European Journal of Combinatorics 138 (2026) 104430

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: https://www.sciencedirect.com/journal/european-journal-of-combinatorics

Twin-width VIIIa: Delineation ☆

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ARTICLE INFO

Article history:
Received 5 December 2025
Accepted 6 July 2026

ABSTRACT

We introduce the notion of delineation. A graph class $\mathcal{C}$ is said *delineated by twin-width* (or simply, *delineated*) if for every hereditary closure $\mathcal{D}$ of a subclass of $\mathcal{C}$, it holds that $\mathcal{D}$ has bounded twin-width if and only if $\mathcal{D}$ is monadically dependent. An effective strengthening of *delineation* for a class $\mathcal{C}$ implies that tractable FO model checking on $\mathcal{C}$ is perfectly understood: On hereditary closures $\mathcal{D}$ of subclasses of $\mathcal{C}$, FO model checking is fixed-parameter tractable (FPT) exactly when $\mathcal{D}$ has bounded twin-width. Ordered graphs [BGOdMSTT, STOC '22], permutation graphs [BKTW, JACM '22] and tournaments [GT, European Journal of Combinatorics '25] are effectively delineated, while subcubic graphs are not. On the one hand, we prove that interval graphs, and even, directed path graphs with boundedly many roots are delineated. On the other hand, we observe or show that segment

☆ This paper comprises of the first part of an extended abstract that was presented at the 17th International Symposium on Parameterized and Exact Computation (IPEC 2022) (Bonnet et al., 2022) [5].

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[1] Supported by the National Research Foundation in Korea (NRF 2025-2029, RS-2025-00563533) and the Institute for Basic Science (IBS-R029-C1).

[2] Supported by HIDSS-0002 DASHH (Data Science in Hamburg - Helmholtz Graduate School for the Structure of Matter).

https://doi.org/10.1016/j.ejc.2026.104430

graphs, directed path graphs (with arbitrarily many roots), and visibility graphs of simple polygons are not delineated.

1. Introduction

Given a graph G and an FO sentence φ as input, the FO model checking problem aims to decide whether φ is satisfied in G. FO model checking is known to be PSPACE-complete [37] and further AW[$*$]-hard on general graphs [14] and thus not likely to be fixed parameter tractable on general graph classes parameterized by the length of the input sentence φ. An active line of research considers the question for which classes of graphs the FO model checking problem is tractable.

For monotone (i.e., closed under removing vertices and edges) classes, the FPT algorithm of Grohe, Kreutzer, and Siebertz [29] for FO model checking on nowhere dense classes, is complemented by W[1]-hardness on classes that are somewhere dense (i.e., *not* nowhere dense) [16], and even AW[$*$]-hardness on classes that are *effectively* somewhere dense [34]. The latter two results imply that, for monotone classes, FO model checking is unlikely to be FPT beyond nowhere dense classes. Thus the classification of monotone classes admitting an FPT FO model checking is over. However such a classification remains an active line of work for the more general hereditary (i.e., closed under removing vertices) classes of graphs and binary structures [17,22,24]. It is conjectured (see for instance [22, Conjecture 8.2]) that:

Conjecture 1. *For every hereditary class $\mathcal{C}$ of structures, FO model checking is FPT on $\mathcal{C}$ if and only if $\mathcal{C}$ is monadically dependent.*[3]

Twin-width was introduced in 2019 by Bonnet et al. [8] and is defined via sequences of vertex contractions. In each step of such a sequence, we identify two vertices u, v (until there is only one vertex left) and recording *errors* in their adjacency with other vertices as red edges, i.e., when only one of the two vertices u, v is adjacent to a third vertex w. The *width* of a contraction sequence roughly measures what the maximum number of errors is that each vertex can be contained in at any point in the sequence and the *twin-width* is the minimum width of any contraction sequence. For our purposes however, the following characterization of classes of bounded twin-width is more useful.

Theorem 2 ([7]). *A class $\mathcal{C}$ has bounded twin-width if and only if there is an integer k such that every graph of $\mathcal{C}$ admits an adjacency matrix* without *rank-k division, i.e., k-division such that every cell has combinatorial rank at least k (i.e. at least k distinct rows or at least k distinct columns).*

Here, a k-division of a matrix is a partition of its column (resp. row) set into k intervals, called *column* (resp. *row*) *parts*, of consecutive columns (resp. rows). A k-division naturally defines k^2 *cells* (contiguous submatrices) made by the entries at the intersection of a column part with a row part. Theorem 2 is effective: There is a computable function f, such that, given a vertex ordering along which the adjacency matrix of a graph G has no rank-k division, one can efficiently find a contraction sequence for G of width $f(k)$, witnessing that $\mathrm{tww}(G) \leqslant f(k)$.

In their seminal work on twin-width, Bonnet et al. provide an FPT FO model checking algorithm on classes of graphs of bounded twin-width in case the input graph is given along with a contraction sequence of small width [8]. This result unifies and extends several known results [20,21,23,25,30]

[3] A model-theoretic notion which roughly says that not every graph G can be built from a nondeterministic $O(1)$-coloring of some $\mathcal{S} \in \mathcal{C}$ by means of a first-order formula $\varphi(x, y)$, in the relations of $\mathcal{S}$ and the added colors, imposing the edge set of G; see Section 2 for a definition.

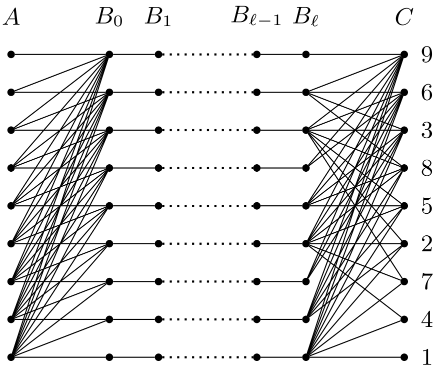

Fig. 1. A generalized transversal pair of half-graphs. The number next to each rightmost vertex v of the transversal pair indicates the height of the neighbor of v in the central column of paths which is not also a neighbor of the vertex just below v.

for hereditary, but not necessarily monotone, classes. The missing element for an FPT FO model-checking algorithm on any class of bounded twin-width is a polynomial-time algorithm and a computable function f, that given a constant integer bound c and a graph G, either finds an $f(c)$-sequence for G, or correctly reports that $\text{tww}(G) > c$. The runtime of the algorithm could be $n^{g(c)}$, for some function g. However to get an FPT algorithm in the combined parameter *size of the sentence + bound on the twin-width*, one would further require that the approximation algorithm takes FPT time in c (now thought of as a parameter), i.e., $g(c)n^{O(1)}$. Such an algorithm exists on ordered graphs (more generally, ordered binary structures) [7], graphs of bounded clique-width, proper minor-closed classes [8], but it is not known whether such an algorithm exist on the class of all graphs. Let us observe that exactly computing the twin-width, and even distinguishing between $\text{tww}(G) = 4$ and $\text{tww}(G) = 5$, is NP-complete [3].

A parallel can be drawn with treewidth and clique-width. Both are NP-hard to compute [1,19], but – unlike twin-width – treewidth admits an FPT algorithm [4], while clique-width can be approximated in FPT time [35]. Celebrated theorems by Courcelle [12] and by Courcelle, Makowsky, and Rotics [13], together with the latter two results, imply that monadic second-order (MSO) model checking is FPT on classes (of incidence graphs) with bounded treewidth, and on classes with bounded clique-width.

In this paper, we consider the question in which classes of graphs the theory of twin-width provides a full picture of when FO model checking is tractable. We consider geometric graph classes with *unbounded* twin-width: interval graphs, (rooted) directed path graphs, segment graphs, visibility graphs of polygons. For all these classes, a vertex-ordering procedure either comes naturally[4] or can be worked out and efficiently computed. Two cases are to be considered. Either the obtained adjacency matrix has no rank-k division –and we get a favorable contraction sequence by Theorem 2– or it does have such a division. In the latter case, a large structured object may be found, such as a *generalized transversal pair of half-graphs* (or *transversal pair*, for short) or some variant of it which constitutes an obstacle to FPT FO model checking; see Fig. 1 for visualization and Section 2 for a formal definition, and for why transversal pairs indeed are such obstacles.

[4] Think, ordering by increasing left endpoints in interval graphs, or following the boundary of polygons.

1.1. Delineation

When, on a class $\mathcal{C}$, a (variant of a) transversal pair can systematically be found as a result of unbounded twin-width, it shows that the classification of FPT FO model checking for hereditary subclasses of $\mathcal{C}$ is entirely settled by the algorithm on graphs of bounded twin-width [8].

More generally if for every hereditary closure[5] $\mathcal{D}$ of a subclass of $\mathcal{C}$, $\mathcal{D}$ has bounded twin-width if and only if $\mathcal{D}$ is monadically dependent, we say that $\mathcal{C}$ is *delineated by twin-width* (or simply, *delineated*). As every class of bounded twin-width is monadically dependent [8], equivalently, we can only require that $\mathcal{D}$ has bounded twin-width if $\mathcal{D}$ is monadically dependent.

