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A short note on minor and major symmetries in linear elasticity

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Abstract Simply applying the directional derivative either twice to the strain-energy density function (hyperelasticity) or once to the stress–strain state (Cauchy elasticity) does not lead to the symmetries of the fourth-order elasticity tensor specified in the literature. Moreover, there are many justifications and arguments for the desired symmetries, which are summarized in this contribution. Thus, a symmetrization operator has to be introduced to guarantee minor symmetry, since the symmetry of the strain tensor is frequently neglected but is needed to obtain results required for particular elasticity relations. A thorough investigation is provided for both Cauchy elasticity and hyperelasticity, and what conclusion can be drawn on by various assumptions.

Keywords Linear elasticity · Elasticity tensor · Minor symmetry · Major symmetry

1 Introduction

It is well known that there are required symmetry conditions for linear elastic materials, i.e. the fourth-order elasticity tensor $\mathcal{C}$ should have specific symmetries, which are called minor symmetries, see, for instance, [1,2], $C_{ijkl} = C_{ijlk} = C_{jikl} = C_{jilk}$, and the major symmetry $C_{ijkl} = C_{klij}$. However, for the case of Cauchy elasticity—no strain-energy density function is assumed—the major symmetry cannot be proven [8]. If the directional derivative is applied twice to the strain-energy function in the case of hyperelasticity, the fourth-order elasticity tensor is obtained. Nevertheless, there are cases where the minor symmetry condition is not directly evident. Moreover, it turns out that, although indicated in some publications, [13], no contradictions arise when particular assumptions are correctly applied in the differentiation process and the resulting matrix representation, i.e. the Voigt notation. This article aims to provide a systematic presentation on the subject, thereby facilitating a more nuanced understanding of the topic. In this context, we will limit ourselves to the theory of small strains.

The article is structured as follows: First, the basic, classical equations for both Cauchy elasticity (Sect. 2) and hyperelasticity (Sect. 3) are provided. In both material model classes, two approaches are investigated,

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namely a general series expansion and a Taylor series. Then, it is possible to recognize which prerequisites lead to which statements of the symmetry properties mentioned. Next, the development of the differentials using the directional derivative is discussed, see Sect. 4, since the required symmetry conditions, if they are necessary, are only obtained using particular assumptions. Afterwards, a general discussion provides a further clarification of the terms minor and major symmetries in Sect. 5.

The notation used is defined as follows: geometric vectors are symbolized by $\vec{a} \in \mathbb{V}^3$, second-order tensors $\mathbf{A} \in \mathbb{L}(\mathbb{V}^3)$ by bold-faced Roman letters, and calligraphic letters $\mathcal{A} \in \mathbb{T}(\mathbb{L})$ define fourth-order tensors. Symmetric second-order tensors, $\mathbf{A} = \mathbf{A}^T$, are elements of the set $\mathbb{S}(\mathbb{V}^3)$. To keep the representation as simple as possible, we restrict ourselves to Cartesian coordinates. All other symbols concerning the notation are explained in the text. In the following, we call the symmetry $C_{ijkl} = C_{jikl}$ the left minor symmetry, and $C_{ijkl} = C_{ijlk}$ the right minor symmetry.

2 Cauchy elasticity

We start with the more general case of Cauchy elasticity, see [23, p. 119], [17, Ch. 4.2], or [8, Ch. 9],

$$\mathbf{T} = \mathbf{h}(\mathbf{E}), \qquad \mathbf{h} : \mathbb{S}(\mathbb{V}^3) \to \mathbb{S}(\mathbb{V}^3). \tag{1}$$

$\mathbf{h}(\mathbf{E})$ defines a symmetric tensor-valued function of the symmetric strain tensor $\mathbf{E} = \big(\operatorname{grad} \vec{u}(\vec{x}) + \operatorname{grad}^T \vec{u}(\vec{x})\big)/2$, $\mathbf{E} = \mathbf{E}^T$, where $\vec{u}(\vec{x})$ is the displacement field, and $\vec{x}$ presents the spatial (material) point—remember, we make no distinction between the reference and actual configuration. In the following, we can expand $\mathbf{h}(\mathbf{E})$ in a general series

$$\mathbf{h}(\mathbf{E}) = \mathbf{T}_0 + \mathcal{C}\mathbf{E} + \mathcal{O}(\|\mathbf{E}\|^2), \tag{2}$$

or in a Taylor series at $\mathbf{E} = \mathbf{0}$,

$$\mathbf{h}(\mathbf{E}) = \mathbf{h}(\mathbf{0}) + \mathsf{D}_{\mathbf{E}}\, \mathbf{h}(\mathbf{E})[\mathbf{H}]\Big|_{\mathbf{E}=\mathbf{0}} + \mathcal{O}(\|\mathbf{E}\|^2), \tag{3}$$

where the higher-order terms are neglected, and where it is assumed that there is a stress-free initial state, $\mathbf{T}_0 = \mathbf{0}$ or $\mathbf{h}(\mathbf{0}) = \mathbf{0}$. Afterwards, $\mathbf{H} = \mathbf{E}$ is set. In this contribution,

$$\mathsf{D}_{\mathbf{E}}\, \mathbf{h}(\mathbf{E})[\mathbf{H}] = \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathbf{h}(\mathbf{E} + \lambda \mathbf{H})\Big|_{\lambda=0} = \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\mathbf{E}} \mathbf{H} \tag{4}$$

is used to denote the directional derivative (differential) also sometimes called Gâteaux-derivative. Two cases are distinguished in the following:
(I) We start with the general expansion (2). In the literature, it is sometimes assumed and sometimes inferred that from the symmetry of both the stress tensor $\mathbf{T} = \mathbf{T}^T$ and the strain tensor $\mathbf{E} = \mathbf{E}^T$ the *minor symmetries*

$$C_{ijkl} = C_{jikl} \quad \text{and} \quad C_{ijkl} = C_{ijlk} \tag{5}$$

of the fourth-order elasticity tensor $\mathcal{C}$ follow, [4,10]. Note that, the minor symmetries (5) can also be described by the transpositions

$$\mathcal{C} = \mathcal{C}^{T_{12}} = \mathcal{C}^{T_{34}}. \tag{6}$$

Here, the symbol $[\cdot]^{T_{ij}}$ denotes the transposition of the i-th and j-th summation index. The linear elasticity relation (2) takes the form:

$$\mathbf{T} = \mathcal{C}\mathbf{E}. \tag{7}$$

The left minor symmetry $(5)_1$ can be proven as follows. Since we assume the symmetry of the stress tensor, $\mathbf{T} = \mathbf{T}^T$, i.e. $T_{ij} = T_{ji}$, the following relations must hold,

$$T_{ij} = T_{ji} \quad \Rightarrow \quad T_{ij} - T_{ji} = 0 \quad \Rightarrow \quad (C_{ijkl} - C_{jikl}) E_{kl} = 0 \quad \forall E_{kl} \quad \Rightarrow \quad C_{ijkl} = C_{jikl}. \tag{8}$$

