The vector space generated by permutations of a trade or a design

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Abstract

Motivated by a classical result of Graver and Jurkat (1973) and Graham, Li, and Li (1980) in combinatorial design theory, which states that the permutations of t-(v,k) minimal trades generate the vector space of all t-(v,k) trades, we investigate the vector space spanned by permutations of an arbitrary trade. We prove that this vector space possesses a decomposition as a direct sum of subspaces formed in the same way by a specific family of so-called total trades. As an application, we demonstrate that for any t-(v,k,λ) design, its permutations can span the vector space generated by all t-(v,k,λ) designs for sufficiently large values of v. In other words, any t-(v,k,λ) design, or even any t-trade, can be expressed as a linear combination of permutations of a fixed t-design. This substantially extends a result by Ghodrati (2019), who proved the same result for Steiner designs.

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1 Introduction

Trades are combinatorial objects introduced and employed in conjunction with t-designs, showcasing rich combinatorial and algebraic properties. During the early 1970s, these objects made their debut in the literature, initially referred to as 't-pods' [10]. Subsequently, various names were adopted by different authors, including 'null designs' [9], and some others in [1, 6]. The term 'trade' had its origins in statistics and was initially coined by Foody and Hedayat [2]. In the application of t-designs in the statistical design of experiments, one may encounter situations where certain blocks become prohibitively expensive for experimentation. For instance, combining particular tasks within a single block may be deemed unacceptable by the experimenter. Imagine that the t-designs available in the literature already include such problematic blocks, and simply renaming them does not eliminate these undesired attributes. How can one tackle this issue? The theory of trade-off [2] offers a solution on how to exchange these undesirable blocks for those that align with the experimenter's criteria. In addition to that, trades have various applications in the construction of signed t-designs, nonisomorphic t-designs, t-designs with repeated blocks, as well as in the determination of the spectrum of support sizes of t-designs and defining sets for certain t-designs (see [11, 16]). In an algebraic perspective, trades constitute a $\mathbb{Z}$-module, and various researchers have explored diverse sets of generators for this module [10, 9, 14]. This approach has allowed researchers to attain profound insights into the realm of t-designs [10, 20, 23].

1.1 Minimal trades and total trades

To formally define trades, let v, k, t be integers such that $v > k > t \geq 0$ and X be a fixed v-set. Suppose that T_+ and T_- are two disjoint collections of k-subsets of X (called *blocks*) such that the number of occurrences of every t-subset of X in T_+ and T_- are the same. Then $T = (T_+, T_-)$ is called a t-(v, k) *trade* (or a t-*trade* when the role of v and k is not important). Among t-trades, there are those that stand out as *minimal*, characterized by having the fewest blocks and employing the smallest number of elements from the set X (a brief account is given in Section 2.2). Minimal trades hold a fundamental role in the algebraic investigation of trades and can be 'algebraically' represented as follows:

$$T = (x_1 - y_1) \cdots (x_{t+1} - y_{t+1}) x_{t+2} \cdots x_k, \tag{1}$$

where $x_i, y_i \in X$ are distinct, and the multiplication among them is to be interpreted as union among sets. In the representation (1), the component $x_{t+2} \cdots x_k$ is called the *tail* of T, and the component $(x_1 - y_1) \cdots (x_{t+1} - y_{t+1})$, the *essential part* of T.

Each t-(v,k) trade indeed serves as a combinatorial representation of a null-vector for the inclusion matrix W_{tk}^{v} (see Section 1.2 below). Consequently, the collection of all t-(v,k) trades forms a vector space, e.g. over the field of rational numbers $\mathbb{Q}$.

We denote the set of all permutations of X by $\mathcal{S}_X$. Every $f \in \mathcal{S}_X$ naturally acts on subsets $B = \{x_1, \ldots, x_k\}$ of X as $f(B) := \{f(x_1), \ldots, f(x_k)\}$, and accordingly on any collection of subsets of X. In particular, if T is a t-(v,k) trade, then $f(T)$, the image of T under the action of f, is also a t-(v,k) trade. Set

$$\Pi(T) := \{f(T) : f \in \mathcal{S}_X\}.$$

Graver and Jurkat [10] proved that for any minimal t-(v,k) trade T, the set $\Pi(T)$ can generate the vector space of t-(v,k) trades. Graham, Li, and Li [9] were the first to introduce the polynomial representation (1) for minimal trades and utilized it to find an explicit basis for the space of trades consisting of minimal trades. Additional explicit constructions for such bases can be found in [3, 14].

Theorem 1 ([10], see also [9]). *Let $v \geq k + t$ and T be an arbitrary t-(v,k) minimal trade. Then $\Pi(T)$ generates the vector space of t-(v,k) trades.*

This motivates us to explore the vector space generated by $\Pi(T)$ for general t-trades T. In this context, a particular type of trades plays an essential role, namely *total trades*. We define a t-(v,k) *total trade* as the sum of all t-(v,k) minimal trades with common essential parts. So a t-(v,k) total trade can be represented as

$$(x_1 - y_1) \cdots (x_{t+1} - y_{t+1}) \sum x_{i_{t+2}} \cdots x_{i_k},$$

where the summation is extended over all $(k-t-1)$-subsets $\{x_{i_{t+2}}, \ldots, x_{i_k}\}$ of $X \setminus \{x_1, y_1, \ldots, x_{t+1}, y_{t+1}\}$. We denote the set of all t-(v,k) total trades by $\mathfrak{T}_{t,k,v}$.

If V is a vector space and $S \subseteq V$, then $\langle S \rangle$ denotes the subspace of V spanned by S. Our main result provides a characterization of the vector space $\langle \Pi(T) \rangle$ for arbitrary trade T.

Theorem 2. *Let T be a t-(v,k) trade. Then*

$$\langle \Pi(T) \rangle = \bigoplus_{i \in I} \langle \mathfrak{T}_{i,k,v} \rangle,$$

where $I \subseteq \{t, \ldots, k-1\}$ consists of integers i such that there is an i-trade in $\langle \Pi(T) \rangle$ which is not an $(i+1)$-trade. In particular, $\langle \Pi(T) \rangle$ is the whole vector space of t-(v,k) trades if and only if $I = \{t, \ldots, k-1\}$.

