Improving Condensation Methods in the Presence of General Masters by a Modified Rayleigh Functional

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Abstract: In the dynamic analysis of structures condensation methods are often used to reduce the number of degrees of freedom to manageable size. The approximation properties of these methods can be enhanced considerably taking advantage of the exactly condensed problem and the corresponding Rayleigh functional or by using general masters. In this note we discuss how to combine these two approaches.

Keywords: generalized eigenvalue problem, condensation, Rayleigh functional, general masters

1 Introduction

In the analysis of the dynamic response of structures using finite element methods very often prohibitively many degrees of freedom are needed to model the behaviour of the system sufficiently accurate. Static condensation is frequently employed to economize the computation of a selected group of eigenvalues and eigenvectors. These methods choose from the degrees of freedom a small number of master variables which appear to be representative. Neglecting inertia terms the remaining variables (termed slaves) are eliminated leaving a much smaller problem for the master variables only.

It has frequently been noted in the literature that the quality of the eigenvalue and eigenvector approximations produced by static condensation is satisfactory only for a very small part of the lower end of the spectrum and several attempts have been made to improve the approximation properties (cf. [6], [7], [8], [10], [11], [12], [13], [14], [15], e.g.). Most of these approaches are very time consuming since every wanted eigenvalue has to be corrected individually by an iterative process and each iteration step requires the solution of a large linear system.

In [13], [15] we took advantage of properties of the exactly condensed eigenvalue problem which is a nonlinear eigenvalue problem $T(\lambda)u = 0$ and which is equivalent to the original problem. For T a Rayleigh functional exists which has similar properties as the Rayleigh quotient of a linear eigenproblem. In particular, the eigenvectors of $T(\cdot)$ are stationary points of p. Hence, evaluating the Rayleigh functional at the eigenvectors of the condensed problem improves the quality of the corresponding eigenvalue approximations substantially. The improvements can be obtained at very low cost if condensation is combined with substructuring and the masters are chosen to be interface degrees of freedom of the substructures.

Incorporating general masters into the condensation process a different enhancement of the approximation properties was derived in [10]. This method has the advantage of being able to choose more suitable generalized coordinates than just the displacements at some master nodes. Hence a priori information of the eigenmodes such as eigenmodes of similar structures considered in reanalysis (cf. [4]) or prolongations of eigenvector approximations obtained on a coarser grid can be implemented into the condensation process. The disadvantage is that usually in the presence of general masters the Rayleigh functional of the corresponding exactly condensed problem can no longer be evaluated at low cost.

In this note we combine the benefits of both approaches. We first determine approximate eigenvalues $\tilde{\lambda}_j$ and eigenvectors $\tilde{u}_j$ from a condensed eigenvalue problem where we take advantage of generalized masters. We restrict the eigenvector approximations to the interface masters of a suitable substructuring, and we evaluate the Rayleigh functional of these restrictions yielding enhancements of some of the $\tilde{\lambda}_j$ at low cost.

The paper is organized as follows: In Section 2 we briefly sketch condensation in the presence of generalized masters. Section 3 introduces the Rayleigh functional of the exactly condensed problem and its efficient evaluation if the masters are chosen as displacements at the interfaces of a substructuring. The improvement of eigenvalue approximations from a condensed problem in the presence of general masters is discussed. Finally, in Section 4 we demonstrate the efficiency of the approach by a numerical example.

2 General Masters in Condensation

We consider the general eigenvalue problem

$$Kx = \lambda M x \tag{1}$$

where $K \in \mathbb{R}^{(n,n)}$ and $M \in \mathbb{R}^{(n,n)}$ are symmetric and positive definite matrices which are usually the stiffness and mass matrix of a finite element model of a structure, respectively.

To reduce the number of degrees of freedom Irons [5] and Guyan [2] suggested to choose a small number of unknowns x_m which are to be retained und to rewrite (1) into the block form

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{bmatrix} x_m \\ x_s \end{bmatrix} = \lambda \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{bmatrix} x_m \\ x_s \end{bmatrix} \tag{2}$$

Neglecting the inertia terms in the second equation, solving for x_s, and substituting x_s into the first equation one obtains the statically condensed eigenproblem

$$\tilde{K}_0 x_m = \lambda \tilde{M}_0 x_m \tag{3}$$

where

$$\begin{aligned} \tilde{K}_0 &:= K_{mm} - K_{ms}K_{ss}^{-1}K_{sm}, \\ \tilde{M}_0 &:= M_{mm} - K_{ms}K_{ss}^{-1}M_{sm} \\ &\quad - M_{ms}K_{ss}^{-1}K_{sm} + K_{ms}K_{ss}^{-1}M_{ss}K_{ss}^{-1}K_{sm}. \end{aligned} \tag{4}$$

This reduction is called nodal condensation. It has the disadvantage that it produces accurate results only for a small part of the lower end of the spectrum. In [10] the approximation properties were enhanced substantially by general masters.