The right minor symmetry $(5)_2$ cannot be proven, see Eq. (73). The major symmetry, $C_{ijkl} = C_{klij}$, is only obtained if a strain-energy density function is assumed, see Sect. 3. For a remark, see [18] as well.
(II) In the second case, we consider the stress–strain relation $\mathbf{h}(\mathbf{E})$, see Eq. (3), and apply a Taylor series. The differential reads

$$\mathsf{D}_{\mathbf{E}}\, \mathbf{h}(\mathbf{E})[\mathbf{H}]\Big|_{\mathbf{E}=\mathbf{0}} = \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\mathbf{E}}\Big|_{\mathbf{E}=\mathbf{0}} \mathbf{H}, \tag{9}$$

and yields the elasticity tensor

$$\mathcal{C} = \left.\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\mathbf{E}}\right|_{\mathbf{E}=\mathbf{0}}, \qquad \mathcal{C} = C_{ijkl}\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l, \tag{10}$$

see [8, Ch. 9.3.1], where we arrive again at relation (7). Obviously, the left minor symmetry is fulfilled again. However, the right minor symmetry is not directly a result of the calculation using the directional derivative, although $\mathbf{h} : \mathbb{S}(\mathbb{V}^3) \to \mathbb{S}(\mathbb{V}^3)$ is a symmetric tensor-valued function of the symmetric strain tensor as well. Subsequently, we will address the problem that the particular symmetries concerned are attained only if the derivatives using index notation and the directional derivative are applied in a specific manner.

3 Hyperelasticity

A material, which is characterized by the existence of a specific strain-energy function $\psi(\mathbf{E})$, $\psi : \mathbb{S}(\mathbb{V}^3) \to \mathbb{R}$ is called hyperelastic (or Green elastic, see [17] or [8, Ch. 9.1]),

$$\mathbf{T} = \rho\frac{\mathrm{d}\overline{\psi}(\mathbf{E})}{\mathrm{d}\mathbf{E}} = \frac{\mathrm{d}\psi(\mathbf{E})}{\mathrm{d}\mathbf{E}} \quad \text{with} \quad \psi = \rho\overline{\psi}. \tag{11}$$

In this relation, ρ defines the mass density. For a more general definition of an elastic material in the context of a fully thermomechanical theory, see [5, Ch. 5.5, pp 168]. In the following, we distinguish between two cases again. For this purpose, the general expansion of the strain-energy function is symbolized by $\hat{\psi}(\mathbf{E})$, and for the Taylor series $\psi(\mathbf{E})$:

(I) In the case of the classical linear approach, an expansion of the strain-energy function (general case) can be assumed,

$$\hat{\psi}(\mathbf{E}) = \hat{\psi}_0 + \mathbf{T}_0 \cdot \mathbf{E} + \frac{1}{2}\mathbf{E}\cdot\hat{\mathcal{C}}\mathbf{E} + \mathcal{O}(\|\mathbf{E}\|^3) \quad \text{with} \quad \mathbf{E} = \mathbf{E}^T. \tag{12}$$

The dot symbolizes the scalar product of second-order tensors, $\mathbf{A}\cdot\mathbf{B} = a_{ij}b_{ij}$, i.e. $(\vec{a}\otimes\vec{b})\cdot\mathbf{A} = \vec{a}\cdot\mathbf{A}\vec{b}$. Again, we assume a stress-free initial state in Eq. (12), $\mathbf{T}_0 = \mathbf{0}$, no initial stored energy, $\hat{\psi}_0 = 0$, and vanishing higher-order terms. We can draw on a first conclusion that under the assumption of arbitrary deformation processes $\mathbf{E}$, the following symmetry condition is obtained,

$$\mathbf{E}\cdot\hat{\mathcal{C}}\mathbf{E} = \hat{\mathcal{C}}^T\mathbf{E}\cdot\mathbf{E} = \mathbf{E}\cdot\hat{\mathcal{C}}^T\mathbf{E} \quad \to \quad \mathbf{E}\cdot\left[\hat{\mathcal{C}} - \hat{\mathcal{C}}^T\right]\mathbf{E} = 0 \quad \to \quad \hat{\mathcal{C}} = \hat{\mathcal{C}}^T, \tag{13}$$

for arbitrary deformation processes $\mathbf{E}$. In this relation, the transposition

$$\mathcal{A}^T := \left[\mathcal{A}^{T_{13}}\right]^{T_{24}} = a_{klij}\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l \tag{14}$$

of an arbitrary fourth-order tensor $\mathcal{A}$ is introduced, $\mathbf{A}\cdot\mathcal{A}\mathbf{B} = \mathcal{A}^T\mathbf{A}\cdot\mathbf{B}$. Now, the stress state (11) is obtained by applying the directional derivative

$$\mathsf{D}_{\mathbf{E}}\,\hat{\psi}(\mathbf{E})[\mathbf{H}] = \frac{\mathrm{d}\hat{\psi}}{\mathrm{d}\mathbf{E}}\cdot\mathbf{H} = \frac{1}{2}\left(\mathbf{H}\cdot\hat{\mathcal{C}}\mathbf{E} + \mathbf{E}\cdot\hat{\mathcal{C}}\mathbf{H}\right) = \left(\frac{1}{2}\left[\hat{\mathcal{C}} + \hat{\mathcal{C}}^T\right]\mathbf{E}\right)\cdot\mathbf{H}, \tag{15}$$

i.e. the stress state reads

$$\mathbf{T} = \frac{1}{2}\left[\hat{\mathcal{C}} + \hat{\mathcal{C}}^T\right]\mathbf{E}, \tag{16}$$

see [20, p. 84] and [22, p. 113]. Again

$$\mathbf{T} = \hat{\mathcal{C}}\mathbf{E}, \tag{17}$$

is obtained using the so-called *major symmetry condition* (13), or in index notation

$$\hat{C}_{ijkl} = \hat{C}_{klij}. \tag{18}$$

The left minor symmetry condition (5)$_1$ has to be fulfilled as well to obtain a symmetric stress tensor. A proof has already been given in Eq. (8). The remaining right minor symmetry can be proven as follows:

$$\hat{\mathcal{C}} = \hat{\mathcal{C}}^T = \hat{\mathcal{C}}^{T_{12}} = \left[\hat{\mathcal{C}}^{T_{12}}\right]^T = \left[\left[\hat{\mathcal{C}}^T\right]^{T_{12}}\right]^T \tag{19}$$

$$\Rightarrow \quad \hat{C}_{ijkl} = \hat{C}_{klij} = \hat{C}_{jikl} = \hat{C}_{klji} = \hat{C}_{ijlk} \tag{20}$$