Although not stated in those terms, permutation graphs were already proven to be delineated [8], as well as ordered graphs [7]. We add interval graphs and directed path graphs with boundedly many roots (see Section 4 for a definition) to the list of delineated classes. For every hereditary subclass of these classes the classification of FPT FO model checking, 1, is now provably settled.[6]

Theorem 3. *Interval graphs, and more generally directed path graphs with boundedly many roots, are delineated.*

Directed path graphs are intersection graphs of directed paths of a directed tree. Clearly, interval graphs which are also the intersection graphs of subpaths of a path, is a subclass of directed path graphs. Directed paths graphs have been studied to find the boundaries of tractability of problems that are polynomial time solvable on interval graphs but NP-hard on chordal graphs (intersection graphs of subtrees of a tree). For example, while the Dominating Set problem is NP-hard on chordal graphs, this problem is polynomial time solvable on directed path graphs [32]. A *root* of directed tree T is a vertex of in-degree 0. A family $\mathcal{F}$ of directed trees has *boundedly many roots* if there is a constant c such that for any $T \in \mathcal{F}$, the number of roots in T is at most c. After showing that interval graphs are delineated, a natural direction of research is to find superclasses that also have the delineation property. Theorem 3 shows that the delineation property holds for directed path graphs with boundedly many roots. In contrast, we rule out delineation for directed path graphs (with unbounded number of roots).

We also rule out delineation of intersection graphs of pure axis-parallel segments with two distinct lengths, and visibility graphs of simple polygons.

Subsequently, Geniet and Thomassé showed that the class of tournaments is delineated [28], answering an open question asked in a preliminary version of this paper.

1.2. Organization

In Section 2 we introduce all important notions and our notation. Additionally, we prove some first results regarding delineation which are used in subsequent sections. In Section 3 we give a proof of delineation of interval graphs. In Section 4 we discuss delineation of directed path graphs. We discuss (non-)delineation of other classes of geometric intersection/visibility graphs in Section 5 and conclude with some related open problems in Section 6.

2. Preliminaries and first results

We may denote the set of integers between i and j by $[i, j]$, and $[k]$ may be used as a short-hand for $[1, k]$. We start this section with the relevant definitions and background in graph theory, finite model theory, and recall some crucial theorems related to twin-width.

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[5] The reason we do not simply quantify over hereditary subclasses of $\mathcal{C}$ is to have a notion that is also meaningful when $\mathcal{C}$ is not hereditary.

[6] We actually need an effective strengthening of delineation that also holds for these classes and will be defined in Section 2.

2.1. Graph theory

We use the standard graph-theoretic definitions and notations. We denote by $V(G)$, and $E(G)$, the vertex set, and the edge set, of a graph G, and by $G[S]$ the subgraph of G induced by $S \subseteq V(G)$. We denote by $\mathrm{Adj}_{\prec}(G)$ the adjacency matrix of G *along the total order* $\prec$ *of* $V(G)$. An *interval along* $\prec$ or $\prec$*-interval* is a set of *consecutive* elements with respect to that order.

A *half-graph* (or *ladder*) *of height* t, denoted by H_t, consists of $2t$ distinct vertices $a_1, \dots, a_t, b_1, \dots, b_t$ such that a_i is adjacent to b_j if and only if $i \leqslant j$. A *generalized transversal pair of half-graphs* consists of $3+\ell$ sets $A = \{a_{i,j} \ : \ i,j \in [t]\}$, $B_0 = \{b^0_{i,j} \ : \ i,j \in [t]\}, \dots, B_\ell = \{b^\ell_{i,j} \ : \ i,j \in [t]\}$, and $C = \{c_{i,j} \ : \ i,j \in [t]\}$ such that there is an edge between $a_{i,j}$ and $b^0_{i',j'}$ if and only if $(i,j) \leqslant_{\mathrm{lex}} (i',j')$, for $k \in [\ell]$ there is an edge between $b^{k-1}_{i,j}$ and $b^k_{i',j'}$ if and only if $(i,j) = (i',j')$ and there is an edge between $b^\ell_{i,j}$ and $c_{i',j'}$ if and only if $(j,i) \leqslant_{\mathrm{lex}} (j',i')$, where $\leqslant_{\mathrm{lex}}$ denotes the lexicographic (left–right) order. We denote this graph by $T_{t,\ell}$, and a *semi-induced* $T_{t,\ell}$ is such a graph with possibly some extra edges between two sets $X, Y \in \{A, B_0, \dots, B_\ell, C\}$ with no predefined edges. Note that $A \cup B_0$ and $B_\ell \cup C$ both induce a half-graph, but the "order" these two half-graphs put on the endpoints of the paths $(b^0_{i,j}, \dots, b^\ell_{i,j})$ is different. We define $T_k := T_{k,0}$ and we call T_k a transversal pair (of half-graphs); see Fig. 1.

We will come back to (generalized) transversal pairs after we introduce interpretations and transductions, and show that they are complex, universal objects as far as closure by FO transductions is concerned.

2.2. First-order logic, interpretations, transductions, and dependence

A relational *signature* σ is a finite set of relation symbols R, each having a specified arity $r \in \mathbb{N}$. A σ*-structure* $\mathbf{A}$ is defined by a set A (the *domain* of $\mathbf{A}$) and a relation $R^{\mathbf{A}} \subseteq A^r$ for each relation symbol $R \in \sigma$ with arity r.

A *binary structure* is a relational structure with symbols of arity at most 2. The syntax and semantics of first-order formulas over σ (or σ*-formulas* for short), are defined as usual. We recall that a *sentence* is a formula without free variable. Most of the time we will consider σ-structures with σ consisting of a single binary relation symbol E, and identify them to graphs. But we will also deal with binary structures that are graphs augmented with a total order (called *totally ordered graphs*, or *ordered graphs* for short) and/or some unary relations.

Interpretations, transductions, and monadic dependence. Let σ, τ be relational signatures. A *simple FO interpretation* (here, *FO interpretation* for short) I of τ-structures in σ-structures consists of the following σ-formulas: a *domain* formula $\nu(x)$, and for each relation symbol $R \in \tau$ of arity r, a formula $\varphi_R(x_1, \dots, x_r)$. If $\mathbf{A}$ is a σ-structure, the τ-structure $\mathsf{I}(\mathbf{A})$ has domain $\nu(\mathbf{A}) = \{v \in A \ : \ \mathbf{A} \models \nu(v)\}$ and the interpretation of a relation symbol $R \in \sigma$ of arity r is $R^{\mathsf{I}(\mathbf{A})} = \{(v_1, \dots, v_r) \in \nu(\mathbf{A})^r \ : \ \mathbf{A} \models \varphi_R(v_1, \dots, v_r)\}$. If $\mathcal{C}$ is a class of σ-structures, we set $\mathsf{I}(\mathcal{C}) = \{\mathsf{I}(\mathbf{A}) \ : \ \mathbf{A} \in \mathcal{C}\}$.

Let $\sigma \subseteq \sigma^+$ be relational signatures. The σ*-reduct* of a σ^+-structure $\mathbf{A}$, denoted by $\mathrm{reduct}_{\sigma^+ \to \sigma}(\mathbf{A})$, is the structure obtained from $\mathbf{A}$ by deleting all the relations not in σ. A *monadic h-lift* of a σ-structure $\mathbf{A}$ is a σ^+-structure $\mathbf{A}^+$, where σ^+ is the union of σ and a set of h unary relation symbols, and $\mathrm{reduct}_{\sigma^+ \to \sigma}(\mathbf{A}^+) = \mathbf{A}$.

A *simple non-copying FO transduction* (here, *FO transduction* for short) T of τ-structures in σ-structures is an interpretation of τ-structures in σ^+-structures, where the σ^+-structures are *monadic h-lifts* of σ-structures for some fixed integer h. As there are many ways of interpreting the extra unary relations, a transduction (contrary to an interpretation) builds on a given input structure several output structures. If $\mathcal{C}$ is a class of σ-structures, $\mathsf{T}(\mathcal{C})$ denotes the class of all the τ-structures output on any σ-structure $\mathbf{A} \in \mathcal{C}$.

