

Model Predictive Control of Thermo-Hydraulic Systems Using Primal Decomposition [★]

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Abstract: Decarbonizing the global energy supply requires more efficient heating and cooling systems. Model predictive control enhances the operation of cooling and heating systems but depends on accurate system models, often based on control volumes. We present an automated framework including time discretization to generate model predictive controllers for such models. To ensure scalability, a primal decomposition exploiting the model structure is applied. The approach is validated on an underground heating system with varying numbers of states, demonstrating the primal decomposition’s advantage regarding scalability.

Keywords: Model predictive control, Thermal systems modeling, Large-scale and networked optimization problems, Primal decomposition, Control and optimization for sustainability and energy systems, Energy management systems, Urban energy distribution systems

1. INTRODUCTION

The European Commission (2019) specified that the heat supply of buildings in the European Union must be decarbonized by 2050. A recent study by Fraunhofer ISE (2021) showed that the share of buildings supplied with heat by district heating networks (DHNs) needs to increase in Germany, in order to meet the targets. This finding is consistent with the results of Lund et al. (2010). To meet the climate targets, DHNs must be decarbonized. According to Lund et al. (2014), an important aspect of decarbonizing DHNs is increasing efficiency, for example through intelligent operation strategies, or decreasing the overall system temperature. In addition, cooling is the fastest-growing energy use in buildings globally. According to the IEA (2018), efficiency measures can reduce the global electricity costs related to cooling demand by approximately 38% compared to a baseline scenario. In summary, the need for efficiency measures for heating and cooling systems will increase in the future.

Efficiency can be increased through optimized planning and more efficient operating procedures. The optimal operation of DHNs using model predictive control (MPC) is studied in Maurer et al. (2023) and Verrilli et al. (2017), while Hering et al. (2021), Wack et al. (2023), and Lambert and Spliethoff (2024) focus on the optimal planning of DHNs. Deng et al. (2017) apply optimization approaches to district heating and cooling networks (DHCNs). These

systems allow for the combined supply of heat and cooling to buildings while using the soil as natural storage. In Taheri et al. (2022), a literature review on MPC for heating, ventilation, and air conditioning (HVAC) systems is conducted, concluding that MPC for HVAC systems has gained attention in recent years.

All of these systems have two things in common: they can be modeled using control volumes (CVs), and a fluid (e.g., water) often circulates in pipes to transport energy. For instance, Westphal et al. (2025) present a library for simulating DHNs that can also be used for cooling systems with minor modifications. This library is used in Vieth et al. (2025) for optimal DHN planning. In Maurer et al. (2021), several pipe models are compared to find the best model for MPC of DHNs. The authors conclude that models based on CVs are a reasonable choice because they are simple to implement and have a constant number of variables, unlike the *Node Method* first presented by Benonysson (1991). However, models based on CVs lead to systems governed by time-continuous differential-algebraic equations (DAEs).

Automated approaches for MPC generation for time-continuous DAEs are presented in Fabien (2010), Houska et al. (2011), and Chen et al. (2019). In Fabien (2010), the optimal control problem is solved using a direct method. According to Olanrewaju and Maciejowski (2017), in direct methods first the time-continuous DAEs are time-discretized, and then an optimization problem is solved based on the discretized system model. In contrast, the approaches presented by Houska et al. (2011) and Chen

[★] his research is supported by the German federal ministry of economic affairs and climate action (BMWK) under the agreement no. 03EWR007O2.

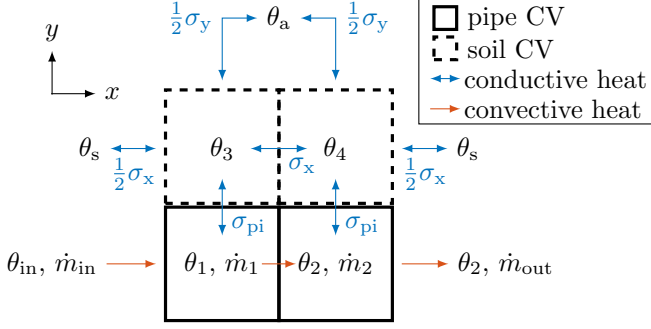


Fig. 1. Simple setup of CVs

et al. (2019) employ multiple shooting techniques to solve the optimal control problem based on the continuous system model.

This work presents a direct method for thermo-hydraulic systems modeled by CVs. One drawback of a direct approach is that the time discretization of nonlinear systems is not trivial. However, since the discretization occurs beforehand, this method may be more computationally efficient. Additionally, focusing on thermo-hydraulic systems allows us to exploit the specific structure of the models using primal decomposition.

Contribution In this work, we present an automated workflow for MPC generation consisting of

- a general dynamic system model for thermo-hydraulic systems (Section 2)
- a procedure for time discretization of nonlinear dynamic systems based on Backward-Differentiation (Section 3)
- a fast and scalable MPC approach for thermo-hydraulic systems using primal decomposition (Section 4)

Notation The identity matrix with n rows and columns is represented by \mathbf{I}_n and $\mathbf{1}_{n \times m}$ is a matrix with n rows and m columns where all elements are equal to one. Vectors are denoted by lowercase bold letters, and matrices by uppercase bold letters.