In other words, left minor symmetry and major symmetry yields right minor symmetry.
(II) If, instead of Eq. (12), a Taylor series of a general strain-energy function is assumed, [8, Ch. 9.3],

$$\psi(\mathbf{E}) = \left(\psi(\mathbf{0}) + \mathsf{D}_{\mathbf{E}}\, \psi(\mathbf{E})[\mathbf{H}]\Big|_{\mathbf{E}=\mathbf{0}} + \frac{1}{2}\mathsf{D}_{\mathbf{E}} \left(\mathsf{D}_{\mathbf{E}}\, \psi(\mathbf{E})[\mathbf{H}]\right)(\mathbf{E})[\mathbf{H}]\Big|_{\mathbf{E}=\mathbf{0}} + \mathcal{O}(\|\mathbf{H}\|^3)\right)\Bigg|_{\mathbf{H}=\mathbf{E}} \tag{21}$$

$$= \psi(\mathbf{0}) + \frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{E}}\Big|_{\mathbf{E}=\mathbf{0}} \cdot \mathbf{E} + \frac{1}{2}\mathbf{E} \cdot \frac{\mathrm{d}^2\psi}{\mathrm{d}\mathbf{E}\mathrm{d}\mathbf{E}}\Big|_{\mathbf{E}=\mathbf{0}} \mathbf{E} + \mathcal{O}(\|\mathbf{E}\|^3), \tag{22}$$

we follow the previous arguments of a strain-energy free and a stress-free initial state, $\psi(\mathbf{0}) = 0$ and $\mathrm{d}\psi/\mathrm{d}\mathbf{E}\big|_{\mathbf{E}=\mathbf{0}} = \mathbf{0}$, respectively, as well as negligible higher-order terms, which lead to

$$\mathbf{T} = \frac{1}{2}\left[\mathcal{C} + \mathcal{C}^T\right]\mathbf{E} \tag{23}$$

with

$$\mathcal{C} := \frac{\mathrm{d}^2\psi}{\mathrm{d}\mathbf{E}\mathrm{d}\mathbf{E}}\Big|_{\mathbf{E}=\mathbf{0}} = \frac{\partial^2\psi}{\partial E_{ij}\partial E_{kl}}\Big|_{\mathbf{E}=\mathbf{0}} \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l. \tag{24}$$

Since the differentiation is interchangeable, the elasticity tensor has a priorly major symmetry

$$C_{ijkl} = C_{klij}, \qquad \frac{\partial^2\psi}{\partial E_{ij}\partial E_{kl}} = \frac{\partial^2\psi}{\partial E_{kl}\partial E_{ij}}. \tag{25}$$

This property, i.e. $\mathcal{C} = \mathcal{C}^T$, automatically yields

$$\mathbf{T} = \mathcal{C}\mathbf{E} \tag{26}$$

from Eq. (23), with major symmetry. Here, however, one has to be careful with the notation since we applied the concept of first determining the derivative and subsequently inserting the symmetry property $\mathbf{E} = \mathbf{E}^T$, i.e. it has to read

$$\mathbf{T} = \frac{\mathrm{d}\psi(\mathbf{E})}{\mathrm{d}\mathbf{E}}\Big|_{\mathbf{E}=\mathbf{E}^T}. \tag{27}$$

This approach has to be extended to definition (24) as well.

Remark 1 A simple example, where a careful treatment of the derivative has to be carried out, is $\psi(\mathbf{E}) = c\operatorname{tr}\mathbf{E}^2 = c\mathbf{E}\cdot\mathbf{E}^T$, reading in the index notation

$$T_{ij} = \frac{\partial\psi}{\partial E_{ij}} \quad \xrightarrow{\text{example}} \quad T_{12} = \frac{\partial\psi}{\partial E_{12}} = 2cE_{21}$$

for $\psi(E_{11}, E_{22}, E_{33}, E_{12}, \ldots, E_{21}, \ldots) = cE_{ij}E_{ji}$. Here, $\operatorname{tr}\mathbf{A} = \mathbf{I}\cdot\mathbf{A} = A_{ii}$ defines the trace operator. The strain-energy function $\psi(\mathbf{E})$ is assumed to be dependent on a general strain tensor $\mathbf{E}$. If $\psi(\mathbf{E}) = \psi^S(\mathbf{E}_{\text{sym}}) = \psi^S(E_{11}, E_{22}, E_{33}, E_{12}, E_{23}, E_{31})$ is assumed to depend only on the independent strain components—if a symmetric strain tensor $\mathbf{E}_{\text{sym}} = \mathbf{E}^T{}_{\text{sym}}$ is assumed, the strain-energy function $\psi(\mathbf{E}) = \psi^S(\mathbf{E}_{\text{sym}})$ reads $\psi^S(\mathbf{E}_{\text{sym}}) = c(E_{11}^2 + E_{22}^2 + E_{33}^2 + 2(E_{12}^2 + E_{23}^2 + E_{31}^2))$. The derivative $T_{ij} = \partial\psi^S/\partial E_{ij}$ yields for $T_{12} = 4cE_{12}$, which is, accordingly, a different result. Similar problems are discussed in matrix calculus in [16, Chp. 18]. A possibility to overcome the problem is provided in [8, Sec. 9.3.2] in a remark, where for derivatives with respect to strain components with different index, the formula $\partial\psi^S/\partial(2E_{ij})$, $i \neq j$, holds.

The terms major and minor symmetries should not be mixed up with definitions in [13]. Further, the condition of supersymmetry in the same reference is a much more restrictive property than only a minor and major symmetry.

4 Application of directional derivative

The properties in question have been thoroughly delineated and summarized in the previous sections. Nevertheless, a salient issue that remains unresolved relates to the forfeiture of minor symmetry that accompanies the application of the directional derivative in an inconsistent manner. This matter is addressed subsequently in the context of Cauchy elasticity and hyperelasticity, accompanied by a proposed solution.