We say that a class $\mathcal{C}$ *interprets* a class $\mathcal{D}$ (or that $\mathcal{D}$ *interprets in* $\mathcal{C}$) if there is an interpretation I such that $\mathcal{D} \subseteq \mathsf{I}(\mathcal{C})$. Further, a class $\mathcal{C}$ *efficiently interprets* $\mathcal{D}$ if additionally a polytime algorithm inputs $\mathbf{A} \in \mathcal{D}$, and outputs a structure $\mathbf{B} \in \mathcal{C}$ such that $\mathsf{I}(\mathbf{B})$ is isomorphic to $\mathbf{A}$. Similarly, we say that a class $\mathcal{C}$ *transduces* a class $\mathcal{D}$ (or that $\mathcal{D}$ *transduces in* $\mathcal{C}$) if there is a transduction T such that $\mathcal{D} \subseteq \mathsf{T}(\mathcal{C})$. Two classes $\mathcal{C}$ and $\mathcal{D}$ are *transduction equivalent* if $\mathcal{C}$ transduces $\mathcal{D}$, and $\mathcal{D}$ transduces

$\mathcal{C}$. We will frequently use the fact that one can compose transductions: If $\mathcal{C}$ transduces $\mathcal{D}$, and $\mathcal{D}$ transduces $\mathcal{E}$, then $\mathcal{C}$ transduces $\mathcal{E}$.

We will not need the original definition of monadic dependence; solely the following characterization:

Theorem 4 (*Baldwin and Shelah [2]*)**.** *$\mathcal{C}$ is monadically dependent if and only if $\mathcal{C}$ does not transduce the class $\mathcal{G}$ of all finite graphs.*

Since FO model checking on the class of all graphs is $\mathsf{AW}[*]$-hard [14], one gets:

Theorem 5 (*[2,14]*)**.** *If $\mathcal{C}$ efficiently interprets the class of all graphs then FO model checking on $\mathcal{C}$ is $\mathsf{AW}[*]$-hard.*

Furthermore, any hereditary class $\mathcal{C}$ which transduces the class of all graphs already efficiently interprets all graphs under the assumption that $\mathsf{AW}[*] \neq$FPT [15]. Hence, if a hereditary class $\mathcal{C}$ is monadically independent, then FO-model checking is AW[*]-hard on $\mathcal{C}$ assuming that $\mathsf{AW}[*] \neq$FPT.

A notion more restrictive to that of monadic dependence is *monadic stability*. A class is said *monadically stable* if it does not transduce the class of all half-graphs. A monadically stable class has in particular to be H_t-free.

Let us finally notice that, throughout the paper, the ambient logic will be first-order. Thus we may for instance drop the "FO" before "interpretation" and "transduction".

2.3. Twin-width and rank divisions

A *division* $\mathcal{D}$ of a matrix M is a pair $(\mathcal{D}^R, \mathcal{D}^C)$, where $\mathcal{D}^R$ (resp. $\mathcal{D}^C$) is a partition of the rows (resp. columns) of M into intervals of consecutive rows (resp. columns). Each element of $\mathcal{D}^R$ (resp. $\mathcal{D}^C$) is called a *row part* (resp. *column part*). A *k-division* is a division satisfying $|\mathcal{D}^R| = |\mathcal{D}^C| = k$. We often list the row (resp. column) parts of D^R (resp. D^C) $R_1, R_2, \dots, R_k$ (resp. $C_1, C_2, \dots, C_k$) when R_i is just below R_{i+1} (resp. C_j is just to the left of C_{j+1}). For every pair $R_i \in \mathcal{D}^R$, $C_j \in \mathcal{D}^C$, the (contiguous) submatrix $R_i \cap C_j$ is called *cell* or *zone* of $\mathcal{D}$. Note that a k-division defines k^2 zones. A *rank-k division* of M is a k-division $\mathcal{D}$ such that for every $R_i \in \mathcal{D}^R$ and $C_j \in \mathcal{D}^C$ the zone $R_i \cap C_j$ has at least k distinct rows or at least k distinct columns (that is, combinatorial rank at least k). For the remainder of the paper, for convenience we use rank to mean the combinatorial rank. By *large rank division*, we informally mean a rank-k division for arbitrarily large values of k. The maximum integer k such that M admits a rank-k division is called *grid rank*, and is denoted by $\mathrm{gr}(M)$.

An adjacency matrix M of a binary structure encodes in any[7] bijective fashion the *atomic type* of every pair of vertices (u, v) (i.e., the set of atomic propositions the pair (u, v) satisfies) at position (u, v) in M. We denote by $\mathrm{Adj}_{\prec}(\mathbf{A})$ the adjacency matrix of $\mathbf{A}$ *along* $\prec$, a total order on A. The *grid rank* of a binary structure $\mathbf{A}$, denoted by $\mathrm{gr}(\mathbf{A})$, is the least integer k such that there is a total order $\prec$ of A satisfying $\mathrm{gr}(\mathrm{Adj}_{\prec}(\mathbf{A})) = k$.

We will not need the original definition of twin-width via contraction sequences (presented in the introduction) generalized to binary structures.[8] So we do *not* reproduce it here. Instead we recall that the twin-width and the grid rank of a binary structure are functionally equivalent, and we encourage the reader to think of the twin-width of $\mathbf{A}$, $\mathrm{tww}(\mathbf{A})$, simply as its grid rank $\mathrm{gr}(\mathbf{A})$. Instead we give the following useful characterization of bounded twin-width, readily generalizable to classes of other binary structures than graphs. The twin-width of the binary structure is then defined as the twin-width of the unordered matrix M, denoted by $\mathrm{tww}(M)$.

Theorem 6 (*[7]*)**.** *There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every binary structure $\mathbf{A}$, the following two implications hold:*

[7] As the chosen encoding will not matter, we do not specify a *canonical* one.

[8] The definition is similar with red edges appearing between the contraction of u and v, and vertex z whenever (u, z) and (v, z) have different atomic types. We refer the curious reader to [8].

- *If* $\mathrm{tww}(\mathbf{A}) \leqslant k$*, then there is a total order* $\prec$ *of* A *such that* $gr(Adj_{\prec}(\mathbf{A})) \leqslant f(k)$*, and*
- *If there is a total order* $\prec$ *of* A *such that* $gr(Adj_{\prec}(\mathbf{A})) \leqslant k$*, then* $\mathrm{tww}(\mathbf{A}) \leqslant f(k)$*.*

Furthermore there are computable functions $g, h : \mathbb{N} \to \mathbb{N}$ *and an algorithm running in time* $h(k) \cdot |A|^{O(1)}$ *which inputs an adjacency matrix* $Adj_{\prec}(\mathbf{A})$ *without rank-*k *division and outputs a* $g(k)$*-sequence of* $\mathbf{A}$*.*

We recall that FO model checking is FPT on classes of bounded twin-width, when the inputs are given with a contraction sequence.

Theorem 7 (*[8]*)**.** *Let* $\mathbf{A}$ *be a binary structure of bounded twin-width given with a d-sequence, and* φ *be an FO sentence of quantifier rank k. Then one can decide* $\mathbf{A} \models \varphi$ *in linear FPT time* $f(d, k) \cdot |A|$*.*

Finally the following is a particularly useful tool.

Theorem 8 (*[8]*)**.** *Every FO transduction of a class with bounded twin-width has bounded twin-width.*

It is known [6,9] that classes of bounded twin-width have exponential growth. Thus by the contrapositive, classes of super-exponential growth, like the following ones, have unbounded twin-width.

Theorem 9 (*[6]*)**.** *The following classes have unbounded twin-width:*

- *the class* $\mathcal{G}_{\leqslant 3}$ *of every subcubic graph;*
- *the class* $\mathcal{B}_{\leqslant 3}$ *of every bipartite subcubic graph.*

Finally, as a useful tool when considering graphs with an associated total order we recall the following theorem.

Theorem 10 (*Erd́os-Szekeres Theorem [18]*)**.** *For every* $k \geq 1$ *any sequence of distinct* $(k-1)^2+1$ *real numbers contains an non-increasing or non-decreasing subsequence of size at least k.*

2.4. Delineation

A class $\mathcal{D}$ is said *delineated by twin-width* (or *delineated* for short) if for every hereditary closure $\mathcal{C}$ of a subclass of $\mathcal{D}$, it holds that $\mathcal{C}$ has bounded twin-width if and only if $\mathcal{C}$ is monadically dependent. A class $\mathcal{D}$ is *effectively delineated by twin-width* (or *effectively delineated*) if further there is an FPT approximation of contraction sequences on $\mathcal{D}$, that is, two computable functions f, g and an algorithm that takes a binary structure $\mathbf{A} \in \mathcal{D}$ and an integer k as input, and in time $f(k) \cdot |A|^{O(1)}$ outputs a $g(k)$-sequence for $\mathbf{A}$ or correctly reports that $\mathrm{tww}(\mathbf{A}) > k$.

Showing that a class $\mathcal{D}$ is effectively delineated establishes that, as far as efficient (that is, FPT) FO model checking is concerned, Theorem 7 gives a complete picture of what happens on $\mathcal{D}$. Indeed it is unlikely that a monadically independent class admits an FPT algorithm for FO model checking (see Section 2.2). Trivially, every class with bounded twin-width is delineated, and every class where $O(1)$-sequences can be found in polynomial time (see [6]) is effectively delineated. We now list some non-trivial examples of (effectively) delineated classes.