2. DYNAMICS OF SYSTEMS MODELED BY CONTROL VOLUMES

Consider the system shown in Figure 1, which could represent any type of heating or cooling system. The differential equations (DEs)

$$\frac{d\boldsymbol{\theta}}{dt} = \mathbf{A}\boldsymbol{\theta} + \mathbf{D}\mathbf{d} + \mathbf{X}_\theta [\dot{\mathbf{m}} \circ (\mathbf{Z}_\theta \boldsymbol{\theta})] + \mathbf{X}_z [\mathbf{Y}_z \dot{\mathbf{m}} \theta_{in}] \quad (1a)$$

describe the temperatures $\boldsymbol{\theta}^\top = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4]$ and the mass flows $\dot{\mathbf{m}}^\top = [\dot{m}_1 \ \dot{m}_2]$ in the CVs. The vector $\mathbf{d}^\top = [\theta_s \ \theta_a]$ comprises the non-controllable inputs: the soil temperature θ_s and the air temperature θ_a . Here, the input temperature θ_{in} is assumed to be an algebraic state (later denoted by z), as in many heating or cooling systems it would be linked with θ_2 and an input variable via an algebraic equation. The constant matrices in (1a) are

$$\mathbf{A} = \begin{bmatrix} -\frac{\sigma_{pi}}{c_w \rho_w V} & 0 & \frac{\sigma_{pi}}{c_w \rho_w V} & 0 \\ 0 & -\frac{\sigma_{pi}}{c_w \rho_w V} & 0 & \frac{\sigma_{pi}}{c_w \rho_w V} \\ \frac{\sigma_{pi}}{c_s \rho_s V} & 0 & -\frac{\sigma_{pi} + \frac{3}{2}\sigma_x + \frac{1}{2}\sigma_y}{c_s \rho_s V} & \frac{\sigma_x}{c_s \rho_s V} \\ 0 & \frac{\sigma_{pi}}{c_s \rho_s V} & \frac{\sigma_x}{c_s \rho_s V} & -\frac{\sigma_{pi} + \frac{3}{2}\sigma_x + \frac{1}{2}\sigma_y}{c_s \rho_s V} \end{bmatrix},$$

$$\mathbf{D} = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ \frac{\sigma_x}{c_s \rho_s V} & \frac{\sigma_y}{c_s \rho_s V} \\ \frac{\sigma_x}{c_s \rho_s V} & \frac{\sigma_y}{c_s \rho_s V} \end{bmatrix}, \mathbf{X}_\theta = \begin{bmatrix} \frac{1}{\rho_w V} & 0 \\ 0 & \frac{1}{\rho_w V} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{X}_z = \begin{bmatrix} \frac{1}{\rho_w V} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{Z}_\theta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \mathbf{Y}_z = [1 \ 0].$$

Here, V is the volume of each CV, σ_x , σ_y , and σ_{pi} are thermal conductivities, and c_i and ρ_i are the heat capacity and density of the soil (s) and the water (w) inside the pipe, respectively, all considered as constant. In (1a), we assume that changes in enthalpy due to changes in pressure are negligible, a condition that is generally met for the systems under consideration. Additionally, algebraic equations (AEs)

$$\dot{m}_{in} = \dot{m}_1 = \dot{m}_2 = \dot{m}_{out} \quad (1b)$$

connect the mass flow variables. Note that inserting (1b) into (1a) would simplify (1a). However, the structure of (1a) leads us to the DAE system

$$\frac{d\boldsymbol{\theta}}{dt} = \mathbf{A}\boldsymbol{\theta} + \mathbf{B}\mathbf{u} + \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{d} + \sum_{v \in \mathcal{V}} \mathbf{X}_v [(\mathbf{Y}_v \dot{\mathbf{m}}) \circ (\mathbf{Z}_v \mathbf{v})], \quad (2a)$$

$$\mathbf{0} = \mathbf{f}_h(\dot{\mathbf{m}}, \mathbf{z}_h, \mathbf{u}_h, \mathbf{d}), \quad (2b)$$

$$\mathbf{0} = \mathbf{f}_t(\boldsymbol{\theta}, \dot{\mathbf{m}}, \mathbf{z}, \mathbf{u}, \mathbf{d}), \quad (2c)$$

$$\mathbf{0} \geq \mathbf{g}_h(\dot{\mathbf{m}}, \mathbf{z}_h, \mathbf{u}_h, \mathbf{d}), \quad (2d)$$

$$\mathbf{0} \geq \mathbf{g}_t(\boldsymbol{\theta}, \dot{\mathbf{m}}, \mathbf{z}, \mathbf{u}, \mathbf{d}), \quad (2e)$$