4.1 Cauchy elasticity

We assume an isotropic elasticity relation of the form

$$\mathbf{h}(\mathbf{E}) = \lambda(\mathbf{I}\cdot\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}, \tag{28}$$

where λ and μ are the Lamé constants, and $\mathbf{I}$ is the second-order identity tensor, $\mathbf{I}\vec{v} = \vec{v}$, $\mathbf{I} = \delta_{ij}\vec{e}_i \otimes \vec{e}_j$. Here, δ_{ij} defines the Kronecker symbol, $\delta_{ij} = \vec{e}_i \cdot \vec{e}_j$. The application of the directional derivative yields:

$$\mathsf{D}_{\mathbf{E}}\,\mathbf{h}(\mathbf{E})[\mathbf{H}] = \lambda(\mathbf{I}\cdot\mathbf{H})\mathbf{I} + 2\mu\mathbf{H} = \big[\lambda\big[\mathbf{I}\otimes\mathbf{I}\big] + 2\mu\mathcal{I}\big]\mathbf{H}, \tag{29}$$

where

$$\mathcal{I} = \big[\mathbf{I}\otimes\mathbf{I}\big]^{T_{23}} = \delta_{ik}\delta_{jl}\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l \tag{30}$$

symbolizes the fourth-order identity tensor, $\mathcal{I}\mathbf{A} = \mathbf{A}$, and the elasticity tensor reads

$$\mathcal{C} = \lambda\mathbf{I}\otimes\mathbf{I} + 2\mu\mathcal{I}. \tag{31}$$

However, the term $2\mu\mathcal{I}$ does not fulfil the minor symmetry conditions (5) due to the fourth-order identity tensor $\mathcal{I}$,

$$C_{ijkl} = C_{ijlk} \quad \rightarrow \quad (\vec{e}_i \otimes \vec{e}_j)\cdot\underbrace{\mathcal{I}(\vec{e}_k \otimes \vec{e}_l)}_{\vec{e}_k\otimes\vec{e}_l} = \delta_{ik}\delta_{jl} \neq (\vec{e}_i \otimes \vec{e}_j)\cdot\underbrace{\mathcal{I}(\vec{e}_l \otimes \vec{e}_k)}_{\vec{e}_l\otimes\vec{e}_k} = \delta_{il}\delta_{jk}. \tag{32}$$

The same holds for

$$C_{ijkl} = C_{jikl} \quad \rightarrow \quad \delta_{ik}\delta_{jl} \neq (\vec{e}_j \otimes \vec{e}_i)\cdot\mathcal{I}(\vec{e}_k \otimes \vec{e}_l) = \delta_{jk}\delta_{il}. \tag{33}$$

In conclusion, the minor symmetry conditions are violated, while the major symmetry (25) is fulfilled (the proof is similar and is left to the reader).

A fundamental reason for the loss of minor symmetry conditions is that no symmetry properties are considered in the directional derivative (4) neither for $\mathbf{E}$ nor for direction $\mathbf{H}$. The problem is the non-uniqueness of the directional derivative, since one can add, e.g. in the derivative of a scalar-valued function with respect to a symmetric second-order tensor, an arbitrary skew-symmetric tensor without changing the result. This is addressed in, for example, [13, Ch. 6.3] and [11,12,14]. The proposal in [12] is to take the Fréchet derivative and symmetrize it afterwards. To avoid the problem, we proceed as follows. First, Eq. (28) is reformulated to

$$\mathbf{h}(\mathbf{E}) = \hat{\mathbf{h}}(\mathbf{E}^{\mathrm{sym}}(\mathbf{E})) = \lambda(\mathbf{I}\cdot\mathbf{E}^{\mathrm{sym}})\mathbf{I} + 2\mu\mathbf{E}^{\mathrm{sym}} \tag{34}$$

with

$$\mathbf{E}^{\mathrm{sym}} = \mathcal{I}^{\mathrm{sym}}\mathbf{E} = \frac{1}{2}(\mathbf{E} + \mathbf{E}^T), \tag{35}$$

and the symmetrizer

$$\mathcal{I}^{\mathrm{sym}} = \frac{1}{2}(\mathcal{I} + \hat{\mathcal{I}}), \quad \mathcal{I}^{\mathrm{sym}} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l, \tag{36}$$

where

$$\hat{\mathcal{I}} = \big[\mathbf{I}\otimes\mathbf{I}\big]^{T_{24}} = \delta_{il}\delta_{kj}\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l \tag{37}$$

defines the transposition of a second-order tensor, $\hat{\mathcal{I}}\mathbf{A} = \mathbf{A}^T$, $(\big[\mathbf{A}\otimes\mathbf{B}\big]^{T_{24}}\mathbf{C} = \mathbf{A}\mathbf{C}^T\mathbf{B})$. Now, the directional derivative implies the application of the chain rule,

$$\mathsf{D}_{\mathbf{E}}\,\mathbf{h}(\mathbf{E})[\mathbf{H}] = \mathsf{D}_{\mathbf{E}^{\mathrm{sym}}}\,\hat{\mathbf{h}}(\mathbf{E}^{\mathrm{sym}})[\mathsf{D}_{\mathbf{E}}\,\mathbf{E}^{\mathrm{sym}}(\mathbf{E})[\mathbf{H}]] = \frac{\mathrm{d}\hat{\mathbf{h}}}{\mathrm{d}\mathbf{E}^{\mathrm{sym}}}\frac{\mathrm{d}\mathbf{E}^{\mathrm{sym}}}{\mathrm{d}\mathbf{E}}\mathbf{H} = \frac{\mathrm{d}\hat{\mathbf{h}}}{\mathrm{d}\mathbf{E}^{\mathrm{sym}}}\mathcal{I}^{\mathrm{sym}}\mathbf{H} \tag{38}$$

with

$$\mathsf{D}_{\mathbf{E}}\, \mathbf{E}^{\text{sym}}(\mathbf{E})[\mathbf{H}] = \mathcal{I}^{\text{sym}}\mathbf{H} \tag{39}$$

see, for example, [7] for the chain rule of differentials. Of course, since $\mathbf{H} \in \mathbb{S}(\mathbb{V}^3)$, we can use $\mathbf{H}^{\text{sym}} = \hat{\mathcal{I}}\mathbf{H}$ in the chain rule as well. However, this does not lead to a different result. For the problem under consideration (34), we obtain

$$\mathcal{C} = \big[\lambda \mathbf{I} \otimes \mathbf{I} + 2\mu\mathcal{I}\big]\mathcal{I}^{\text{sym}} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu\mathcal{I}^{\text{sym}} = \lambda \mathbf{I} \otimes \mathbf{I} + \mu\big[\mathcal{I} + \hat{\mathcal{I}}\big], \tag{40}$$

or, in index notation,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}), \tag{41}$$

see [15, p. 98] or [19, p. 24].

