete E, which is known from the scalar-valued case, where $s(\mathbb{N}_0^d,E)$ is the space of rapidly decreasing sequences on $\mathbb{N}_0^d$ with values in E.

Now, the strength of a generator and a weaker concept to define a natural E-valued version of $\mathcal{FV}(\Omega)$ come into play to answer the question on the surjectivity of our key map S. Let $\mathcal{FV}(\Omega)$ be a dom-space. We define the vector space of E-valued weak $\mathcal{FV}$-functions by

$$\mathcal{FV}(\Omega,E)_\sigma := \{f:\Omega\to E \mid \forall\, e'\in E':\ e'\circ f\in\mathcal{FV}(\Omega)\}.$$

Moreover, for $f\in\mathcal{FV}(\Omega,E)_\sigma$ we define the linear map

$$R_f: E'\to\mathcal{FV}(\Omega),\ R_f(e'):=e'\circ f,$$

and the dual map

$$R_f^t:\mathcal{FV}(\Omega)'\to E'^{\star},\ f'\longmapsto\left[e'\mapsto f'\left(R_f(e')\right)\right],$$

where $E'^{\star}$ is the algebraic dual of E'. Furthermore, we set

$$\mathcal{FV}(\Omega,E)_\kappa := \{f\in\mathcal{FV}(\Omega,E)_\sigma \mid \forall\,\alpha\in\mathfrak{A}:\ R_f(B_\alpha^\circ)\text{ relatively compact in }\mathcal{FV}(\Omega)\}$$

where $B_\alpha:=\{x\in E\mid p_\alpha(x)<1\}$ for $\alpha\in\mathfrak{A}$. Next, we give a sufficient condition for the inclusion $\mathcal{FV}(\Omega,E)\subset\mathcal{FV}(\Omega,E)_\sigma$ by means of the family $(T_m^E,T_m^{\mathbb{K}})_{m\in M}$.

Lemma 12 *If $(T_m^E,T_m^{\mathbb{K}})_{m\in M}$ is a strong generator for $(\mathcal{FV},E)$, then we have $\mathcal{FV}(\Omega,E)\subset\mathcal{FV}(\Omega,E)_\sigma$ and*

$$\sup_{e'\in B_\alpha^\circ}|R_f(e')|_{j,m}=|f|_{j,m,\alpha} \tag{3}$$

for every $f \in \mathcal{FV}(\Omega, E)$, $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$ with the set $N_{j,m}(f)$ from Definition 3.

Proof Let $f \in \mathcal{FV}(\Omega, E)$. We have $e' \circ f \in \mathcal{F}(\Omega)$ for every $e' \in E'$ since $(T^E_m, T^{\mathbb{K}}_m)_{m \in M}$ is a strong generator. Moreover, we have

$$\begin{aligned} |R_f(e')|_{j,m} = |e' \circ f|_{j,m} &= \sup_{x \in \omega_m} \left| T^{\mathbb{K}}_m(e' \circ f)(x) \right| \nu_{j,m}(x) \\ &= \sup_{x \in \omega_m} \left| e' \left(T^E_m(f)(x) \right) \right| \nu_{j,m}(x) = \sup_{x \in N_{j,m}(f)} |e'(x)| \end{aligned} \tag{4}$$

for every $j \in J$ and $m \in M$. We note that $N_{j,m}(f)$ is bounded in E by Definition 3 and thus weakly bounded, implying that the right-hand side of (4) is finite. Hence we conclude $f \in \mathcal{FV}(\Omega, E)_\sigma$. Further, we observe that

$$\sup_{e' \in B^\circ_\alpha} |R_f(e')|_{j,m} = |f|_{j,m,\alpha}$$

for every $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$ due to [28, Proposition 22.14, p. 256]. □

Now, we phrase some sufficient conditions for $\mathcal{FV}(\Omega, E) \subset \mathcal{FV}(\Omega, E)_\kappa$ to hold which is one of the key points regarding the surjectivity of S.