Theorem 11 (*[7,8,28,33]* + *This Paper*)**.** *The following classes of binary structures are effectively delineated:*

- *permutation graphs [8]; and even,*
- *circle graphs [33];*[9]

[9] Hliněný and Pokrývka [33] show that any hereditary subclass of circle graphs excluding a permutation graph has (effectively) bounded twin-width, which implies delineation, since the class of all permutation graphs have unbounded twin-width.

- *ordered graphs [7];*
- *interval graphs (see Section 3); and even,*
- *directed path graphs with boundedly many roots (see Section 4);*
- *tournaments [28].*

Our proofs of (effective) delineation will follow the same path. Either an $O(1)$-sequence of the graph is found (bounded twin-width) or an arbitrarily large semi-induced generalized transversal pair is detected. We now see that the latter implies monadic independence (hence, in particular, unbounded twin-width).

Lemma 12. *Let ℓ be a fixed non-negative integer. Let $\mathcal{C}$ be a hereditary class containing a semi-induced generalized transversal pair of half-graphs $T_{n,\ell}$, for every positive integer n. Then $\mathcal{C}$ is monadically independent.*

Proof. It is folklore that the class $\mathcal{M}_b$ of all totally ordered matchings is monadically independent (see for instance [7,9]). By *totally ordered bipartite matching*, we mean two sets X, Y of same cardinality, with a total order over $X \cup Y$ such that X and Y are each an interval along that order, and a matching between X and Y. We shall just argue that $\mathcal{M}_b$ transduces in $\mathcal{C}$. We first show the lemma when $\ell = 0$, that is, $\mathcal{C}$ contains a semi-induced $T_{n,0} = T_n$ for every n.

Let $(G = (X, Y, E(G)), \prec)$ be any member of $\mathcal{M}_b$. Let $x_1 \prec x_2 \prec \ldots \prec x_n$ be the elements of X, and $y_1 \prec y_2 \prec \ldots \prec y_n$, the elements of Y. Finally, let π be the permutation such that, for every $i \in [n]$, $x_i y_j \in E(G)$ if and only if $j = \pi(i)$.

Let (A, B, C) be the tripartition of a semi-induced T_n in $\mathcal{C}$. The transduction T guesses the tripartition (A, B, C) with 3 corresponding unary relations. Eventually (X, Y) will be a subset of (A, B). We interpret a total order on $A \cup B$ by

$$x \prec y \equiv (A(x) \wedge B(y)) \vee \big(x \neq y \wedge A(x) \wedge A(y) \wedge \forall z (B(z) \wedge E(x,z)) \rightarrow E(y,z)\big)$$
$$\vee \big(x \neq y \wedge B(x) \wedge B(y) \wedge \forall z (C(z) \wedge E(x,z)) \rightarrow E(y,z)\big).$$

We then interpret a matching between A and B by

$$\varphi(x, y) \equiv A(x) \wedge B(y) \wedge E(x, y) \wedge \forall z (z \prec x \rightarrow \neg E(z, y)).$$

Observe that φ and $\prec$ define a universal structure for totally ordered bipartite matchings on $2n$ vertices.

In particular, a fourth unary relation can guess the domain ($X \subseteq A$, $Y \subseteq B$), by picking the rows and columns of the biadjacency matrix $\mathrm{Adj}_{\prec}(A, B, \{ab \ : \ T_n \models \varphi(a, b)\})$ corresponding to the 1 entries, which, in the n-division with all row and column parts of size n falls in the $(i, \pi(i))$-cells with $i \in [n]$. Thus $\mathsf{T}(T_n)$ outputs $(G, \prec)$.

We now deal with the general case by reducing it to $\ell = 0$. For that, we transduce a semi-induced T_n in a semi-induced $T_{n,\ell}$. The transduction is simply based on the definition of generalized transversal pairs. It uses $3 + \ell$ unary relations $A, B_0, \ldots, B_\ell, C$, redefines the domain as $A \cup B_0 \cup C$, keeps the edges between A and B_0, and adds an edge between $x \in B_0$ and $y \in C$ if and only if there is a path from x to y going through $B_1, B_2, \ldots, B_\ell$, in this order. All of this is easily expressible in first-order logic. □

We deduce the following.

Lemma 13. *Let ℓ be a fixed non-negative integer. Let $f : \mathbb{N} \to \mathbb{N}$ be any computable function, and $\mathcal{C}$ be a graph class. If for every natural k and $G \in \mathcal{C}$, either G admits an $f(k)$-sequence or G has a semi-induced generalized transversal pair $T_{k,\ell}$, then $\mathcal{C}$ is delineated.*

Furthermore, if the contraction sequence can be found in time $g(k) \cdot |V(G)|^{O(1)}$ for some computable function g, then $\mathcal{C}$ is effectively delineated.

Proof. Let $\mathcal{D}$ be a hereditary subclass of $\mathcal{C}$. Let $k \in \mathbb{N} \cup \{+\infty\}$ be the maximum generalized natural such that $\mathcal{D}$ admits a semi-induced generalized transversal pair $T_{k,\ell}$. If $k = +\infty$, then we conclude that $\mathcal{D}$ is monadically independent by Lemma 12, and thus has unbounded twin-width. If k is finite, then $\mathcal{D}$ has bounded twin-width (by $f(k)$), and hence is monadically dependent. □

By Theorem 6, in the previous theorem, the $f(k)$-*sequence of* G can be replaced by an adjacency matrix of G of grid rank at most $f(k)$.

On the contrary, the class of subcubic graphs is *not* delineated. Indeed the whole class is monadically dependent (see for instance [36]), even monadically stable, but has unbounded twin-width [6]. We will see that the classes of segment graphs (even with some further restrictions) and visibility graphs of simple polygons are also *not* delineated. In some sense, what we do is to reduce to the easy case of subcubic graphs.

Lemma 14. *If $\mathcal{C}$ admits a subclass which is transduction equivalent to $\mathcal{G}_{\leqslant 3}$ or to $\mathcal{B}_{\leqslant 3}$, then $\mathcal{C}$ is not delineated.*

Proof. It is easy to check that $\mathcal{G}_{\leqslant 3}$ and $\mathcal{B}_{\leqslant 3}$ are transduction equivalent, so we assume without loss of generality that $\mathcal{C}$ has a subclass $\mathcal{D}$ transduction equivalent to the class of all subcubic graphs. As $\mathcal{D}$, and a fortiori its hereditary closure $\mathcal{D}'$, transduces $\mathcal{G}_{\leqslant 3}$, the class $\mathcal{D}'$ cannot have bounded twin-width by Theorems 8 and 9. As $\mathcal{G}_{\leqslant 3}$ transduces $\mathcal{D}$ and thus, its hereditary closure $\mathcal{D}'$, the class $\mathcal{D}'$ is monadically dependent, and even monadically stable [36]. □

We will prove that the above mentioned classes are transduction equivalent to the class of all (bipartite) subcubic graphs and invoke Lemma 14. Again, that shows something stronger: The considered classes contain a subclass whose hereditary closure is monadically stable and of unbounded twin-width.

3. Delineation of interval graphs

Let G be an interval graph and $\mathcal{I}_G = \{I_v \ : \ v \in V(G)\}$ be an interval representation of G over the real line, where the interval I_v is of the form $[i, j]$ for some integers $1 \leq i \leq j$. When the graph G is clear from the context, we omit the subscript of $\mathcal{I}_G$. For every $v \in V(G)$ and the associated interval I_v, the first and the second entries of I_v are called the left and the right endpoint of v, and denoted as ℓ_v and r_v, respectively.

This subsection is devoted to proving that interval graphs are delineated and intended as a preamble before considering the directed path graphs. Moreover, interval graphs appear as a subcase in the proof for directed path graphs with boundedly many roots.

Proposition 15. *The class of all interval graphs is effectively delineated.*

We recall that every class of bounded twin-width is monadically dependent. To show that the hereditary closure of every monadically dependent subclass of interval graphs has bounded twin-width, we shall henceforth find a semi-induced transversal pair of half-graphs T_t in any interval graph G of twin-width at least $f(t)$. The delineation of interval graphs then follows from Lemma 12.

For each interval graph G with an interval representation $\mathcal{I}$, we associate a total order $\prec$, following a lexicographic order on $\mathcal{I}$, defined in the following way. We say $u \prec v$ if and only if $\ell_u < \ell_v$ or, $\ell_u = \ell_v$ and $r_u < r_v$. If $\ell_u = \ell_v$ and $r_u = r_v$, we choose $u \prec v$ arbitrarily. We assume that $\mathcal{I}$ is a minimal representation in the sense that the sum of interval lengths is as small as possible. A consequence of the minimal representation assumption is that if $\ell_u < \ell_w$ for vertices $u, w \in V(G)$, there exists a vertex $v \in V(G)$ (possibly $v = u$) such that $\ell_u \leq r_v < \ell_w$ since otherwise, the interval I_u can be shortened by setting the new value for ℓ_u to ℓ_w. This is the only property we use from the minimality assumption on $\mathcal{I}$. Note that the order $\prec$ can be computed in linear time from the interval representation which is essential for obtaining effective delineation.