Alternatively, we might assume that the direction $\mathbf{H} \in \mathbb{S}(\mathbb{V}^3)$ of the directional derivative explicitly has to be expressed as a symmetric tensor,

$$\mathsf{D}_{\mathbf{E}}\, \mathbf{h}(\mathbf{E})[\mathbf{H}^{\text{sym}}] = \mathsf{D}_{\mathbf{E}}\, \mathbf{h}(\mathbf{E})[\mathcal{I}^{\text{sym}}\mathbf{H}] = \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\mathbf{E}}\mathcal{I}^{\text{sym}}\mathbf{H} = \left[\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\mathbf{E}}\mathcal{I}^{\text{sym}}\right]\mathbf{H}, \tag{42}$$

yielding the same result as shown in Eq. (38).

These considerations can be extended to anisotropic formulations as well. Exemplarily, transversal isotropy is chosen, see [21],

$$\mathbf{T} = \mathbf{h}(\mathbf{E}) = (\Lambda \mathrm{I}_{\mathbf{E}} + \alpha \mathrm{IV}_{\mathbf{E}})\mathbf{I} + 2\mu_T\mathbf{E} + (\alpha \mathrm{I}_{\mathbf{E}} + \beta \mathrm{IV}_{\mathbf{E}})\mathbf{M} + 2(\mu_L - \mu_T)(\mathbf{EM} + \mathbf{ME}) \tag{43}$$

implying

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\mathbf{E}} = \Lambda \mathbf{I} \otimes \mathbf{I} + 2\mu_T\mathcal{I} + \alpha[\mathbf{I} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}] + \beta\mathbf{M} \otimes \mathbf{M} + 2(\mu_L - \mu_T)[\mathbf{I} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}]^{T_{23}}. \tag{44}$$

Here, the abbreviations of the invariants $\mathrm{I}_{\mathbf{E}} = \operatorname{tr}\mathbf{E}$, $\mathrm{IV}_{\mathbf{E}} = \operatorname{tr}(\mathbf{EM}) = \mathbf{E}\cdot\mathbf{M}$, and the structural tensor $\mathbf{M} = \vec{a}\otimes\vec{a}$ are chosen, where $\vec{a}$, $|\vec{a}| = 1$, defines the preferred direction. This fourth-order tensor has to be applied to the symmetrizer

$$\mathcal{I}^{\text{sym}} = \frac{1}{2}(\mathcal{I} + \hat{\mathcal{I}}) = \frac{1}{2}\left[[\mathbf{I} \otimes \mathbf{I}]^{T_{23}} + [\mathbf{I} \otimes \mathbf{I}]^{T_{24}}\right]. \tag{45}$$

In Appendix A, particular products are provided, which are necessary for the calculation in the following. These products lead to

$$\begin{aligned}\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\mathbf{E}}\mathcal{I}^{\text{sym}} = {} & \Lambda \mathbf{I} \otimes \mathbf{I} + 2\mu_T\mathcal{I}^{\text{sym}} + \alpha[\mathbf{I} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}] + \beta\mathbf{M} \otimes \mathbf{M} \\ & + (\mu_L - \mu_T)\left[[\mathbf{I} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}]^{T_{23}} + [\mathbf{I} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}]^{T_{24}}\right],\end{aligned} \tag{46}$$

with $\mathbf{M} = \mathbf{M}^T$. Here, only the fourth-order identity tensor—second term—and the last term in Eq. (44) are influenced by the symmetrizer.

4.2 Hyperelasticity

In the following, the results of the previous investigations are applied to the case of the hyperelastic model, i.e. if we consider the direction $\mathbf{H}^{\text{sym}} = \mathcal{I}^{\text{sym}}\mathbf{H}$ (version 2), then

$$g(\mathbf{E}, \mathbf{H}) := \mathsf{D}_{\mathbf{E}}\, \psi(\mathbf{E})[\mathcal{I}^{\text{sym}}\mathbf{H}] = \frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{E}} \cdot \mathcal{I}^{\text{sym}}\mathbf{H} = \mathcal{I}^{\text{sym}T}\frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{E}} \cdot \mathbf{H} = \mathcal{I}^{\text{sym}}\frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{E}} \cdot \mathbf{H} \tag{47}$$

is obtained, since $\mathcal{I}^{\text{sym}} = \mathcal{I}^{\text{sym}T}$. Version 1 is derived in Appendix B. The second derivative (differential), see Eq. (21), reads

$$\mathsf{D}_{\mathbf{E}}\, g(\mathbf{E}, \mathbf{H})[\mathcal{I}^{\text{sym}}\mathbf{H}] = \mathcal{I}^{\text{sym}T}\mathsf{D}_{\mathbf{E}}\,\frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{E}}[\mathcal{I}^{\text{sym}}\mathbf{H}] \cdot \mathbf{H} = \left[\mathcal{I}^{\text{sym}T}\frac{\mathrm{d}^2\psi}{\mathrm{d}\mathbf{E}\mathrm{d}\mathbf{E}}\mathcal{I}^{\text{sym}}\right]\mathbf{H} \cdot \mathbf{H}, \tag{48}$$

i.e. the elasticity tensors can be written as

$$\mathcal{C} = \mathcal{I}^{\mathrm{sym}T} \frac{\mathrm{d}^2\psi}{\mathrm{d}\mathbf{E}\mathrm{d}\mathbf{E}} \mathcal{I}^{\mathrm{sym}}, \tag{49}$$

(where the transposition of the symmetrizer can be omitted). In index notation, we have

$$C_{ijkl} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{mj}) \frac{\partial^2\psi}{\partial E_{mn}\partial E_{op}} \frac{1}{2}(\delta_{ok}\delta_{pl} + \delta_{ol}\delta_{kp}). \tag{50}$$

The application of the procedure to the classical strain-energy function

$$\psi(\mathbf{E}) = \frac{\lambda}{2}(\mathbf{E}\cdot\mathbf{I})^2 + \mu\mathbf{E}\cdot\mathbf{E} \tag{51}$$

is as follows. We take the first

$$\frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{E}} = \lambda(\mathbf{E}\cdot\mathbf{I})\mathbf{I} + 2\mu\mathbf{E} \tag{52}$$

and the second derivatives

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\mathbf{E}\mathrm{d}\mathbf{E}} = \lambda(\mathbf{I}\otimes\mathbf{I}) + 2\mu\mathcal{I}, \tag{53}$$

respectively, and multiply with $\mathcal{I}^{\mathrm{sym}}$ from the left-hand and the right-hand sides. The first tensor in Eq. (53) remains unaffected (since it already exhibits both minor and major symmetries), and for the second one, we obtain

$$\mathcal{I}^{\mathrm{sym}}\mathcal{I}\mathcal{I}^{\mathrm{sym}} = \mathcal{I}^{\mathrm{sym}}\mathcal{I}^{\mathrm{sym}} = \frac{1}{4}\big[\mathcal{I}+\hat{\mathcal{I}}\big]\big[\mathcal{I}+\hat{\mathcal{I}}\big] = \frac{1}{4}\big[\mathcal{I}+\hat{\mathcal{I}}+\hat{\mathcal{I}}+\underbrace{\hat{\mathcal{I}}\hat{\mathcal{I}}}_{\mathcal{I}}\big] = \mathcal{I}^{\mathrm{sym}}. \tag{54}$$