which represents an arbitrary system of interconnected CVs with the controllable inputs $\mathbf{u}^\top = [\mathbf{u}_h^\top \ \mathbf{u}_t^\top]$ and the algebraic variables $\mathbf{z}^\top = [\mathbf{z}_h^\top \ \mathbf{z}_t^\top]$. The subscript h denotes variables related to the hydraulics (e.g. pressure differences in pipes) and t to the thermal-model (e.g. power input at heat exchangers). All matrices are constant, \mathbf{f}_h is a function representing the hydraulic AEs, \mathbf{f}_t all other AEs, \mathbf{g}_h the hydraulic inequality constraints, \mathbf{g}_t all other inequality constraints, and $\mathcal{V} = \{\boldsymbol{\theta}, \mathbf{z}, \mathbf{u}, \mathbf{d}\}$. The reason for extracting the hydraulic constraints from the other constraints is solely for notational purposes and does not lead to any modeling restrictions, since all variables can occur in \mathbf{f}_t and \mathbf{g}_t . This approach is motivated by the primal decomposition scheme, as discussed in Section 4.2.

3. TIME DISCRETIZATION OF SYSTEMS MODELED BY CONTROL VOLUMES

The goal of this section is to find a time-discrete representation of (2a). Common techniques for the time discretization of nonlinear control systems include the Taylor approximation, as presented in Kazantzis and Kravaris (1999) and Park et al. (2004), Runge-Kutta integration, see Herty et al. (2013) and Frey et al. (2024), and backward differentiation, as performed in Vaclavek and Blaha (2013). As shown in Vaclavek and Blaha (2013), Taylor approximation and Runge-Kutta integration can result in time-discrete system dynamics with additional complicating nonlinear terms. Furthermore, as the discretization

order increases, the general structure of (2a) may change due to the discretization. This has the disadvantage of making the time-discrete form of (2a) non-bilinear. Thus, the complexity of optimization within MPC is probably increased. Additionally, the ability to obtain linear optimization problems using primal decomposition disappears if nonlinear equations that are not bilinear result from the time-discretization approach.

For this reason, second-order backward differentiation is used. According to Alikhani et al. (2016), the time-discrete system dynamics are given by

$$\begin{aligned} \mathbf{0} &= \mathbf{f}_{\text{de},k}(\boldsymbol{\theta}_k, \dot{\mathbf{m}}_k, \mathbf{z}_k, \mathbf{u}_k, \mathbf{d}_k) \\ &= \left[\mathbf{I}_{|\boldsymbol{\theta}_k|} - \frac{2}{3}\Delta t \mathbf{A} \right] \boldsymbol{\theta}_k - \frac{4}{3}\boldsymbol{\theta}_{k-1} + \frac{1}{3}\boldsymbol{\theta}_{k-2} \\ &\quad - \frac{2}{3}\Delta t \left[\mathbf{B}\mathbf{u}_k + \mathbf{C}\mathbf{z}_k + \mathbf{D}\mathbf{d}_k \right. \\ &\quad \left. + \sum_{v \in \mathcal{V}} \mathbf{X}_v \left[(\mathbf{Y}_v \dot{\mathbf{m}}_k) \circ (\mathbf{Z}_v \mathbf{v}_k) \right] \right] \end{aligned} \quad (3)$$

with the constant timestep Δt . Note that the implementation of higher-order backward differentiation is similar.

4. PRIMAL-DECOMPOSITION OF SYSTEMS MODELED BY CONTROL VOLUMES

4.1 General MPC optimization problem

The goal of the MPC is to minimize an objective function

$$\begin{aligned} J_k(\hat{\boldsymbol{\theta}}_k, \hat{\mathbf{m}}_k, \hat{\mathbf{z}}_k, \hat{\mathbf{u}}_k) &= J_{\text{h},k}(\hat{\mathbf{m}}_k, \hat{\mathbf{z}}_{\text{h},k}, \hat{\mathbf{u}}_{\text{h},k}) \\ &\quad + J_{\text{t},k}(\hat{\boldsymbol{\theta}}_k, \hat{\mathbf{m}}_k, \hat{\mathbf{z}}_k, \hat{\mathbf{u}}_k), \end{aligned} \quad (4a)$$

with the notation convention

$$\hat{\mathbf{a}}_k = [\mathbf{a}_k^\top \ \mathbf{a}_{k+1}^\top \ \dots \ \mathbf{a}_{k+n_c-1}^\top]^\top,$$

and with

$$J_{\text{h},k}(\hat{\mathbf{m}}_k, \hat{\mathbf{z}}_{\text{h},k}, \hat{\mathbf{u}}_{\text{h},k}) = \sum_{\kappa \in \mathcal{K}} j_{\text{h},\kappa}(\dot{\mathbf{m}}_{\kappa}, \mathbf{z}_{\text{h},\kappa}, \mathbf{u}_{\text{h},\kappa}), \quad (4b)$$

$$J_{\text{t},k}(\hat{\boldsymbol{\theta}}_k, \hat{\mathbf{m}}_k, \hat{\mathbf{z}}_k, \hat{\mathbf{u}}_k) = \sum_{\kappa \in \mathcal{K}} j_{\text{t},\kappa}(\boldsymbol{\theta}_{\kappa}, \dot{\mathbf{m}}_{\kappa}, \mathbf{z}_{\kappa}, \mathbf{u}_{\kappa}) \quad (4c)$$

