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ON SHIP MANEUVERABILITY AND COLLISION AVOIDANCE[1]

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Summary

The paper describes some work conducted within a subproject of the newly formed Special Research Pool for Ship Technology (SFB 98) at Hamburg and Hanover. Motivated by the increasing need of assessing the economic costs and benefits of design features pertaining to ship safety, the author's team has developed an analytical model for computing collision rates by pitting collision-avoidance capability versus demand in a statistically characterized traffic environment. The model accounts for both deterministic and random aspects of capability and demand. Specially the deterministic aspects of collision avoidance making optimal use of maneuvering capability are evaluated in detail using Isaacs' theory of differential games. Here ships are modelled as circular discs moving with constant speeds and freely controllable but bounded turning rates. Erratic use of maneuverability owing to misjudgement is modelled by solving nine different games ranging from optimal collaborative evasion to optimal collaborative rendezvous between two ships. Possible generalizations of the model are discussed.

[1]This work was done within the framework of the *Sonderforschungsbereich* 98 *"Schiffstechnik und Schiffbau"* at the *Institut für Schiffbau,* Hamburg, with the financial support of the *Deutsche Forschungsgemeinschaft.*

1. MOTIVATION

The purpose of this paper is to present some results (mainly by-products) of an attempt to relate the maneuvering qualities of a ship with its capacity to avoid collisions at sea. Before coming to the subject proper it will help to devote a few thoughts to the motivation for this investigation. During the past ten to twenty years naval architects have begun to apply the so-called systems approach to ship design. The essence of systems approach is to try to drop as many arbitrary (empirical or intuitive) constraints as possible in the interests of optimizing a single, comprehensive, ultimate utility measure (usually some economic criterion). This development seems to be part of a current fundamental trend in the art of selling technology. Apparently, it suffices no more to accomplish impressive technological feats. The engineer is being increasingly required to justify his actions in terms of benefit to society. But how do we measure such benefit? Despite certain obvious faults the most workable measure available today is still the profitability of an enterprise. The classical argument is that under ideal circumstances in a free-market economy the price of goods or services should be the money equivalent of the gross benefit derived by the society in its role as consumer, while the cost should reflect the resources expended by society in its role as producer. Hence the difference, viz. profit should ideally be a measure of net benefit accrued from the activity. If we accept this argument, the naval architect should strive to relate as many technical features of the ship as possible to the economics of ship operation and produce a design to maximize profitability. This is indeed already being done, at least with respect to those features whose relation to income and expenditure is easy to establish. Thus it is becoming common practice to choose ship speed and size (even main dimensions) on the basis of a formal quantitative optimization (as distinguished from a vague verbal claim) of an appropriate economic criterion of merit. An excellent up-to-date review of the state of the art has recently been given by Benford (1975).

However, there exist important design features, especially those connected with ship safety, whose costs are all too obvious but benefits are annoyingly difficult to quantify in economic terms. It is therefore not surprising that they are in practice still being treated as fixed constraints (either imposed by official rules and regulations or set judiciously by the designer himself) rather than as independent variables in an optimization procedure. Of course, the first step in a rational approach to ship safety is for the designer to stop pretending that his ship is as safe as technologically feasible. The next step is to try to quantify the safety of a proposed design with respect to various relevant risks. At least conceptually, this can be accomplished by defining safety as the probability of nonoccurrence of failure, although the practical difficulties involved in the numerical evaluation of such probabilities are often formidable. The final and most difficult step is to establish rational standards of safety, i.e. to determine either an optimum or at least a socially acceptable level of safety. For a further discussion of these general problems see for instance Krappinger and Sharma (1974). Notwithstanding all aforesaid difficulties some brave attempts have been made to incorporate the economic (and even ethical) benefits of ship safety features explicitly and quantitatively in objective design procedures. In a pioneering paper Abrahamsen (1962) tried to optimize the structural design of a ship by minimizing the total mortality resulting from accidents in all activities associated with the building and operation of a ship! More recently, the relation between the seakeeping qualities of a ship and its safety against structural failure has been examined by Mansour (1972) among others. In a similar vein the newly formed special research pool for ship technology (*Sonderforschungsbereich* 98) at Hamburg and Hanover has undertaken to generate conceptual models for assessing the costs and benefits of various measures for improving ship safety with a view to developing a more rational design procedure. In particular, the author's team has been trying to relate the maneuvering qualities of a ship to its safety against collisions following a suggestion by Krappinger (1972) to use the mean collision rate as a methodical link between technical features and economic evaluation.

2. AN ANALYTICAL MODEL FOR COMPUTING COLLISION RATES

2.1. Concept

The expected average number of collisions per unit time under suitably characterized operational conditions is called the collision rate. It is an essentially probabilistic concept quantifying the collision risk arising from some degree of randomness inherent both in the traffic environment and in the response of any single ship. Combined with an estimate of the damage resulting from collisions, it could be used to assess the expected economic costs of collisions and hence to evaluate the economic benefits of measures designed to enhance the safety of ships against collisions. We shall now construct in a few simple steps following Krappinger (1973) an analytical model for computing the collision rate of a given ship (called here own ship O) moving through a statistically characterized traffic pattern.

2.2. Encounter rate

Consider first a single encounter of own ship O moving at speed V_o and course θ_o with another ship A moving at speed V_a and course θ_a (see Fig. 1) so that the relative heading angle and relative speed may be written as

$$\theta = \theta_a - \theta_o \tag{1}$$

$$V_r = \sqrt{V_o^2 + V_a^2 - 2V_oV_a\cos\theta} \tag{2}$$

respectively. In a typical innocuous encounter both ships will maintain course and speed so that the range r (measured center to center) will steadily decrease up to the point of closest approach (conventionally abbreviated CPA, although PCA would be more to the point) and then increase. For any initial state defined by the range r and bearing α of A relative to O the CPA can easily be predicted from the kinematics of the encounter, as indicated by the geometrical construction of points A_f, O_f in Fig. 1. Let us denote the anticipated range at CPA (usually called the miss distance) by r_{fNN} and the relative bearing at CPA by α_{fNN}, where the subscript f stands for *final* in the sense that the approach phase of the encounter terminates at CPA. The additional subscripts NN indicate that the quantities have been predicted on the premise that neither 0 nor A shall maneuver before CPA (see also explanation of symbols at the end of the paper). Now visualize own ship 0 as crossing a stream of traffic comprising other ships A_1, A_2, etc. all moving with the same velocity (V_a, θ_a) but randomly distributed with a uniform density of ρ ships per unit area as indicated schematically in Fig. 2. We wish to count all encounters of 0 with anticipated miss distances r_{fNN} less than some specified value r_m. The expected average number of such encounters per unit time, the so-called encounter rate, is evidently

$$\lambda = 2\rho r_m \, V_r(V_o,V_a,\theta) \tag{3}$$

as can be seen by imagining the other ships to be stationary and the own ship as sweeping a sea area of $2r_mV_r$ per unit time. From here we can easily generalize to a traffic pattern comprising all possible speeds and course angles if only it can be characterized by a bivariate frequency function $f(V_a,\theta_a)$ expressing the relative frequency of occurrence of each velocity. The encounter rate for 0 is then the weighted integral:

$$\lambda = 2\rho r_m \int_0^{2\pi} \int_0^{\infty} V_r(V_o,V_a,\theta)\, f(V_a,\theta_a)\, dV_a d\theta_a \tag{4}$$

2.3. Collision rate

Now suppose that r_m is a threshold value so chosen that an anticipated miss distance $r_{fNN} < r_m$ would constitute a threat of collision thereby causing either 0 or A to execute an evasive maneuver at some range r_e. In general, hopefully, the evasive maneuver will lead to a safe passing, but we are interested in just those few events in which for some reason or another the evasive maneuver fails and collision occurs. If we could assign to each encounter characterized by given (V_o,V_a,θ,r_{fNN}) a probability of failure $P\{\text{fail}\}$, then the collision rate would follow as the weighted integral:

$$\lambda_c = \rho \int_{-r_m}^{r_m} \int_0^{2\pi} \int_0^{\infty} V_r(V_o,V_a,\theta)\, f(V_a,\theta_a)\, P\{\text{fail}\}\, dV_a d\theta_a dr_{fNN} \tag{5}$$

The question now is how to estimate the probability $P\{\text{fail}\}$? Instead of trying to cover all the multifarious possible modes of failure we focus our attention - in accordance with our primary purpose - on collisions related to maneuverability. For this we invoke the principle of pitting capability versus demand, a standard technique in reliability theory. We note parenthetically that its first successful application in naval architecture was by Wendel (1960) for calculating the probability of surviving collision damage! In our case demand may be represented by range at evading r_e and the capability by some critical range r_c in the approach phase after which the evasive maneuver is doomed to failure.