The proof of $\hat{\mathcal{I}}\hat{\mathcal{I}} = \mathcal{I}$ is provided in Appendix A. Thus, we obtain

$$\mathcal{C} = \lambda\mathbf{I}\otimes\mathbf{I} + 2\mu\mathcal{I}^{\mathrm{sym}} = \lambda\mathbf{I}\otimes\mathbf{I} + \mu\big[\mathcal{I}+\hat{\mathcal{I}}\big]. \tag{55}$$

5 Further remarks

Equation (28) can be reformulated using $\big[\mathbf{A}\otimes\mathbf{B}\big]\mathbf{C} = (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$, i.e. $(\mathbf{I}\cdot\mathbf{E})\mathbf{I} = \big[\mathbf{I}\otimes\mathbf{I}\big]\mathbf{E}$, and $\mathcal{I}\mathbf{E} = \mathbf{E}$,

$$\mathbf{h}(\mathbf{E}) = \lambda(\mathbf{I}\cdot\mathbf{E})\mathbf{I} + 2\mu\mathbf{E} = \big[\lambda\mathbf{I}\otimes\mathbf{I} + 2\mu\mathcal{I}\big]\mathbf{E}. \tag{56}$$

Since $\mathbf{E} = \mathbf{E}^T$ holds, no principal mistake has been done. However, the right minor symmetry $C_{ijkl} = C_{ijlk}$ is not fulfilled, see Eq. (32). Here, it is common to state the symmetry condition as an assumption without the loss of generality, [20,22]. Even from a pure component derivation, see Appendix C, one can only derive representation (73), i.e. an additional assumption has to be considered. However, if we carry out a differentiation process incorporating the symmetrizer for both the Cauchy elasticity and hyperelasticity, the right minor symmetry condition is fulfilled, and one obtains Eq. (74).

Another approach can be found in [3]. They first define what right minor symmetry means, [3, p. 100]. This symmetry exists if the statement $\mathcal{C}\mathbf{A} = \mathcal{C}\mathbf{A}^T$ is fulfilled for any tensor $\mathbf{A} \in \mathbb{L}(\mathbb{V}^3)$,

$$\mathcal{C}\mathbf{A} - \mathcal{C}\mathbf{A}^T = (C_{ijkl}A_{kl} - C_{ijkl}A_{lk})\vec{e}_i\otimes\vec{e}_j = (C_{ijkl} - C_{ijlk})A_{kl}\vec{e}_i\otimes\vec{e}_j = \mathbf{0},$$

i.e. $C_{ijkl} = C_{ijlk}$. Later, [3, pp.167-169], the elasticity tensor is defined as a mapping of the displacement gradient $\mathbf{A} := \operatorname{grad}\vec{u}$, implying that the skew-symmetric part $skw\,\mathbf{A} := (\mathbf{A}-\mathbf{A}^T)/2$ has to vanish, $\mathcal{C}\,skw\,\mathbf{A} = \mathbf{0}$. However, if one assumes that the elasticity tensor is a mapping of the symmetric strain tensor, a further assumption has to be made, [3, p. 100].

For example, in [9], the derivative of the components of a symmetric tensor $\mathbf{A} = \mathbf{A}^T$ with respect to the components is assumed to be

$$\frac{\partial A_{ij}}{\partial A_{kl}} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

which is similar to Eq. (39). The correct transition from tensor to matrix notation, however, does not require such assumptions, see procedure in Appendix C.

6 Conclusions

The symmetry conditions, which include minor and major symmetries in the context of linear elasticity, have been an ongoing subject of research. This article attempts to address these questions by examining the various reasons associated with them. The reason for this is that in part a mixture of statements in index representation and representation as linear mappings are given, and on the other hand, statements of Cauchy elasticity are mixed with results of hyperelasticity. In addition, there is a loss of information in differentiation processes based on directional derivatives (sometimes also referred to as non-uniqueness).

In the context of hyperelasticity, the statement of major symmetry, $C_{ijkl} = C_{klij}$, is valid regardless of whether a general series expansion or a Taylor series is used. The left minor symmetry $C_{ijkl} = C_{jikl}$ can mathematically be proven by the symmetry of the stress tensor. The right minor symmetry $C_{ijkl} = C_{ijlk}$ does not come directly (on the basis of the fourth-order representation) but can be proven if major symmetry is given as well. A proof that Cauchy elasticity and hyperelasticity based on a Taylor expansion have a right minor symmetry can only be provided when using a directional derivative that considers the symmetry of the argument or the direction in a specific manner.

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Appendix A. Fourth-order tensor products

In Sect. 3, the following products are required:

$$[\mathbf{A}\otimes\mathbf{B}]\,[\mathbf{C}\otimes\mathbf{D}]^{T_{24}}\,, \quad [\mathbf{A}\otimes\mathbf{B}]^{T_{24}}\,[\mathbf{C}\otimes\mathbf{D}], \quad [\mathbf{A}\otimes\mathbf{B}]^{T_{23}}\,[\mathbf{C}\otimes\mathbf{D}]^{T_{24}}\,, \quad [\mathbf{A}\otimes\mathbf{B}]^{T_{24}}\,[\mathbf{C}\otimes\mathbf{D}]^{T_{23}}\,. \tag{57}$$

These expressions are derived using the properties $[\mathcal{A}\mathcal{B}]\mathbf{C} = \mathcal{A}(\mathcal{B}\mathbf{C})$, and

$$[\mathbf{A}\otimes\mathbf{B}]^{T_{24}}\,\mathbf{C} = \mathbf{A}\mathbf{C}^T\mathbf{B} \tag{58}$$

$$[\mathbf{A}\otimes\mathbf{B}]^{T_{23}}\,\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^T \tag{59}$$

$$[\mathbf{A}\otimes\mathbf{B}]\mathbf{C} = (\mathbf{B}\cdot\mathbf{C})\mathbf{A} \tag{60}$$

for the first expression,

$$\begin{aligned}[\mathbf{A}\otimes\mathbf{B}]\,[\mathbf{C}\otimes\mathbf{D}]^{T_{24}}\,\mathbf{E} &= [\mathbf{A}\otimes\mathbf{B}]\mathbf{C}\mathbf{E}^T\mathbf{D} = (\mathbf{B}\cdot\mathbf{C}\mathbf{E}^T\mathbf{D})\mathbf{A}\\ &= (\mathbf{C}^T\mathbf{B}\mathbf{D}^T\cdot\mathbf{E}^T)\mathbf{A} = (\mathbf{D}\mathbf{B}^T\mathbf{C}\cdot\mathbf{E})\mathbf{A} = [\mathbf{A}\otimes\mathbf{D}\mathbf{B}^T\mathbf{C}]\mathbf{E}\end{aligned}$$

