

## RESEARCH ARTICLE

# $r$ -Cross $t$ -intersecting families via necessary intersection points

Pranshu Gupta<sup>1</sup> | Yannick Mogge<sup>1</sup> | Simón Piga<sup>2</sup> |  
Bjarne Schülke<sup>3,4</sup>

<sup>1</sup>Hamburg University of Technology,  
Institute of Mathematics (E-10), Chair  
Discrete Mathematics, Hamburg,  
Germany

<sup>2</sup>School of Mathematics, Watson Building,  
University of Birmingham, Edgbaston,  
Birmingham, UK

<sup>3</sup>Caltech, The Division of Physics,  
Mathematics and Astronomy, Pasadena,  
California, USA

<sup>4</sup>University of Hamburg, Department of  
Mathematics, Hamburg, Germany

**Correspondence**

Bjarne Schülke, Caltech, The Division of  
Physics, Mathematics and Astronomy,  
1200 E California Blvd, Pasadena CA  
91125, USA.  
Email: [schuelke@caltech.edu](mailto:schuelke@caltech.edu)

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**Abstract**

Given integers  $r \geq 2$  and  $n, t \geq 1$  we call families  $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathcal{P}([n])$   $r$ -cross  $t$ -intersecting if for all  $F_i \in \mathcal{F}_i$ ,  $i \in [r]$ , we have  $|\bigcap_{i \in [r]} F_i| \geq t$ . We obtain a strong generalisation of the classic Hilton–Milner theorem on cross-intersecting families. In particular, we determine the maximum of  $\sum_{j \in [r]} |\mathcal{F}_j|$  for  $r$ -cross  $t$ -intersecting families in the cases when these are  $k$ -uniform families or arbitrary subfamilies of  $\mathcal{P}([n])$ . Only some special cases of these results had been proved before. We obtain the aforementioned theorems as instances of a more general result that considers measures of  $r$ -cross  $t$ -intersecting families. This also provides the maximum of  $\sum_{j \in [r]} |\mathcal{F}_j|$  for families of possibly mixed uniformities  $k_1, \dots, k_r$ .

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## 1 | INTRODUCTION

For a set  $A$ , we denote the power set of  $A$  as  $\mathcal{P}(A) = \{B : B \subseteq A\}$ . Let  $j \in \mathbb{N}$ , then set  $[j] = \{1, \dots, j\}$ ,  $[j]_0 = [j] \cup \{0\}$ , and for  $i \in [j]_0$  set  $[i, j] = \{i, i + 1, \dots, j\}$ . For a set with a single element, say  $\{i\}$ , we sometimes just write  $i$ . Given a set  $A$ , we write  $A^{(k)}$  for the set of  $k$ -element subsets of  $A$  and similarly  $A^{(\leq k)}$  for the set containing all subsets of  $A$  that are of size at most  $k$ .

One of the main themes in extremal set theory are intersecting families. Given some  $n \in \mathbb{N}$ , a family  $\mathcal{F} \subseteq \mathcal{P}([n])$  is said to be *intersecting* if for all  $F, F' \in \mathcal{F}$  we have  $F \cap F' \neq \emptyset$ . The

following well-known theorem by Erdős, Ko and Rado [13] is one of the earliest results in extremal set theory.

**Theorem 1.1.** *Let  $k, n \in \mathbb{N}$  with  $2k \leq n$  and let  $\mathcal{F} \subseteq [n]^{(k)}$  be an intersecting family. Then  $|\mathcal{F}| \leq \binom{n-1}{k-1}$  and this bound is sharp.*

Observe that this maximum is attained by a family which contains all the sets of size  $k$  that contain one fixed element, for instance,  $\mathcal{F} = \{F \in [n]^{(k)} : 1 \in F\}$ .

As a variation *cross-intersecting* families can be considered. For  $r, t, n \in \mathbb{N}$  we say that families  $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathcal{P}([n])$  are *r-cross t-intersecting* if for all  $F_1 \in \mathcal{F}_1, \dots, F_r \in \mathcal{F}_r$  we have  $|\bigcap_{i \in [r]} F_i| \geq t$ . Now it is natural to ask for the maximum of  $\sum_{i \in [r]} |\mathcal{F}_i|$  taken over all non-empty *r-cross t-intersecting* families  $\mathcal{F}_1, \dots, \mathcal{F}_r$ . In this regime there are several partial results concerning the maximum sum of sizes of *r-cross t-intersecting* families for specific instances of  $r$  and  $t$ , starting with theorems by Hilton [25] and by Hilton and Milner [26] and continued, for instance, in [6, 20, 21, 24, 31, 32, 34, 35] (also see the references therein). We determine  $\sum_{i \in [r]} |\mathcal{F}_i|$  for every  $r \geq 2$  and  $t \geq 1$  for both uniform families and non-uniform families (see Corollary 1.3 and Corollary 1.5), generalising a result by Frankl and Wong H.W. [24].

In fact, we show these results in the more general setting of measures. Here, one can ask for the maximum measure of families instead of their sizes. More formally, consider a function  $\mu : \mathcal{P}([n]) \rightarrow \mathbb{R}_{\geq 0}$ , that is, a map assigning a weight to each set in  $\mathcal{P}([n])$ . Now, instead of asking for the maximum size of an intersecting family, we ask for the maximum measure of an intersecting family, where the measure of a family  $\mathcal{F} \subseteq \mathcal{P}([n])$  is defined as  $\mu(\mathcal{F}) = \sum_{F \in \mathcal{F}} \mu(F)$ . Two commonly considered measures are the product measure  $\varrho_p$  and the uniform measure  $\nu_k$ . For  $p \in [0, 1]$  we define the product measure as  $\varrho_p(F) = p^{|F|}(1-p)^{n-|F|}$ , where  $F \in \mathcal{P}([n])$ . Note that this can be interpreted as the probability that a specific set  $F$  is the result of a random experiment which includes each element from  $[n]$  with probability  $p$  in  $F$ . The uniform measure  $\nu_k$ , with  $k \in [n]$ , is defined as  $\nu_k(F) = 1/\binom{n}{k}$  if  $|F| = k$  and  $\nu_k(F) = 0$  if  $|F| \neq k$ .