That on account of the inertia of the ships and their finite maneuverability such critical ranges must exist is intuitively obvious. This is illustrated schematically in Fig. 2 where own ship O may now be visualized as trying to keep other ships A (coming with constant V_a, θ_a but on random tracks) out of a terminal circle of radius r_m. Clearly if the other ship is detected outside a band formed by tangents to the terminal circle (from B and C) parallel to V_r, i.e. on a track such as A_2, then there is no need to maneuver. The inside of the band is the *risk* zone. Here we may expect limiting curves BD and CE such that if A is still on the far side (say at A_1) at the moment of evasion it can be kept out of the terminal circle by a hard right or left turn respectively. But if A has already reached a position such as A_3 it is too late to keep it out of the terminal circle. The contour BFC formed by the critical range r_c as a function of bearing α thus encloses what may be called the *collision* zone around O (for given V_a, θ) in the sense that for initial states of A inside this zone a miss distance greater than r_m cannot be achieved by O. If O and A were circular discs and r_m equal to the sum of their radii, a miss distance $r_f < r_m$ would correspond to a physical collision.

In reality, of course, both r_e and r_c will have deterministic as well as random aspects. The latter may arise due to instrumental or observational inaccuracies, ambiguities in the Rules of the Nautical Road, inconsistent or even erratic decision-making by mariners, equipment failure, environmental factors such as current and wind, etc. Without going into details let us assume that the random aspects can be expressed symbolically by independent frequency functions $f_e(r_e)$ and $f_c(r_c)$ with the deterministic aspects entering through implicit parameters. Then by standard rules for combining independent probabilities we get

$$P\{\text{fail}\} = \int_0^\infty f_e(r_e) \int_{r_e}^\infty f_c(r_c)\, dr_c\, dr_e \tag{6}$$

Note that smaller r_e implies more stringent demand and smaller r_c implies higher capability so that failure (i.e. excess of demand over capability) corresponds to the case $r_e < r_c$. Substitution of Eq. (6) in Eq. (5) completes in principle our analytical model for computing collision rates.

2.4. Required input

Admittedly our model is based on several simplifying assumptions. In order to test its practical usefulness the unknown functions $f(V_a,\theta_a)$, $f_e(r_e)$ and $f_c(r_c)$ must be estimated, and numerical examples must be worked and compared with actuarial collision rates derived from statistics of past collisions. We have not yet reached that stage but following is an indication of our efforts in this direction. For lack of suitable published data we have started conducting in collaboration with the German Association of Shipowners (*Verband Deutscher Reeder*) surveys of traffic density ρ and distribution $f(V_a,\theta_a)$ on typical sea routes. Details of the method and first results have been published in two papers by Kwik and Stecher (1976). Another member of our team has compiled data on $f_e(r_e)$ by interrogation and radar-simulator testing of mariners, cf. Limbach (1977). As regards the distribution of collision avoidance capability $f_c(r_c)$, we have focussed our attention primarily on the deterministic aspects of the problem, i.e. on the objective determination of critical ranges r_c and the associated critical maneuvers, cf. Miloh (1974), and Miloh and Sharma (1975a,b). The rest of this paper will be devoted mainly to a presentation and discussion of some results of this last effort.

3. A METHOD FOR DETERMINING COLLISION ZONES

3.1. Foreword

We shall now describe an analytical method for determining the objective collision-avoidance capability (of two ships involved in a last-minute situation) as a function of encounter geometry and ship dynamics assuming mathematically optimal maneuvering. In order to be able to cover the full spectrum of real-life behavior, which is understandably not always mathematically optimal, we shall artificially ascribe to our ships such extreme alternative objectives as optimal evasion and optimal pursuit! The mathematical technique ideally suited to this problem is the theory of Differential Games, a brainchild of Isaacs (1965) sprung from an alliance of game theory with the calculus of variations. We shall qualitatively describe the basic concepts of this theory, formulate our problem as a differential game, outline the technique of solution and present numerical examples but omit the mathematical details of the analysis for which the reader is referred to Isaacs' book (specifically the *game of two cars*) or to our previous papers already cited.

3.2. Basic concepts of the theory of differential games

The essential requisite of a differential game is a dynamic system controlled by two (or more) players. State variables characterize any instantaneous state of the system. Its dynamic behavior is defined by kinematic equations relating the time rate of change of state variables to the state variables themselves and to control variables, which are at the volition of the players. Quantities which do not change with time during a particular play of the game, but which we might wish to vary from play to play, are called parameters of the game. The end of any play is marked by predefined terminal conditions of the system. The prize in the game is a payoff comprising a terminal component as a function of the terminal conditions and/or an integral component as a function of the path along which the terminal conditions are reached. Typically, the objective of one player is to maximize and of the other to minimize the payoff. A function relating a control variable to the state variables is called a strategy. Both players are assumed to have complete information on the current state of the system and the range of controls available to each. For solvable games there exist for both players mutually optimal equilibrium strategies, which assign to every starting state of the system a conceptually predetermined unique payoff called the value. The theoretical solution of the game comprises the optimal strategies, the optimal paths leading to the terminal conditions and the value as a function of the starting state of the system. A continuous payoff function yields a game of degree with a continuous value function in state space. A payoff function defined to have only discrete values yields a game of kind with the state space subdivided into zones of different value separated by barriers. If only one player is active or both follow a common objective the game degenerates to a conceptually simpler problem of optimal control of a continuous process.

3.3. Formulation of our problem as a differential game

Our dynamic system consists of two ships O and A maneuvering in open, undisturbed water. We choose the simplest plausible kinematic equations (involving three state variables, two control variables and four parameters) describing the apparent motion of A in a coordinate system attached to O:

$$dx/dt = V_a\cos\theta + y\phi V_o/R_o - V_o \qquad (7)$$

$$dy/dt = V_a\sin\theta - x\phi V_o/R_o \qquad (8)$$

$$d\theta/dt = \psi V_a/R_a - \phi V_o/R_o \qquad (9)$$

The state variables are the rectilinear coordinates x,y and the heading angle θ of A relative to O, see Fig. 2. The coordinates x,y transform into the usual range r and bearing α by

$$x = r\cos\alpha, \qquad y = r\sin\alpha \qquad (10)$$

The two control variables are the normalized rates of turn $-1 \leqslant \phi \leqslant 1$ and $-1 \leqslant \psi \leqslant 1$ of O and A respectively. The four parameters are the speeds V_o,V_a and minimum turning radii R_o,R_a of O and A respectively. Thus our kinematic model implies: 1) that ship positions and headings (x,y,θ) are continuous functions of time t, 2) that forward speeds V_o, V_a do not change during the maneuver, and 3) that the turning rates can be instantaneously changed within prescribed bounds $\pm V_o/R_o$, $\pm V_a/R_a$. At first sight the last two postulates might appear rather restrictive. But we recall that almost all previous nautical literature on anti-collision maneuvers - so comprehensively and competently reviewed by Kemp (1974) - allowed abrupt changes of heading (sic), the two notable exceptions being Vincent (1972) and Merz (1973) who both used kinematic models equivalent to ours.

We consider the resolution of any encounter as a play of the game and let it terminate with the closest approach at first pass defined kinematically by

$$dr/dt = 0, \qquad d^2r/dt^2 > 0, \tag{11}$$

for what happens after that is practically irrelevant to the collision avoidance problem. Our payoff is purely terminal and equal to the miss distance, i.e. range r_f at time t_f of closest approach. Each of our players O and A may be either neutral (i.e. standing on with $\phi,\psi = 0$) or active. In the latter event his objective may be either optimal evasion (i.e. to maximize r_f) or optimal pursuit (i.e. to minimize r_f). This yields nine versions of the game, conveniently abbreviated as O_EA_E, O_NA_E, O_PA_E, O_EA_N, O_NA_N, O_PA_N, O_EA_P, O_NA_P, and O_PA_P, with the subscripts E for evasion, N for neutrality and P for pursuit. We can thus encompass a wide class of problems stretching from the trivial game of both standing on (O_NA_N) across the degenerate but useful games of collaborative evasion (O_EA_E) and collaborative rendezvous (O_PA_P) up to the genuine game of optimal evasion in face of optimal pursuit (O_EA_P or O_PA_E). We shall use, wherever necessary, the subscripts E,N,P in the sequence O,A to identify a quantity with a particular version of the game. Thus the miss distance in game O_EA_E is denoted by r_{fEE}, in O_NA_E by r_{fNE}, and so on.