where Eq. (60) and $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}^T\cdot\mathbf{B}^T$ are exploited.
For the second term in Eq. (57), we have

$$[\mathbf{A}\otimes\mathbf{B}]^{T_{24}}\,[\mathbf{C}\otimes\mathbf{D}]\mathbf{E} = (\mathbf{D}\cdot\mathbf{E})\,[\mathbf{A}\otimes\mathbf{B}]^{T_{24}}\,\mathbf{C} = (\mathbf{D}\cdot\mathbf{E})\mathbf{A}\mathbf{C}^T\mathbf{B} = [\mathbf{A}\mathbf{C}^T\mathbf{B}\otimes\mathbf{D}]\mathbf{E}.$$

The product $(57)_3$ yields

$$[\mathbf{A}\otimes\mathbf{B}]^{T_{23}}\,[\mathbf{C}\otimes\mathbf{D}]^{T_{24}}\,\mathbf{E} = [\mathbf{A}\otimes\mathbf{B}]^{T_{23}}\,\mathbf{C}\mathbf{E}^T\mathbf{D} = \mathbf{A}\mathbf{C}\mathbf{E}^T\mathbf{D}\mathbf{B}^T = \left[\mathbf{A}\mathbf{C}\otimes\mathbf{D}\mathbf{B}^T\right]^{T_{24}}\mathbf{E}$$

and $(57)_4$ leads to

$$[\mathbf{A}\otimes\mathbf{B}]^{T_{24}}\,[\mathbf{C}\otimes\mathbf{D}]^{T_{23}}\,\mathbf{E} = [\mathbf{A}\otimes\mathbf{B}]^{T_{24}}\,\mathbf{C}\mathbf{E}\mathbf{D}^T = \mathbf{A}\mathbf{D}\mathbf{E}^T\mathbf{D}^T\mathbf{B} = \left[\mathbf{A}\mathbf{D}\otimes\mathbf{C}^T\mathbf{B}\right]^{T_{24}}\mathbf{E}.$$

In conclusion, the following products are equivalent:

$$[\mathbf{A}\otimes\mathbf{B}]\,[\mathbf{C}\otimes\mathbf{D}]^{T_{24}} = \mathbf{A}\otimes\mathbf{D}\mathbf{B}^T\mathbf{C} \tag{61}$$

$$[\mathbf{A}\otimes\mathbf{B}]^{T_{24}}\,[\mathbf{C}\otimes\mathbf{D}] = \mathbf{A}\mathbf{C}^T\mathbf{B}\otimes\mathbf{D} \tag{62}$$

$$[\mathbf{A}\otimes\mathbf{B}]^{T_{23}}\,[\mathbf{C}\otimes\mathbf{D}]^{T_{24}} = \left[\mathbf{A}\mathbf{C}\otimes\mathbf{D}\mathbf{B}^T\right]^{T_{24}} \tag{63}$$

$$[\mathbf{A}\otimes\mathbf{B}]^{T_{24}}\,[\mathbf{C}\otimes\mathbf{D}]^{T_{23}} = \left[\mathbf{A}\mathbf{D}\otimes\mathbf{C}^T\mathbf{B}\right]^{T_{24}} \tag{64}$$

Finally, for Eqns. (58) and (59), we obtain

$$\begin{aligned}\left[\mathbf{A}\otimes\mathbf{B}\right]^{T_{24}}\left(\left[\mathbf{C}\otimes\mathbf{D}\right]^{T_{24}}\mathbf{E}\right) &= \left[\mathbf{A}\otimes\mathbf{B}\right]^{T_{24}}(\mathbf{C}\mathbf{E}^T\mathbf{D})\\ &= \mathbf{A}\mathbf{D}^T\mathbf{E}\mathbf{C}^T\mathbf{B} = \left[\mathbf{A}\mathbf{D}^T\otimes\mathbf{B}^T\mathbf{C}\right]^{T_{23}}\mathbf{E}.\end{aligned} \tag{65}$$

Thus, we have the relation

$$\left[\mathbf{A}\otimes\mathbf{B}\right]^{T_{24}}\left[\mathbf{C}\otimes\mathbf{D}\right]^{T_{24}} = \left[\mathbf{A}\mathbf{D}^T\otimes\mathbf{B}^T\mathbf{C}\right]^{T_{23}}. \tag{66}$$

If we apply this to pure identity tensors, it results

$$\hat{\mathcal{I}}\hat{\mathcal{I}} = \left[\mathbf{I}\otimes\mathbf{I}\right]^{T_{24}}\left[\mathbf{I}\otimes\mathbf{I}\right]^{T_{24}} = \left[\mathbf{I}\otimes\mathbf{I}\right]^{T_{23}} = \mathcal{I}. \tag{67}$$

The proof can be carried out in index notation as well.

Appendix B. Derivative of strain-energy function with respect to symmetric tensor

Regarding Eq. (47), we assume here $\psi(\mathbf{E}^{\mathrm{sym}}) = \psi(\mathbf{E})$, yielding the directional derivative

$$\begin{aligned}g(\mathbf{E}^{\mathrm{sym}},\mathbf{H}) &:= \mathsf{D}_{\mathbf{E}}\,\psi(\mathbf{E})[\mathbf{H}]\\ &= \mathsf{D}_{\mathbf{E}^{\mathrm{sym}}}\,\psi(\mathbf{E}^{\mathrm{sym}})[\mathsf{D}_{\mathbf{E}}\,\mathbf{E}^{\mathrm{sym}}(\mathbf{E})[\mathbf{H}]] = \frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{E}^{\mathrm{sym}}}\cdot\mathcal{I}^{\mathrm{sym}}\mathbf{H} = \mathcal{I}^{\mathrm{sym}\,T}\frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{E}^{\mathrm{sym}}}\cdot\mathbf{H}\end{aligned} \tag{68}$$