Let $\mathcal{C}$ be a class of interval graphs of unbounded twin-width. Fix an arbitrary integer t, and let $G \in \mathcal{C}$ be an interval graph such that the adjacency matrix $\mathrm{Adj}_{\prec}(G)$ of G with rows and columns ordered by $\prec$ has a rank-$8t^2$ division. Such a graph exists in $\mathcal{C}$ by Theorem 6. Let $\mathcal{I}$ be an interval representation of G. We remark that an interval representation can be computed in linear time [10,31]. A rank-$8t^2$ division of $\mathrm{Adj}_{\prec}(G)$ can be viewed as a $(2,2)$-division of $\mathrm{Adj}_{\prec}(G)$, each cell having a rank-$4t^2$ division. By choosing a suitable cell of this $(2,2)$-division, one can extract two disjoint vertex sets X and Y of $V(G)$ such that

(i) $X \prec Y$,
(ii) X admits a division $\mathcal{X} = \{X_1, \ldots, X_{2t^2}\}$ into $2t^2$ (non-empty) sets with $X_1 \prec \cdots \prec X_{2t^2}$,
(iii) Y admits a division $\mathcal{Y} = \{Y_1, \ldots, Y_{4t^2}\}$ into $4t^2$ (non-empty) sets with $Y_1 \prec \cdots \prec Y_{4t^2}$,
(iv) The rank of $X_i \cap Y_j$ is at least 2 for every $i \in [2t^2]$, $j \in [4t^2]$.

Next we do some cleaning up to derive that we can assume that there are pairwise disjoint intervals covering the starting points of each cell. This will enable us subsequently to find a semi-induced transversal pair T_t in G.

Exclusive region covering left endpoints of each A_i and B_j. For vertices $u, v, w \in V(G)$ with $u \prec v \prec w$, observe that $(u, w) \in E(G)$ implies $(u, v) \in E(G)$. This means that in $\mathrm{Adj}_{\prec}(G)$ the zone $X \cap Y$, i.e., the submatrix of $\mathrm{Adj}_{\prec}(G)$ obtained by taking rows indexed by X and columns indexed by Y, the 1 entry at (u, w) imposes 1 at all the preceding entries in the same row. Since each zone $X_i \cap Y_j$ has rank at least 2, we infer that for each $i \in [2t^2]$, $j \in [4t^2]$ there exists a row containing 10 as two consecutive entries within the zone $X_i \cap Y_j$.

For each part $P \in \mathcal{X} \cup \mathcal{Y}$, let $\mathsf{start}(P)$ be the minimal interval on the real line which covers the left endpoints $\{\ell_v : v \in P\}$, and we call $\mathsf{start}(P)$ the *start interval* for P. Furthermore, for $P, P' \in \mathcal{X} \cup \mathcal{Y}$ we write $\mathsf{start}(P) \leq \mathsf{start}(P')$ if $p \leq p'$ for every two points $p \in \mathsf{start}(P)$ and $p' \in \mathsf{start}(P')$ (and equivalently for $<$). While it holds that $\mathsf{start}(Y_1) \leq \cdots \leq \mathsf{start}(Y_j) \leq \cdots \leq \mathsf{start}(Y_{4t^2})$ due to the lexicographic order on $V(G)$, two consecutive start intervals may share an endpoint. To extract a sub-collection with disjoint start intervals, we argue that $\mathsf{start}(Y_j) < \mathsf{start}(Y_{j+2})$. Consider two consecutive (w.r.t $\prec$) vertices y, y' of Y_{j+1} such that for some $x \in X$ the entry (x, y) equals 1 and (x, y') equals 0. The existence of such y and y' was argued in the previous paragraph. The mismatch of the two entries means that $\ell_y < \ell_{y'}$ as $r_x \geq \ell_y$ and $r_x < \ell_{y'}$ by assumption. It remains to observe that $\mathsf{start}(Y_j) \leq \ell_y < \ell_{y'} \leq \mathsf{start}(Y_{j+2})$.

To see that $\mathsf{start}(X_i) < \mathsf{start}(X_{i+2})$, consider the zones $X_i \cap Y_{4t^2-2i+1}$ for all $i \in [2t^2]$. For every $i \in [2t^2]$, consider a vertex $x_i \in X_i$ for which "10" appears as two consecutive entries in the row indexed by x_i within the zone $X_i \cap Y_{4t^2-2i+1}$; the existence of such x_i and the entries "10" on the row of x_i were argued previously. Note that the entries "10" represent that x_i is adjacent with some vertex of Y_{4t^2-2i+1} and non-adjacent with another vertex of Y_{4t^2-2i+1}. Therefore, the right endpoint r_{x_i} of the interval for x_i is contained in $\mathsf{start}(Y_{4t^2-2i+1})$. As we have $\mathsf{start}(Y_{4t^2-2j+1}) < \mathsf{start}(Y_{4t^2-2i+1})$ for every $1 \leq i < j \leq 2t^2$, we conclude $r_{x_j} < r_{x_i}$. As $x_i \prec x_j$, this implies $\ell_{x_i} < \ell_{x_j}$ due to the construction of $\prec$. Now, observe that $\mathsf{start}(X_i) \leq \ell_{x_i} < \ell_{x_{i+1}} \leq \mathsf{start}(X_{i+2})$, as claimed.

Therefore, after removing vertices associated with every other index, we assume that the collections $\mathcal{X}$ and $\mathcal{Y}$ consist of t^2 sets each, and their start intervals satisfy

$$\mathsf{start}(X_1) < \cdots < \mathsf{start}(X_i) < \cdots < \mathsf{start}(X_{t^2}) \leq \mathsf{start}(Y_1) < \cdots < \mathsf{start}(Y_j) < \cdots \mathsf{start}(Y_{t^2}).$$

Finding a transversal pair of half-graphs. The following statement implies that the graph G contains a semi-induced T_t for our setting and finishes the proof of delineation of interval graphs. Lemma 16 will be reused in the later section on directed path graphs. Note that in the lemma we do not use the assumption that the indexing of the adjacency matrix follows the order $\prec$. Indeed the only property we require of the ordering of the adjacency matrix is that the left endpoints of parts in the partition can be covered by exclusive regions.

Lemma 16. *Let G be an interval graph and $Adj(G)$ an adjacency matrix of G. Let $\mathcal{X} = \{X_1, \ldots, X_{t^2}\}$ and $\mathcal{Y}_2 = \{Y_1, \ldots, Y_{t^2}\}$ be two classes of pairwise disjoint vertex sets such that the following properties hold.*

1. $\mathsf{start}(X_1) < \cdots < \mathsf{start}(X_{t^2}) \leq \mathsf{start}(Y_1) < \cdots < \mathsf{start}(Y_{t^2})$,
2. *The submatrix of $Adj(G)$ induced by rows X_i and columns Y_j has rank at least 2,*

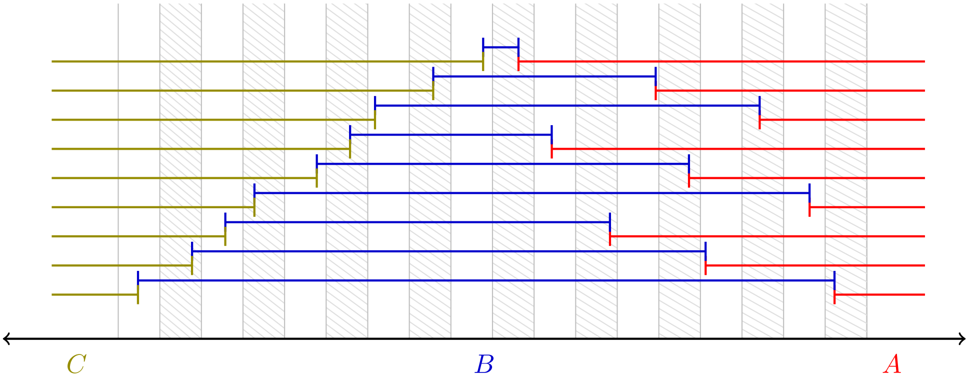


Fig. 2. Schematic illustration of the interval representation of a semi-induced transversal pair. Note that the left endpoints of intervals in C could lie to the right of left endpoints in B.

where the starting intervals $\mathsf{start}(X_i)$ *and* $\mathsf{start}(Y_j)$ *are defined with respect to a minimal interval representation of* G*. Then* G *contains a semi-induced transversal pair* T_t*.*

Proof. For obtaining a transversal pair (A, B, C) in G we use a suitable subset of $\bigcup_{i\in[t^2]} Y_i$ to represent the vertices in A and we use a suitable subset of $\bigcup_{i\in[t^2]} X_i$ to represent the vertices in B while vertices in C will be found among the remainder of vertices. For an illustration see Fig. 2.