Let $z_1, \ldots, z_m$ be linearly independent master vectors, and supplement it by a basis $y_{m+1}, \ldots, y_n$ of the orthogonal complement of $\operatorname{span}\{z_1, \ldots, z_m\}$. If we define $Z := (z_1, \ldots, z_m) \in \mathbb{R}^{(n,m)}$ and $Y := (y_{m+1}, \ldots, y_n) \in \mathbb{R}^{(n,n-m)}$ then x has the unique representation

$$x = Zx_m + Yx_s.$$

If we insert this into the original problem (1) and premultiply it by $(Z, Y)^T$ we obtain the following eigenvalue problem

$$\begin{bmatrix} K_{zz} & K_{zy} \\ K_{yz} & K_{yy} \end{bmatrix} \begin{bmatrix} x_m \\ x_s \end{bmatrix} = \lambda \begin{bmatrix} M_{zz} & M_{zy} \\ M_{yz} & M_{yy} \end{bmatrix} \begin{bmatrix} x_m \\ x_s \end{bmatrix} \tag{5}$$

where for $L \in \{K, M\}$

$$L_{zz} := Z^T L Z,\ L_{zy} := Z^T L Y =: L_{yz}^T,\ L_{yy} := Y^T L Y. \tag{6}$$

Therefore, the stiffness and the mass matrix have been decomposed with respect to the spaces Z and Y in a similar way as in (2), and indeed it covers the special case of nodal condensation by setting $(Z, Y) := I_n$.

In principle equation (5) could be employed to reduce the eigenvalue problem (1) using $\{z_1, \ldots, z_m\}$ as master degrees of freedom. However, since in practice only the small set of masters is available, but the large set of slave vectors $\{y_{m+1}, \ldots, y_n\}$ is definitely not the matrices $K_{zy}, K_{yy}, M_{zy}, M_{yy}$ are usually not at hand. Hence, the straightforward transfer of (3), (4) to problem (5) to perform the reduction in the presence of general masters does not apply. In [10] it has been shown how to generate the condensed problem corresponding to the decomposition (5) with the basis $z_1, \ldots, z_m$ only.

Theorem 1 *Let $Z = (z_1, \ldots, z_m) \in \mathbb{R}^{(n,m)}$ have maximal rank. Then the condensed eigenvalue problem with general masters $z_1, \ldots, z_m$ is given by*

$$P^T K P x_m = \lambda P^T M P x_m \tag{7}$$

with the projection matrix

$$P = K^{-1} Z \left(Z^T K^{-1} Z\right)^{-1} Z^T Z. \tag{8}$$

Since $\left(Z^T K^{-1} Z\right)^{-1} Z^T Z \in \mathbb{R}^{(m,m)}$ is a nonsingular matrix the condensed problem is equivalent to the projection of problem (1) to the space spanned by the columns of $K^{-1}Z$. Hence, equation (8) demonstrates that condensation is nothing else but one step of simultaneous inverse iteration with initial space $X := M^{-1}Z$. This observation indicates that we can expect good eigenvalue and eigenvector approximations from the condensed problem (7) if we introduce general masters $z_j := Mx_j$ where x_j denotes approximate eigenvectors of (1). Here we have in mind known eigenvectors of a similar structure as in eigenreanalysis (cf. [4], [9], [18]) or known vibration modes of substructures as in component mode synthesis (cf.[1], [16], [17]) or prolongations of known approximate eigenvectors obtained on a coarser grid (cf. Section 4).