The second derivative (differential) reads

$$\begin{aligned}\mathsf{D}_{\mathbf{E}}\,g(\mathbf{E}^{\mathrm{sym}},\mathbf{H})[\mathbf{H}] &= \mathsf{D}_{\mathbf{E}^{\mathrm{sym}}}\,g(\mathbf{E}^{\mathrm{sym}})[\mathsf{D}_{\mathbf{E}}\,\mathbf{E}^{\mathrm{sym}}(\mathbf{E})[\mathbf{H}]]\\ &= \left[\mathcal{I}^{\mathrm{sym}\,T}\frac{\mathrm{d}^2\psi}{\mathrm{d}\mathbf{E}^{\mathrm{sym}}\mathrm{d}\mathbf{E}^{\mathrm{sym}}}\mathcal{I}^{\mathrm{sym}}\right]\mathbf{H}\cdot\mathbf{H},\end{aligned} \tag{69}$$

i.e. the elasticity tensor is given by

$$\mathcal{C} = \mathcal{I}^{\mathrm{sym}\,T}\frac{\mathrm{d}^2\psi}{\mathrm{d}\mathbf{E}^{\mathrm{sym}}\mathrm{d}\mathbf{E}^{\mathrm{sym}}}\mathcal{I}^{\mathrm{sym}}. \tag{70}$$

Appendix C. Reduction of fourth-order tensor to Voigt notation

Since the minor and major symmetry conditions are related to the Voigt notation of matrices, we recall some basics: first, we sort all entries of the second- and fourth-order tensors into column vectors and a matrix, respectively:

$$\begin{Bmatrix} T_{11}\\ T_{22}\\ T_{33}\\ T_{12}\\ T_{23}\\ T_{31}\\ T_{13}\\ T_{21}\\ T_{32} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} & C_{1113} & C_{1121} & C_{1132}\\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} & C_{2213} & C_{2221} & C_{2232}\\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} & C_{3313} & C_{3321} & C_{3332}\\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} & C_{1213} & C_{1221} & C_{1232}\\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} & C_{2313} & C_{2321} & C_{2332}\\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131} & C_{3113} & C_{3121} & C_{3132}\\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1323} & C_{1331} & C_{1313} & C_{1321} & C_{1332}\\ C_{2111} & C_{2122} & C_{2133} & C_{2112} & C_{2123} & C_{2131} & C_{2113} & C_{2121} & C_{2132}\\ C_{3211} & C_{3222} & C_{3233} & C_{3212} & C_{3223} & C_{3231} & C_{3213} & C_{3221} & C_{3232} \end{bmatrix} \begin{Bmatrix} E_{11}\\ E_{22}\\ E_{33}\\ E_{12}\\ E_{23}\\ E_{31}\\ E_{13}\\ E_{21}\\ E_{32} \end{Bmatrix}. \tag{71}$$

Using Eq. (8), we arrive at

$$\begin{Bmatrix} T_{11}\\ T_{22}\\ T_{33}\\ T_{12}\\ T_{23}\\ T_{31} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} & C_{1113} & C_{1121} & C_{1132}\\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} & C_{2213} & C_{2221} & C_{2232}\\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} & C_{3313} & C_{3321} & C_{3332}\\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} & C_{1213} & C_{1221} & C_{1232}\\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} & C_{2313} & C_{2321} & C_{2332}\\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131} & C_{3113} & C_{3121} & C_{3132} \end{bmatrix} \begin{Bmatrix} E_{11}\\ E_{22}\\ E_{33}\\ E_{12}\\ E_{23}\\ E_{31}\\ E_{13}\\ E_{21}\\ E_{32} \end{Bmatrix} \tag{72}$$

If we consider the symmetry of the strain tensor, $\mathbf{E} = \mathbf{E}^T$, i.e. $E_{ij} = E_{ji}$, we arrive at

$$\begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{31} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} + C_{1121} & C_{1123} + C_{1132} & C_{1131} + C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} + C_{2221} & C_{2223} + C_{2232} & C_{2231} + C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} + C_{3321} & C_{3323} + C_{3332} & C_{3331} + C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} + C_{1221} & C_{1223} + C_{1232} & C_{1231} + C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} + C_{2321} & C_{2323} + C_{2332} & C_{2331} + C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3112} + C_{3121} & C_{3123} + C_{3132} & C_{3131} + C_{3113} \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ E_{12} \\ E_{23} \\ E_{31} \end{Bmatrix} \qquad (73)$$

by evaluating the sums in Eq. (72). Thus, the symmetry of the strain tensor does not automatically yield an elasticity matrix containing components with right minor symmetry. In [6, p. 51] it is stated "Since $E_{kl} = E_{lk}$, it is at least possible to choose $C_{ijkl} = C_{ijlk}$, assigning to each of the two equal strain components half of the influence which they exert together on T_{ij}". Thus, an additional assumption is introduced.

If the right minor symmetry is present or assumed, $C_{ijkl} = C_{ijlk}$, the following representation results

$$\begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{31} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131} \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{Bmatrix}, \qquad (74)$$

i.e. we have a maximum of 36 independent entries in the elasticity matrix. In the case of hyperelasticity, major symmetry, $C_{ijkl} = C_{klij}$ leads to a symmetric elasticity matrix which contains at most 21 different components.

We will now return to the fact that a correct conversion of the (9×9)-representation to the (6×6) matrix representation does not lead to any problems. If we insert the tensor components of the fourth-order identity tensor $\mathcal{I}$, the (9×9)-representation reads

$$\mathcal{I} \quad \rightarrow \quad \begin{bmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ &&& 1 &&&&& \\ &&&& 1 &&&& \\ &&&&& 1 &&& \\ &&&&&& 1 && \\ &&&&&&& 1 & \\ &&&&&&&& 1 \end{bmatrix} \qquad (75)$$

and for the symmetrizer

$$\mathcal{I}^{\text{sym}} = \frac{1}{2}(\mathcal{I} + \hat{\mathcal{I}}) \quad \rightarrow \quad \begin{bmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ &&& \frac{1}{2} &&&& \frac{1}{2} & \\ &&&& \frac{1}{2} &&&& \frac{1}{2} \\ &&&&& \frac{1}{2} & \frac{1}{2} && \\ &&&&& \frac{1}{2} & \frac{1}{2} && \\ &&& \frac{1}{2} &&&& \frac{1}{2} & \\ &&&& \frac{1}{2} &&&& \frac{1}{2} \end{bmatrix}. \qquad (76)$$

If these matrices are applied to a symmetric tensor, see Eq. (73), we obtain in both cases the (6×6)-representation

$$\mathcal{I} \quad \text{or} \quad \mathcal{I}^{\text{sym}} = \frac{1}{2}(\mathcal{I} + \hat{\mathcal{I}}) \quad \rightarrow \quad \begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{bmatrix}. \qquad (77)$$

Considering an isotropic linear elastic material, we have

$$\begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{31} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda &&& \\ \lambda & \lambda + 2\mu & \lambda &&& \\ \lambda & \lambda & \lambda + 2\mu &&& \\ &&& \mu && \\ &&&& \mu & \\ &&&&& \mu \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{Bmatrix}, \qquad (78)$$

see Eq. (55).

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