First, note that $\mathsf{start}(X_1) < \cdots \mathsf{start}(X_{t^2}) \leq \mathsf{start}(Y_1) < \cdots < \mathsf{start}(Y_{t^2})$ implies the following property

(P1) If for some $b \in X_i$ and $a \in Y_j$ the entry (b, a) of $\mathrm{Adj}(G)$ is 1 then the entry (b, a') is also 1 for every $a' \in Y_k$ for $k < j$.
(P2) If for some $b \in X_i$ and $a \in Y_j$ the entry (b, a) of $\mathrm{Adj}(G)$ is 0 then the entry (b, a') is also 0 for every $a' \in Y_k$ for $k > j$.

For each $i, j \in [t]$, choose a vertex $b_{i,j} \in X_{(j-1)t+i}$ and two vertices $a_{i,j}, \bar{a}_{i,j} \in Y_{(i-1)t+j}$ such that the entry $(b_{i,j}, a_{i,j})$ equals 1 and the entry $(b_{i,j}, \bar{a}_{i,j})$ equals 0. Note that this choice is possible because the submatrix of $\mathrm{Adj}(G)$ induced by rows X_i and columns Y_j has rank at least 2 and hence must contain 10 or 01 as consecutive entries. As the entry $(b_{i,j}, a_{i,j})$ is 1, we get that the entry $(b_{i,j}, a_{i',j'})$ is also 1 whenever $(i'-1)t + j' < (i-1)t + j$ by property (P1) and hence $b_{i,j}$ is adjacent to $a_{i',j'}$ whenever $(i', j') \leqslant_{\text{lex}} (i, j)$. On the other hand, the entry $(b_{i,j}, \bar{a}_{i,j})$ being 0 implies that the entry $(b_{i,j}, a_{i',j'})$ is 0 if $(i'-1)t + j' > (i-1)t + j$ by property (P2) and hence $b_{i,j}$ is not adjacent to $a_{i',j'}$ whenever $(i', j') >_{\text{lex}} (i, j)$.

To choose set C consider the left endpoints $\ell_{b_{i,j}}$ for all $i, j \in [t]$. Note that by the disjointness of the start intervals $\ell_{b_{i,j}} < \ell_{b_{i',j'}}$ if and only if $(j, i) <_{\text{lex}} (j', i')$ as $b_{i,j} \in X_{(j-1)t+i}$ and $b_{i',j'} \in X_{(j'-1)t+i'}$ and $(j-1)t + i \leq (j'-1)t + i'$ if and only if $(j, i) <_{\text{lex}} (j', i')$. By the minimality assumption on the representation $\mathcal{I}$, there exists a vertex $c_{i,j}$ for each $i, j \in [t]$ for which $\ell_{b_{i,j}} \leq r_{c_{i,j}} < \ell_b$ for every b for which $\ell_b > \ell_{b_{i,j}}$. Note that in particular $c_{i,j} \neq b_{i,j}$ as $r_{b_{i,j}} \geq \ell_{a_{i,j}} \geq \ell_b$. Furthermore, $c_{i,j}$ is adjacent to $b_{i,j}$ and non-adjacent to every b for which $\ell_b > \ell_{b_{i,j}}$. In addition, $c_{i,j}$ is adjacent to every $b_{i',j'}$ for which $\ell_{b_{i'j'}} \leq \ell_{b_{i,j}}$ (even if $\ell_{c_{i,j}} > \ell_{b_{i',j'}}$) as $r_{b_{i',j'}} \geq \ell_{a_{1,1}} \geq \ell_{b_{i'',j''}}$ for every $i'', j'' \in [t]$. Hence $b_{i,j}$ is adjacent to $c_{i',j'}$ if and only if $(j, i) \leqslant_{\text{lex}} (j', i')$.

We obtain that for $A := \{a_{i,j} : i, j \in [t]\}$, $B := \{b_{i,j} : i, j \in [t]\}$, and $C := \{c_{i,j} : i, j \in [t]\}$ the triple (A, B, C) is a transversal pair of half-graphs T_t. $\square$

4. Delineation of directed path graphs with boundedly many roots

A *tree model* of a graph G consists of a tree T together with a collection $\{P_v : v \in V(G)\}$ of subtrees of T such that $(u, w) \in E(G)$ if and only if $V(P_u) \cap V(P_w) \neq \emptyset$. Gavril [26,27] showed that a graph

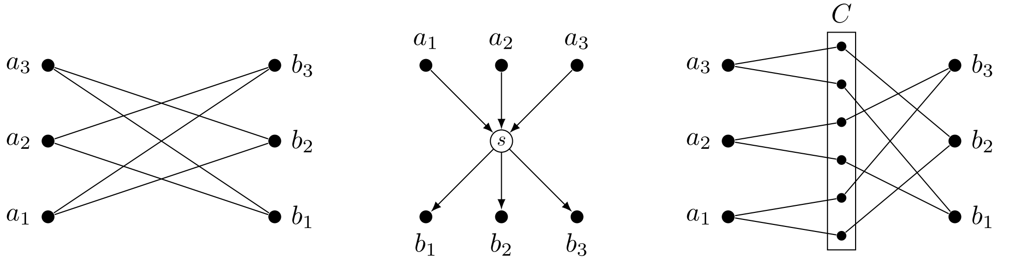


Fig. 3. From left to right: a subcubic bipartite graph G, the star S, and the directed path graph $\Pi(G)$ (the edges in C are omitted).

G is chordal if and only if G can be represented by a tree model, and that a tree model of G can be constructed in polynomial time. To avoid ambiguities, we refer to $V(T)$ as the *nodes of* T.

If every P_v in a tree model of a graph G is a path, we say that G is an *undirected path graph*. If T is oriented and every P_v is a directed path, we say that G is a *directed path graph*. Finally, a directed path graph with tree model $(T, \{P_v : v \in V(G)\})$ such that T has ρ roots (nodes of in-degree 0) is called a ρ*-rooted directed path graph*. If ρ is 1 we omit ρ and obtain *rooted directed path graphs*. Notice that interval graphs form a subclass of rooted directed path graphs for which T is a directed path. We say that a class of directed path graphs has *boundedly many roots* if there is $\rho_{\max} \in \mathbb{N}$ such that every graph $G \in \mathcal{P}$ is a ρ-rooted directed path graph for some $\rho \leq \rho_{\max}$.

In the remainder of the section, we prove the following result.

Theorem 17. *Any class $\mathcal{P}$ of directed path graphs with boundedly many roots is effectively delineated.*

In contrast, directed path graphs in general are *not* delineated.

Theorem 18. *There is a class $\mathcal{P}$ of directed path graphs with tree model comprising of paths of a directed star which is transduction equivalent to $\mathcal{B}_{\leq 3}$. In particular, the class of directed path graphs is not delineated.*

Proof. Given a bipartite graph G with parts A, B by subdividing each edge of G and making a clique of the set C of newly added subdivision vertices, we create a directed path graph $\Pi(G)$ that encodes G. A tree model of $\Pi(G)$ is easy to construct. We start with a star S with central vertex s and leaves $A \cup B$, orient edges touching A towards s, and edges touching B away from s. For $u \in A \cup B$ we let P_u be the trivial path containing only u in S. Additionally, for each edge $ab \in E(G)$ we associate the path in S from a to b with the vertex used to subdivide the edge ab in $\Pi(G)$. More precisely, for each $c \in C$ with neighbors a, b in $\Pi(G)$ we let P_c be the path from a to b in S. Now $(S, \{P_u : u \in A \cup B\} \cup \{P_c : c \in C\})$ is a tree model of $\Pi(G)$. See Fig. 3 for an example of this construction.

The class $\mathcal{P} = \{\Pi(G) : G \in \mathcal{B}_{\leq 3}\}$ is transduction equivalent to $\mathcal{B}_{\leq 3}$ and thus by Lemma 14 we conclude that directed path graphs are not delineated. To see that $\mathcal{P}$ transduces $\mathcal{B}_{\leq 3}$ we give a transduction which recovers G from $\Pi(G)$. We use three extra unary predicates A, B, C to recover the original tripartition of $\Pi(G)$ into sets A, B and C. We can recover the original set of vertices of G with the predicate $\nu(x) \equiv A(x) \vee B(x)$ and interpret the edge set of G by predicate

$$\varphi(x, y) \equiv A(x) \wedge B(y) \wedge \exists z(C(z) \wedge E(x, z) \wedge E(z, y)).$$

On the other hand, the class $\mathcal{B}_{\leq 3}$ transduces $\mathcal{P}$. To see this, observe that the graph $\Pi(G)$ omitting the clique edges of C is a subcubic bipartite graph G'. Hence, given G' our transduction guesses set C using a unary predicate and then adds all edges between pairs of vertices from C. □

We remark that this construction generates graphs that are modeled on trees with many roots. Thus delineation for any subclass of directed path graphs can be expected only if such a class can be modeled by trees with a bounded number of roots. We show that this is indeed the case.