For these measures analogues of the Erdős–Ko–Rado theorem can be considered. Indeed, we can reformulate Theorem 1.1 as follows: For  $k, n \in \mathbb{N}$  with  $2k \leq n$  and an intersecting family  $\mathcal{F} \subseteq [n]^{(k)}$  it follows that  $\nu_k(\mathcal{F}) \leq \frac{k}{n}$ . For the product measure the following analogous result was first proved in [1]. For  $p \leq 1/2$  and an intersecting family  $\mathcal{F} \subseteq \mathcal{P}([n])$  we have  $\varrho_p(\mathcal{F}) \leq p$ .

Several results for specific measures and (cross) intersecting families are known, see [4, 5, 11, 14, 15, 33]. For a more thorough overview we recommend [23, Chapter 12] and [9]. In particular, a result due to Borg [8] determines the maximum product of *r-cross t-intersecting* uniform families and also considers a variant with measures.

Since normally the measures considered depend only on the size of the sets, we will introduce the following abuse of notation. For a function  $\mu : [n]_0 \rightarrow \mathbb{R}_{\geq 0}$  and a set  $F \subseteq [n]$  we consider the measure  $F \mapsto \mu(|F|)$  but write  $\mu(F)$  instead of  $\mu(|F|)$  and we refer to  $\mu$  as a measure. Further, for  $\mathcal{F} \subseteq \mathcal{P}([n])$  we write  $\mu(\mathcal{F}) = \sum_{F \in \mathcal{F}} \mu(F)$ .

In our main results we determine the maximum sum of measures of *r-cross t-intersecting* families. Given  $n, a, t \in \mathbb{N}$  with  $n \geq a \geq t$  consider the families

$$\mathcal{A}(n, a, t) = \{F \in \mathcal{P}([n]) : |F \cap [a]| \geq t\}$$

$$\mathcal{B}(n, a) = \{F \in \mathcal{P}([n]) : [a] \subseteq F\}.$$

Essentially, our main results state that the maximum is attained by families ‘derived’ from  $\mathcal{A}(n, a, t)$  and  $\mathcal{B}(n, a)$ , even when we consider different kinds of measures (including  $\nu_k$

and  $g_p$  when  $p \leq 1/2$ ). For  $\mathcal{F} \subseteq \mathcal{P}([n])$  and  $k \in \mathbb{N}$ , we set  $\mathcal{F}^k = \{F \in \mathcal{F} : |F| = k\}$  and  $\mathcal{F}^{\leq k} = \{F \in \mathcal{F} : |F| \leq k\}$ . Let  $r$  be an integer with  $r \geq 2$ , for every  $i \in [r]$  let  $k_i \in \mathbb{N}$  and let  $j \in [r]$  such that  $k_j = \min_{i \in [r]} k_i$ . Then we write  $\text{secmin}_{i \in [r]} k_i = \min_{i \in [r] \setminus j} k_i$ . Let us now state our first result which in particular determines the maximum of  $\sum_{i \in [r]} |\mathcal{F}_i|$  for  $k$ -uniform families.

**Theorem 1.2.** *Let  $r, t, n \in \mathbb{N}$  with  $r \geq 2$ . For  $i \in [r]$  let  $k_i \in [n]$ ,  $\mu_i : [n]_0 \rightarrow \mathbb{R}_{\geq 0}$ , and  $\mathcal{F}_i \subseteq [n]^{\leq k_i}$  such that  $n \geq 2 \max_{i \in [r]} k_i + \text{secmin}_{i \in [r]} k_i - t$ . If  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are non-empty  $r$ -cross  $t$ -intersecting families, then*

$$\sum_{j \in [r]} \mu_j(\mathcal{F}_j) \leq \max \left\{ \mu_\ell(\mathcal{A}(n, a, t)^{\leq k_\ell}) + \sum_{j \in [r] \setminus \ell} \mu_j(\mathcal{B}(n, a)^{\leq k_j}) \right\}, \tag{1.1}$$

where the maximum is taken over  $\ell \in [r]$  and  $a \in [t, \min_{i \in [r] \setminus \ell} k_i]$ .

Note that  $\mathcal{A}(n, i, t)$  together with  $r - 1$  copies of  $\mathcal{B}(n, i)$  are  $r$ -cross  $t$ -intersecting for every  $i \geq t$ . Thus, this result is sharp in the sense that there are  $r$ -cross  $t$ -intersecting families which attain the bound.

As mentioned above, applying Theorem 1.2 with  $k_i = k$  and the measure  $\mu_i = \nu_k \binom{n}{k}$  for every  $i \in [r]$ , we obtain the following result for  $k$ -uniform families.

**Corollary 1.3.** *Let  $r \geq 2$  and  $n, t \geq 1$  be integers,  $k \in [n]$ , and for  $i \in [r]$  let  $\mathcal{F}_i \subseteq [n]^{(k)}$ . If  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are non-empty  $r$ -cross  $t$ -intersecting families and  $n \geq 3k - t$ , then*

$$\sum_{j \in [r]} |\mathcal{F}_j| \leq \max_{m \in [t, k]} \left\{ \sum_{i \in [t, k]} \binom{m}{i} \cdot \binom{n - m}{k - i} + (r - 1) \binom{n - m}{k - m} \right\},$$

and this bound is attained.

Some special cases of this result were obtained before. For  $r = 2$  and  $t \geq 1$  Corollary 1.3 was proved by Frankl and Kupavskii [20]. For  $t = 1$  and  $r \geq 2$  Corollary 1.3 was shown very recently in independent work by Shi, Frankl and Qian [31], where they deduce it from a result about two families by an elegant application of the Kruskal–Katona theorem [27, 28].