The strategies in our game are nothing but turn instructions for O and A as functions of system state and parameters, i.e. symbolically

$$\phi = \phi(x,y,\theta;V_o,V_a,R_o,R_a) \tag{12}$$

$$\psi = \psi(x,y,\theta;V_o,V_a,R_o,R_a) \tag{13}$$

Clearly, any given pair of strategies $\phi(\ ),\psi(\ )$ assigns via Eq. (7-11) to every starting state a certain miss distance (our payoff), i.e. symbolically

$$r_f = r_f(x,y,\theta;V_o,V_a,R_o,R_a;\phi(\ ),\psi(\ )) \tag{14}$$

The crux of the problem is to find a pair of optimal strategies $\bar{\phi}(\ ),\bar{\psi}(\ )$ of a game. For instance, the optimal strategies $\bar{\phi}_{EP}(\ ),\bar{\psi}_{EP}(\ )$ of game O_EA_P must fulfil the max-min condition for every permissible $\phi(\ ),\psi(\ )$:

$$r_f(x,,;V_o,,,;\phi(\ ),\bar{\psi}(\ ))<r_f(x,,;V_o,,,;\bar{\phi}(\ ),\bar{\psi}(\ ))<r_f(x,,;V_o,,,;\bar{\phi}(\ ),\psi(\ )) \tag{15}$$

(Here most of the function arguments and the subscripts EP on $\bar{\phi},\bar{\psi}$ have been omitted to avoid repetition.) Hence in a game with conflict of objectives the mutually optimal strategies are so defined that any unilateral departure from his optimal strategy by a player leads to a less satisfactory result for him if the adversary adheres to his optimal strategy! Therefore rational players are logically compelled to play their mutually optimal (also called equilibrium or saddle point) strategies. For degenerate games without a conflict of objectives the meaning of the term optimal should be self-evident.

The rest of the problem is conceptually predetermined in terms of optimal strategies. Thus optimal paths are time integrals of the kinematic equations (7-9) with $\bar{\phi}(\ ),\bar{\psi}(\ )$ substituted for ϕ,ψ. The game value at any starting state is just the terminal value r_f along the optimal path passing through that starting state.

3.4. Salient features of the solution

Lack of space prohibits a description of how the variational problem formulated above can be reduced to a set of differential equations and solved. However, certain intuitively graspable salient features of the solution deserve mention before passing on to a presentation of illustrative numerical examples.

First, it is characteristic of game theory that the solution makes use of the *retrogression* principle. It is not possible to solve for a given initial state in isolation. But a complete solution for a game can be derived by working backward in time from all permissible terminal conditions, thereby filling the state space with optimal paths and retrogressively assigning to every point along the path an optimal control and a game value. (Where different local solutions emerging retrogressively from different terminal points intersect in state space global considerations usually dictate a choice.) In our problem it can be shown that the terminal condition (11) is equivalent to the following relation between relative bearing and heading:

$$\sin\alpha_f = \varepsilon(V_o - V_a\cos\theta_f)/V_r, \quad \cos\alpha_f = \varepsilon V_a \sin\theta_f / V_r, \quad \varepsilon = \pm 1 \tag{16}$$

so that the set of all permissible terminal conditions:

$$x = r_f\cos\alpha_f, \quad y = r_f\sin\alpha_f, \quad \theta = \theta_f \tag{17}$$

is the two-parametric manifold $0 < r_f < \infty$, $0 < \theta_f < 2\pi$ in a three-dimensional state space (x, y, θ or equivalently r, α, θ).

Second, it is a fortunate (but not accidental) consequence of our kinematic equations (7-9) being linear in the control variables ϕ, ψ that optimal controls turn out to be piecewise constant and either (almost always) extremal ($\bar{\phi}, \bar{\psi} = \pm 1$) or (in some singular cases) neutral ($\bar{\phi}, \bar{\psi} = 0$), although we had postulated no restrictions on the variation of ϕ, ψ with time! In concrete terms, the mathematically optimal maneuvers generally turn out to be steady hard right or left turns with an occasional straight course, and possibly multi-step. This is the famous *bang-bang* principle of control theory. It has two rewards for us. One is that thanks to the piecewise constancy of $\bar{\phi}, \bar{\psi}$ our kinematic equations can be integrated analytically, thereby significantly reducing the labor involved in finding solutions, and the other is that the theoretical solutions are also rather practicable solutions. (It is a bit difficult to imagine how a solution requiring complicated variations of turning rate with time could be implemented, say in a practical collision avoidance system.)

Third, it is found that optimal *terminal* controls of O and A are mutually independent and in compliance with intuition. Explicitly, one can write

$$\bar{\phi}_f = \gamma \operatorname{sgn}(-\sin\alpha_f) \tag{18}$$

$$\bar{\psi}_f = \delta \operatorname{sgn}[\sin(\alpha_f - \theta_f)] \tag{19}$$

using the shorthand:

$$\gamma = 1,0,-1 \text{ for E,N,P by O}, \quad \delta = 1,0,-1 \text{ for E,N,P by A.} \tag{20}$$

A little inspection will reveal the following simple and intuitively appealing interpretation: Optimal maneuvers always terminate in such a way that at CPA the evader is turning away from/and the pursuer turning toward his target! Unfortunately, no such simple rule can be devised in terms of *initial* conditions which is really what is ultimately needed. Tracing optimal paths retrogressively from terminal conditions as outlined above then yields the optimal strategies, which as expected depend on initial state (x,y,θ), on game parameters (V_o,V_a,R_o,R_a) and on the objectives (γ,δ) ascribed to each of the two players. These multivariate optimal strategies are often counter-intuitive and devious, e.g. requiring the evader to turn initially toward the target in order to finally achieve the maximum possible miss distance! This is also why examples can easily be constructed to invalidate any simple *progressive* maneuvering rule purporting to be universal.

4. EXAMPLES OF COMPUTED COLLISION ZONES

4.1. Prelude

We have just seen that each of our nine games involves three state variables and four game parameters. Hence, although we had chosen the simplest plausible kinematic model of ship dynamics, even the mere graphical display of the multivariate solution is a problem. A complete presentation would require nine sets of charts covering in each set the full practical range of all seven independent variables. Obviously we shall have to be content with a few illustrative simple examples. It is expedient to use a nondimensional representation based on own ship speed V_o and minimum turning radius R_o as fundamental units. Then the solution can be displayed in a nondimensional state space $(x/R_o, y/R_o, \theta)$, effectively reducing the number of game parameters from four to two ($V_a/V_o, R_a/R_o$). As usual this has the disadvantage that the graphs do not directly convey a feeling for absolute numbers such as physical ranges and times. Here it will help to occasionally use the graphs to evaluate dimensional quantities for an illustrative own ship O of say 150 m length and V_o = 20 kn, R_o = 500 m, corresponding to a maximum turning rate of about 1.2°/s. Moreover, it will be convenient to work with retrograde time $\tau = t_f - t$ instead of forward time t. The retrograde time associated with any initial state is, of course, part of the solution of a game. Only τ_{NN} for the trivial game $O_N A_N$ is easy to predict; it represents in our notation what is ordinarily called time to CPA.

4.2. Optimal maneuvers and miss-distances

Fig. 3 shows the optimal strategies and miss distances in a plane $\theta = 180^o$ of our three-dimensional state space attached to O. The game being played is $O_E A_N$ with parameters $V_a/V_o = 1$ and R_a/R_o irrelevant since A is neutral. It is a very simple situation and optimal maneuvers fully conform to intuition. Ship A is on reciprocal course. So O evades optimally by turning hard left (symbol $O_L A_N$) or hard right (symbol $O_R A_N$) depending on whether A is initially in the quadrant $0^o < \alpha < 90^o$ or $270^o < \alpha < 360^o$. The optimal miss distance r_{fEN} achievable from any starting state in the x,y-plane can be read from the solid contours, while the amount of turn (and hence duration of maneuver up to CPA) can be read from the orthogonal dashed contours. Let us interpret the chart at an initial state (marked A in Fig. 3) for our illustrative ship. The current range r is 1.69× 500 = 845 m, the anticipated miss distance r_{fNN} = 0.17×500 = 85 m, and time to CPA $\tau_{NN} = x/(V_o+V_a)$ = 41 s. If O evades optimally by turning hard left, it can achieve a miss distance r_{fEN} = 0.5×500 = 250 m. The change of heading up to maneuver CPA will be −45° in time τ_{EN} = 45/1.2 = 38 s. For practical purposes a miss distance less than one ship length will be tantamount to collision, so for the illustrative ship the solid contour r_f/R_o = 0.3 can be regarded as a barrier bounding the collision zone. The following diagrams show how various factors influence the shape and size of the collision zone using $r_f/R_o = 0.3$ as a typical value for defining "collision".