Defining a linear order for directed path graphs with boundedly many roots. First, we adopt notation related to trees to describe relationships between nodes of a tree T. We say that an oriented tree is *rooted* at $r \in V(T)$ if r is the only root of T. We let T_p be the maximal subtree of T rooted at p for $p \in V(T)$. If there is a directed path from p to q in T, we say that p is an *ancestor* of q and that q is a *descendant* of p. If $p \neq q$ we also say that p (resp. q) is a *proper ancestor* (resp. *proper descendant*) of q (resp. p). If there is an edge oriented from p to q in T, then we say that q is a *child* of p and that p is the *parent* of q. If p and q share the same parent, then we say that they are *siblings*. If T is a rooted tree, then the *least common ancestor* of p and q is the node $s \in V(T)$ that is furthest from the root of T and such that s is an ancestor of both p and q.

For the remaining of this section, and unless stated otherwise, we assume that G is a ρ-rooted directed path graph and that $(T, \{P_v : v \in V(G)\})$ is a minimal tree model of G, where every P_v is a directed path of T. For $v \in V(G)$ we denote by $\mathsf{high}(v)$ and $\mathsf{low}(v)$ the start and end point of P_v, respectively. Notice they are not necessarily distinct. We extend this notation to sets of vertices by defining $\mathsf{high}(X) = \{\mathsf{high}(v) : v \in X\}$ and $\mathsf{low}(X) = \{\mathsf{low}(v) : v \in X\}$ for $X \subseteq V(G)$.

We define an order $\leq_T$ on the nodes of T as follows. We recursively define a partition of the nodes of T into ρ rooted trees $T^1, \ldots, T^\rho$ and then order the nodes of T^ρ to appear before the nodes of $T^{\rho-1}$ and so on. Let r^1 be an arbitrarily chosen root of T, T^1 the subtree rooted at r^1 and $\leq^1$ the ancestor/descendant relationship in T^1. We now choose a root $r^2 \neq r^1$ arbitrarily among all remaining roots. We define T^2 to be the subtree obtained from the subtree of T rooted at r^2 by removing all nodes contained in T^1. We let $\leq^2$ be the order obtained from $\leq^1$ and the ancestor/descendant relationship on T^2 by adding the additional relationship $p \leq^2 q$ whenever $p \in V(T^2)$ and $q \in V(T^1)$. We continue this process until we have partitioned T. That is in the ith step we choose among the remaining roots a root r^i and set T^i to be subtree of T obtained from the subtree of T rooted at r^i by removing all nodes in $\bigcup_{j=i-1}^{1} V(T^j)$. We define $\leq^i$ to be the order obtained from $\leq^{i-1}$ and the ancestor/descendant relationship on T^i by adding the additional relationship $p \leq^i q$ whenever $p \in V(T^i)$ and $q \in \bigcup_{j=i-1}^{1} T^j$. We define $\leq_T$ to be the final ordering $\leq^\rho$ we obtain by this process. Note that $\leq_T$ is different from the ancestor/descendant relationship on T. More precisely, for every node q and ancestor p of q we have that $p \leq_T q$ but not every pair of nodes p, q for which $p \leq_T q$ we have that p is an ancestor of q. Additionally, paths in T do not necessarily appear as an interval in $\leq_T$. We sometimes compare subpaths with $\leq_T$, i.e., for two (node disjoint) subpaths P, P' of T we write $P <_T P'$ if for every $p \in V(P)$, $p' \in V(P')$ we have $p <_T p'$.

Let $p, p' \in V(T)$. We denote by V_p the set $\{u \in V(G) : p \in V(P_u)\}$, and we say that the model $(T, \{P_v : v \in V(G)\})$ is *minimal* if there are no adjacent $p, q \in V(T)$ such that $V_p \subseteq V_q$. It is known that one can always generate a minimal tree model of a directed path graph G whenever a tree model of G is given [27] and hence we assume that the $(T, \{P_v : v \in V(G)\})$ is minimal from now on.

We now use the tree model $(T, \{P_v : v \in V(G)\})$ of G and the order $\leq_T$ to obtain a total order $\prec$ of the vertex set of G by a lex-DFS style search on T which we describe in the following. We traverse T with depth-first search (from each root). We start the first depth-first search in r^1, the second in r^2 and so on. Hence, the depth first search visits all nodes in T^1 first, then the nodes in T^2 and so on. We say that we *processed* a node q when q has been visited and every node in $V(T_q)$ other than q has been processed by the search (leaves get processed immediately once they are visited). We define a total order $\prec_{\mathrm{DFS}}$ on $V(T)$ which orders the nodes of T by the time in the depth-first search they were processed, i.e., $p \prec_{\mathrm{DFS}} q$ if there is a point in the depth-first search when T_p has been fully processed but T_q has not yet been fully processed. In this depth-first search we additionally use the following tie breaking rule. If p, q are siblings then we explore T_p before T_q if the symmetric difference of the two sets $\{\mathsf{high}(v) : \mathsf{low}(v) \in V(T_p)\}$ and $\{\mathsf{high}(v) : \mathsf{low}(v) \in V(T_q)\}$ has a minimum $\mathsf{high}(v)$ with respect to $\leq_T$ and $\mathsf{low}(v) \in V(T_p)$ (in case the symmetric difference of these two sets has no minimum we choose arbitrarily). From the order $\prec_{\mathrm{DFS}}$ we obtain $\prec$ using the following rules. We let $u \prec v$ if

- $\mathsf{low}(u) \prec_{\mathrm{DFS}} \mathsf{low}(v)$ or
- $\mathsf{low}(u) = \mathsf{low}(v)$ and $\mathsf{high}(u) <_T \mathsf{high}(v)$.

In case $\mathsf{low}(u) = \mathsf{low}(v)$ and $\mathsf{high}(u) = \mathsf{high}(v)$ we choose an arbitrary order of u and v.

The order $\prec$ has the following basic properties. By following a DFS on T, using the aforementioned tie breaking rule to decide which subtree to explore first, it is not hard to see that $\prec$ can be generated in polynomial time. Each time a node p is processed, we add to a stack S every $v \in V(G)$ with $\mathsf{low}(v) = p$, following the tie breaking rules for this decision when needed. In the end, the order that the nodes appear in S is the reverse order of $\prec$.

Using the construction of $\prec$ we immediately obtain the following three observations. We restrict the last two observations to nodes within one subtree T^i and remark that this restriction is not necessary, but sufficient for our purposes.

Observation 19. *Let $V^i \subseteq V(G)$ be the vertices v for which $\mathsf{low}(v) \in V(T^i)$. Then it holds that $V^1 \prec V^2 \prec \cdots \prec V^\rho$. Furthermore, P_v cannot contain any node in T^j for any vertex $v \in V^i$ and $i > j$.*

Observation 20. *Let $i \in [\rho]$ and x be an internal node of T^i and $y, z \in V(T^i)$ be two distinct children of x. For any vertices $v_y^1, v_y^2, v_z \in V(G)$ with $\mathsf{low}(v_y^1), \mathsf{low}(v_y^2), \mathsf{low}(v_z) \in V(T^i)$, if $\mathsf{low}(v_y^i) \geq_T y$ and $\mathsf{low}(v_z) \geq_T z$ for $i = 1, 2$, then either $v_y^i \prec v_z$ for both $i = 1, 2$ or $v_y^i \succ v_z$ for both $i = 1, 2$.*

Observation 21. *Let $u, v, w \in V(G)$ with $u \prec v \prec w$ and $\mathsf{low}(u), \mathsf{low}(v), \mathsf{low}(w) \in V(T^i)$ for some $i \in [\rho]$. Then the least common ancestor in T^i of $\mathsf{low}(u)$ and $\mathsf{low}(w)$ is an ancestor of the least common ancestor of $\mathsf{low}(v)$ and $\mathsf{low}(w)$. Likewise, it is also an ancestor of the least common ancestor of $\mathsf{low}(u)$ and $\mathsf{low}(v)$.*

Proof. Suppose not, i.e., the former is a strict descendant of the latter. Choose x to be the least common ancestor of $\mathsf{low}(v)$ and $\mathsf{low}(w)$. By assumption, x must have two distinct children y, z such that $\mathsf{low}(w) \geq_T y$ and $\mathsf{low}(v) \geq_T z$. As the least common ancestor of $\mathsf{low}(u)$ and $\mathsf{low}(z)$ is a strict descendant of x, it must in particular be a descendant of z and therefore $\mathsf{low}(u) \geq_T z$. Since $u \prec v \prec w$ this contradicts Observation 20. $\square$

Overview of the upcoming proof. We now show that directed paths graphs with boundedly many roots are effectively delineated. Let G be a directed path graph with ρ many roots such that $\mathsf{Adj}_\prec(G)$ has large grid rank (depending on ρ and the size of the substructure we aim to find). We first observe that $\mathsf{Adj}_\prec(G)$ will still have large grid rank after restricting both column and row parts to vertices v with $\mathsf{low}(v)$ appearing in an appropriate rooted subtree T_i of T. We further restrict to two disjoint collections of row parts $\mathcal{X} = \{X_1, \ldots, X_{f(t)}\}$ and column parts $\mathcal{Y} = \{Y_1, \ldots, Y_{f(t)}\}$.