Note that, in fact, Theorem 1.2 also determines the maximum of  $\sum_{i \in [r]} |\mathcal{F}_i|$  for families of different uniformities  $k_1, \dots, k_r$ . Sometimes, a weaker definition of ‘cross intersecting’ is used in the literature: for  $r, t, n \in \mathbb{N}$ , we say that families  $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathcal{P}([n])$  are pairwise  $r$ -cross  $t$ -intersecting if for all  $i, j \in [r]$  and all  $F_i \in \mathcal{F}_i, F_j \in \mathcal{F}_j$ , we have  $|F_i \cap F_j| \geq t$ . In [31], Shi, Frankl and Qian posed three problems regarding the maximum sum of sizes of pairwise  $r$ -cross  $t$ -intersecting families. Theorem 1.2 solves all of these problems for  $r$ -cross  $t$ -intersecting families (and slightly larger  $n$ ). To some extent, our proof method also applies to the pairwise cross intersecting setting and we plan to address this in future work.

In the context of non-uniform families, one of the results of a very recent work by Frankl and Wong H.W. [24] establishes the maximum possible size of 2-cross  $t$ -intersecting families. The following theorem generalises their result for all  $r \geq 2$  and for measures.

**Theorem 1.4.** Let  $r, t, n \in \mathbb{N}$  with  $r \geq 2$ . For  $i \in [r]$  let  $\mu_i : [n]_0 \rightarrow \mathbb{R}_{\geq 0}$  be non-increasing, and let  $\mathcal{F}_i \subseteq \mathcal{P}([n])$ . If  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are non-empty  $r$ -cross  $t$ -intersecting families, then

$$\sum_{j \in [r]} \mu_j(\mathcal{F}_j) \leq \max \left\{ \mu_\ell(\mathcal{A}(n, a, t)) + \sum_{j \in [r] \setminus \ell} \mu_j(\mathcal{B}(n, a)) \right\}, \tag{1.2}$$

where the maximum is taken over  $\ell \in [r]$  and  $a \in [t, n]$ .

As before, this bound is attained. Note that by taking  $\mu_i \equiv 1$  for every  $i \in [r]$  we obtain the maximum of  $\sum_{i \in [r]} |\mathcal{F}_i|$  for  $r$ -cross  $t$ -intersecting families.

Independently, Borg and Feghali [10] proved the special case of Theorem 1.4 when  $r = 2, t = 1$ , and  $\mu_i(k) = \mathbb{1}_{k \leq k_i}, i \in \{1, 2\}$ , for some  $k_1, k_2 \in [n]$ .

**Corollary 1.5.** Let  $r \geq 2, n, t \geq 1$  be integers and let  $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathcal{P}([n])$  be non-empty  $r$ -cross  $t$ -intersecting families. Then,

$$\sum_{j \in [r]} |\mathcal{F}_j| \leq \max_{m \in [t, n]} \left\{ 2^{n-m} \sum_{i \in [t, m]} \binom{m}{i} + (r-1)2^{n-m} \right\},$$

and this bound is attained.

For a further application, note that Theorem 1.4 also provides the maximum for the product measure  $\varrho_p$ , if  $p \leq 1/2$ .

As it turns out, we can essentially prove Theorems 1.2 and 1.4 simultaneously and we derive them as special cases of the more general Proposition 3.7. This proposition even considers the case of  $r$ -cross  $t$ -intersecting families  $\mathcal{F}_1, \dots, \mathcal{F}_r$  when some of them satisfy the conditions of Theorem 1.2 and some others satisfy the conditions of Theorem 1.4.

### 1.1 | Idea of the proof

Our proof is based on what we call *necessary intersection points* (see Definition 3.1). Roughly speaking we say that a vertex  $a \in [n]$  is a necessary intersection point for  $r$ -cross  $t$ -intersecting families  $\mathcal{F}_1, \dots, \mathcal{F}_r$  if there are sets in the families which ‘depend’ on this vertex to fulfil their intersection property. For example, if we consider the 2-cross 1-intersecting families  $\mathcal{A}(n, 2, 1)$  and  $\mathcal{B}(n, 2)$ , the vertex 2 is a necessary intersection point because there are pairs of sets that intersect only in 2. In this case, 1 and 2 are the only necessary intersection points of these families. The idea is to ‘decrease’ the largest necessary intersection point as long as possible, that is, replace the presently considered  $r$ -cross  $t$ -intersecting families by  $r$ -cross  $t$ -intersecting families whose sum of measures is not smaller but which have a smaller largest necessary intersection point.

Let  $\mathcal{F}_1, \dots, \mathcal{F}_r$  be some  $r$ -cross  $t$ -intersecting families and let  $a \in [n]$  be their largest necessary intersection point. To construct the new families we first remove all sets that ‘depend’ on  $a$  in one family, say  $\mathcal{F}_r$ ; we call the family of these sets  $\mathcal{F}_r(a)$ . Then  $a$  will no longer be a necessary intersection point. Potentially, there are some subsets of  $[n]$  which could not be in any of the other families because they would not intersect ‘correctly’ with some set in  $\mathcal{F}_r(a)$ . However, after

removing  $\mathcal{F}_r(a)$  from  $\mathcal{F}_r$  and depending on how such a set relates with  $\mathcal{F}_r \setminus \mathcal{F}_r(a)$ , it may be added to one of the other families without breaking the intersection property.

There are some structural properties that follow from  $a$  being the largest necessary intersection point and the fact that the families are shifted. These will help us to analyse which new sets can actually be added to the families  $\mathcal{F}_1, \dots, \mathcal{F}_{r-1}$  and to prove that in fact the measure of the newly added sets is at least as large as the measure of the removed sets. Moreover, this analysis guarantees that the new largest necessary intersection point is at most  $a - 1$ .

We can iterate this construction and decrease the largest necessary intersection point in every step. This process has to stop at a certain point, and we show that then the resulting families are contained in families with the desired structure (namely  $\mathcal{A}(n, a, t)$  and  $\mathcal{B}(n, a)$ ).