1.2 Space generated by t-designs

A t-(v, k, λ) *design* is a pair $(X, \mathcal{B})$ where X is our fixed v-set and $\mathcal{B}$ is a collection of k-subsets of X such that every t-subset of X occurs exactly λ times in blocks $B \in \mathcal{B}$. If $\mathcal{B}$ has no repeated blocks, then the design is called *simple*. The condition that

$$\binom{k-i}{t-i} \mid \binom{v-i}{t-i}, \quad \text{for } i = 0, 1, \ldots, t, \tag{2}$$

is necessary for the existence of a t-(v, k, λ) design (see [15]). Remarkably, t-designs can be described in a linear-algebraic language by means of inclusion matrices. The *inclusion matrix* W_{tk}^v is a $(0,1)$-matrix whose rows and columns are indexed by t-subsets and k-subsets of X, respectively, and $W_{tk}^v(T, K) = 1$ if and only if $T \subseteq K$. The characteristic vector of length $\binom{v}{k}$ for the block set of a t-(v, k, λ) design is a solution for the system of linear equations

$$W_{tk}^v \mathbf{x} = \lambda \mathbf{j}, \tag{3}$$

where $\mathbf{j}$ is the all-1 vector (here of length $\binom{v}{t}$). This is a motivation to generalize the concept of t-designs to signed t-designs: Every integral solution of (3) is called a *signed* t-(v, k, λ) *design*. We remark that the condition (2) is necessary and sufficient for the existence of an integral solution for (3). This is proved by Graver and Jurkat [10], and Wilson [20], see also [5, 17, 23] for alternative proofs.

The 'Existence Conjecture' from the 19th century asserts that for given t, k, and λ, there exists a $v_0(t, k, \lambda)$ such that if $v \geq v_0(t, k, \lambda)$ satisfies the divisibility conditions (2), then a t-(v, k, λ) design exists. Over a century later, a breakthrough result of Wilson [18, 19, 21, 22] resolved the case $t = 2$. In a recent breakthrough, Keevash [21] proved the Existence Conjecture in general, based on the method of 'Randomised Algebraic Constructions'. This method is inspired by Wilson's algebraic approach to the case $t = 2$ as well as results on signed designs by Graver and Jurkat [10]. Very recently, Glock, Kühn, Lo, and Osthus [7] provided a new proof for the Existence Conjecture.

It is a well-known fact (first proved by Gottlieb [8]) that the rank of W_{tk} with $v \geq k+t$ over $\mathbb{Q}$ is equal to its number of rows. So the nullity of W_{tk} is $\binom{v}{k} - \binom{v}{t}$ implying that the dimension of the vector space spanned by all rational solutions of (3) (as a subspace of $\mathbb{Q}^{\binom{v}{k}}$) is at most $\binom{v}{k} - \binom{v}{t} + 1$, and the bound is met when at least one solution exists. Ghodrati [4] proved that in the case of $\lambda = 1$, the space spanned by $(0,1)$-solutions of (3), provided that at least one of them exists, reaches the same dimension. In other words, he proved that if a Steiner design, namely a t-$(v, k, 1)$ design, exists, then the characteristic vectors of t-$(v, k, 1)$ designs span a vector space of dimension $\binom{v}{k} - \binom{v}{t} + 1$.

As an application of Theorem 2, we establish a more general result for permutations of a fixed signed design.

Theorem 3. *Let $\mathcal{D}$ be a t-(v,k,λ) signed design. Then*

$$\dim\langle\Pi(\mathcal{D})\rangle = 1 + \sum_{i\in I}\left(\binom{v}{i+1} - \binom{v}{i}\right),$$

for some $I \subseteq \{t,\dots,k-1\}$.

The set I in Theorem 3 can be explicitly determined (refer to Corollary 13 below). However, for the sake of simplicity, we avoid detailing it here.

When the design $\mathcal{D}$ is simple, or more generally, a design with non-negative components, we prove that the value given in Theorem 3 reaches its maximum provided that v is greater than a mild lower bound. This additional condition, however, is not particularly restrictive, given that the existence of designs holds for sufficiently large v.

Theorem 4. *Let $\mathcal{D}$ be a t-(v,k,λ) design which is not a $(t+1)$-design. If $v \geq (2^t\lambda + 1)(k-t-1) + 2(t+1)$, then $\dim\langle\Pi(\mathcal{D})\rangle = \binom{v}{k} - \binom{v}{t} + 1$.*

The remainder of this paper is organized as follows. In Section 2, we provide a recap of key definitions and introduce some notation. Section 3 is dedicated to characterizing the space generated by permutations of a trade and includes the proof of Theorem 2. In Section 4, we consider the space generated by permutations of a t-design and will prove Theorems 3 and 4.

2 Preliminaries

In this section, we provide some preliminaries on trades and vector representation of designs. All trades and designs considered in the following sections are based on our fixed v-set X. We use the notation $\binom{X}{\ell}$ to represent the set of all ℓ-subsets of X.

2.1 Vector representation of a collection of k-subsets

We refer to any member of $\mathbb{Z}^{\binom{v}{k}}$ as a (v,k)-*vector*. Let $X_1,\dots,X_{\binom{v}{k}}$ (in a fixed order) be all the k-subsets of X. Let $\mathbf{a}$ be a (v,k)-vector. We can formally represent $\mathbf{a}$ as $a_1X_1 + \cdots + a_{\binom{v}{k}}X_{\binom{v}{k}}$. For instance, the all-1 vector $\mathbf{j}$ represents the complete design: $\mathbf{j} = X_1 + \cdots + X_{\binom{v}{k}}$. We say that X_i is a block of $\mathbf{a}$ if $a_i \neq 0$. For $x \in X$, we define the $(v-1,k-1)$-vector $\mathbf{a}_x$ as

$$\mathbf{a}_x := \sum_{i\colon x\in X_i} a_i X_i',$$

where $X_i' = X_i \setminus \{x\}$. Additionally, we set $\mathbf{a}_{\overline{x}}$ as the $(v-1,k)$-vector:

$$\mathbf{a}_{\overline{x}} := \sum_{i:\, x \notin X_i} a_i X_i.$$

We will use the above two notations and their combinations iteratively, which should make expressions such as $\mathbf{a}_{x_1 \ldots x_t}$, $\mathbf{a}_{x\overline{y}}$, or $\mathbf{a}_{\overline{x}y}$ clear. We say that $x \in X$ *appears in* $\mathbf{a}$ if $\mathbf{a}_x \neq \mathbf{0}$, or in other words if some block of $\mathbf{a}$ contains x. Suppose that $x \in X$ does not appear in $\mathbf{a}$. We define the $(v,k+1)$-vector $x\mathbf{a}$ (not to be confused with the scalar product $c\mathbf{a}$ for $c \in \mathbb{Q}$) as

$$x\mathbf{a} := \sum_{i=1}^{\binom{v}{k}} a_i (X_i \cup \{x\}),$$

and extend it linearly:

$$(x-y)\mathbf{a} = x\mathbf{a} - y\mathbf{a}.$$

This is compatible with the representation (1).