4.3. Three-dimensional collision zone

As we have a three-dimensional state space (counting the relative heading θ as an abstract third dimension) we also have a three-dimensional collision zone around O. The most natural way to exhibit it on paper is to plot cross-sections θ = constant in the physical x,y-plane. This is illustrated in Fig. 4 for seven values of θ from 0^o to 180^o in steps of 30^o. By virtue of symmetry cross-sections for $180^o<\theta<360^o$ may be obtained by taking mirror images about the plane $y = 0$. Although not marked in the figure, each cross-section is here composed of two segments O_LA_N and O_RA_N meeting at a corner which in general lies neither along own heading ($\alpha = 0$) nor along the perfect collision track ($r_{fNN} = 0$), thus illustrating the point made earlier about optimal maneuvers being often non-intuitive. The shape and size of the cross-section vary considerably with θ. Specially the longest critical range $\max_\alpha r_c(\alpha,\theta)$ varies widely with θ, namely from 150 m at $\theta = 0^o$ to 785 m at $\theta = 180^o$ for our illustrative ship. But interestingly the corresponding time to CPA τ_{NN} is very nearly constant, varying only from 38 s at $\theta = 180^o$ to 44 s at $\theta = 90^o$. In other words, time to CPA is a pretty good measure of achievable miss distance and so possibly a more efficient way of calculating collision probabilities may be to match demand versus capability in terms of retrograde time rather than range as was done in Section 2.3. Incidentally, the envelope of all cross-sections θ = constant yields a two-dimensional figure akin to what Kenan (1972) and Webster (1974) have previously called the "critical zone". It is also an objective counterpart of the subjective "domain" that the master would *like* to keep clear of other ships, cf. Goodwin (1975).

4.4. Influence of speed

The parameter V_a/V_o has a substantial effect on the shape and size of the collision zone as illustrated in Fig. 5 for seven values of the speed ratio. We are still playing game O_EA_N and only the limiting cross-sections $\theta = 0^o$ and 180^o are shown for the sake of simplicity. As expected the collision zone lies ahead of 0 in the meeting situation ($\theta = 180^o$) and when 0 is overtaking ($V_a/V_o < 1$), but abaft of 0 when 0 is being overtaken ($V_a/V_o > 1$). It is instructive to tabulate the critical ranges and retrograde times along the perfect collision track ($\alpha = 0$, $\theta = 180^o$) for our illustrative ship:

V_a/V_o	0	0.25	0.5	1	1.5	2	2.5	3
r_c [m]	415	505	595	780	975	1170	1370	1565
τ_{NN} [s]	40	39	39	38	38	38	38	38

It is striking that the critical time to CPA needed to achieve a specified miss distance by optimal maneuver is almost independent of speed ratio whereas the critical range is very sensitive. This suggests the wisdom of using a fixed time to CPA rather than a fixed range as an alarm threshold in automatic radar-driven collision warning systems. An even more precise measure of threat, albeit much more difficult to compute, would be the achievable miss disstance itself, as proposed recently also by Ciletti and Merz (1976).

4.5. Different versions of the game

The collision zone depends dramatically on which of the nine versions of the game is being played as demonstrated in Fig. 6 for the simplest possible case of identical ships ($V_a/V_o = R_a/R_o = 1$) on initially reciprocal courses $\theta = 180^o$ for the ease of interpretation. Collaborative evasion 0_EA_E naturally leads to the smallest collision zone. Unilateral evasion produces a somewhat larger collision zone being identical for 0_EA_N and 0_NA_E due to the symmetry of this particular situation. The pursuit-evasion games 0_PA_E and (with reversed roles) 0_EA_P generate even larger but still finite collision zones, while the trivial game 0_NA_N of course produces a collision band of width $2r_f$ extending to infinity in the x-direction. Unilateral pursuits 0_PA_N and 0_NA_P produce an ever widening collision zone ahead of 0 but the state space is still subdivided into regions of collision and escape. (For general values of game parameters a relation between V_a/V_o, R_a/R_o and r_f/R_o can be derived as a criterion for closure of the collision zone. An open zone implies collision from every starting state under optimal pursuit.) Collaborative rendezvous is, of course, possible from every starting state. The open contour marked $O_P A_P$ in Fig. 6 delineates a barrier in the sense that collision cannot occur at first pass from initial states on the far side of the barrier. With reference to our mathematical model for computing collision rates (Section 2.3), one way of estimating the frequency function $f_c(r_c)$ would be to calculate r_c for all nine versions of the game and to assign suitable probabilities to each paying due attention to the rules of the nautical road and to the actual observed behavior of mariners on board ships.

4.6. Slower ship's dilemma

Fig. 7 shows a more interesting comparison of four game versions in a crossing situation ($\theta = 90^o$) between two dissimilar ships of which one has a significantly higher speed ($V_a/V_o = 0.5$) and the other a significantly lower minimum turning radius ($R_a/R_o = 0.6$). It is striking how much more effective unilateral evasion by the faster ship (see $O_E A_N$) is compared to unilateral evasion by the slower ship (see $O_N A_E$). With O evading, as expected the collision zone expands if A pursues ($O_P A_E$) and shrinks if A evades ($O_E A_E$). But the differences are so small that the faster ship O practically dominates the game. Note also that in none of the four games the critical range is a maximum along the perfect collision track (dashed line in figure). In order to compare numbers we adjoin to our illustrative ship O a companion A of $V_a = 10$ kn, $R_a = 300$ m and get:

Game		$O_E A_E$	$O_E A_N$	$O_E A_P$	$O_N A_E$
$\max_\alpha r_c(\alpha)$	[m]	410	465	510	1005
τ_{NN}	[s]	35	40	44	87

See how critically the time needed to achieve a specified miss distance (here 150 m) depends on the speed of the evader. Halving the speed roughly doubles the time. This figure also serves to illustrate a dilemma which can arise from a strict observance of the International Regulations for Preventing Collisions at Sea (1960). By Rule 19 here O is privileged and A is burdened. Suppose A fails to do his duty. By Rule 21 then O must stand on until A reaches the boundary of zone $O_N A_E$. Then and only then is O allowed to evade what in our example he can easily do since the collision zone $O_E A_N$ is much smaller and fully contained within zone $O_N A_E$. However, the situation changes radically, if we take the mirror image ($\theta = 270^o$) so that the slower ship A becomes privileged. If a slower privileged ship lets a negligent faster burdened ship approach according to Rule 21 until the latter demonstrably cannot avoid collision, then in general it is also too late for the former! So slow ships also get their fair share of suspense at sea.

5. POSSIBLE GENERALIZATIONS

The theory of differential games has opened up new vistas to the collision avoidance problem of which the preceding examples could catch but a few glimpses. Many meaningful modifications and generalizations of the game formulated here suggest themselves. First, consider the kinematic equations (7-9). In his first go at the problem Miloh (1974) had already allowed in addition to turning rate control (simulating rudder commands) also linear thrust control (simulating engine commands) and discovered beside the succinct terminal turning rule of Section 3.4 the equally esthetic terminal thrust rule that at closest approach the evader should be accelerating away from (and the pursuer toward) his target! It was then sacrificed in favor of analytical elegance and because in practice voluntary acceleration is generally not feasible and deceleration not advisable due to attendant loss of rudder effectiveness. However, the involuntary loss of speed due to augmented resistance in a turn is an effect which ought to be included. In previous papers we have indicated how this can be approximately accounted for and in unpublished work it has been examined more rigorously.

Second, the terminal condition of zero range rate can also be gainfully generalized. Measuring the miss distance from center to center makes strict sense only for circular objects. For slender ships it is better defined as the shortest distance between any pair of points one of which is on the boundary of O and the other on that of A. For convex contours such as ellipses the mathematics is still tractable. Eccentric circle and ogival shapes have been tried by Vincent and Peng (1973) and Vincent (1974) for simulating optimal evasion of torpedoes.