## 2 | PRELIMINARIES

In this section we introduce some well-known facts about shifting. For  $F \subseteq [n]$  and  $i, j \in [n]$  we set

$$\sigma_{ij}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\} & \text{if } j \in F \text{ and } i \notin F \\ F & \text{otherwise,} \end{cases}$$

and note that  $|\sigma_{ij}(F)| = |F|$ . Moreover, for a given  $\mathcal{F} \subseteq \mathcal{P}([n])$  we define the family  $\sigma_{ij}(\mathcal{F}) = \{\sigma_{ij}(F) : F \in \mathcal{F}\} \cup \{F \in \mathcal{F} : \sigma_{ij}(F) \in \mathcal{F}\}$  and note that  $|\sigma_{ij}(\mathcal{F})| = |\mathcal{F}|$ . Further, it can easily be checked that if  $\mathcal{F} \subseteq \mathcal{P}([n])$  is intersecting, then  $\sigma_{ij}(\mathcal{F})$  is also intersecting. We say that  $\mathcal{F} \subseteq \mathcal{P}([n])$  is *shifted* if for all  $i, j \in [n]$  with  $i < j$  we have  $\sigma_{ij}(\mathcal{F}) = \mathcal{F}$ , that is, for all  $F \in \mathcal{F}$  we have that  $\sigma_{ij}(F) \in \mathcal{F}$ . By shifting an intersecting family  $\mathcal{F} \subseteq \mathcal{P}([n])$  repeatedly, that is, replacing  $\mathcal{F}$  by  $\sigma_{ij}(\mathcal{F})$  repeatedly for all  $i, j \in [n]$  with  $i < j$ , we obtain an intersecting family  $\mathcal{G}$  that is shifted and for which we have  $|\mathcal{G}| = |\mathcal{F}|$  and  $|\mathcal{G}^k| = |\mathcal{F}^k|$ . Thus, to determine the maximum size of an intersecting family, one can restrict themselves to shifted families.

Moreover, for the sake of completeness, we prove the following fact.

**Fact 2.1.** *Let  $a, b \in [n]$ . If  $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathcal{P}([n])$  are  $r$ -cross  $t$ -intersecting, then the families  $\sigma_{ab}(\mathcal{F}_1), \dots, \sigma_{ab}(\mathcal{F}_r)$  are  $r$ -cross  $t$ -intersecting.*

*Proof.* Assume the contrary and let  $F'_1 \in \sigma_{ab}(\mathcal{F}_1), \dots, F'_r \in \sigma_{ab}(\mathcal{F}_r)$  for every  $i \in [r]$  such that  $|\bigcap_{i \in [r]} F'_i| < t$ . For every  $i \in [r]$ , let  $F_i = F'_i$  if  $F'_i \in \mathcal{F}_i$ . If  $F'_i \notin \mathcal{F}_i$ , we know that  $a \in F'_i$  and  $b \notin F'_i$  and we set  $F_i = \sigma_{ba}(F'_i) \in \mathcal{F}_i$ . Since  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are  $r$ -cross  $t$ -intersecting, we have  $|\bigcap_{i \in [r]} F_i| \geq t$  and so there is some  $j \in [r]$  such that  $F_j = \sigma_{ba}(F'_j) \neq F'_j$ . But then we have  $a \notin F_j$ , and thus,  $a \notin \bigcap_{i \in [r]} F_i$ . This yields that

$$t - 1 \leq \left| \bigcap_{i \in [r]} F_i \setminus \{a, b\} \right| = \left| \bigcap_{i \in [r]} F'_i \setminus \{a, b\} \right|. \tag{2.1}$$

Note that the assumption  $|\bigcap_{i \in [r]} F'_i| < t$  tells us that in fact the left side inequality above is an equality. This in turn implies that  $b \in \bigcap_{i \in [r]} F_i$ .

Our assumption together with (2.1) also give some  $\ell \in [r]$  such that  $a \notin F'_\ell$ . Then it follows by definition that  $\sigma_{ab}(F'_\ell) \in \mathcal{F}_\ell$  because  $b \in \bigcap_{i \in [r]} F_i$ . Hence,  $|\sigma_{ab}(F'_\ell) \cap \bigcap_{i \in [r] \setminus \ell} F_i| < t$  contradicts  $\mathcal{F}_1, \dots, \mathcal{F}_r$  being  $r$ -cross  $t$ -intersecting.  $\square$

This allows us to restrict ourselves to shifted families when looking for the maximum sum of measures of  $r$ -cross  $t$ -intersecting families if the measure of a set  $F$  depends only on the size of  $F$ .

### 3 | PROOF OF THEOREMS 1.2 AND 1.4

We begin by introducing necessary intersection points which are central to our proofs.

**Definition 3.1.** Let  $\mathcal{F}_1 \subseteq \mathcal{P}([n]), \dots, \mathcal{F}_r \subseteq \mathcal{P}([n])$  be  $r$ -cross  $t$ -intersecting families. We say that  $a \in [n]$  is a *necessary intersection point* of  $\mathcal{F}_1, \dots, \mathcal{F}_r$  if for all  $j \in [r]$  there is an  $F_j \in \mathcal{F}_j$  such that

$$|[a] \cap \bigcap_{j \in [r]} F_j| = t \quad \text{and} \quad a \in \bigcap_{j \in [r]} F_j. \tag{3.1}$$

The following easy lemma is one of the useful properties of necessary intersection points used together with shifting.

**Lemma 3.2.** Let  $\mathcal{F}_1 \subseteq \mathcal{P}([n]), \dots, \mathcal{F}_r \subseteq \mathcal{P}([n])$  be shifted  $r$ -cross  $t$ -intersecting families and let  $a$  be their largest necessary intersection point. If  $i \in [r], F \in \mathcal{F}_i$ , and  $F_j \in \mathcal{F}_j$  for  $j \in [r] \setminus i$  are such that  $|[a - 1] \cap F \cap \bigcap_{j \in [r] \setminus i} F_j| < t$ , then  $[a - 1] \subseteq F \cup \bigcap_{j \in [r] \setminus i} F_j$ .