Lemma 13 *If $(T^E_m, T^{\mathbb{K}}_m)_{m \in M}$ is a strong generator for $(\mathcal{FV}, E)$ and one of the following conditions is fulfilled, then $\mathcal{FV}(\Omega, E) \subset \mathcal{FV}(\Omega, E)_\kappa$.*

a) $\mathcal{FV}(\Omega)$ is a semi-Montel space.
b) E is a semi-Montel space.
c) $\forall\, f \in \mathcal{FV}(\Omega, E), j \in J, m \in M\ \exists\, K \in \gamma(E) :\ N_{j,m}(f) \subset K$.

Proof Let $f \in \mathcal{FV}(\Omega, E)$. By virtue of Lemma 12 we already have $f \in \mathcal{FV}(\Omega, E)_\sigma$.

a) For every $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$ we derive from

$$\sup_{e' \in B^\circ_\alpha} |R_f(e')|_{j,m} \underset{(3)}{=} |f|_{j,m,\alpha} < \infty$$

that $R_f(B^\circ_\alpha)$ is bounded and thus relatively compact in the semi-Montel space $\mathcal{FV}(\Omega)$.

c) It follows from (4) that $R_f \in L(E'_\gamma, \mathcal{FV}(\Omega))$. Further, the polar B°_α is relatively compact in E'_γ for every $\alpha \in \mathfrak{A}$ by the Alaoğlu-Bourbaki theorem. The continuity of R_f implies that $R_f(B^\circ_\alpha)$ is relatively compact as well.

b) Let $j \in J$ and $m \in M$. The set $K := N_{j,m}(f)$ is bounded in E by Definition 3. We deduce that K is already precompact in E since it is relatively compact in the semi-Montel space E. Hence the statement follows from c).

□

Finally, we can state our main theorem. We identify E with a linear subspace of the algebraic dual $E'^{\star}$ of E' by the canonical injection $x \longmapsto [e' \mapsto e'(x)] =: \langle x, e' \rangle$.

Theorem 14 *Let $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ be a strong, consistent generator for $(\mathcal{FV}, E)$. If*

(i) *$\mathcal{FV}(\Omega)$ is a semi-Montel space and E complete, or*
(ii) *E is a semi-Montel space, or*
(iii) $\forall\, f \in \mathcal{FV}(\Omega, E),\, j \in J,\, m \in M\; \exists\, K \in \kappa(E):\; N_{j,m}(f) \subset K,$

then $\mathcal{FV}(\Omega, E) \cong \mathcal{FV}(\Omega)\varepsilon E$ via S and the inverse fulfils

$$S^{-1}(f)(f') = R^t_f(f') \in E \subset E'^{\star}, \quad f \in \mathcal{FV}(\Omega, E),\ f' \in \mathcal{FV}(\Omega)'.$$

Proof First, we show that $S(\mathcal{FV}(\Omega)\varepsilon E) \subset \mathcal{FV}(\Omega, E)$. Let $u \in \mathcal{FV}(\Omega)\varepsilon E$. Due to the consistency of $(T^E_m, T^{\mathbb{K}}_m)_{m\in M}$ we have $S(u) \in \mathcal{F}(\Omega, E)$ and

$$\big(T^E_m S(u)\big)(x) = u(T^{\mathbb{K}}_{m,x}), \quad m \in M\ x \in \omega_m.$$

Furthermore, we get by Lemma 7 for every $j \in J$, $m \in M$ and $\alpha \in \mathfrak{A}$

$$|S(u)|_{j,m,\alpha} = \sup_{x\in\omega_m} p_\alpha\big(T^E_m(S(u))(x)\big)v_{j,m}(x) = \|u\|_{j,m,\alpha} < \infty, \tag{5}$$

implying $S(u) \in \mathcal{FV}(\Omega, E)$ and the continuity of S. Moreover, we deduce from (5) that S is injective and that the inverse of S on the range of S is also continuous.