Third, the payoff is prolific of variety. Maximizing miss distance is the proper goal at the eleventh hour when collision is threatening but still avoidable. If collision is already inevitable but still a few precious moments away, it may be more rational to play for minimum damage, say by bringing the boats bow to bow together. Unfortunately, in marginal cases it can create a predicament as the optimal maneuvers of this game are often the opposite of those for maximizing miss distance. At the other end, for timely evasion at long range, miss distance is a constraint rather than the payoff, which is then better related to time lost in maneuver as proposed by Sorensen (1975). We have concentrated on the eleventh hour situation because only there is maneuverability of crucial consequence.

Finally, it would surely be of practical interest to formulate collision avoidance games in bounded state space (confined waters), with more than two players (congested traffic), and with incomplete or noisy information (real-life case). But the conceptual and formal difficulties then grow to such an extent that closed form theoretical solutions are not yet in sight. Here it seems more expedient to follow the computer simulation approach of Kenan (1972) and Webster (1974) who used a very elaborate nonlinear model of ship dynamics and an almost exact terminal condition for physical collision but sought optimal maneuvers from an arbitrary set of discrete alternatives by trial and error rather than from complete continua by variational principles basic to the theory of differential games. No wonder they did not detect the important principle (not explicitly discussed here) that the slower ship must often switch controls, i.e. stop turning and adopt straight course *before* reaching the condition of closest approach. A promising compromise would be to solve complex games through digital simulation of trajectories corresponding to a finite set of imaginative trial strategies inspired by theoretical solutions of simpler cognate games.

6. CONCLUDING REMARKS

An analytical model for predicting the collision rate of a ship by statistically matching the maneuvering capability available to the ship versus the evasive capability demanded by the traffic environment has been presented. Pending evaluation of numerical examples and comparison with actuarial collision rates the practical usefulness of the model cannot be ascertained. However, quantitative results have been derived and displayed regarding the objective collision avoidance capability of a ship under optimal use of maneuverability. The complexity of the problem and the furcate structure of the solution will no doubt surprise those who continue to seek simple universal anti-collision rules. While our original motivation was naval architectural, it now seems likely that a more immediate application of these results will be nautical, namely in the software of computerized radar-based collision avoidance systems and possibly in developing new standards for documentation of ship maneuverability for use by the master.

Alternative models for estimating collision rates have been proposed by others. The slightly lop-sided model of Fujii and Shiobara (1971) focuses on traffic demand and ignores maneuverability, see also Fujii and Yamanouchi (1973). Macduff (1974) has offered a model for calculating stranding rates in a channel due to randomly executed stopping maneuvers by bizarre analogy to Buffon's needle problem and Hara (1974) has worked out a model for calculating collision probability in dense traffic by application of queuing theory. In fact Hara's model explicitly predicts a more realistic nonlinear dependence of collision rate on traffic density whereas ours is apparently linear. However, nonlinearity can be introduced indirectly into our model by letting the frequency functions of evasion demand and capability depend also on traffic density. Whichever model ultimately emerges, it will require as input statistical data on sea-traffic density and velocity and an algorithmic description of the mariner's use of maneuverability. Therefore, the author would like to close by pleading with all researchers in the area to join hands in an international effort to collect this urgently needed information so that considerations of safety against collisions in the design and operation of ships can be put on a more rational footing.

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SYMBOLS

Abbreviations

A	The other ship
CPA	Condition of closest approach ($dr/dt = 0$)
E	Optimal evasion
L	Hard left turn
$\max_\alpha$	Maximum with respect to α
N	Neutrality (No maneuver)
O	Own ship (Origin of coordinates x,y and r,α)
P	Optimal pursuit
R	Hard right turn

Variables

$f(V_a,\theta_a)$	Frequency function of velocity in ambient traffic
$f_c(r_c)$	Frequency function of r_c (collision avoidance capability)
$f_e(r_e)$	Frequency function of r_e (collision avoidance demand)

$P\{\ \}$	Probability of { }
R	Minimum turning radius of O or A
r	Range between A and O, polar coordinate of A (see Fig. 1)
r_c	Critical (=minimum) range required for avoiding collision
r_e	Range at initiation of evasive maneuver
r_f	Final range, i.e. range at CPA (Miss distance)
r_m	Some specified value of range or miss-distance
t	Time
V	Speed of O or A (see Fig. 1)
V_r	Speed of A relative to O (see Fig. 1)
x,y	Coordinates of A relative to O (see Fig. 2)
α	Relative bearing of A taken clockwise from O (see Fig. 1)
γ	Game identifier = 1,0,-1 for E,N,P by O
δ	Game identifier = 1,0,-1 for E,N,P by A
θ	With subscript: Course angle of O or A (see Fig. 1) Without subscript: Heading of A relative to O (see Fig. 2)
λ	Encounter rate
λ_c	Collision rate
ρ	Ambient traffic density
τ	Retrograde time (t_f-t), i.e. time to CPA
ϕ,ψ	Control variable of O,A (Normalized turning rate, positive to right)
$\bar{\phi},\bar{\psi}$	Optimal control of O,A

<u>General subscripts</u>

a	Of ship A, applies to V,R,θ
E	Optimal evasion, applies to A,O,$r_f,\alpha,\theta,\tau,\bar{\phi},\bar{\psi}$
f	Final (i.e. at CPA) value, applies to $r,t,\alpha,\theta,\bar{\phi},\bar{\psi}$
L	Hard left turn, applies to A,O
N	Neutral (No turn), applies to A,O,$r_f,\alpha,\theta,\tau,\bar{\phi},\bar{\psi}$
o	Of ship O, applies to V,R,θ
P	Optimal pursuit, applies to A,O,$r_f,\alpha,\theta,\tau,\bar{\phi},\bar{\psi}$
R	Hard right turn, applies to A,O

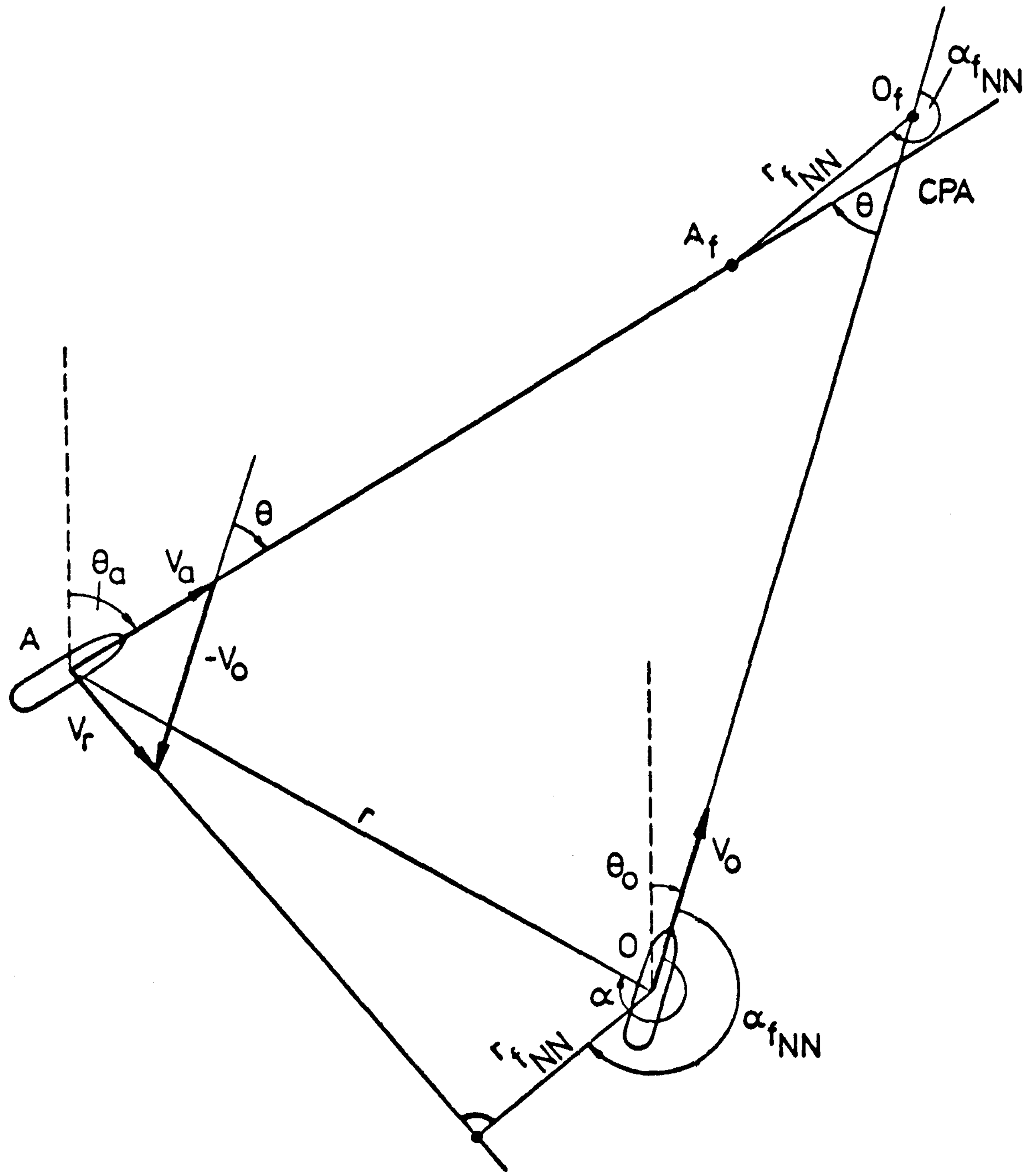

Fig. 1 Kinematics of two-ship encounter and condition of closest approach.