*Proof.* We will assume that there is a  $b \in [a - 1] \setminus (F \cup \bigcap_{j \in [r] \setminus i} F_j)$  and derive a contradiction. Suppose  $a \notin F$ . Then  $|[a] \cap F \cap \bigcap_{j \in [r] \setminus i} F_j| < t$ . Thus, since  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are  $r$ -cross  $t$ -intersecting, there is a necessary intersection point larger than  $a$ . This contradicts the assumption that  $a$  is the largest necessary intersection point of  $\mathcal{F}_1, \dots, \mathcal{F}_r$ . We conclude that  $a \in F$ .

Further, we know that  $\sigma_{ba}(F) \in \mathcal{F}_i$  since  $\mathcal{F}_i$  is shifted and  $b < a$ . But then we have  $|[a] \cap \sigma_{ba}(F) \cap \bigcap_{j \in [r] \setminus i} F_j| < t$ , which again contradicts  $\mathcal{F}_1, \dots, \mathcal{F}_r$  being  $r$ -cross  $t$ -intersecting with the largest necessary intersection point  $a$ . □

Roughly speaking, the proof proceeds by iteratively decreasing the largest necessary intersection point, that is, replacing the currently considered families by families with a smaller largest necessary intersection point. In this ‘updating’ process we need to be careful with those sets which need  $a$  fulfil the intersection property. To make this more precise, we introduce the following notation.

Let  $\mathcal{F}_1 \subseteq \mathcal{P}([n]), \dots, \mathcal{F}_r \subseteq \mathcal{P}([n])$  be  $r$ -cross  $t$ -intersecting families and let  $a$  be their largest necessary intersection point. For every  $j \in [r]$  define  $\mathcal{F}_j(a)$  to be the set of all  $F \in \mathcal{F}_j$  for which there exist  $F_i \in \mathcal{F}_i$  for every  $i \in [r] \setminus j$  such that (3.1) holds. We also refer to the sets in  $\mathcal{F}_j(a)$  as the sets in  $\mathcal{F}_j$  *depending on*  $a$ . Further, for  $A \subseteq [a - 1]$  set  $\mathcal{F}_j(A, a) = \{F \in \mathcal{F}_j(a) : F \cap [a - 1] = A\}$ .

The following lemma is the key of our proof. It will allow us to ‘push down’ the largest necessary intersection point of the families considered in case that we are not already done. Since we will prove Theorems 1.2 and 1.4 simultaneously (by proving Proposition 3.7), we phrase this lemma in a general setting. The families with indices in  $[r_1]$  are families as in Theorem 1.2 and the remaining families are as in Theorem 1.4.

**Lemma 3.3.** Let  $r, t, n \in \mathbb{N}, r_1 \in \mathbb{N}_0$  with  $r \geq r_1, r \geq 2$ , and  $r_1 \neq 1$ , and let  $a \in [n]$ . If  $r_1 \geq 2$ , suppose that  $k_1, \dots, k_{r_1} \in [n]$  are such that  $n \geq 2 \max_{i \in [r_1]} k_i + \text{secmin}_{i \in [r_1]} k_i - t$ , and let  $\mu_1, \dots, \mu_{r_1} :$

$[n]_0 \rightarrow \mathbb{R}_{\geq 0}$ . For  $i \in [r_1 + 1, r]$ , set  $k_i = n$  and let  $\mu_i : [n]_0 \rightarrow \mathbb{R}_{\geq 0}$  be non-increasing. For  $i \in [r]$ , let  $\mathcal{F}_i \subseteq [n]^{(\leq k_i)}$ . If  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are shifted  $r$ -cross  $t$ -intersecting families with largest necessary intersection point  $a \geq t + 1$  such that for all  $i \in [r]$ , the family  $\mathcal{F}_i \setminus \mathcal{F}_i(a)$  is non-empty, then there are non-empty families  $\mathcal{H}_1, \dots, \mathcal{H}_r$  such that

- (a) for  $i \in [r]$  we have  $\mathcal{H}_i \subseteq [n]^{(\leq k_i)}$ ,
- (b)  $\mathcal{H}_1, \dots, \mathcal{H}_r$  are  $r$ -cross  $t$ -intersecting with largest necessary intersection point at most  $a - 1$  and
- (c)  $\sum_{j \in [r]} \mu_j(\mathcal{H}_j) \geq \sum_{j \in [r]} \mu_j(\mathcal{F}_j)$ .

*Proof.* Roughly speaking, the families  $\mathcal{H}_1, \dots, \mathcal{H}_r$  will be obtained from  $\mathcal{F}_1, \dots, \mathcal{F}_r$  by deleting  $\mathcal{F}_i(a)$  from some of them and adding new sets to the others. More precisely, define for every  $i \in [r]$  the family

$$\mathcal{F}_i^{\text{add}} = \bigcup_{k \in [k_i]} \bigcup_{\substack{A \subseteq [a-1]: \\ \mathcal{F}_i(A, a)^k \neq \emptyset}} \{A \cup T : T \in [a + 1, n]^{(k - |A|)}\}, \tag{3.2}$$

and for  $i \in [r_1 + 1, r]$  define the family  $\mathcal{F}_i^{\text{add}} = \{F \setminus a : F \in \mathcal{F}_i(a)\}$ . Next, for  $i \in [r]$  we set  $\mathcal{F}_i^- = \mathcal{F}_i \setminus \mathcal{F}_i(a)$  and  $\mathcal{F}_i^+ = \mathcal{F}_i \cup \mathcal{F}_i^{\text{add}}$ . Note that for all  $i \in [r]$  we have  $\mathcal{F}_i^+, \mathcal{F}_i^- \subseteq [n]^{(\leq k_i)}$  and, hence, they satisfy (a).