Now, we only have to show that S is surjective. We set

$$p_{B^\circ_\alpha}(y) := \sup_{e'\in B^\circ_\alpha} |y(e')| \le \infty, \quad y \in E'^{\star}, \tag{6}$$

for every $\alpha \in \mathfrak{A}$ and we remark that $p_\alpha(x) = p_{B^\circ_\alpha}(\langle x, \cdot\rangle)$ for all $x \in E$. Let $f \in \mathcal{FV}(\Omega, E)$. We consider the dual map R^t_f and claim that $R^t_f \in L(\mathcal{FV}(\Omega)'_\kappa, E)$. Indeed, we have

$$p_{B^\circ_\alpha}\Big(R^t_f(f')\Big) = \sup_{e'\in B^\circ_\alpha} \Big|f'\big(R_f(e')\big)\Big| = \sup_{x\in R_f(B^\circ_\alpha)} |f'(x)| \le \sup_{x\in K_\alpha} |y(x)|, \quad f' \in \mathcal{FV}(\Omega)', \tag{7}$$

where $K_\alpha := \overline{R_f(B^\circ_\alpha)}$. Since $\mathcal{FV}(\Omega, E) \subset \mathcal{FV}(\Omega, E)_\kappa$ by Lemma 13, the set $R_f(B^\circ_\alpha)$ is absolutely convex and relatively compact, implying that K_α is absolutely convex and compact in $\mathcal{FV}(\Omega)$ by [17, 6.2.1 Proposition, p. 103]. What remains to be shown is that $R^t_f(f') \in E$ for all $f' \in \mathcal{FV}(\Omega)'$. We have for all $e' \in E'$ and $x \in \Omega$

$$R^t_f(\delta_x)(e') = \delta_x(e' \circ f) = e'(f(x)) = \langle f(x), e'\rangle \tag{8}$$

and thus $R^t_f(\delta_x) \in E$.

(i) Let E be complete and $f' \in \mathcal{FV}(\Omega)'$. Since the span of $\{\delta_x \mid x \in \Omega\}$ is dense in $\mathcal{F}(\Omega)'_\kappa$ by the bipolar theorem, there is a net $(f'_\tau)_\tau$ converging to f' in $\mathcal{FV}(\Omega)'_\kappa$ with $R^t_f(f'_\tau) \in E$ by (8). As

$$p_{B^\circ_\alpha}\left(R^t_f(f'_\tau) - R^t_f(f')\right) \underset{(7)}{\leq} \sup_{x \in K_\alpha} |(f'_\tau - f')(x)| \to 0, \tag{9}$$

for all $\alpha \in \mathfrak{A}$, we gain that $(R^t_f(f'_\tau))_\tau$ is a Cauchy net in the complete space E. Hence it has a limit $g \in E$ which coincides with $R^t_f(f')$ since

$$\begin{aligned} p_{B^\circ_\alpha}\left(g - R^t_f(f')\right) &\leq p_{B^\circ_\alpha}\left(g - R^t_f(f'_\tau)\right) + p_{B^\circ_\alpha}\left(R^t_f(f'_\tau) - R^t_f(f')\right) \\ &\underset{(9)}{\leq} p_{B^\circ_\alpha}\left(g - R^t_f(f'_\tau)\right) + \sup_{x \in K_\alpha} \left|(f'_\tau - f')(x)\right| \to 0. \end{aligned}$$

We conclude that $R^t_f(f') \in E$ for every $f' \in \mathcal{FV}(\Omega)'$.

(iii) Let condition (iii) be fulfilled. For every $f' \in \mathcal{FV}(\Omega)'$ there are $j \in J$, $m \in M$ and $C > 0$ such that

$$|R^t_f(f')(e')| \leq C|R_f(e')|_{j,m} \underset{(4)}{=} C \sup_{x \in N_{j,m}(f)} |e'(x)|, \quad e' \in E',$$

because $(T^E_m, T^{\mathbb{K}}_m)_{m \in M}$ is a strong generator. Since there is $K \in \kappa(E)$ such that $N_{j,m}(f) \subset K$, we have

$$|R^t_f(f')(e')| \leq C \sup_{x \in K} |e'(x)|, \quad e' \in E',$$

implying $R^t_f(f') \in (E'_\kappa)' = E$ by the Mackey-Arens theorem.

(ii) If E is a semi-Montel space, then $K := \overline{\mathrm{acx}}(N_{j,m}(f))$ is absolutely convex and compact in E by [17, 6.2.1 Proposition, p. 103] and [17, 6.7.1 Proposition, p. 112] for every $j \in J$ and $m \in M$. Thus (ii) is a special case of (iii).