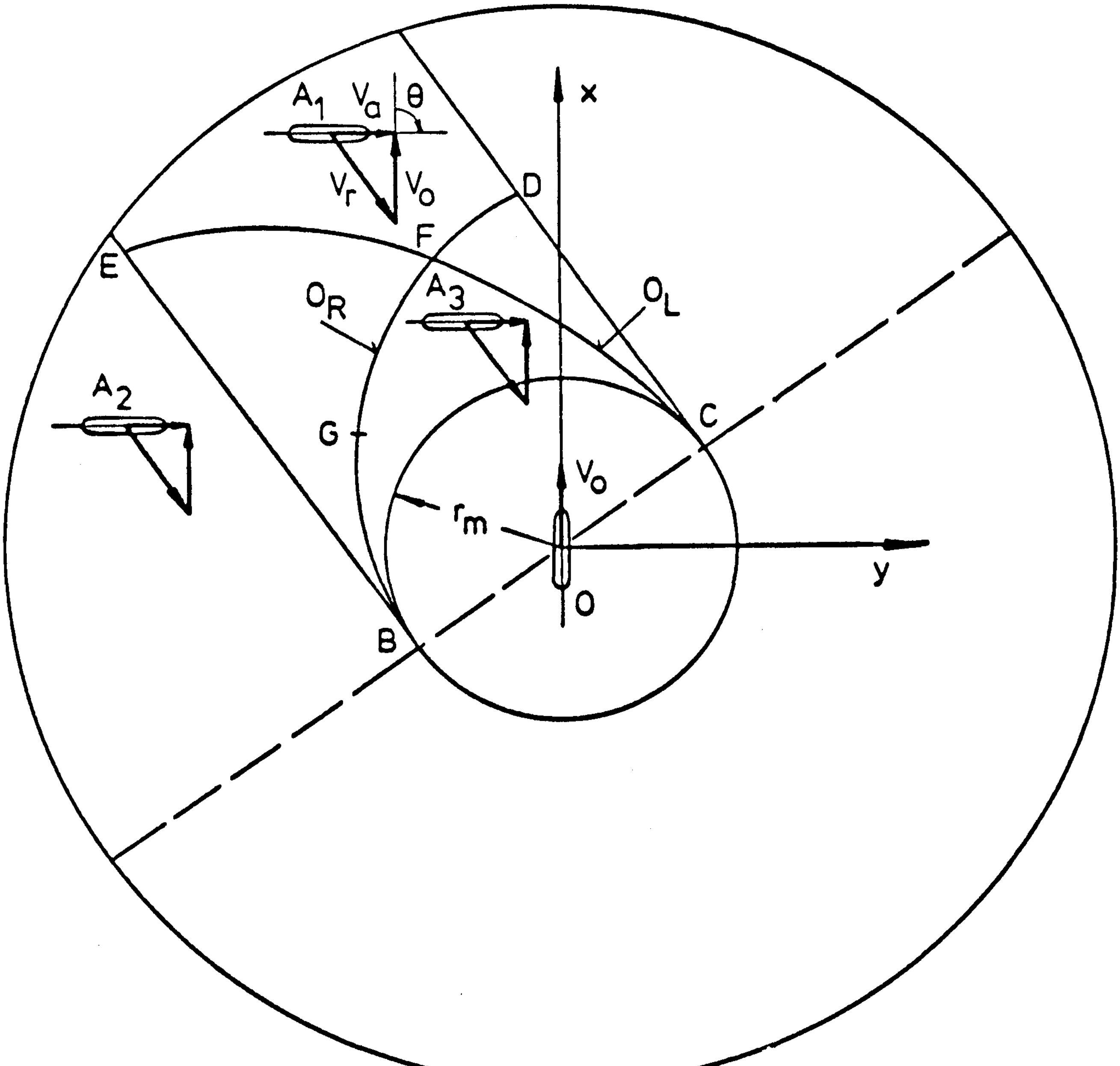

Fig. 2 Schematic illustration of traffic environment and collision zone around own ship O.

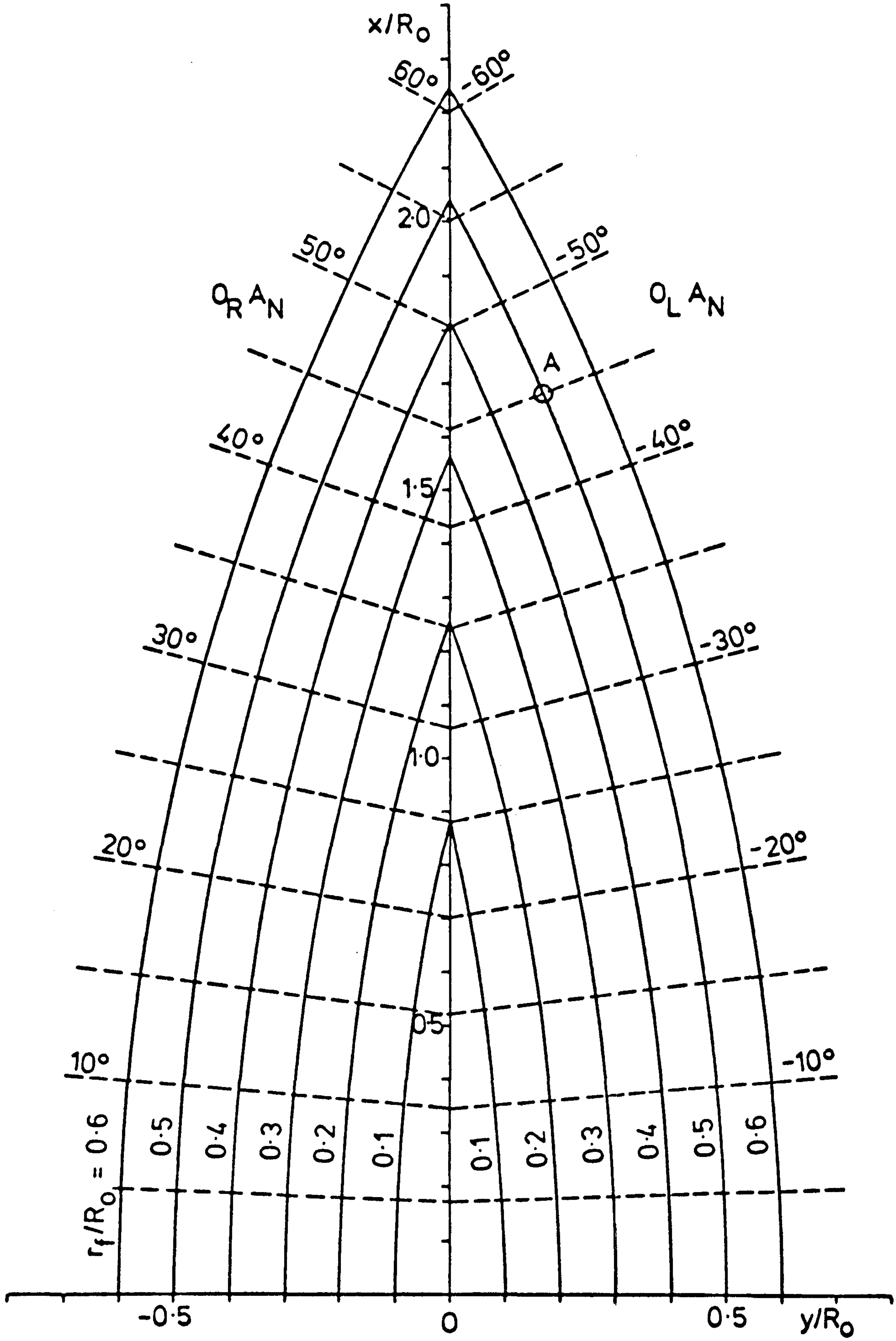

Fig. 3 Optimal maneuvers and miss-distances for $\theta = 180^{o}$ in game $O_E A_N$ with $V_a/V_o = 1$.