3 Improvement by Rayleigh Functional

A different approach for improving condensation is to take advantage of the properties of the exactly condensed problem which is obtained by solving the second equation of (2) for x_s and substituting x_s in the first equation. One gets a nonlinear eigenvalue problem

$$T(\lambda)x_m = 0. \tag{9}$$

It is well known that $T(\lambda)$ can be given a more convenient form if modal properties of the slave problem are exploited. Let $\Phi \in \mathbb{R}^{(s,s)}$ and $\Omega := \operatorname{diag}\{\omega_j\} \in \mathbb{R}^{(s,s)}$ be the modal matrix and the spectral matrix of the slave eigenvalue problem

$$K_{ss}\phi = \omega M_{ss}\phi, \tag{10}$$

respectively, such that $\Phi^T M_{ss}\Phi = I$ and $\Phi^T K_{ss}\Phi = \Omega$. Then $T(\lambda)$ can be rewritten as (cf. Leung [8])

$$T(\lambda) = -K_0 + \lambda M_0 + SD(\lambda)S^T \tag{11}$$

where K_0 and M_0 are the reduced stiffness and mass matrix of the statically condensed problem, and

$$S := M_{ms}\Phi - K_{ms}\Phi\Omega^{-1},\ D(\lambda) := \operatorname{diag}\left\{\frac{\lambda^2}{\omega_j - \lambda}\right\}.$$

By the way, this representation demonstrates that the statically condensed problem is the linearization of the exactly condensed problem at $\hat{\lambda} = 0$.

Let $\underline{\omega}$ be the smallest eigenvalue of the slave eigenproblem (10), and let $J := (0, \underline{\omega})$. Then for every fixed vector $u \in \mathbb{R}^m$, $u \neq 0$, the real valued function

$$f(\cdot, u)\ :\ J \to \mathbb{R},\ \lambda \mapsto f(\lambda, u) := u^T T(\lambda)u,$$

i.e.

$$f(\lambda, u) = -u^T K_0 u + \lambda u^T M_0 u + \sum_{j=1}^{s} \frac{\alpha_j^2 \lambda^2}{\omega_j - \lambda} \tag{12}$$

with

$$\alpha_j := \phi_j^T M_{sm} u - \frac{1}{\omega_j}\phi_j^T K_{sm} u$$

is strictly monotonely increasing. Hence, the nonlinear equation $f(\lambda, u) = 0$ has at most one solution in J. Therefore, it implicitly defines a functional

$$p\ :\ \mathbb{R}^m \supset D(p) \to J,\ f(p(u), u) = 0,$$

which is called the Rayleigh functional of the nonlinear eigenproblem (9).

The Rayleigh functional has similar properties as the Rayleigh quotient for linear eigenproblems. In particular, the eigenvalues of problem (1) contained in J which are identical to the eigenvalues of the nonlinear eigenvalue problem (9) in J can be characterized as minmax values of p, and the eigenvectors of $T(\cdot)$ are stationary vectors of p (cf. [15]). Thus, if $\bar{u} \in D(p)$ is a first order approximation of an eigenvector then $p(\bar{u})$ will be a second order approximation of the corresponding eigenvalue. Evaluating the Rayleigh functional at the eigenvectors of the statically condensed problem (which are actually contained in $D(p)$) therefore should improve the corresponding eigenvalue approximations considerably.

At a first glance this observation seems to be of doubtful use since all eigenvalues and eigenvectors of the slave eigenvalue problem (10) are needed to evaluate the Rayleigh functional at some vector $u \in \mathbb{R}^m$, and the dimension s of the slave problem usually will be nearly as big as n, the dimension of the original problem. However, if we combine condensation with substructuring, i.e. if we decompose the structure under consideration into r substructures and if we choose the masters as the interface degrees of freedom of the substructures such that the substructures connect to each other through master variables only then (numbering the slave variables appropriately) the stiffness matrix is given by

$$K = \begin{bmatrix} K_{mm} & K_{ms1} & K_{ms2} & \dots & K_{msr} \\ K_{sm1} & K_{ss1} & O & \dots & O \\ K_{sm2} & O & K_{ss2} & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{smr} & O & O & \dots & K_{ssr} \end{bmatrix}$$

and the mass matrix M has the same block form. Hence, the slave problem (10) splits into r independent eigenvalue problems

$$K_{ssj}\phi_{ji} = \omega_{ji} M_{ssj}\phi_{ji},\ j = 1, \dots, r.$$

The coefficients α_j of the rational function in (12) can be determined substructurewise, and f obtains the form

$$\begin{aligned} f(\lambda, u) &:= -u^T K_0 u + \lambda u^T M_0 u \\ &+ \sum_{j=1}^{r}\sum_{i=1}^{s_j} \frac{\lambda^2}{\tilde{\omega}_{ji} - \lambda}\Big(\phi_{ji}^T M_{smj} u - \frac{1}{\omega_{ji}}\phi_{ji}^T K_{smj} u\Big)^2 \\ &=: \ -\kappa_0 + \lambda\kappa_1 + \sum_{j=1}^{r}\sum_{i=1}^{s_j} \sigma_{ji}\frac{\lambda^2}{\tilde{\omega}_{ji} - \lambda} \qquad (13) \end{aligned}$$

where s_j denotes the number of slave degrees of freedom in the j-th substructure.