Therefore we obtain that $R^t_f \in L(\mathcal{FV}(\Omega)'_\kappa, E) = \mathcal{FV}(\Omega)\varepsilon E$ (as vector spaces) from (7) and

$$S(R^t_f)(x) = R^t_f(\delta_x) \underset{(8)}{=} f(x)$$

for every $x \in \Omega$, proving the surjectivity of S. □

Remark 15 If J, M and $\mathfrak{A}$ are countable, then S is an isometry with respect to the induced metrics on $\mathcal{FV}(\Omega, E)$ and $\mathcal{FV}(\Omega)\varepsilon E$ by (5).

The basic idea for Theorem 14 was derived from analysing the proof of an analogous statement for Bierstedt's weighted spaces $\mathcal{CV}(\Omega, E)$ and $\mathcal{CV}_0(\Omega, E)$ of continuous functions already mentioned in the introduction (see [2, 4.2 Lemma, 4.3 Folgerung, p. 199–200] and [3, 2.1 Satz, p. 137]). Further sufficient conditions for S being a topological isomorphism can be found in [24, 3.7 Proposition, p. 8].

Let us apply our preceding results to our weighted spaces of k-times continuously partially differentiable functions on an open set $\Omega \subset \mathbb{R}^d$ with $k \in \mathbb{N}_\infty$.

Example 16 Let E be an lcHs, $k \in \mathbb{N}_\infty$, $\mathcal{V}^k$ be a directed family of weights which is locally bounded away from zero on an open set $\Omega \subset \mathbb{R}^d$.

a) $\mathcal{CV}^k(\Omega, E) \cong \mathcal{CV}^k(\Omega)\varepsilon E$ if E is a semi-Montel space and $\mathcal{CV}^k(\Omega)$ barrelled.
b) $\mathcal{CV}^k_{P(\partial)}(\Omega, E) \cong \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$ if E is a semi-Montel space and $\mathcal{CV}^k_{P(\partial)}(\Omega)$ barrelled.
c) $\mathcal{CV}^k(\Omega, E) \cong \mathcal{CV}^k(\Omega)\varepsilon E$ if E is complete and $\mathcal{CV}^k(\Omega)$ a Montel space.
d) $\mathcal{CV}^k_{P(\partial)}(\Omega, E) \cong \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$ if E is complete and $\mathcal{CV}^k_{P(\partial)}(\Omega)$ a Montel space.

Proof The generator of $(\mathcal{CV}^k, E)$ and $(\mathcal{CV}^k_{P(\partial)}, E)$ is strong and consistent by Proposition 9. From Theorem 14 (ii) we deduce part a) and b) and from (i) part c) and d). $\square$

The spaces $\mathcal{CV}^k(\Omega)$ from Example 8 a)(i) with $\omega_m := M_m \times \Omega$ for all $m \in \mathbb{N}_0$, where $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq \min(m, k)\}$, are Fréchet spaces and thus barrelled if the J in $\mathcal{V}^k := (v_{j,m})_{j\in J, m\in\mathbb{N}_0}$ is countable by [25, 3.7 Proposition p. 240]. In the case of ultradifferentiable functions, i.e. Example 8 a)(ii) with $\omega_m := \mathbb{N}_0^d \times \Omega$ for all $m \in \mathbb{N}_0$ and $k = \infty$, conditions for being a Montel space may be found in [19, Theorem 2.6, p. 44] (Beurling) and [19, Theorem 5.12, p. 65-66] (Roumieu). The special case of example d) of holomorphic functions, i.e. $P(\partial) = \overline{\partial}$, with exponential growth on strips is handled in [23, 3.11 Theorem, p. 31]. For the Schwartz space $\mathcal{S}(\mathbb{R}^d, E)$ an improvement of c) to quasi-complete E is known, see e.g. [31, Proposition 9, p. 108, Théorème 1, p. 111], which we extend to more general spaces from Example 8 a)(i) and d). We call $\mathcal{V}^k$ *locally bounded* on Ω if

$$\forall\, K \subset \Omega \text{ compact},\ j \in J,\ m \in \mathbb{N}_0,\ \beta \in M_m :\ \sup_{x\in K} v_{j,m}(\beta, x) < \infty.$$