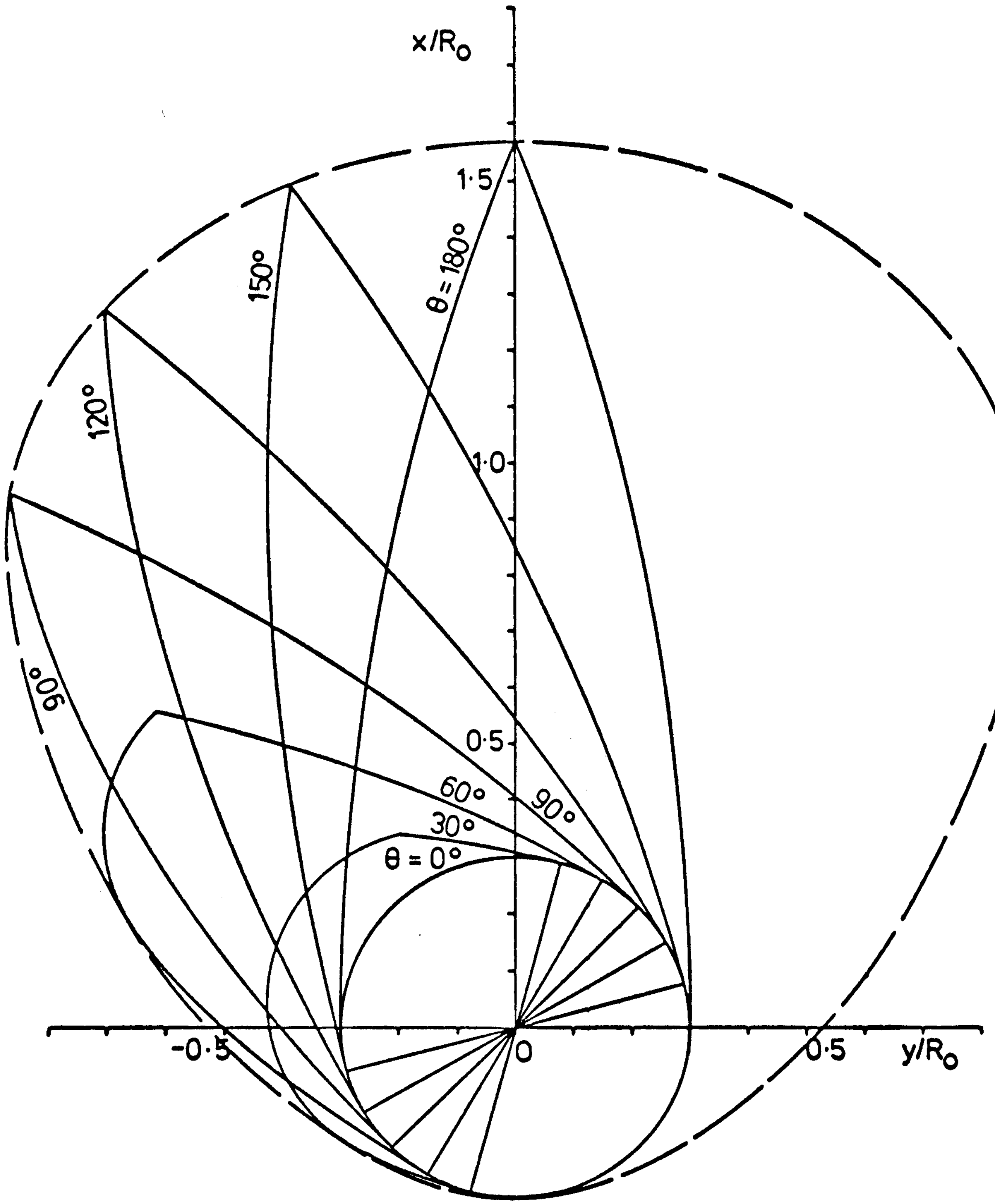

Fig. 4 Cross-sections through the three-dimensional collision zone for $r_f/R_o = 0.3$ in game $O_E A_N$ with $V_a/V_o = 1$.

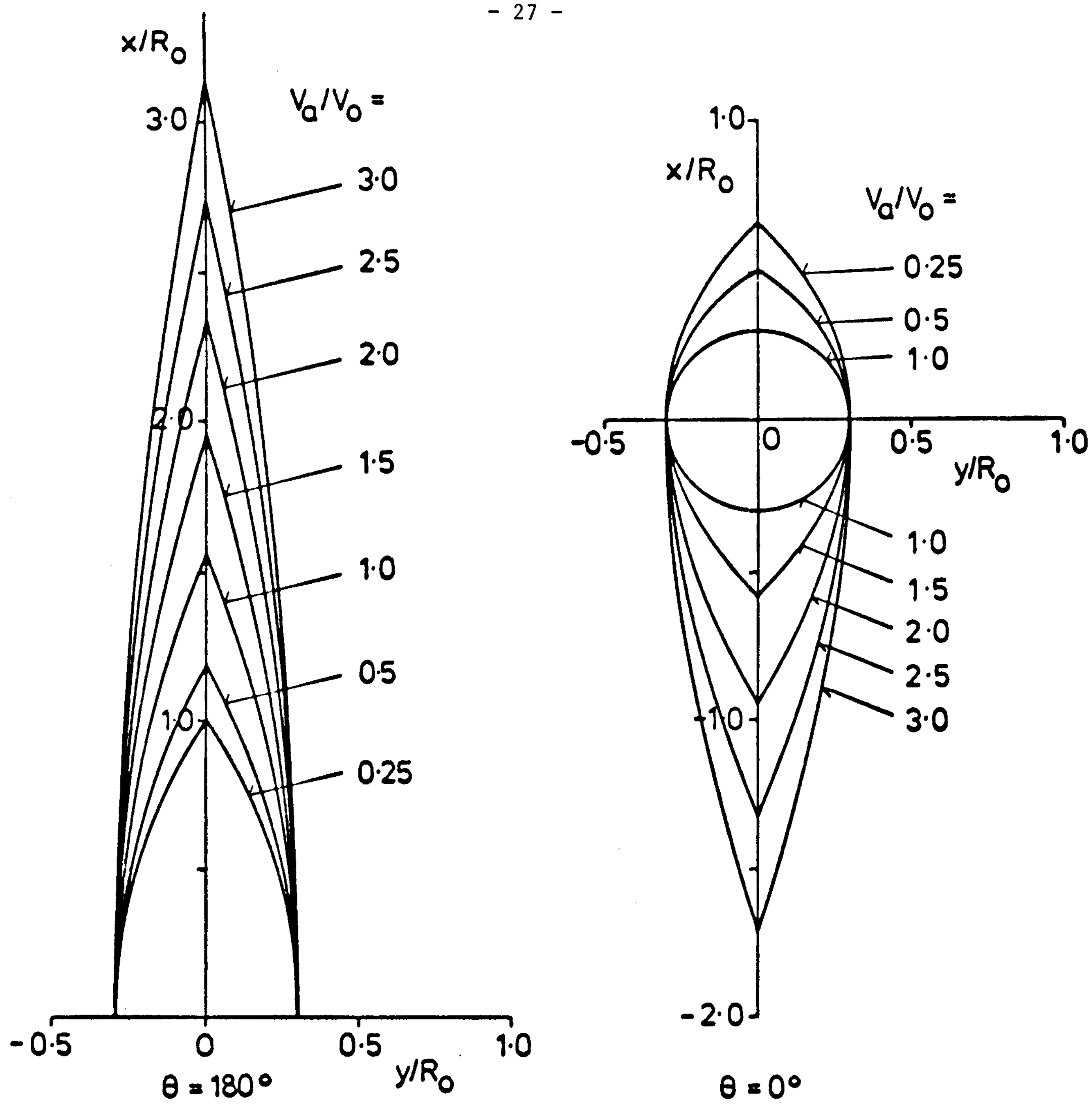

Fig. 5 Limiting cross-sections $\theta = 180^\circ$ and $\theta = 0^\circ$ of the collision zones ($r_f/R_o = 0.3$) in game $O_E A_N$ for seven values of speed ratio V_a/V_o.