We aim to show that considering  $\mathcal{F}_i^-$  for some indices and  $\mathcal{F}_j^+$  for the other indices will yield families as desired. To this end let us now observe the following claim, ensuring that such a collection will fulfil (b).

*Claim 3.4.* Let  $i \in [r]$ .

- (1) The families  $\mathcal{F}_1^-, \dots, \mathcal{F}_{i-1}^-, \mathcal{F}_i^+, \mathcal{F}_{i+1}^-, \dots, \mathcal{F}_r^-$  are  $r$ -cross  $t$ -intersecting with largest necessary intersection point at most  $a - 1$ .
- (2) The families  $\mathcal{F}_1^+, \dots, \mathcal{F}_{i-1}^+, \mathcal{F}_i^-, \mathcal{F}_{i+1}^+, \dots, \mathcal{F}_r^+$  are  $r$ -cross  $t$ -intersecting with largest necessary intersection point at most  $a - 1$ .

*Proof.*

- (1) Assume the contrary and let  $F_j \in \mathcal{F}_j^-$  for  $j \in [r] \setminus i$  and  $F_i \in \mathcal{F}_i^+$  such that  $|[a - 1] \cap \bigcap_{j \in [r]} F_j| < t$ . Since  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are  $r$ -cross  $t$ -intersecting, this means that there is some  $F' \in \mathcal{F}_i(a)$  (potentially  $F' = F_i$ ) with  $F_i \cap [a - 1] = F' \cap [a - 1]$ . But then  $|[a - 1] \cap F' \cap \bigcap_{j \in [r] \setminus i} F_j| < t$ , which is a contradiction because  $F_j \in \mathcal{F}_j^- = \mathcal{F}_j \setminus \mathcal{F}_j(a)$ .
- (2): Assume the contrary and let  $F_j \in \mathcal{F}_j^+$  for  $j \in [r] \setminus i$  and  $F_i \in \mathcal{F}_i^-$  such that  $|[a - 1] \cap \bigcap_{j \in [r]} F_j| < t$ . Since  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are  $r$ -cross  $t$ -intersecting, this means that for all  $j \in [r] \setminus i$  there is an  $F'_j \in \mathcal{F}_j(a)$  with  $F_j \cap [a - 1] = F'_j \cap [a - 1]$ . But then  $|[a - 1] \cap F_i \cap \bigcap_{j \in [r] \setminus i} F'_j| < t$ , which is a contradiction because  $F_i \in \mathcal{F}_i^- = \mathcal{F}_i \setminus \mathcal{F}_i(a)$ . □

Now, let us show that the updated families will still have maximum measure, that is, that (c) holds. This essentially follows from the next two claims.

*Claim 3.5.* For  $i \in [r]$  we have  $\mu_i(\mathcal{F}_i^{\text{add}}) \geq \mu_i(\mathcal{F}_i(a))$ .

*Proof.* If  $i \in [r_1 + 1, r]$ , note that the definition of  $\mathcal{F}_i^{\text{add}}$  implies an injection  $\varphi : \mathcal{F}_i(a) \rightarrow \mathcal{F}_i^{\text{add}}$  with  $|\varphi(F)| = |F| - 1$ . Thus, recalling that  $\mu_i$  is non-increasing, the claim is proved.

If  $i \in [r_1]$ , we need to work a bit more. If  $r_1 = 0$ , there is nothing else to show, so assume that  $r_1 \geq 2$ . First, we want to get an upper bound on  $a$ . Let  $s$  be the minimal integer such that there is some  $m_* \in [r_1]$  and  $A_* \in [a - 1]^{(s)}$  such that  $\mathcal{F}_{m_*}(A_*, a) \neq \emptyset$ . By definition we know that for  $F \in \mathcal{F}_{m_*}(A_*, a)$  there are  $F_j \in \mathcal{F}_j$  for all  $j \in [r] \setminus m_*$  such that  $|[a - 1] \cap F \cap \bigcap_{j \in [r] \setminus m_*} F_j| < t$ . Thus, Lemma 3.2 yields that  $[a - 1] \subseteq F \cup \bigcap_{j \in [r] \setminus m_*} F_j$ . Since  $|F \cap [a - 1]| = |A_*| = s$  and  $r_1 \geq 2$ , this entails  $a \leq s + 1 + \min_{j \in [r_1] \setminus m_*} k_j - t$ .

To show  $\mu_i(\mathcal{F}_i^{\text{add}}) \geq \mu_i(\mathcal{F}_i(a))$  it is enough to show that for all  $k \in [k_i]$  we have  $|(\mathcal{F}_i^{\text{add}})^k| \geq |(\mathcal{F}_i(a))^k|$ .

Further, it is easy to see that for all  $k \in [k_i]$

$$(\mathcal{F}_i(a))^k \subseteq \bigcup_{\substack{A \subseteq [a-1]: \\ \mathcal{F}_i(A,a)^k \neq \emptyset}} \{A \cup a \cup T : T \in [a + 1, n]^{(k-1-|A|)}\}.$$

Hence, in view of (3.2), to show  $|(\mathcal{F}_i^{\text{add}})^k| \geq |(\mathcal{F}_i(a))^k|$  it is enough to show that for every  $A \subseteq [a - 1]$  with  $\mathcal{F}_i(A, a)^k \neq \emptyset$  we have  $\binom{n-a}{k-1-|A|} \leq \binom{n-a}{k-|A|}$ , which in turn holds if  $\frac{n-a}{2} > k - 1 - |A|$ . And indeed, the bounds on  $a$  and  $n$  entail

$$\frac{n-a}{2} \geq \frac{n-s-1-\min_{j \in [r_1] \setminus m_*} k_j + t}{2} \geq \frac{2 \max_{i \in [r_1]} k_i - s - 1}{2} > k - 1 - |A|. \quad \square$$

Further let us observe the following.