Example 17 Let E be an lcHs, $k \in \mathbb{N}_\infty$, $\omega_m := M_m \times \Omega$ for $m \in \mathbb{N}_0$ and $\mathcal{V}^k := (v_{j,m})_{j\in J, m\in\mathbb{N}_0}$ be a directed family of weights which is locally bounded and locally bounded away from zero on an open set $\Omega \subset \mathbb{R}^d$ and fulfils the condition (V_∞), i.e.

$$\forall\, m \in \mathbb{N}_0, j \in J\ \exists\, n \in \mathbb{N}_{\geq m}, i \in J\ \forall\, \varepsilon > 0\ \exists\, K \subset \Omega \text{ compact}\ \forall\, \beta \in M_m, x \in \Omega \setminus K :$$
$$v_{j,m}(\beta, x) \leq \varepsilon v_{i,n}(\beta, x).$$

Then $\mathcal{CV}^k(\Omega, E) \cong \mathcal{CV}^k(\Omega)\varepsilon E$ resp. $\mathcal{CV}^k_{P(\partial)}(\Omega, E) \cong \mathcal{CV}^k_{P(\partial)}(\Omega)\varepsilon E$ if E is quasi-complete and $\mathcal{CV}^k(\Omega)$ resp. $\mathcal{CV}^k_{P(\partial)}(\Omega)$ barrelled.

Proof The generator of $(\mathcal{CV}^k, E)$ resp. $(\mathcal{CV}^k_{P(\partial)}, E)$ is strong and consistent by Proposition 9. We want to apply Theorem 14 (iii). Thus we have to prove that $N_{j,m}(f)$ is contained in an absolutely convex compact subset of E for every $f \in \mathcal{CV}^k(\Omega, E)$, $j \in J$ and $m \in \mathbb{N}_0$. First, we prove that $N_{j,m}(f)$ is precompact in E. Let U be a

0-neighbourhood in E. Then there are $\alpha \in \mathfrak{A}$ and $\varepsilon > 0$ such that $B_{\varepsilon,\alpha} \subset U$ where $B_{\varepsilon,\alpha} := \{x \in E \mid p_\alpha(x) < \varepsilon\}$. For $m \in \mathbb{N}_0$ and $j \in J$ there are $n \in \mathbb{N}_{\geq m}$ and $i \in J$ such that for $\varepsilon_0 > 0$ with $\varepsilon_0 |f|_{i,n,\alpha} < \varepsilon$ there is a compact set $K \subset \Omega$ with $v_{j,m}(x) \leq \varepsilon_0 v_{i,n}(x)$ for all $x \in \Omega \setminus K$. We decompose the set $N_{j,m}(f) = N_{K,j,m}(f) \cup N_{\Omega\setminus K,j,m}(f)$ with

$$N_{K,j,m}(f) := \{(\partial^\beta)^E f(x) v_{j,m}(\beta, x) \mid \beta \in M_m,\ x \in K\}$$

and

$$N_{\Omega\setminus K,j,m}(f) := \{(\partial^\beta)^E f(x) v_{j,m}(\beta, x) \mid \beta \in M_m,\ x \in \Omega \setminus K\}.$$

From condition (V_∞) it follows that

$$\sup_{e \in N_{\Omega\setminus K,j,m}(f)} p_\alpha(e) = \sup_{\substack{x \in \Omega \setminus K \\ \beta \in M_m}} p_\alpha\big((\partial^\beta)^E f(x)\big) v_{j,m}(\beta, x) \leq \varepsilon_0 |f|_{i,n,\alpha} < \varepsilon$$

and so $N_{\Omega\setminus K,j,m}(f) \subset B_{\varepsilon,\alpha}$. Writing

$$N_{K,j,m}(f) = \bigcup_{\beta \in M_m} (\partial^\beta)^E f(K) v_{j,m}(\beta, K),$$

we see that $N_{K,j,m}(f)$ is precompact if the sets $(\partial^\beta)^E f(K) v_{j,m}(\beta, K)$ are precompact since $N_{K,j,m}(f)$ is a finite union of these sets. But this is a consequence of the proof of [1, §1, 16. Lemma, p. 15] using the continuity of $(\partial^\beta)^E f$ and the boundedness of $v_{j,m}(\beta, K)$.