Let $f \in \mathcal{S}_X$. Recall the action of f on k-subsets and (v,k)-vectors: If $B = \{x_1, \ldots, x_k\} \subseteq X$, then $f(B) = \{f(x_1), \ldots, f(x_k)\}$ and

$$f(\mathbf{a}) := a_1 f(X_1) + \cdots + a_{\binom{v}{k}} f\left(X_{\binom{v}{k}}\right).$$

With these conventions, $\Pi(\mathbf{a}) = \{f(\mathbf{a}) : f \in \mathcal{S}_X\}$. We specifically employ the transposition that replaces x and y, denoting it as f_{xy}.

For $L \subseteq X$, we denote the total number of appearances of L in the blocks of $\mathbf{a}$ by $m_{\mathbf{a}}(L)$. To be more precise,

$$m_{\mathbf{a}}(L) = \sum_{i:\, L \subseteq X_i} a_i.$$

2.2 Trades

Historically, t-trades have been defined and utilized in connection with t-designs. Their significance in this context is grounded in the fundamental property that if $\mathcal{B}_1$ and $\mathcal{B}_2$ represent the block sets of two t-designs with identical parameters and the same ground set, then $(\mathcal{B}_1 \setminus \mathcal{B}_2, \mathcal{B}_2 \setminus \mathcal{B}_1)$ forms a t-trade. In other words, if $\mathbf{s}$ and $\mathbf{s}'$ are two t-(v,k,λ) designs, represented as (v,k)-vectors, then $\mathbf{s} - \mathbf{s}'$ is a t-(v,k) trade. This becomes evident through the linear-algebraic interpretation of t-designs as solutions to the system (3) and t-trades as null-vectors of the inclusion matrix W_{tk}^{v}.

In a t-(v,k) trade $T = (T_+, T_-)$, two essential conditions are satisfied. Firstly, the cardinality of the collections T_+ and T_- is equal, which is referred to as the *volume* of T. Additionally, both T_+ and T_- must encompass the same subset of X, meaning that $\bigcup_{B\in T_+} B = \bigcup_{B\in T_-} B$. This common set is called the *foundation* of T. The smallest possible values for the volume and foundation size of T are 2^t and $k+t+1$, respectively, as proved in [12]. When a t-(v,k) trade has a volume of 2^t and a foundation size of $k+t+1$, it is referred to as a *minimal trade*. Any minimal trade possesses a representation as given in (1). For instance,

$$T = (x_1 - y_1)(x_2 - y_2)(x_3 - y_3)x_4x_5,$$

is a 2-$(v,5)$ minimal trade for any $v \geq 8$, for which

$$\begin{aligned} T_+ &= \{x_1x_2x_3x_4x_5,\ x_1y_2y_3x_4x_5,\ y_1x_2y_3x_4x_5,\ y_1y_2x_3x_4x_5\},\\ T_- &= \{y_1x_2x_3x_4x_5,\ x_1y_2x_3x_4x_5,\ x_1x_2y_3x_4x_5,\ y_1y_2y_3x_4x_5\}. \end{aligned}$$

A t-(v,k) *total trade* is the sum of all t-(v,k) minimal trades with common essential parts. More precisely, it is a trade of the form $(x_1 - y_1)\cdots(x_{t+1} - y_{t+1})\mathbf{j}$. Here $\mathbf{j}$ is the all-1 $(v-2t-2, k-t-1)$-vector, which serves as the characteristic vector of all $(k-t-1)$-subsets of $X \setminus \{x_1, y_1, \ldots, x_{t+1}, y_{t+1}\}$. For example, consider a 2-$(10,5)$ total trade as the following:

$$T = (x_1 - y_1)(x_2 - y_2)(x_3 - y_3)(x_4x_5 + x_4x_6 + \cdots + x_6x_7),$$

for which

$$\begin{aligned} T_+ &= \bigcup_{4\leqslant i<j\leqslant 7} \{x_1x_2x_3x_ix_j,\ x_1y_2y_3x_ix_j,\ y_1x_2y_3x_ix_j,\ y_1y_2x_3x_ix_j\},\\ T_- &= \bigcup_{4\leqslant i<j\leqslant 7} \{y_1x_2x_3x_ix_j,\ x_1y_2x_3x_ix_j,\ x_1x_2y_3x_ix_j,\ y_1y_2y_3x_ix_j\}. \end{aligned}$$

3 The space generated by permutations of a trade

In this section, we address the vector space generated by permutation of a trade. We obtain a structural characterization for such a space. Notably, t-(v,k) total trades play a crucial role as the space spanned by their permutations serve as the 'building blocks' for the spaces generated by the permutations of general t-(v,k) trades. We conclude this section with the proof of Theorem 2.

We begin with the following useful lemma.

Lemma 5. *Let* $\mathbf{s}$ *be a* (v,k)*-vector such that* $W_{tk}^{v}\mathbf{s} \in \langle \mathbf{j} \rangle$ *but* $W_{t+1,k}^{v}\mathbf{s} \notin \langle \mathbf{j} \rangle$ *for some* $0 \leq t < k$*. Then* $\langle \Pi(\mathbf{s}) \rangle$ *contains a trade of the form*

$$\mathbf{w} = (x - y)\mathbf{s}',$$

for some $x, y \in X$ *such that* $\mathbf{s}'$ *is a* $(t-1)$*-trade but not a* t*-trade.*

Proof. Given that $W^v_{tk}\mathbf{s} \in \langle \mathbf{j} \rangle$, we have $W^v_{tk} f(\mathbf{s}) = W^v_{tk}\mathbf{s}$ for any permutation $f \in \mathcal{S}_X$. So $\mathbf{s} - f(\mathbf{s})$ is a t-trade. Considering the transposition f_{xy} for $x, y \in X$, we have the relation

$$\mathbf{s} - f_{xy}(\mathbf{s}) = (x - y)(\mathbf{s}_{x\overline{y}} - \mathbf{s}_{\overline{x}y}). \tag{4}$$