*Claim 3.6.* For  $i \in [r]$  we have  $\mathcal{F}_i \cap \mathcal{F}_i^{\text{add}} = \emptyset$ .

*Proof.* Assume that there is some  $F \in \mathcal{F}_i \cap \mathcal{F}_i^{\text{add}}$ . Then, because  $F \in \mathcal{F}_i^{\text{add}}$ , there is some  $F' \in \mathcal{F}_i(a)$  with  $[a - 1] \cap F = [a - 1] \cap F'$ . For  $F'$  on the other hand, there are  $F_j \in \mathcal{F}_j$  for all  $j \in [r] \setminus i$  such that  $|[a] \cap F' \cap \bigcap_{j \in [r] \setminus i} F_j| = t$  and  $a \in F' \cap \bigcap_{j \in [r] \setminus i} F_j$ . But since  $F \in \mathcal{F}_i^{\text{add}}$ , we know that  $a \notin F$  and thus we have  $|[a] \cap F \cap \bigcap_{j \in [r] \setminus i} F_j| < t$ . This gives us a contradiction since  $F \in \mathcal{F}_i$  and  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are  $r$ -cross  $t$ -intersecting with largest necessary intersection point  $a$ .  $\square$

Finally, we can ‘update’ the collection of families. If  $\mu_r(\mathcal{F}_r(a)) \leq \sum_{i \in [r-1]} \mu_i(\mathcal{F}_i(a))$ , we consider the families  $\mathcal{H}_i = \mathcal{F}_i^+$  for  $i \in [r - 1]$  and  $\mathcal{H}_r = \mathcal{F}_r^-$ . Recall that we have  $\mathcal{H}_i \subseteq [n]^{\leq k_i}$  for  $i \in [r]$  and that they are non-empty by the condition that  $\mathcal{F}_i \setminus \mathcal{F}_i(a) \neq \emptyset$  for all  $i \in [r]$ . By Claim 3.4 these families are  $r$ -cross  $t$ -intersecting with their largest necessary intersection point at most  $a - 1$  and by Claims 3.5 and 3.6, and  $\mu_r(\mathcal{F}_r(a)) \leq \sum_{i \in [r-1]} \mu_i(\mathcal{F}_i(a))$  we have  $\sum_{i \in [r]} \mu_i(\mathcal{F}_i) \leq \sum_{i \in [r-1]} \mu_i(\mathcal{F}_i^+) + \mu_r(\mathcal{F}_r^-)$ . Together, this yields (a)–(c) in the conclusion of the lemma.

If  $\mu_r(\mathcal{F}_r(a)) \geq \sum_{i \in [r-1]} \mu_i(\mathcal{F}_i(a))$ , we consider the families  $\mathcal{F}_1^-, \dots, \mathcal{F}_{r-1}^-, \mathcal{F}_r^+$ . Similarly as before, it follows that these will satisfy (a)–(c).  $\square$

Both Theorems 1.2 and 1.4 can be obtained from the following more general result by setting  $r_1 = r$  and  $r_1 = 0$ , respectively. Moreover, this result also provides the maximum of  $\sum_{i \in [r]} \mu_i(\mathcal{F}_i)$  in the case when some of the families and measures satisfy the conditions in Theorem 1.2 and the others satisfy those in Theorem 1.4.

**Proposition 3.7.** *Let  $r, t, n \in \mathbb{N}$ ,  $r_1 \in \mathbb{N}_0$  with  $r \geq r_1$ ,  $r \geq 2$ , and  $r_1 \neq 1$ , and let  $a \in [n]$ . If  $r_1 \geq 2$ , for  $i \in [r_1]$  let  $k_i \in [n]$  be such that  $n \geq 2 \max_{i \in [r_1]} k_i + \text{secmin } k_i - t$ , and let  $\mu_i : [n]_0 \rightarrow \mathbb{R}_{\geq 0}$ .*

*For  $i \in [r_1 + 1, r]$  set  $k_i = n$  and let  $\mu_i : [n]_0 \rightarrow \mathbb{R}_{\geq 0}$  be non-increasing. For  $i \in [r]$  let  $\mathcal{F}_i \subseteq [n]^{(\leq k_i)}$ . If  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are non-empty r-cross t-intersecting families with largest necessary intersection point at most a, then*

$$\sum_{j \in [r]} \mu_j(\mathcal{F}_j) \leq \max \left\{ \mu_\ell(\mathcal{A}(n, a_*, t)^{\leq k_\ell}) + \sum_{j \in [r] \setminus \ell} \mu_j(\mathcal{B}(n, a_*)^{\leq k_j}) \right\}, \tag{3.3}$$

where the maximum is taken over  $\ell \in [r]$  and  $a_* \in [t, \min\{a, \min_{i \in [r] \setminus \ell} k_i\}]$ .

*Proof.* We perform an induction on  $r$ . The beginning is the same for the induction start and the induction step. Let all the parameters and  $\mu_i$  be given as in the statement of the theorem and note that without restriction  $t \leq \min_{i \in [r]} k_i$ . Further, let  $\mathcal{F}_1, \dots, \mathcal{F}_r$  be such that

- (1) for  $i \in [r]$  we have  $\mathcal{F}_i \subseteq [n]^{(\leq k_i)}$ ,
- (2) they are r-cross t-intersecting with largest necessary intersection point at most a,
- (3) they maximise  $\sum_{j \in [r]} \mu_j(\mathcal{F}_j)$  among all families satisfying (1) and (2),
- (4) their largest necessary intersection point is minimal among those families that fulfil (1)–(3).

Since the properties (1)–(4) are preserved when shifting, we may assume that  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are shifted. Denote the largest necessary intersection point of  $\mathcal{F}_1, \dots, \mathcal{F}_r$  by  $a_*$  and observe that if  $a_* = t$ , we are done. So we assume that  $a_* \geq t + 1$ .