The precompactness of $N_{K,j,m}(f)$ implies that there exists a finite set $P \subset E$ such that $N_{K,j,m}(f) \subset P + U$. Hence we conclude

$$\begin{aligned} N_{j,m}(f) &= \big(N_{\Omega\setminus K,j,m}(f) \cup N_{K,j,m}(f)\big) \\ &\subset \big(B_{\varepsilon,\alpha} \cup (P + U)\big) \subset (U \cup (P + U)) = (P \cup \{0\}) + U, \end{aligned}$$

which means that $N_{j,m}(f)$ is precompact. Since E is quasi-complete, $N_{j,m}(f)$ is relatively compact as well by [17, 3.5.3 Proposition, p. 65]. This yields that $K := \overline{\operatorname{acx}}(N_{j,m}(f))$ is absolutely convex and compact because the quasi-complete space E has ccp. Thus we can apply Theorem 14 (iii). □

The condition (V_∞) implies that the functions in $\mathcal{CV}^k(\Omega, E)$ and $\mathcal{CV}^k_{P(\partial)}(\Omega, E)$ vanish, when weighted, at infinity (cf. [25, Remark 3.4, p. 239]). It is easy to check that the family of weights of the Schwartz space fulfils this condition (see [25, Example 3.5, p. 239]). The technique of Example 17 works as well if $\mathcal{V}^k = \mathcal{W}^k$, i.e. $\mathcal{C}^k(\Omega, E)$ is equipped with its usual topology of uniform convergence of all partial derivatives up to order k on compact subsets of Ω. For $\Omega = \mathbb{R}^d$ this can also be found in [31, Proposition 9, p. 108, Théorème 1, p. 111] and for general open $\Omega \subset \mathbb{R}^d$ it is already mentioned in [18, (9), p. 236] (without a proof) that $\mathcal{CW}^k(\Omega, E) \cong \mathcal{CW}^k(\Omega)\varepsilon E$ for $k \in \mathbb{N}_\infty$ and quasi-complete E. For $k = \infty$ we even have $\mathcal{CW}^\infty(\Omega, E) \cong \mathcal{CW}^\infty(\Omega)\varepsilon E$ for locally complete E by [6, p. 228]. Our technique allows us to generalise the first result and to get back the second result.

Example 18 Let E be an lcHs, $k \in \mathbb{N}_\infty$ and $\Omega \subset \mathbb{R}^d$ open. If $k < \infty$ and E has metric ccp or if $k = \infty$ and E is locally complete, then

a) $\mathcal{CW}^k(\Omega, E) \cong \mathcal{CW}^k(\Omega)\varepsilon E$, and
b) $\mathcal{CW}^k_{P(\partial)}(\Omega, E) \cong \mathcal{CW}^k_{P(\partial)}(\Omega)\varepsilon E$ if $\mathcal{CW}^k_{P(\partial)}(\Omega)$ is closed in $\mathcal{CW}^k(\Omega)$.

Proof We recall from Example 8 b) that $\mathcal{W}^k$ is the family of weights given by $\nu_{K,m}(\beta, x) := \chi_K(x)$, $(\beta, x) \in M_m \times \Omega$, for all $m \in \mathbb{N}_0$ and compact $K \subset \Omega$ where $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq \min(m, k)\}$ and χ_K is the characteristic function of K. We already know that the generator for $(\mathcal{CW}^k, E)$ and $(\mathcal{CW}^k_{P(\partial)}, E)$ is strong and consistent by Proposition 9 because $\mathcal{W}^k$ is locally bounded away from zero on Ω, $\mathcal{CW}^k(\Omega)$ and its closed subspace $\mathcal{CW}^k_{P(\partial)}(\Omega)$ are Fréchet spaces. Let $f \in \mathcal{CW}^k(\Omega, E)$, $K \subset \Omega$ be compact, $m \in \mathbb{N}_0$ and consider

$$N_{K,m}(f) = \{(\partial^\beta)^E f(x)\nu_{K,m}(\beta, x) \mid x \in \Omega,\ \beta \in M_m\} = \{0\} \cup \bigcup_{\beta \in M_m} (\partial^\beta)^E f(K).$$