Note that $\mathbf{s}' := \mathbf{s}_{x\overline{y}} - \mathbf{s}_{\overline{x}y}$ is a $(v-2, k-1)$-vector. The left-hand side of (4) is a t-trade, and so is its right-hand side, demanding that $\mathbf{s}'$ should be a $(t-1)$-trade. To conclude the proof, we show that x and y can be chosen so that $\mathbf{s}'$ is not a t-trade. From the assumption $W^v_{t+1,k}\mathbf{s} \notin \langle \mathbf{j} \rangle$, it follows that there exist two $L, L' \in \binom{X}{t+1}$ such that $|L \cap L'| = t$ and $m_{\mathbf{s}}(L) \neq m_{\mathbf{s}}(L')$. (If for any such pair L, L', one has $m_{\mathbf{s}}(L) = m_{\mathbf{s}}(L')$, then this property holds for any pair of $(t+1)$-subsets of X.) It turns out that for some $x, y \in X$ and $L_0 \in \binom{X}{t}$, we have $L = L_0 \cup \{x\}$ and $L' = L_0 \cup \{y\}$. Consequently,

$$m_{\mathbf{s}_{x\overline{y}}}(L_0) + m_{\mathbf{s}_{xy}}(L_0) = m_{\mathbf{s}_x}(L_0) = m_{\mathbf{s}}(L) \neq m_{\mathbf{s}}(L') = m_{\mathbf{s}_y}(L_0) = m_{\mathbf{s}_{\overline{x}y}}(L_0) + m_{\mathbf{s}_{xy}}(L_0),$$

which means that $m_{\mathbf{s}'}(L_0) = m_{\mathbf{s}_{x\overline{y}}}(L_0) - m_{\mathbf{s}_{\overline{x}y}}(L_0) \neq 0$, implying that $\mathbf{s}'$ is not a t-trade. □

If a (v,k)-vector $\mathbf{a} \notin \langle \mathbf{j} \rangle$ is a signed design, then there is a positive integer $t < k$ such that $W^v_{tk}\mathbf{a} \in \langle \mathbf{j} \rangle$ but $W^v_{t+1,k}\mathbf{a} \notin \langle \mathbf{j} \rangle$. For vectors in $\langle \mathbf{j} \rangle$ such t does not exist. Furthermore, any (v,k)-vector $\mathbf{a}$ satisfies $W^v_{0,k}\mathbf{a} \in \langle \mathbf{j} \rangle$. So it is inferred that if $\mathbf{a} \notin \langle \mathbf{j} \rangle$, then it satisfies the condition of Lemma 5 for some $0 \leq t \leq k-1$. In particular, we can deduce the following lemma:

Lemma 6. *Let $\mathbf{a} \notin \langle \mathbf{j} \rangle$ be a (v,s)-vector. Then $\langle \Pi(\mathbf{a}) \rangle$ contains a trade $(x-y)\mathbf{a}'$ for some $x, y \in X$ and $(v-2, s-1)$-vector $\mathbf{a}'$.*

Remark 7. If x, y do not appear in $\mathbf{a}$ and f is a permutation on X fixing x, y, then $f((x-y)\mathbf{a}) = (x-y)f(\mathbf{a})$.

For a (v,k)-vector $\mathbf{a} = (a_1, \dots, a_{\binom{v}{k}})$, we set

$$\sigma(\mathbf{a}) := \sum_{i=1}^{\binom{v}{k}} a_i.$$

Note that by the definition of a t-trade, $\mathbf{a}$ is a 0-trade if and only if $\sigma(\mathbf{a}) = 0$.

The following lemma serves as one of the main ingredients in our proof of Theorem 2.

Lemma 8. *Let $\mathbf{s}$ be a (v,k)-vector such that $W^v_{tk}\mathbf{s} \in \langle \mathbf{j} \rangle$ but $W^v_{t+1,k}\mathbf{s} \notin \langle \mathbf{j} \rangle$ for some $0 \leq t < k$. Then $\langle \Pi(\mathbf{s}) \rangle$ contains a trade of the form*

$$\mathbf{w} = (x_1 - y_1) \cdots (x_{t+1} - y_{t+1})\mathbf{a},$$

with $x_1, \dots, x_{t+1}, y_1, \dots, y_{t+1} \in X$ such that $\mathbf{a}$ is not a 0-trade. Furthermore, $\mathbf{w}$ is a linear combination of 2^{t+1} vectors in $\Pi(\mathbf{s})$ with ± 1 coefficients.

Proof. By Lemma 5, we can choose $x, y \in X$ such that $\mathbf{w} = \mathbf{s} - f_{xy}(\mathbf{s}) = (x-y)\mathbf{s}' \in \Pi(\mathbf{s})$ and $\mathbf{s}'$ is a $(t-1)$-trade but not a t-trade. We proceed by induction on t to prove the assertion. For $t = 0$, it holds evidently by the above argument.

Let $t \geq 1$. By the induction hypothesis, applied on $\mathbf{s}'$, there exist $x_2, \ldots, x_{t+1}, y_2, \ldots, y_{t+1} \in X$ and a non-0-trade vector $\mathbf{a}$, such that

$$\mathbf{w}' = (x_2 - y_2) \cdots (x_{t+1} - y_{t+1})\mathbf{a}$$

is a linear combination of 2^t vectors in $\Pi(\mathbf{s}')$, that is $\mathbf{w}' = \sum_{i=1}^{2^t} c_i g_i(\mathbf{s}')$ where g_i are permutations on $X \setminus \{x_1, y_1\}$ and $c_i = \pm 1$. Let $\tilde{g}_i$ be the extension of g_i to X which fixes x_1, y_1 and $h_i = \tilde{g}_i \circ f_{x_1 y_1}$. By Remark 7, we observe that

$$(x_1 - y_1)g_i(\mathbf{s}') = \tilde{g}_i\left((x_1 - y_1)\mathbf{s}'\right) = \tilde{g}_i\left(\mathbf{s} - f_{x_1 y_1}(\mathbf{s})\right) = \tilde{g}_i(\mathbf{s}) - h_i(\mathbf{s}).$$

It follows that

$$(x_1 - y_1)\mathbf{w}' = \sum_{i=1}^{2^t} c_i (x_1 - y_1) g_i(\mathbf{s}') = \sum_{i=1}^{2^t} c_i \tilde{g}_i(\mathbf{s}) - \sum_{i=1}^{2^t} c_i h_i(\mathbf{s}).$$

So $\mathbf{w} = (x_1 - y_1) \cdots (x_{t+1} - y_{t+1})\mathbf{a}$ is a linear combination of 2^{t+1} vectors in $\Pi(\mathbf{s})$ with ± 1 coefficients, as desired. $\square$

Although, as stated in Theorem 1, the dimension of the space of trades generated by t-(v,k) minimal trades is $\binom{v}{k} - \binom{v}{t}$, our next result reveals that this quantity for t-(v,k) total trades is $\binom{v}{t+1} - \binom{v}{t}$.