over all time-steps $\mathcal{K} = \{k, k+n_c-1\}$ within the control-horizon n_c with respect to the hydraulic constraints

$$\mathbf{f}_{\text{h},\kappa}(\dot{\mathbf{m}}_{\kappa}, \mathbf{z}_{\text{h},\kappa}, \mathbf{u}_{\text{h},\kappa}, \mathbf{d}_{\kappa}) = 0, \quad \forall \kappa \in \mathcal{K}, \quad (5a)$$

$$\mathbf{g}_{\text{h},\kappa}(\dot{\mathbf{m}}_{\kappa}, \mathbf{z}_{\text{h},\kappa}, \mathbf{u}_{\text{h},\kappa}, \mathbf{d}_{\kappa}) \leq 0, \quad \forall \kappa \in \mathcal{K}, \quad (5b)$$

and the non-hydraulic constraints

$$\mathbf{f}_{\text{de},\kappa}(\boldsymbol{\theta}_{\kappa}, \dot{\mathbf{m}}_{\kappa}, \mathbf{z}_{\kappa}, \mathbf{u}_{\kappa}, \mathbf{d}_{\kappa}) = 0, \quad \forall \kappa \in \mathcal{K}, \quad (6a)$$

$$\mathbf{f}_{\text{t},\kappa}(\boldsymbol{\theta}_{\kappa}, \dot{\mathbf{m}}_{\kappa}, \mathbf{z}_{\kappa}, \mathbf{u}_{\kappa}, \mathbf{d}_{\kappa}) = 0, \quad \forall \kappa \in \mathcal{K}, \quad (6b)$$

$$\mathbf{g}_{\text{t},\kappa}(\boldsymbol{\theta}_{\kappa}, \dot{\mathbf{m}}_{\kappa}, \mathbf{z}_{\kappa}, \mathbf{u}_{\kappa}, \mathbf{d}_{\kappa}) \leq 0, \quad \forall \kappa \in \mathcal{K}. \quad (6c)$$

Note that splitting the objective into a hydraulic objective j_{h} and a temperature objective j_{t} does not impose any restrictions, since all optimization variables can still be part of j_{t} . Therefore, the optimization problem that must be solved at each time-step k is given by

$$\begin{aligned} J_k^* &= \min_{\boldsymbol{\theta}_k, \dot{\mathbf{m}}_k, \mathbf{z}_k, \hat{\mathbf{u}}_k} J_k(\hat{\boldsymbol{\theta}}_k, \hat{\mathbf{m}}_k, \hat{\mathbf{z}}_k, \hat{\mathbf{u}}_k) \\ &\quad \text{subject to (5) and (6),} \end{aligned} \quad (7)$$

where J_k^* is the optimal function value of (4a).