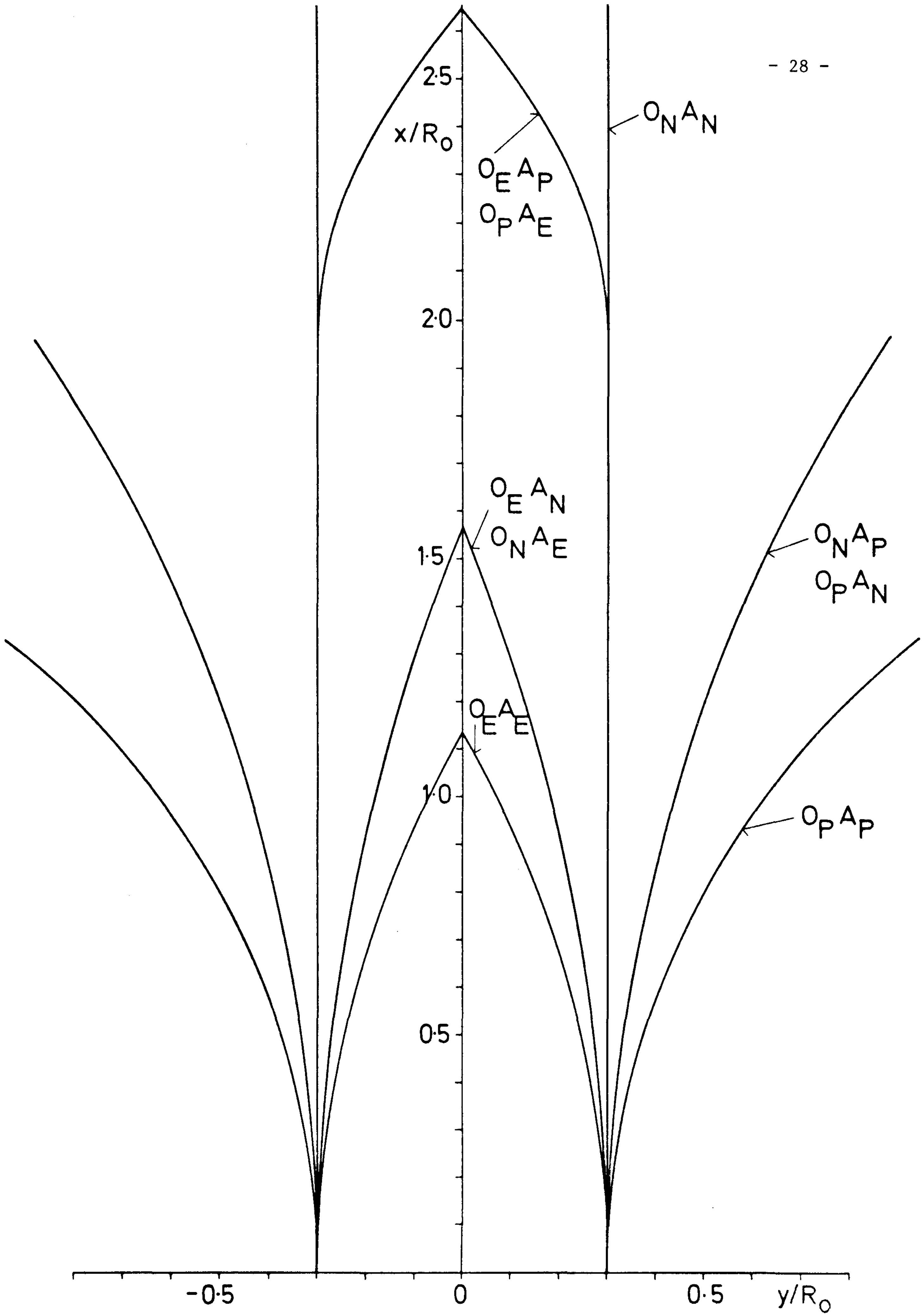

Fig. 6 Cross-sections $\theta = 180^\circ$ of the collision zones ($r_f/R_o = 0.3$) for all nine versions of the game between twin ships $V_a/V_o = R_a/R_o = 1$.

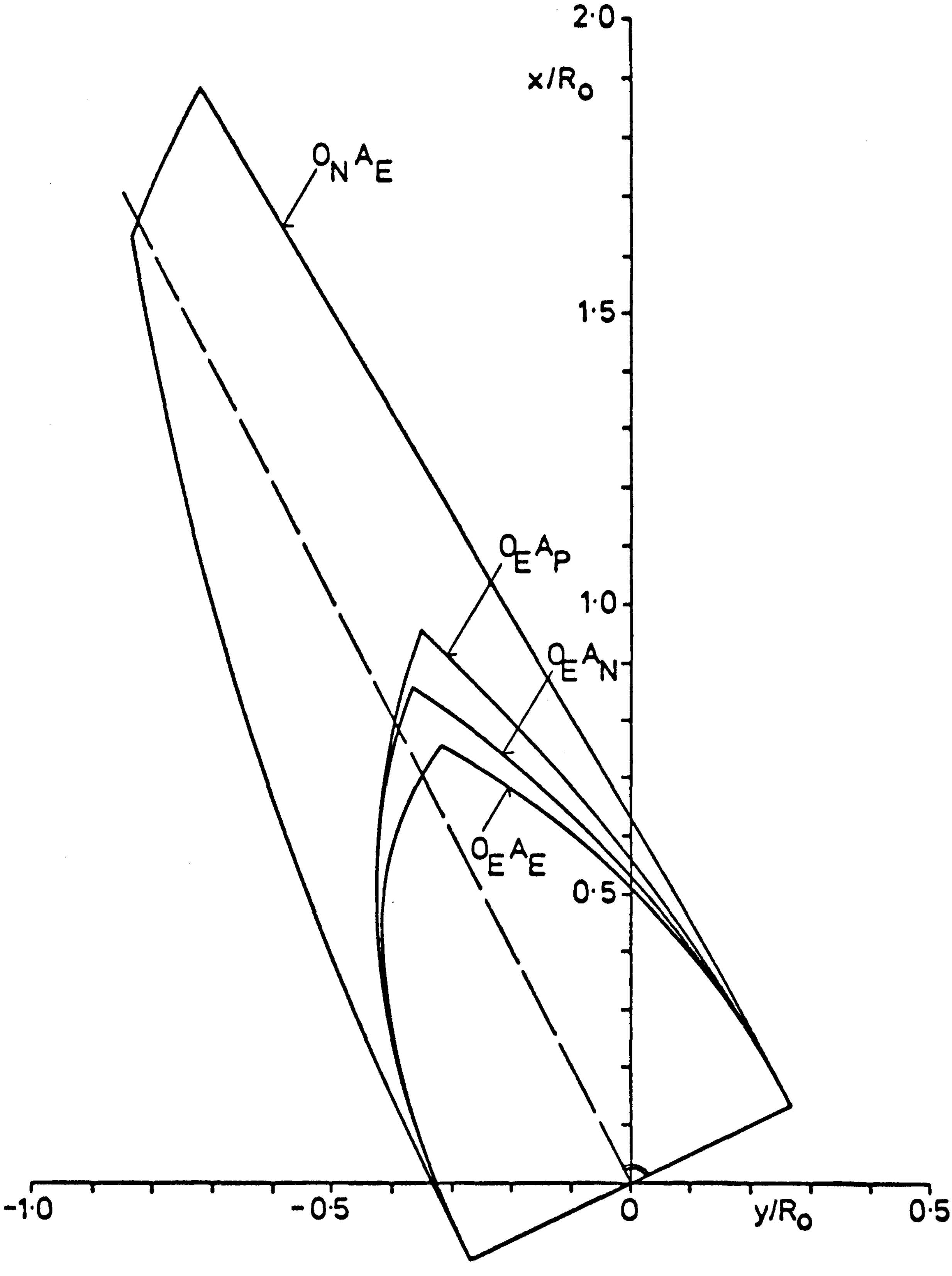

Fig. 7 Cross-sections $\theta = 90^{\circ}$ of the collision zones ($r_f/R_o = 0.3$) for four different games between dissimilar ships $V_a/V_o = 0.5$, $R_a/R_o = 0.6$.