Theorem 9. *For integers $k > t > 0$ and $v \geq k + t$, $\dim\langle \mathfrak{T}_{t,k,v} \rangle = \binom{v}{t+1} - \binom{v}{t}$.*

Proof. For $k = t + 1$, the trades in $\mathfrak{T}_{t,k,v}$ have no tail, that is they are t-$(v, t+1)$ minimal trades and the assertion follows from Theorem 1. So we assume that $k \geq t + 2$. Consider the 'natural' bijection from $\mathfrak{T}_{t,k,v}$ to $\mathfrak{T}_{t,t+1,v}$ that removes tails, in other words, it maps $T = (x_1 - y_1) \cdots (x_{t+1} - y_{t+1})\mathbf{j}$ to $\widetilde{T} = (x_1 - y_1) \cdots (x_{t+1} - y_{t+1})$. Our goal is to demonstrate that this map preserves both linear dependence and independence, from which it evidently follows that $\dim\langle \mathfrak{T}_{t,k,v} \rangle = \dim\langle \mathfrak{T}_{t,t+1,v} \rangle = \binom{v}{t+1} - \binom{v}{t}$.

First, we show that for any $S \in \binom{X}{t+1}$,

$$m_T(S) = \binom{v - 2t - 2}{k - t - 1} m_{\widetilde{T}}(S). \tag{5}$$

To verify this, note that any block of $\widetilde{T}$ and T has an intersection of size 1 with every $\{x_i, y_i\}$ for $i = 1, \ldots, t+1$. Therefore, if $|S \cap \{x_i, y_i\}| = 2$ for some i, then $m_T(S) =$

$m_{\widetilde{T}}(S) = 0$. Similarly, if $|S \cap \{x_i, y_i\}| = 0$ for some i, then $m_{\widetilde{T}}(S) = 0$. In T, half of the blocks contain x_i and the other half contain y_i with opposite signs. Thus, the total number of blocks containing S in either of the above two parts cancels out, implying $m_T(S) = 0$. It remains to consider the case where $|S \cap \{x_i, y_i\}| = 1$ for every $i = 1, \ldots, t+1$. In this case, either S or $-S$ is a block of $\widetilde{T}$, which means that either $S \cup L$ for all $L \in \binom{X \setminus \{x_1, y_1, \ldots, x_{t+1}, y_{t+1}\}}{k-t-1}$, or $-(S \cup L)$ for all such L, are blocks of T. This justifies (5).

The next task is to establish a relationship between the number of appearances of k-subsets in T and that of $(t+1)$-subsets in $\widetilde{T}$. We show that for an arbitrary $K \in \binom{X}{k}$,

$$m_T(K) = \sum_{L \in \binom{K}{t+1}} m_{\widetilde{T}}(L). \tag{6}$$

The left-hand side is either 0 or ± 1, and it is ± 1 if and only if $|\{x_i, y_i\} \cap K| = 1$ for all $i = 1, \ldots, t+1$. First, consider the case where $m_T(K) = 1$. So, without loss of generality, we assume that $\{x_i, y_i\} \cap K = \{x_i\}$ for all $i = 1, \ldots, t+1$. For any $L \in \binom{K}{t+1}$, if $L \neq \{x_1, \ldots, x_{t+1}\}$, then some $\{x_j, y_j\}$ has an empty intersection with L, and thus $m_{\widetilde{T}}(L) = 0$. Clearly $m_{\widetilde{T}}(\{x_1, \ldots, x_{t+1}\}) = 1$, so the right-hand side of (6) is also 1. Now, consider the case where $m_T(K) = 0$. Then for some j, $|\{x_j, y_j\} \cap K| = 0$ or 2. If the former occurs, then for all $L \in \binom{K}{t+1}$, $|\{x_j, y_j\} \cap L| = 0$ and so $m_{\widetilde{T}}(L) = 0$. Now, suppose that the latter is the case. We only need to consider those $(t+1)$-subsets L such that $|\{x_j, y_j\} \cap L| = 1$, which means they are of the form $L' \cup \{x_j\}$ or $L' \cup \{y_j\}$ for some t-subset L' of $K \setminus \{x_j, y_j\}$. More precisely,

$$\sum_{L \in \binom{K}{t+1}} m_{\widetilde{T}}(L) = \sum_{L' \in \binom{K \setminus \{x_j, y_j\}}{t}} \left(m_{\widetilde{T}}(L' \cup \{x_j\}) + m_{\widetilde{T}}(L' \cup \{y_j\}) \right) = 0,$$

where the last equality follows from the fact that $m_{\widetilde{T}}(L' \cup \{x_j\}) = -m_{\widetilde{T}}(L' \cup \{y_j\})$. Thus (6) is established.

We are now ready to establish our objective: For any $a_1, \ldots, a_m \in \mathbb{Q}$ and $T_1, \ldots, T_m \in \mathfrak{T}_{t,k,v}$, we have $T = \sum_{i=1}^{m} a_i T_i = \mathbf{0}$ if and only if $T^* = \sum_{i=1}^{m} a_i \widetilde{T}_i = \mathbf{0}$.

If $T = \mathbf{0}$, then for any $S \in \binom{X}{t+1}$, we have $m_T(S) = 0$. This in view of (5) implies that

$$0 = \sum_{i=1}^{m} a_i m_{T_i}(S) = \binom{v-2t-2}{k-t-1} \sum_{i=1}^{m} a_i m_{\widetilde{T}_i}(S) = \binom{v-2t-2}{k-t-1} m_{T^*}(S). \tag{7}$$

This means that the $(v, t+1)$-vector T^* is a $(t+1)$-trade, and this is only possible when $T^* = \mathbf{0}$.