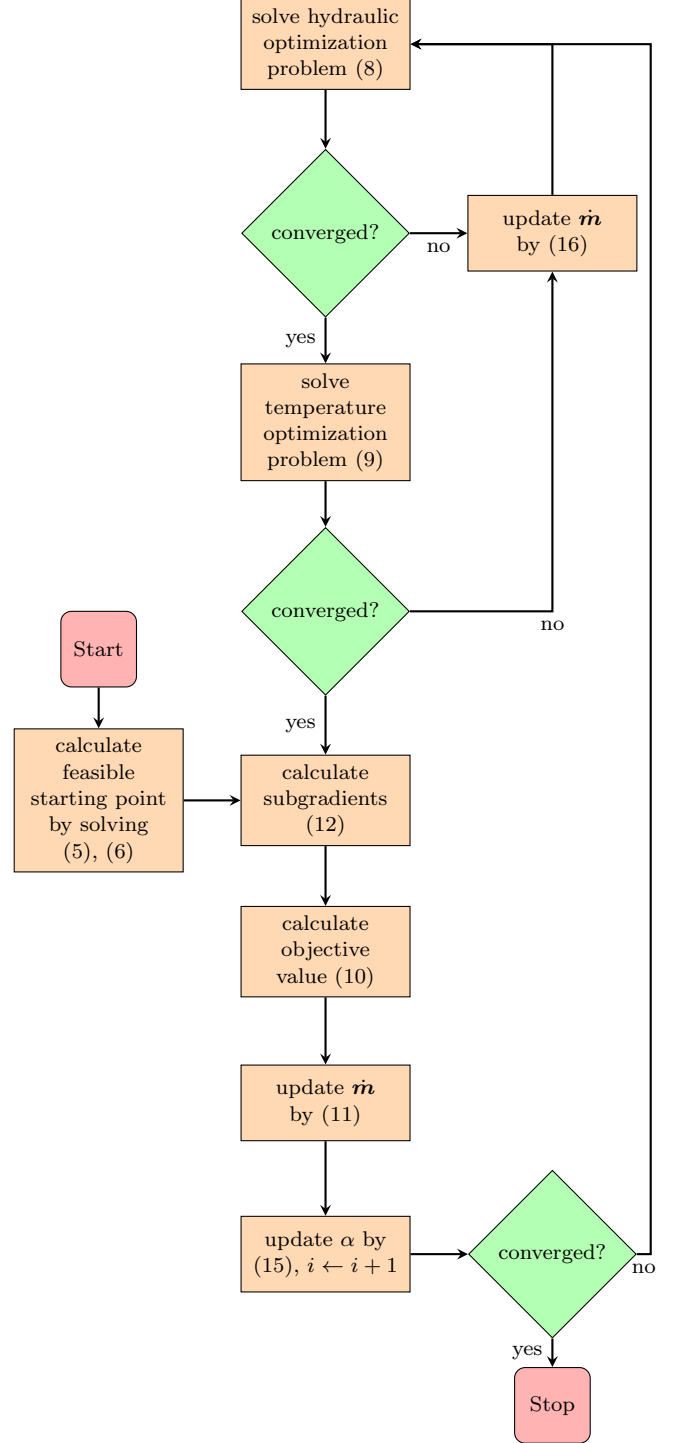


Fig. 2. Procedure of the primal decomposition scheme

4.2 Primal decomposition

As can be seen in (3), assuming $\dot{\mathbf{m}}_k$ is constant makes (3) a linear system of equations. If the same holds true for (2b)–(2e), then *primal decomposition* can be used to efficiently solve the optimization problem (7). Figure 2 displays a procedure of the primal decomposition scheme.

First, a feasible starting point is calculated by solving the system of nonlinear equations (5) and (6), where the control variables are taken from the MPC results of the previous time-step. The algorithm works by iteratively

altering $\hat{\mathbf{m}}_k$. The first step in each iteration is to solve the hydraulic optimization problem

$$J_{h,k}^i = \min_{\hat{\mathbf{z}}_{h,k}^i, \hat{\mathbf{u}}_{h,k}^i} J_{h,k} \left(\hat{\mathbf{m}}_k^i, \hat{\mathbf{z}}_{h,k}^i, \hat{\mathbf{u}}_{h,k}^i \right) \quad (8)$$

subject to (5)

with i being the primal decomposition iteration index. Solving (8) yields the optimal values for $\hat{\mathbf{z}}_{h,k}^i$ and $\hat{\mathbf{u}}_{h,k}^i$.

Next, the thermal optimization problem

$$J_{t,k}^i = \min_{\hat{\boldsymbol{\theta}}_k^i, \hat{\mathbf{m}}_k^i, \hat{\mathbf{z}}_{h,k}^i, \hat{\mathbf{z}}_{t,k}^i, \hat{\mathbf{u}}_{h,k}^i, \hat{\mathbf{u}}_{t,k}^i} J_{t,k} \left(\hat{\boldsymbol{\theta}}_k^i, \hat{\mathbf{m}}_k^i, \hat{\mathbf{z}}_{h,k}^i, \hat{\mathbf{z}}_{t,k}^i, \hat{\mathbf{u}}_{h,k}^i, \hat{\mathbf{u}}_{t,k}^i \right) \quad (9)$$

subject to (6)

is solved, resulting in $\hat{\boldsymbol{\theta}}_k^i$, $\hat{\mathbf{z}}_{t,k}^i$, and $\hat{\mathbf{u}}_{t,k}^i$. The objective value after iteration i is given by

$$J_k^i = J_{h,k}^i + J_{t,k}^i. \quad (10)$$

A convergence criterion, e. g. based on $\Delta J = J_k^i - J_k^{i-1}$, can then be used. If the convergence criterion is not met, the primal variable is updated by Boyd et al. (2008)

$$\hat{\mathbf{m}}_k^{i+1} = \hat{\mathbf{m}}_k^i - \alpha^i \mathbf{s}_k^i \quad (11)$$

with the subgradient

$$\mathbf{s}_k = \frac{\partial L_k \left(\hat{\boldsymbol{\theta}}_k, \hat{\mathbf{m}}_k, \hat{\mathbf{z}}_k, \hat{\mathbf{u}}_k, \hat{\boldsymbol{\lambda}}_k \right)}{\partial \hat{\mathbf{m}}_k}, \quad (12)$$

the Lagrangian as in Palomar and Chiang (2006)

$$L_k = J_k - \sum_{\kappa \in \mathcal{K}} \sum_{\mathbf{h} \in \mathcal{H}} \boldsymbol{\lambda}_{h,\kappa}^\top \mathbf{h}_\kappa, \quad (13)$$

the set of constraint functions $\mathcal{H} = \{\mathbf{f}_h, \mathbf{g}_h, \mathbf{f}_{de}, \mathbf{f}_t, \mathbf{g}_t\}$, and the optimal Lagrange multipliers

$$\hat{\boldsymbol{\lambda}}_k^\top = \left[\hat{\boldsymbol{\lambda}}_{f_h,k}^\top, \hat{\boldsymbol{\lambda}}_{g_h,k}^\top, \hat{\boldsymbol{\lambda}}_{f_{de},k}^\top, \hat{\boldsymbol{\lambda}}_{f_t,k}^\top, \hat{\boldsymbol{\lambda}}_{g_t,k}^\top \right]. \quad (14)$$

Once again the notation convention from (4) is used, e. g. $\hat{\boldsymbol{\lambda}}_{f_h,k}^\top = [\boldsymbol{\lambda}_{f_h,k}^\top \dots \boldsymbol{\lambda}_{f_h,k+n_c-1}^\top]$. At the end of each iteration, α^i is updated using the nonsummable, diminishing step-size rule Boyd (2014)

$$\alpha^i = \frac{\alpha}{\sqrt{i}} \quad (15)$$

which is in accordance with the conditions specified in Bertsekas and Tsitsiklis (1997).