Firstly, consider the case that for all  $i \in [r]$  we have that  $\mathcal{F}_i^- \neq \emptyset$ . Then Lemma 3.3 yields families  $\mathcal{H}_1, \dots, \mathcal{H}_r$  satisfying (1)–(3) with a largest necessary intersection point smaller than  $a_*$ . This is a contradiction to the choice of the families (see (4)) and thereby completes the proof of both the induction start and the induction step.

Secondly, consider the case that for some  $j \in [r]$ , without loss of generality  $r$ , it holds that  $\mathcal{F}_r \setminus \mathcal{F}_r(a_*) = \emptyset$ . That is to say, all sets in  $\mathcal{F}_r$  depend on  $a_*$ .

Assume that there is a  $b \in [a_* - 1]$  and  $F \in \mathcal{F}_r$  such that  $b \notin F$ . As  $\mathcal{F}_r$  is shifted, we have that  $\sigma_{ba_*}(F) \in \mathcal{F}_r$ , but this set does not depend on  $a_*$ . Hence, for every  $F \in \mathcal{F}_r$  we have  $[a_*] \subseteq F$ , in other words  $\mathcal{F}_r \subseteq \mathcal{B}(n, a_*)^{\leq k_r}$ .

For  $r = 2$  notice that since  $a_*$  is the largest necessary intersection point, every  $F_1 \in \mathcal{F}_1$  has at least  $t$  elements in  $[a_*]$ . This yields  $\mathcal{F}_1 \subseteq \mathcal{A}(n, a_*, t)^{\leq k_1}$  and hence

$$\mu_1(\mathcal{F}_1) + \mu_2(\mathcal{F}_2) \leq \mu_1(\mathcal{A}(n, a_*, t)^{\leq k_1}) + \mu_2(\mathcal{B}(n, a_*)^{\leq k_2}),$$

which finishes the proof of the induction start.

For  $r \geq 3$  observe that the families  $\mathcal{F}_1, \dots, \mathcal{F}_{r-1}$  are  $(r - 1)$ -cross t-intersecting families with largest necessary intersection point at most  $a_*$  which maximise  $\sum_{j \in [r-1]} \mu_j(\mathcal{F}_j)$  (among all  $(r - 1)$ -cross t-intersecting families  $\mathcal{G}_i \subseteq [n]^{(\leq k_i)}$  with the largest necessary intersection point at most  $a_*$ ). Thus, the induction hypothesis implies that there is an  $\ell \in [r - 1]$  and an  $a_{**} \in [a_*]$  such that

$$\sum_{j \in [r-1]} \mu_j(\mathcal{F}_j) \leq \mu_\ell(\mathcal{A}(n, a_{**}, t)^{\leq k_\ell}) + \sum_{j \in [r-1] \setminus \ell} \mu_j(\mathcal{B}(n, a_{**})^{\leq k_j}).$$

Since  $\mathcal{F}_r \subseteq \mathcal{B}(n, a_*)^{\leq k_r} \subseteq \mathcal{B}(n, a_{**})^{\leq k_r}$ , this entails

$$\sum_{j \in [r]} \mu_j(\mathcal{F}_j) \leq \mu_\ell(\mathcal{A}(n, a_{**}, t)^{\leq k_\ell}) + \sum_{j \in [r] \setminus \ell} \mu_j(\mathcal{B}(n, a_{**})^{\leq k_j}),$$

which finishes the induction step. □

#### 4 | CONCLUDING REMARKS

Observe that the maxima in our results are attained for different  $i$  (and  $\ell$ ), depending on the measures and  $r, t$  and  $n$ . However, we remark the following.

*Remark 4.1.* For given  $t, n, k \in \mathbb{N}$  and a measure  $\mu$  there is an  $r_0$  such that if  $r \geq r_0$ , the maximum in Theorems 1.2 and 1.4 is always attained for  $i = t$  if  $\mu = \mu_j$  (and  $k_j = k$ ) for all  $j \in [r]$ .

One can also ask for the maximum of the product of sizes or, more generally, the product of measures of  $r$ -cross  $t$ -intersecting families, instead of the sum. More precisely, for given measures  $\mu_1, \dots, \mu_r$  find the maximum possible value of

$$\prod_{i \in [r]} \mu_i(\mathcal{F}_i) \tag{4.1}$$

for  $\mathcal{F}_1, \dots, \mathcal{F}_r$  being  $r$ -cross  $t$ -intersecting families.

There are some partial results concerning this problem ([8, 19, 29, 30]). Frankl and Tokushige [22] determined the maximum product of the sizes of  $r$ -cross 1-intersecting families. In [8], Borg determined the maximum of (4.1) for  $r$ -cross  $t$ -intersecting uniform families and measures with certain properties (which include the product measure, the uniform measure and the constant measure) (see also [9] for a general result). Moreover, in [7], Borg also determined the maximum product of sizes for  $r$ -cross 1-intersecting families in which the size of the sets is bounded, that is, each of the families is of the form  $[n]^{\leq k}$ , where  $n$  and  $k$  may be distinct across the different families. It is well known that for  $a_1, \dots, a_r \in \mathbb{R}_{\geq 0}$  with  $\sum_{i \in [r]} a_i \leq a$  the product  $\prod_{i \in [r]} a_i$  is maximised if  $a_i = \frac{a}{r}$  for all  $i \in [r]$ . Therefore, considering Remark 4.1, given  $n$ , measures  $\mu_i = \mu$  (and  $k_i = k$ ) with  $\mu$  (and  $k$  and  $n$ ) satisfying the conditions in Theorem 1.2 or Theorem 1.4, there is an  $r_0$  such that for  $r \geq r_0$  these theorems actually also yield that the maximum of (4.1) is  $(\mu(\mathcal{B}(n, t)^{\leq k}))^r$ . This particularly includes the product measure, the uniform measure and the constant measure, and solves a few instances of the Problems 12.10 and 12.11, and of the Conjectures 12.12 and 12.13 posed by Frankl and Tokushige in [23].

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