If $T^* = \mathbf{0}$, then for every $S \in \binom{X}{t+1}$, $m_{T^*}(S) = 0$. Consider an arbitrary $K \in \binom{X}{k}$, then using (6),

$$m_T(K) = \sum_{i=1}^{m} a_i m_{T_i}(K) = \sum_{i=1}^{m} a_i \sum_{L \in \binom{K}{t+1}} m_{\widetilde{T}_i}(L)$$

$$= \sum_{L \in \binom{K}{t+1}} \sum_{i=1}^{m} a_i m_{\widetilde{T}_i}(L) = \sum_{L \in \binom{K}{t+1}} m_{T^*}(L) = 0.$$

This implies T is a k-trade and so $T = \mathbf{0}$. □

From the above proof, we can deduce the following crucial property of $\langle \mathfrak{T}_{j,k,v} \rangle$.

Lemma 10. *For every $j = 1, \ldots, k-1$, $\langle \mathfrak{T}_{j,k,v} \rangle$ contains no non-zero $(j+1)$-trade.*

Proof. For a contradiction, suppose $\mathbf{0} \neq T \in \langle \mathfrak{T}_{j,k,v} \rangle$ is a $(j+1)$-trade. Let $T^* \in \langle \mathfrak{T}_{j,j+1,v} \rangle$ be as defined in the proof of Theorem 9. By (7), T^* is a $(j+1)$-trade. From the property established in the proof of Theorem 9 that $T = \mathbf{0}$ if and only if $T^* = \mathbf{0}$, we conclude that $T^* \neq \mathbf{0}$. This contradiction implies the result. □

The next theorem provides a decomposition of the space of trades as a direct sum of the subspaces generated by total trades.

Theorem 11. *The vector space of t-(v,k) trades can be decomposed into the direct sum $\langle \mathfrak{T}_{t,k,v} \rangle \oplus \cdots \oplus \langle \mathfrak{T}_{k-1,k,v} \rangle$.*

Proof. In our argument v and k are fixed, so for the sake of simplicity, let us write $\mathfrak{T}_j$ for $\mathfrak{T}_{j,k,v}$.

First we show that $\langle \mathfrak{T}_t \rangle + \cdots + \langle \mathfrak{T}_{k-1} \rangle$ is a direct sum. To establish this, it is sufficient to demonstrate that if $\mathbf{s}_t + \cdots + \mathbf{s}_{k-1} = \mathbf{0}$ for $\mathbf{s}_i \in \langle \mathfrak{T}_i \rangle$, then $\mathbf{s}_i = \mathbf{0}$ for $i = t, \ldots, k-1$. For a contradiction, let r be the smallest index with $\mathbf{s}_r \neq 0$. Then we have

$$\mathbf{s}_r \in \langle \mathfrak{T}_{r+1} \rangle + \cdots + \langle \mathfrak{T}_{k-1} \rangle,$$

implying that $\mathbf{s}_r$ is a $(r+1)$-trade which is a contradiction by Lemma 10 (as $\mathbf{s}_r \in \langle \mathfrak{T}_r \rangle$).

Now by Theorem 9,

$$\sum_{i=t}^{k-1} \dim \langle \mathfrak{T}_i \rangle = \sum_{i=t}^{k-1} \left(\binom{v}{i+1} - \binom{v}{i} \right) = \binom{v}{k} - \binom{v}{t}.$$

This is the dimension of the space of t-(v,k) trades, and so the result follows. □

Let us now recall our main theorem from Introduction:

Theorem 2. *Let T be a t-(v,k) trade. Then*

$$\langle \Pi(T) \rangle = \bigoplus_{i \in I} \langle \mathfrak{T}_{i,k,v} \rangle ,$$

where $I \subseteq \{t, \ldots, k-1\}$ consists of integers i such that there is an i-trade in $\langle \Pi(T) \rangle$ which is not an $(i+1)$-trade. In particular, $\langle \Pi(T) \rangle$ is the whole vector space of t-(v,k) trades if and only if $I = \{t, \ldots, k-1\}$.

Proof. Let $\mathbf{s}$ be the corresponding (v,k)-vector of T. Without loss of generality we may assume that $\mathbf{s}$ is not a $(t+1)$-trade. From Theorem 11, it follows that

$$\mathbf{s} = \sum_{i \in J} \mathbf{s}_i,$$

for some $J \subseteq \{t, \dots, k-1\}$ and $\mathbf{0} \neq \mathbf{s}_i \in \langle \mathfrak{T}_i \rangle$. Thus $\langle \Pi(\mathbf{s}) \rangle \subseteq \left\langle \bigcup_{i \in J} \Pi(\mathbf{s}_i) \right\rangle \subseteq \bigoplus_{i \in J} \langle \mathfrak{T}_i \rangle$.

To show the reverse inclusion $\bigoplus_{i \in J} \langle \mathfrak{T}_i \rangle \subseteq \langle \Pi(\mathbf{s}) \rangle$, we proceed by induction on $|J|$. If $|J| = 1$, say $J = \{j\}$, then as $\mathfrak{T}_j = \Pi(\mathbf{s}_j)$, we have $\langle \mathfrak{T}_j \rangle = \langle \Pi(\mathbf{s}_j) \rangle = \langle \Pi(\mathbf{s}) \rangle$, and we are done. Let $|J| \geq 2$. We first prove that there is some $j \in \{t, \dots, k-1\}$ such that $\langle \mathfrak{T}_j \rangle \subseteq \langle \Pi(\mathbf{s}) \rangle$. By Lemma 8, there exists some trade $\mathbf{w} = (x_1 - y_1) \cdots (x_{t+1} - y_{t+1}) \mathbf{a} \in \langle \Pi(\mathbf{s}) \rangle$ for some $(v - 2t - 2, k - t - 1)$-vector $\mathbf{a}$, and that $\mathbf{a}$ is not a 0-trade. If $\mathbf{a}$ is a scalar multiple of $\mathbf{j}$, then $\langle \Pi(\mathbf{w}) \rangle = \langle \mathfrak{T}_t \rangle$ is contained in $\langle \Pi(\mathbf{s}) \rangle$. Otherwise, by Lemma 6, there exist x_{t+2} and y_{t+2} such that $(x_{t+2} - y_{t+2}) \mathbf{a}' \in \langle \Pi(\mathbf{a}) \rangle$, and thus by Remark 7,

$$\mathbf{w}' = (x_1 - y_1) \cdots (x_{t+2} - y_{t+2}) \mathbf{a}' \in \langle \Pi(\mathbf{s}) \rangle.$$