$N_{K,m}(f)$ is compact since it is a finite union of compact sets. Furthermore, the compact sets $\{0\}$ and $(\partial^\beta)^E f(K)$ are metrisable by [8, Chap. IX, §2.10, Proposition 17, p. 159] and thus their finite union $N_{K,m}(f)$ is metrisable as well by [33, Theorem 1, p. 361] since the compact set $N_{K,m}(f)$ is collectionwise normal and locally countably compact by [11, 5.1.18 Theorem, p. 305]. If E has metric ccp, then the set $\overline{\operatorname{acx}}(N_{K,m}(f))$ is absolutely convex and compact. Thus Theorem 14 (iii) settles the case for $k < \infty$. If $k = \infty$ and E is locally complete, we observe that $K_\beta := \overline{\operatorname{acx}}((\partial^\beta)^E f(K))$ for $f \in \mathcal{CW}^\infty(\Omega, E)$ is absolutely convex and compact by [7, Proposition 2, p. 354]. Then we have

$$N_{K,m}(f) \subset \operatorname{acx}\left(\bigcup_{\beta \in M_m} K_\beta\right)$$

and the set on the right-hand side is absolutely convex and compact by [17, 6.7.3 Proposition, p. 113]. Again, the statement follows from Theorem 14 (iii). □

In the context of differentiability on infinite dimensional spaces the preceding example a) remains true for an open subset Ω of a Fréchet space or DFM-space and quasi-complete E by [27, 3.2 Corollary, p. 286]. Like here this can be generalised to E with [metric] ccp. A special case of example b) is already known to be a consequence of [6, Theorem 9, p. 232], namely, if $k = \infty$ and $P(\partial)$ is hypoelliptic with constant coefficients. In particular, this covers the space of holomorphic functions and the space of harmonic functions. Holomorphy on infinite dimensional spaces is treated in [9, Corollary 6.35, p. 332-333] where $\mathcal{V} = \mathcal{W}^0$, Ω is an open subset of a locally convex Hausdorff k-space and E a quasi-complete locally convex Hausdorff space, both over $\mathbb{C}$, which can be generalised to E with [metric] ccp in a similar way.

Now, we direct our attention to spaces of continuously partially differentiable functions on an open bounded set such that all derivatives can be continuously extended to the boundary. Let E be an lcHs, $k \in \mathbb{N}_\infty$ and $\Omega \subset \mathbb{R}^d$ open and bounded. The space $\mathcal{C}^k(\overline{\Omega}, E)$ is given by

$$\mathcal{C}^k(\overline{\Omega}, E) := \{f \in \mathcal{C}^k(\Omega, E) \mid (\partial^\beta)^E f \text{ cont. extendable on } \overline{\Omega} \text{ for all } \beta \in \mathbb{N}_0^d,\ |\beta| \leq k\}$$

and equipped with the system of seminorms given by

$$|f|_\alpha := \sup_{\substack{x \in \Omega \\ \beta \in \mathbb{N}_0^d, |\beta| \leq k}} p_\alpha\big((\partial^\beta)^E f(x)\big), \quad f \in \mathcal{C}^k(\overline{\Omega}, E),$$

for $\alpha \in \mathfrak{A}$ if $k < \infty$ and by

$$|f|_{m,\alpha} := \sup_{\substack{x \in \Omega \\ \beta \in \mathbb{N}_0^d, |\beta| \leq m}} p_\alpha\big((\partial^\beta)^E f(x)\big), \quad f \in \mathcal{C}^\infty(\overline{\Omega}, E),$$

for $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$ if $k = \infty$. We prepare the proof of consistency of its generator. We write $\mathcal{C}^{ext}(\Omega, E)$ for the space of functions $f \in \mathcal{C}(\Omega, E)$ which have a continuous extension to $\overline{\Omega}$ and set $\mathcal{C}^{ext}(\Omega) := \mathcal{C}^{ext}(\Omega, \mathbb{K})$. We equip $\mathcal{C}^{ext}(\Omega)$ with the topology of uniform convergence on compact subsets of Ω.