Moreover, a considerable saving of work can be made by the observation that usually the substructures are much stiffer than the entire structure. Hence, only very few substructure modes have to be considered in the evaluation of the Rayleigh functional to achieve an eigenvalue approximation of good accuracy. Consequently we replace the Rayleigh functional by the solution $\tilde{p}(u) \in J$ of a curtailed rational function

$$\tilde{f}(\lambda, u) =: -\kappa_0 + \lambda\kappa_1 + \sum_{j=1}^{r}\sum_{i=1}^{\tilde{s}_j} \sigma_{ji}\frac{\lambda^2}{\tilde{\omega}_{ji} - \lambda}$$

where $\tilde{s}_j \ll s_j,\ j = 1, \dots, r$. Notice, that for a given eigenvector approximation u the function $\lambda \mapsto \tilde{f}(\lambda, \tilde{u})$ is monotonely increasing and convex, and therefore the solution $\tilde{p}(u)$ of $\tilde{f}(\lambda, u) = 0$ can be computed easily with Newton's method.

Of course these considerations hold if we replace the decomposition (2) by (5). However, in the presence of general masters the matrices K_{yy} and M_{yy} are not at hand and therefore the Rayleigh functional can not be evaluated using the representation of $T(\lambda)$ in (11). In [3] it has been shown that the following iteration

$$\begin{aligned} w_k &:= (K - \lambda_k M)^{-1} X (X^T (K - \lambda_k M)^{-1} X)^{-1} u \\ \lambda_{k+1} &:= \frac{w_k^T K w_k}{w_k^T M w_k} \end{aligned}$$

converges locally to the value $p(u)$ of the Rayleigh functional. However, this iteration requires the solution of a large linear system in every step similarly as in the approaches in [7], [8], [11] or [14].

The way out is to combine the merits of two condensation methods, i.e. to use preinformations on eigenvectors of problem (1) as general masters to determine reasonable eigenvector approximations $u_1, \dots, u_k$ of a reduced problem, to determine restrictions $\tilde{u}_j$ of u_j to the interface degrees of freedom of a suitable substructuring, and to improve the eigenvalue approximations by $\tilde{p}(\tilde{u}_j)$ where $\tilde{p}(\tilde{u}_j)$ denotes the root of the curtailed rational function $\tilde{f}(\cdot, \tilde{u}_j)$ corresponding to the chosen substructuring.

In [9] and [4] we developed parallel condensation methods for the case that the interface masters of a substructuring are complemented by a small number of approximate eigenvectors as general masters. In this case the quadratic forms $\kappa_0 := \tilde{u}^T K_0 \tilde{u}$ and $\kappa_1 := \tilde{u}^T M_0 \tilde{u}$ in (13) can obtained easily from the data produced when determining the reduced problem, and the additional cost to evaluate the modified curtailed Rayleigh functional are negligible.

4 A numerical example

We consider the free vibration problem of a uniform thin clamped plate covering the rectangular region $\Omega := (0,4) \times (0,3)$ which are governed by the eigenvalue problem

$$\Delta^2 u = \lambda u \quad \text{in } \Omega,\ u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

We discretized this problem by Bogner-Fox-Schmidt elements (with node variables u, u_x, u_y and u_{xy}) on a quadratic mesh of meshsize $h = 0.1$ and obtained a discrete problem of dimension $n = 4524$. Dividing Ω into 12 identical substructures each of them being a square of sidelength 1 and choosing all interface degrees of freedom as masters we obtained a reduced problem of dimension $m = 636$.

For the 10 smallest eigenvalues of the discrete problem Fig. 1 displays the relative errors of the approximations of the nodally condensed problem, its improvement by the Rayleigh functional, and the chopped Rayleigh functional where only 3 eigenmodes of each of the 12 substructures were considered.

We discretized the plate problem with Bogner-Fox-Schmidt elements with stepsize $h = 1$ resulting in a matrix eigenvalue problem of dimension 24. We prolongated the eigenvectors corresponding to the 4 smallest eigenvalues to the fine grid by bi-cubic interpolation, multiplied them by the mass matrix M, and used these vectors as additional general masters. Table Fig. 2 contains the relative errors of the condensed problem, of the Rayleigh functional and of the modified Rayleigh functional considering only the nodal masters on the interfaces of the substructures. The eigenvalue approximations of the smallest 4 eigenvalues are improved by 3 orders of magnitude whereas the higher eigenvalues are effected not very much. Notice that the Rayleigh functionals and its modification are of the same quality.