Again if $\mathbf{a}'$ is a scalar multiple of $\mathbf{j}$, then $\langle \Pi(\mathbf{w}') \rangle = \langle \mathfrak{T}_{t+1} \rangle \subseteq \langle \Pi(\mathbf{s}) \rangle$. Otherwise, continuing this process, we find some $j \in \{t, \dots, k-1\}$ such that $\langle \mathfrak{T}_j \rangle \subseteq \langle \Pi(\mathbf{s}) \rangle$. (Note that this process stops at some point: in the worst case we arrive at $\mathbf{w}' = (x_1 - y_1) \cdots (x_k - y_k) \in \mathfrak{T}_{k-1}$.) It follows that $\langle \mathfrak{T}_j \rangle \subseteq \langle \Pi(\mathbf{s}) \rangle \subseteq \bigoplus_{i \in J} \langle \mathfrak{T}_i \rangle$, and, by the properties of direct sums, this is possible only if $j \in J$. Then we consider $\mathbf{s}' = \mathbf{s} - \mathbf{s}_j \in \langle \Pi(\mathbf{s}) \rangle$, for which we have $\mathbf{s}' = \sum_{i \in J'} \mathbf{s}_i$, where $J' = J \setminus \{j\}$. By the induction hypothesis, $\bigoplus_{i \in J'} \langle \mathfrak{T}_i \rangle \subseteq \langle \Pi(\mathbf{s}') \rangle$, and so we are done.

To complete the proof, we need to show that $I = J$. It is clear that $J \subseteq I$. By the assumption that $\mathbf{s}$ is a t-trade but not a $(t+1)$-trade, we know that $t \in I$. If $t \notin J$, then all the trades in $\bigoplus_{i \in J} \langle \mathfrak{T}_i \rangle$ are $(t+1)$-trades, which is impossible. So $t \in J$. Let $j = \min(Y \setminus \{t\}) \geq \min(J \setminus \{t\})$. Then $\bigoplus_{i \in J \setminus \{t\}} \langle \mathfrak{T}_i \rangle$ must contains a j-trade which is not $(j+1)$-trade. If $j \notin J$, then all the members of $\bigoplus_{i \in J \setminus \{t\}} \langle \mathfrak{T}_i \rangle$ are $(j+1)$-trade, a contradiction. So $j \in J$. $\square$

Remark 12. The equality of the sets I and J in the proof of Theorem 2 demonstrates that to specify the summands appearing in the decomposition of $\langle \Pi(\mathbf{s}) \rangle$, it is sufficient to obtain the representation of $\mathbf{s}$ in $\langle \mathfrak{T}_{t,k,v} \rangle \oplus \cdots \oplus \langle \mathfrak{T}_{k-1,k,v} \rangle$. This justifies that computationally, the decomposition can be efficiently extracted. Furthermore, if the t-trade $\mathbf{s}$ is not a $(t+1)$-trade, then I contains t.

4 Dimension of the space generated by permutations of a t-design

In this section, we deal with the dimension of the vector space spanned by permutations of a signed t-design. We begin with the following corollary of Theorem 2, from which Theorem 3 readily follows.

Corollary 13. *Let* $\mathbf{s}$ *be a* t-(v,k,λ) *signed design. Then*

$$\langle \Pi(\mathbf{s})\rangle = \langle \mathbf{j}\rangle \oplus \bigoplus_{i\in I} \langle \mathfrak{T}_{i,k,v}\rangle ,$$

where the set $I \subseteq \{t,\dots,k-1\}$ *is determined by the non-zero summands of the representation of the* t*-trade* $\mathbf{s}-\alpha\mathbf{j}$*, with* $\alpha = \lambda/\binom{v-t}{k-t}$*, in the direct sum* $\langle \mathfrak{T}_{t,k,v}\rangle \oplus \cdots \oplus \langle \mathfrak{T}_{k-1,k,v}\rangle$*.*

Proof. Note that $\mathbf{j} \in \langle \Pi(\mathbf{s})\rangle$ because the sum of all vectors of $\Pi(\mathbf{s})$ is a non-zero scalar multiple of $\mathbf{j}$. Since $\alpha\mathbf{j}$ is a rational solution of (3), it follows that $\mathbf{s}-\alpha\mathbf{j}$ is a t-trade in $\langle \Pi(\mathbf{s})\rangle$. By Theorem 11, $\mathbf{s}-\alpha\mathbf{j} = \sum_{i\in I}\mathbf{s}_i$ for some $I \subseteq \{t,\dots,k-1\}$ and $\mathbf{0} \neq \mathbf{s}_i \in \langle \mathfrak{T}_{i,k,v}\rangle$. It follows that

$$\langle \Pi(\mathbf{s})\rangle = \left\langle \Pi\left(\alpha\mathbf{j} + \sum_{i\in I}\mathbf{s}_i\right)\right\rangle \subseteq \langle \mathbf{j}\rangle \oplus \left\langle \Pi\left(\sum_{i\in I}\mathbf{s}_i\right)\right\rangle .$$

The above inclusion is indeed an equality as the right-hand side is a subsapce of $\langle \Pi(\mathbf{s})\rangle$. By the proof Theorem 2, $\left\langle \Pi\left(\sum_{i\in I}\mathbf{s}_i\right)\right\rangle = \bigoplus_{i\in I}\langle \mathfrak{T}_{i,k,v}\rangle$, and we are done. □

The rest of this section is devoted to the proof of Theorem 4, demonstrating that when the design $\mathbf{s}$ in Corollary 13 is non-negative and v is large enough, $\langle \Pi(\mathbf{s})\rangle$ attains its maximum dimension.