If (5) or (6) become infeasible due to the primal variables update by (11), then the primal variables are re-updated by

$$\hat{\mathbf{m}}_k^i = \hat{\mathbf{m}}_k^{i-1} - b\alpha^{i-1} \mathbf{s}^{i-1}, \quad (16)$$

and procedure (16) is repeated with a decreasing $b < 1$ until (5) and (6) are fulfilled. This is similar to the backtracking approach presented by Krishnamoorthy et al. (2019). Engelmann et al. (2025) present a different approach, using relaxation techniques and auxiliary variables to find a feasible α^i .

5. UNDERGROUND HEATING SYSTEM

5.1 System model

To validate the MPC approach and the scalability of the primal decomposition, we use three different versions of a simple underground heating system. Such system setups are frequently used in geothermal or underfloor heating

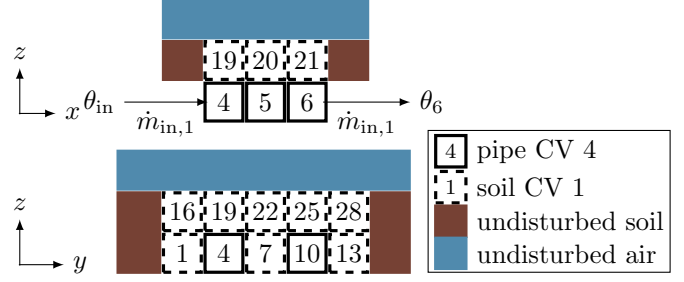


Fig. 3. underground heating system with two parallel pipes and 30 CVs

systems. Furthermore, such systems are used at airports and sports fields to maintain a snow-free surface. The three versions vary in system size regarding number of pipes n_{pi} , number of CVs n_{cv} and number of CVs in x -direction n_x which is equal to the number of CVs per pipe. Figure 3 shows two perspectives of the system with $n_{pi} = 2$, $n_{cv} = 30$, and $n_x = 3$. For all three cases with different n_{pi} values, two layers of CVs are assumed in the z -direction, and soil CVs are assumed at both ends of the bottom layers in the y -direction. The objective is to minimize energy costs while keeping soil CV temperatures between 278.15 K and 313.15 K.

System variables The states of the system are the CV temperatures $\boldsymbol{\theta}^\top = [\theta_1 \dots \theta_{n_{cv}}]$. The algebraic states are the CV mass flows $\hat{\mathbf{m}}^\top = [\hat{m}_1 \dots \hat{m}_{n_x}]$, the pipe pressure losses $\Delta \mathbf{p}^\top = [\Delta p_1 \dots \Delta p_{n_{pi}}]$, the input temperature of the pipes θ_{in} and the temperature θ_{out} of the mass flow after mixing all pipe exit temperatures (see (20)). The controllable inputs of the system consist of the mass flows in the pipes $\hat{\mathbf{m}}_{in}^\top = [\hat{m}_{in,1} \dots \hat{m}_{in,n_{cv}}]$ and the temperature increase $\Delta \theta$ in between pipe out- and input. The uncontrollable inputs are soil temperature θ_s and air temperature θ_a . Table 1 depicts all variables together with their assignment to the vectors contained in \mathcal{V} .

Objective The goal of the controller is to minimize the cost of the required electrical energy for the pumps and the heating system given the electricity price $c_{el,k}$. The cost of electricity for the pumps per time step is given by

$$j_{h,k} = \frac{c_{el,k}}{\rho_w \eta_{pu}} \mathbf{z}_{h,k}^\top \mathbf{u}_{h,k} \quad (17a)$$

with the pump efficiency $\eta_{pu} = 0.8$. Assuming an electric boiler is used, the cost of the electricity required to heat the water between the pipe inputs and outputs is given by

$$j_{t,k} = c_{el,k} c_w u_{t,k} \mathbf{1}_{1 \times n_{pi}} \mathbf{u}_{h,k}. \quad (17b)$$

System dynamics The system dynamics are based on (1) and are not repeated here due to space limitations. The parameters are set to $V = 3.75 \text{ m}^3$, $\sigma_x = 0.025 \text{ W K}^{-1}$, $\sigma_y = \sigma_z = 22.5 \text{ W K}^{-1}$, $\sigma_{pi} = 44.86 \text{ W K}^{-1}$, $c_w = 4200 \text{ J kg}^{-1} \text{ K}^{-1}$, $\rho_w = 1000 \text{ kg m}^{-3}$, and $c_s \rho_s = 1.5 \times 10^6 \text{ J m}^{-3} \text{ K}$.