Proposition 19 *Let $\Omega \subset \mathbb{R}^d$ be open, $\mathcal{FV}(\Omega)$ a* dom-*space and $T \in L(\mathcal{FV}(\Omega), \mathcal{C}^{ext}(\Omega))$. Then $\delta \circ T \in \mathcal{C}^{ext}(\Omega, \mathcal{FV}(\Omega)_\gamma')$ if $\mathcal{FV}(\Omega)$ is barrelled.*

Proof From Proposition 10 a) we derive that $\delta \circ T \in \mathcal{C}(\Omega, \mathcal{FV}(\Omega)_\gamma')$. Let $x \in \partial\Omega$ and (x_n) a sequence in Ω with $x_n \to x$. Then $(\delta_{x_n} \circ T)$ is a sequence in $\mathcal{FV}(\Omega)'$ and

$$\lim_{n\to\infty} (\delta_{x_n} \circ T)(f) = \lim_{n\to\infty} T(f)(x_n) =: (\delta_x^{ext} \circ T)(f)$$

in $\mathbb{K}$ for every $f \in \mathcal{FV}(\Omega)$, which implies that $(\delta_{x_n} \circ T)$ converges to $\delta_x^{ext} \circ T$ pointwise in f because $T(f) \in \mathcal{C}^{ext}(\Omega)$. As a consequence of the Banach-Steinhaus theorem we get $(\delta_x^{ext} \circ T) \in \mathcal{FV}(\Omega)'$ and the convergence in $\mathcal{FV}(\Omega)_\gamma'$. □

Example 20 Let E be an lcHs, $k \in \mathbb{N}_\infty$ and $\Omega \subset \mathbb{R}^d$ open and bounded. Then $\mathcal{C}^k(\overline{\Omega}, E) \cong \mathcal{C}^k(\overline{\Omega}) \varepsilon E$ if E has metric ccp.

Proof The generator coincides with the one of Example 8 a)(i). Due to Proposition 10 we have $S(u) \in \mathcal{C}^k(\Omega, E)$ and

$$(\partial^\beta)^E S(u)(x) = u(\delta_x \circ (\partial^\beta)^{\mathbb{K}}), \quad \beta \in \mathbb{N}_0^d,\ |\beta| \leq k,\ x \in \Omega,$$

for all $u \in \mathcal{C}^k(\overline{\Omega}) \varepsilon E$ since $\mathcal{C}^k(\overline{\Omega})$ is a Banach space if $k < \infty$ and a Fréchet space if $k = \infty$, in particular, both are barrelled. As a consequence of Proposition 19 with $T = (\partial^\beta)^{\mathbb{K}}$ for $\beta \in \mathbb{N}_0^d$, $|\beta| \leq k$, we obtain that $(\partial^\beta)^E S(u) \in \mathcal{C}^{ext}(\Omega, E)$ for all $u \in \mathcal{C}^k(\overline{\Omega}) \varepsilon E$. Hence the generator is consistent. It is easy to check that it is strong too.

Let $f \in \mathcal{C}^k(\overline{\Omega}, E)$, $J := \{1\}$, $m \in \mathbb{N}_0$ and set $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq k\}$ if $k < \infty$ and $M_m := \{\beta \in \mathbb{N}_0^d \mid |\beta| \leq m\}$ if $k = \infty$. We denote by f_β the continuous extension of $(\partial^\beta)^E f$ on the compact metrisable set $\overline{\Omega}$. The set

$$N_{1,m}(f) = \{(\partial^\beta)^E f(x) \mid x \in \Omega,\ \beta \in M_m\} \subset \bigcup_{\beta \in M_m} f_\beta(\overline{\Omega})$$

is relatively compact and metrisable since it is a subset of a finite union of the compact metrisable sets $f_\beta(\overline{\Omega})$ like in Example 18. Due to Theorem 14 (iii) we obtain our statement as E has metric ccp. □

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