Fig. 3 shows the relative errors of the approximations by the curtailed modified Rayleigh functional if we take into account 1, 3, 4, 8 and 16 slave eigenmodes of each substructure, respectively. We did not consider the case of 2 slave eigenmodes since the second eigenvalue of each substructure has multiplicity 2. The higher eigenvalue approximations gain a good deal of accuracy even for a small number of slave eigenmodes considered in the curtailed Rayleigh functional wheres the outstanding approximation properties for the smallest 4 eigenvalues are deteriorated. Before determining the eigenvector approximation one should choose the minimum of the Rayleigh functional and the corresponding eigenvalue of the condensed problem as eigenvalue approximation.

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Fig.1

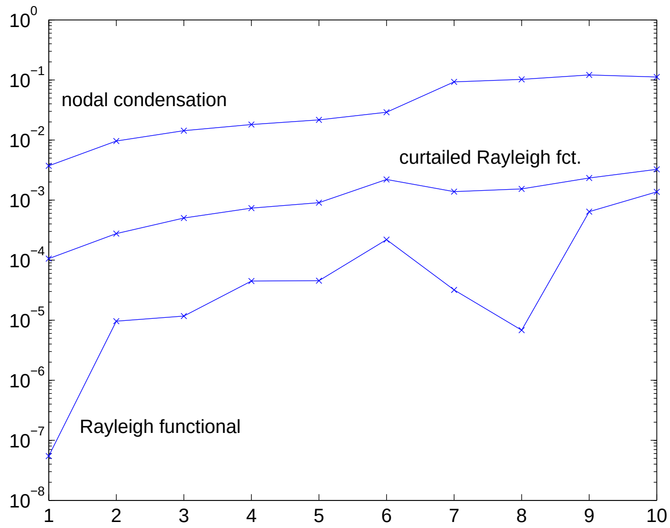


Fig.2

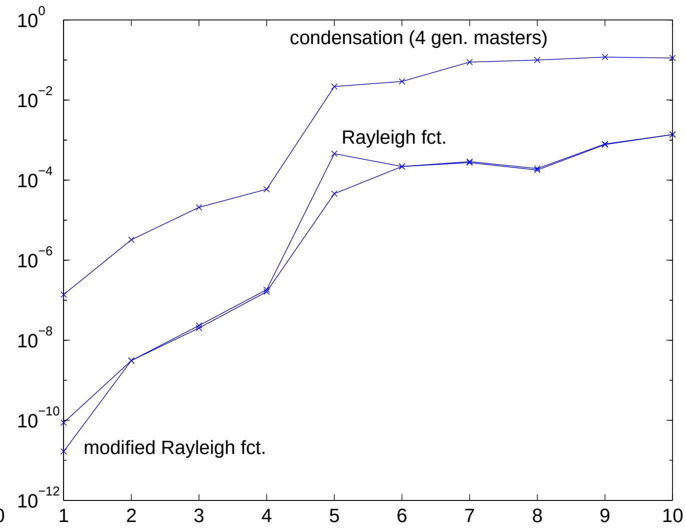
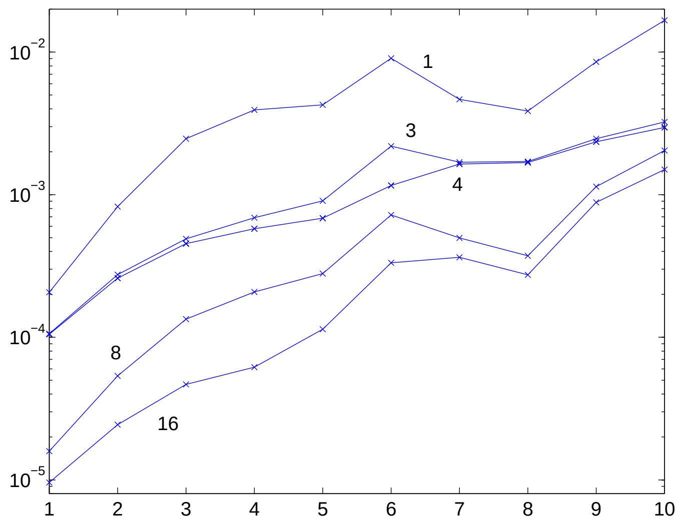


Fig.3



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