Algebraic equations In series with each pipe p , there is a pump that controls the mass flow rate $\hat{m}_{in,p}$. The first set of algebraic equations are the pressure loss relations

Table 1. System variables

states			controllable inputs		uncontrollable inputs	
θ	z_h	z_t	\dot{m}	u_h	u_t	d
θ_1	Δp_1	θ_{in}	\dot{m}_1	$\dot{m}_{in,1}$		θ_s
\vdots	\vdots	\vdots	\vdots	\vdots	$\Delta\theta$	
$\theta_{n_{cv}}$	$\Delta p_{n_{pi}}$	θ_{out}	\dot{m}_{n_x}	$\dot{m}_{in,n_{pi}}$		θ_a

Table 2. Numerical results of the two MPCs

number of pipes	number of variables	average $j_h + j_t$ per time step in €		average calc. time per time step in s	
		IPM MPC	PD MPC	IPM MPC	PD MPC
1	19	116.01	117.91	1.18	6.03
2	43	271.53	274.4	12.51	7.81
7	202	1074.7	1298.5	3706	124.29

$$\Delta p_p (\dot{m}_{in,p}) = \frac{\Delta p_{nom}}{\dot{m}_{nom}^2} \dot{m}_{in,p}^2, \quad (18a)$$

where $\Delta p_{nom} = 8 \times 10^5$ Pa and $\dot{m}_{nom} = 30 \text{ kg s}^{-1}$ are nominal values. The input temperature is given by

$$\theta_{in} = \theta_{out} + \Delta\theta. \quad (19)$$

and θ_{out} is governed by

$$\left(\sum_{p=1}^{n_{pi}} \dot{m}_{in,p} \right) \theta_{out} = \left(\sum_{p=1}^{n_{pi}} \dot{m}_{in,p} \theta_{out,p} \right), \quad (20)$$

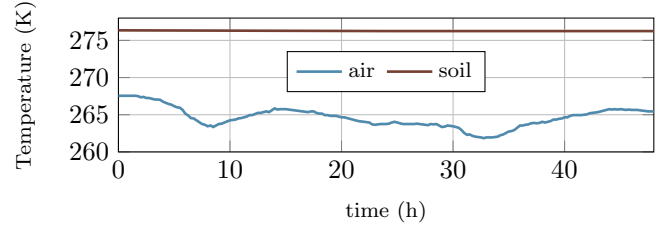
where $\theta_{out,p}$ is the temperature of the last CV of pipe p , for example θ_6 for $p = 1$ and θ_{12} for $p = 2$ in Figure 3.

5.2 Time discretization

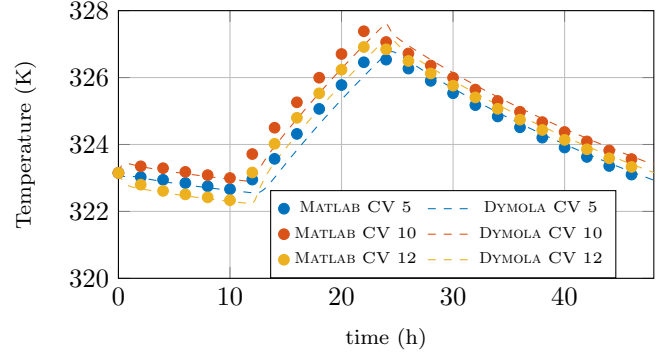
One version of the underground heating system was implemented in MATLAB. It was time-discretized using (3) with a time step of $\Delta t = 2$ h and simulated by solving the resulting system of equations. A second version was created using the library from Westphal et al. (2025) in DYMOLA. DYMOLA also uses backward differentiation. In DYMOLA, however, the solver selects Δt and the order of the backward differentiation depending on the state of the system. Due to the validated models from Westphal et al. (2025), as well as the highly sophisticated time discretization in DYMOLA, the DYMOLA model is used as a benchmark. Figure 4 shows a comparison of the two simulations. As can be seen, the MATLAB model is more accurate for soil CVs, with a maximum error of 0.473 K, than for pipe CVs, with a maximum error of 0.936 K, due to the slower dynamics. The highest deviations in the pipes occur after increasing $\dot{m}_{in,2}$ from 2 kg s^{-1} to 5 kg s^{-1} after 10 h and after decreasing $\Delta\theta$ from 1 K to 0.5 K. Figure 3 shows ambient temperature data, and $\dot{m}_{in,1}$ is kept constant at 2 kg s^{-1} . If higher model accuracy is required, Δt must be decreased. For $\Delta t = 15$ min, the maximum pipe temperature error is 0.23 K.