Lemma 14. *Let* $\mathbf{a}$ *be a* (v,k)*-vector with* $\sigma(\mathbf{a}) \neq 0$ *such that there exist at least* k *elements of* X *not appearing in* $\mathbf{a}$*. Then* $\dim\langle \Pi(\mathbf{a})\rangle = \binom{v}{k}$*.*

Proof. For $k=1$, with no loss of generality, we may assume that $\mathbf{a} = a_1x_1 + \cdots + a_sx_s$ for some $x_1,\dots,x_s \in X$ with $a_s \neq 0$ and $s < v$. Consider

$$\mathbf{w} = \mathbf{a} - f_{x_s x_{s+1}}(\mathbf{a}) = a_s(x_s - x_{s+1}) \in \langle \Pi(\mathbf{a})\rangle,$$

that is a 0-(v,k) minimal trade. By Theorem 1, $\dim\langle \Pi(\mathbf{w})\rangle = \binom{v}{k} - \binom{v}{0}$. As $\sigma(\mathbf{a}) \neq 0$, $\mathbf{a}$ is not a trade which in turn implies that $\langle \Pi(\mathbf{w})\rangle$ is a proper subspace of $\langle \Pi(\mathbf{a})\rangle$ and so $\dim\langle \Pi(\mathbf{a})\rangle \geq \dim\langle \Pi(\mathbf{w})\rangle + 1 = \binom{v}{k}$, as desired. Now, suppose that $k \geq 2$. For some $x \in X$, $\mathbf{a}_x$ is not a 0-trade (otherwise $\mathbf{a}$ would be a 1-trade). This means $\sigma(\mathbf{a}_x) \neq 0$.

Let us choose some $y \in X$ not appearing in $\mathbf{a}$. Then $\sigma(\mathbf{a}_{x\overline{y}}) = \sigma(\mathbf{a}_x) \neq 0$. By the assumption, there are at least $k-2$ elements of $X \setminus \{x, y\}$ that do not appear in $\mathbf{a}_{x\overline{y}}$. Hence by the induction hypothesis, $\dim\langle \Pi(\mathbf{a}_{x\overline{y}})\rangle = \binom{v-2}{k-1}$. In particular, $\langle \Pi(\mathbf{a}_{x\overline{y}})\rangle$ contains a vector with all 0 components except one non-zero component corresponding to the set $\{x_1, \ldots, x_{k-1}\}$. Note that $\mathbf{a}_{\overline{x}y} = \mathbf{0}$. Thus, by (4), we have $\mathbf{a} - f_{xy}(\mathbf{a}) = (x-y)\mathbf{a}_{x\overline{y}}$. In view of Remark 7, it becomes apparent that $\mathbf{w} = (x-y)x_1 \cdots x_{k-1} \in \langle \Pi(\mathbf{a})\rangle$. Again by Theorem 1, $\dim\langle \Pi(\mathbf{w})\rangle = \binom{v}{k} - \binom{v}{0}$ and thus $\dim\langle \Pi(\mathbf{a})\rangle = \binom{v}{k}$. □

We now establish Theorem 4 in the broader context of signed designs. Subsequently, as a corollary, we derive the result for designs with non-negative components. By $\|\mathbf{a}\|$ we mean the 1-norm of $\mathbf{a}$, that is

$$\|\mathbf{a}\| = \sum_{i=1}^{\binom{v}{k}} |a_i|.$$

Theorem 15. *Let $\lambda \geq 0$ and $\mathbf{s}$ be a t-(v, k, λ) signed design which is not a $(t+1)$-design and*

$$\mu = \mu(\mathbf{s}) = \max\left\{ \|\mathbf{s}_{x_1 \ldots x_t}\| : \{x_1, \ldots, x_t\} \in \binom{X}{t} \right\}.$$

If $v \geq (2^t\mu + 1)(k-t-1) + 2(t+1)$, then $\dim\langle \Pi(\mathbf{s})\rangle = \binom{v}{k} - \binom{v}{t} + 1$.

Proof. By Lemma 8, there exists some trade $\mathbf{w} = (x_1 - y_1) \cdots (x_{t+1} - y_{t+1})\mathbf{a} \in \langle \Pi(\mathbf{s})\rangle$. Then $\|\mathbf{w}_{x_1 \ldots x_t}\| = \|(x_{t+1} - y_{t+1})\mathbf{a}\| = 2\|\mathbf{a}\|$. On the other hand, $\mathbf{w}$ is a linear combination of 2^{t+1} permutations of $\mathbf{s}$ with ± 1 coefficients. For any permutation f, $\|f(\mathbf{s})_{x_1 \ldots x_t}\| \leq \mu$. It follows that $\|\mathbf{w}_{x_1 \ldots x_t}\| \leq 2^{t+1}\mu$ and thus $\|\mathbf{a}\| \leq 2^t\mu$. Note that the size of blocks of $\mathbf{a}$ is $k-t-1$ and so at most $2^t\mu(k-t-1)$ elements of X appear in $\mathbf{a}$. So we have at least $v - 2^t\mu(k-t-1) - 2(t+1)$ elements not appearing in $\mathbf{w}$. Taking into account the assumption that $v \geq (2^t\mu + 1)(k-t-1) + 2(t+1)$, it follows that at least $k-t-1$ elements do not appear in $\mathbf{w}$. This means that $k-t-1$ elements of $X \setminus \{x_1, \ldots, x_{t+1}, y_1, \ldots, y_{t+1}\}$ do not appear in $\mathbf{a}$. Hence $\mathbf{a}$ as a $(v-2t-2, k-t-1)$-vector satisfies the assumptions of Lemma 14, and subsequently $\langle \Pi(\mathbf{a})\rangle$ contains a vector with all 0 components except one non-zero component corresponding to the set $\{x_{t+2}, \ldots, x_k\}$. Using Remark 7, it is inferred that $(x_1 - y_1) \cdots (x_{t+1} - y_{t+1})x_{t+2} \cdots x_k \in \langle \Pi(\mathbf{w})\rangle \subseteq \langle \Pi(\mathbf{s})\rangle$. Thus by Theorem 1, $\dim \Pi(\mathbf{w}) = \binom{v}{k} - \binom{v}{t}$. Since $\mathbf{s}$ is not a trade, we have $\dim\langle \Pi(\mathbf{s})\rangle \geq \dim\langle \Pi(\mathbf{w})\rangle + 1$, implying the result. □

In particular, if $\mathbf{s}$ is a t-(v, k, λ) design with non-negative components, then for any $\{x_1, \ldots, x_t\} \in \binom{X}{t}$, we have $\|\mathbf{s}_{x_1 \ldots x_t}\| = \lambda$, and so $\mu(\mathbf{s}) = \lambda$. Therefore, Theorem 4 follows as a special case of Theorem 15.

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