5.3 Model predictive controllers

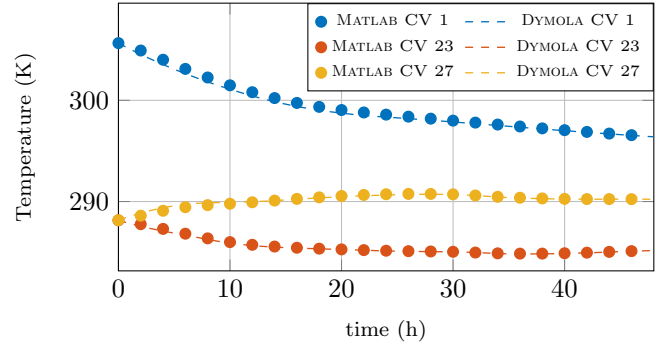
Two MPCs were used to control each system. Both controllers were implemented in MATLAB. One MPC solved



(a) Ambient temperatures



(b) Pipe temperatures



(c) Soil temperatures

Fig. 4. Comparison of the DYMOLA (dashed lines) and the MATLAB (dots) simulations for the system depicted in Figure 3

optimization problem (7) using the standard interior-point method (IPM) of MATLAB. The second MPC solved (7) using the primal decomposition (PD) scheme presented in this work with the vector of primal variables u_h . The control-horizon was set to $n_c = 12$, which spans 24 h, since $\Delta t = 2$ h. Hence, the total number of optimization variables of each MPC is given by $n_c (n_{cv} + n_x + 2n_{pi} + 3)$ (see Table 1).

5.4 Results of the model predictive controller

As shown in Table 2, the IPM outperformed the PD in all three system setups in terms of objective value. However, the PD outperformed the IPM in terms of calculation time for all but the smallest system setup. For $n_{pi} = 2$, the PD MPC calculation time is 62.4% of the IPM MPC calculation time; for $n_{pi} = 7$, it is 3.35%. In conclusion, the PD MPC is more scalable than the IPM MPC. For the two smallest system setups, the average cost per time step is nearly equal for both MPCs, and for the system with seven pipes, the IPM MPC outperforms the PD MPC by 17.2%.

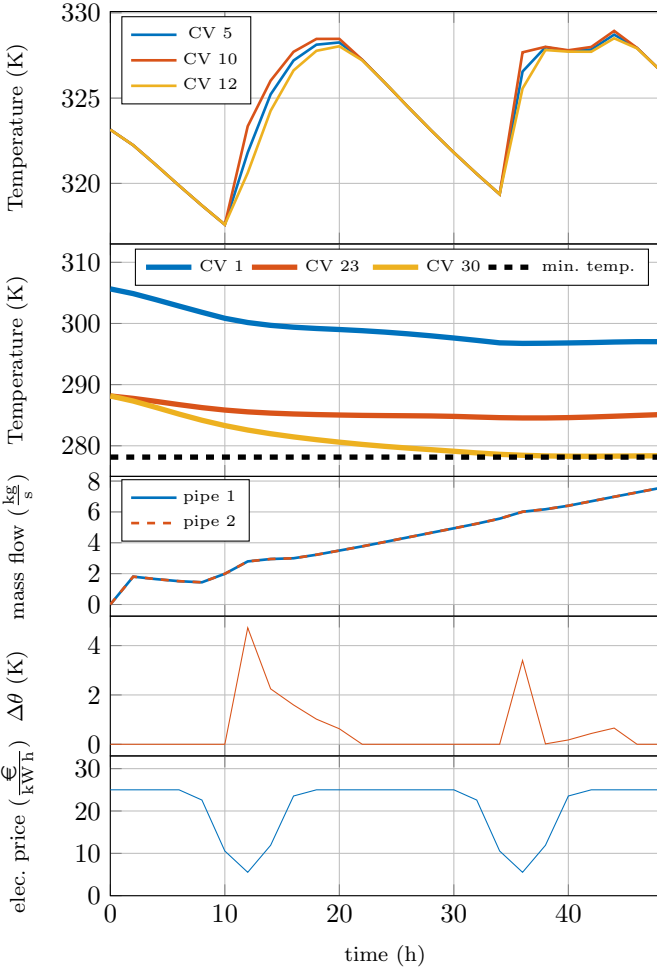


Fig. 5. Results for the primal MPC for the system displayed in Figure 3

Figure 5 shows the simulation results for the system controlled by the PD MPC. When electricity prices are low, the MPC increases $\Delta\theta$, resulting in higher pipe temperatures. Otherwise, using only the mass flow rate is sufficient for distributing the stored heat at the beginning of the simulation without raising the temperature. As the soil temperature decreases, the mass flow increases. Note that the same mass flow is used for both pipes due to the symmetry of the system. The simulation results clearly demonstrate that the PD MPC can control the system according to changing electricity prices and leverage the system's flexibility to minimize operating costs.

6. CONCLUSION

This work presents a direct method for model predictive controller design for thermo-hydraulic systems modeled by control volumes. Using primal-decomposition increases the approach's scalability massively compared to a standard MPC approach. Future work should include a more extensive analysis of more sophisticated system setups to provide a more robust evaluation of the presented approach. Additionally, comparisons with other automated MPC frameworks, as presented in Houska et al. (2011) and Chen et al. (2019), are necessary.

DECLARATION OF GENERATIVE AI AND AI-ASSISTED TECHNOLOGIES IN THE WRITING PROCESS

During the preparation of this work the authors used DeepL Write and GPT4.1-Mini in order to improve readability. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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