We now choose for every $i \in [f(t)]$ one representative x_i among all vertices in X_i and one representative y_i among all vertices in Y_i. We set $\mathsf{REP}_\mathcal{X}$ to be the set of all x_i and $\mathsf{REP}_\mathcal{Y}$ to be the set of all y_i. The goal is to use $\mathsf{REP}_\mathcal{X}$ and $\mathsf{REP}_\mathcal{Y}$ to distinguish between two cases in the proof: one where we can organize the relevant vertices appearing in sets in $\mathcal{X}$ and $\mathcal{Y}$ (that is, the vertices defining adjacency in zones $X_i \cap Y_j$) in such way that a large interval graph appears and then apply Lemma 16, and one where we have a good interval-like organization for one of $\mathcal{X}, \mathcal{Y}$ but not for the other. In the latter, we inductively show that we either obtain a semi-induced $T_{t,2}$ or (after $g(t)$ inductive steps) fall back to the first case.

To obtain the desired layout of $\mathcal{X}$ and $\mathcal{Y}$ the first crucial observation is that there is a path $P_\forall$ of T which intersects P_u for every vertex u, with the exception of vertices in parts of $\mathcal{X}$ that are not adjacent with any vertex in a part of $\mathcal{Y}$ and similarly for vertices in parts of $\mathcal{Y}$ that are not adjacent to any vertex in a part in $\mathcal{X}$. We observe that $P_\forall$ defines an order $\leq_{P_\forall}$ on the relevant vertices as follows. We denote by $\mathsf{last}_{P_\forall}(u)$ the node in $V(P_u) \cap V(P_\forall)$ that is closest to $\mathsf{low}(u)$ and say that $u \leq_{P_\forall} v$ if and only if $\mathsf{last}_{P_\forall}(u) \leq_T \mathsf{last}_{P_\forall}(v)$. Using the order $\prec$ we show that in the order $\leq_{P_\forall}$ for vertices v from a part X_i or Y_j the position of $\mathsf{last}_{P_\forall}(v)$ cannot diverge too much from the respective position of their representative.

When both $\mathsf{REP}_\mathcal{X}$ and $\mathsf{REP}_\mathcal{Y}$ contain sufficiently large chains with respect to $\leq_{P_\forall}$. Since $\leq_{P_\forall}$ may not agree with $\prec$, we apply the Erdős-Szekeres Theorem 10 to extract a large monotone sequence of both chains, and keep only the vertices appearing in those sequences in $\mathsf{REP}_\mathcal{X}$ and $\mathsf{REP}_\mathcal{Y}$. We then use those sequences to define, for each X_i associated with a vertex in the new set $\mathsf{REP}_{\mathcal{X}}$, an exclusive subpath I_i of $P_\forall$ that contains $\mathsf{last}_{P_\forall}(x)$ for every $x \in X_i$, and do the same for each Y_j. With these subpaths we construct an interval graph G_I which allows us to apply Lemma 16 to resolve this case.

We can use the large grid rank of $\mathrm{Adj}_\prec(G)$ to show that either $\mathsf{REP}_{\mathcal{X}}$ or $\mathsf{REP}_{\mathcal{Y}}$ must contain a large chain with respect to $\leq_{P_\forall}$. Without loss of generality we assume that the former is true. In this case, there must be a node $p \in P_\forall$ concentrating a large set of $\{\mathsf{last}_{P_\forall}(y_i) : y_i \in \mathsf{REP}_{\mathcal{Y}}\}$. We distinguish the two cases where either there is a child q of p which is still contained in P_{y_i} for most of the representatives y_i that are concentrated in p or the parts in $\mathcal{Y}$ whose representatives are concentrated in p distribute nicely over the children of p. In the former case we proceed inductively while in the latter case we show that there is a subset $\mathcal{Y}' \subseteq \mathcal{Y}$ such that for any $Y \in \mathcal{Y}'$ all nodes $\mathsf{low}(y)$ for $y \in Y$ are contained in the subtrees rooted in a private set of children of p. We use this configuration to find the paths connecting the one half-graph to the other half-graph in the $T_{t,2}$. Roughly speaking, while the first half graph can be found using the large grid rank, the second half graph is obtained as a consequence of the tie breaking rule used to obtain $\prec$.

Theorem 22. *Let $h(t) = 3(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}$ and define $g_{(4t^2+1)^2+1}(t)$ to be constant 1, $g_i(t) = (g_{i+1}(t)+\rho_{\max}+4)\cdot h(t)$ for every $i \in [0, (4t^2+1)^2]$ and $g(t) = (g_0(t)+\rho_{\max}+4)\cdot((8t^2\cdot\rho_{\max}+1)^2+1)$. Define*[10] *$f(t) = 2\cdot g(t)\cdot\rho_{\max}+\rho_{\max}-1$. For every natural t, if G is a directed path graph with at most $\rho_{\max}$ roots and the grid rank of $\mathrm{Adj}_\prec(G)$ is at least $f(t)$ then G has a semi-induced T_t or $T_{t,2}$.*

Proof. Let $(T, \{P_v : v \in V(G)\})$ be a minimal tree model of G, where each P_v is a path, and $M = \mathrm{Adj}_\prec(G)$. Assume T has $\rho \leq \rho_{\max}$ many roots and let $T^1, \ldots, T^\rho$ be the partition used to obtain $\prec$ with $V^i \subseteq V(G)$ being the vertices v for which $\mathsf{low}(v) \in V(T^i)$. Notice that each row and each column of M is associated with a vertex of G and so we interchangeably refer to vertices of G as rows and/or columns of M, and vice-versa.

Assume that M contains a rank-$f(t)$ division $(\mathcal{D}^R, \mathcal{D}^C)$ with $\mathcal{D}^R = R_1, \ldots, R_{f(t)}$ and $\mathcal{D}^C = C_1, \ldots, C_{f(t)}$. We first remove all row parts R_i such that R_i contains vertices from two distinct V^j. Symmetrically, we remove all column parts C_i for which C_i contains vertices from two different V^j. As there are ρ distinct V^j which appear as an interval of $\prec$, we remove at most $\rho-1$ row and $\rho-1$ column parts and are left with a rank-$(2\cdot g(t)\cdot\rho_{\max})$ division. Hence, there must be a subtree T^i such that V^i is partitioned into at least $2\cdot g(t)$ row parts in the rank-$(2\cdot g(t)\cdot\rho_{\max})$ division. Similarly, there is a subtree T^j such that V^j is partitioned into at least $2\cdot g(t)$ column parts in the rank-$(2\cdot g(t)\cdot\rho_{\max})$ division. We remove all row parts not consisting of vertices in V^i and all column parts not consisting of vertices in V^j and are left with a rank-$(2\cdot g(t))$ division $(\widehat{\mathcal{D}}^R, \widehat{\mathcal{D}}^C)$. For the remainder of the proof we let T^i be T' and T^j be T'' (in particular, we will reuse indices i and j throughout the proof).

As in the interval graph case, by choosing $X_1, \ldots, X_{g(t)}$ to be the first/last $g(t)$ parts of $\widehat{\mathcal{D}}^R$ and $Y_1, \ldots, Y_{g(t)}$ be the last/first $g(t)$ parts of $\widehat{\mathcal{D}}^C$ we can assume that $X_1, \ldots, X_{g(t)}, Y_1, \ldots, Y_{g(t)}$ are pairwise disjoint. By definition of rank divisions, for every $i, j \in [g(t)]$ the rank of $X_i \cap Y_j$ is at least 4 and hence there are vertices $x \in X_i$ and $y \in Y_j$ such that x and y are adjacent in G. We denote by X_i' the subset of vertices of X_i which have a neighbor in some Y_j, and by Y_i' the subset of vertices of Y_i which have a neighbor in some X_j. Note that for the restriction M' of M to rows from $\bigcup_{i\in g(t)} X_i'$ and columns $\bigcup_{j\in g(t)} Y_j'$ the division into row parts X_i' and column parts Y_j' still yields a large rank division (we delete only rows and columns that are constant zero within the relevant zones). For the rest of the proof, we use matrix M' with row division $X_1', \ldots, X_{g(t)}'$ and column division $Y_1', \ldots, Y_{g(t)}'$ and the property that $X_i' \cap Y_j'$ must have large rank, in particular rank larger than $2t^2\cdot\rho_{\max}$, (deleting rows constant zero will reduce the combinatorial rank by at most 1). Without loss of generality, we assume that $Y_1' \prec \cdots \prec Y_{g(t)}' \prec X_1' \prec \cdots \prec X_{g(t)}'$ (nodes in $\mathsf{low}(X_i')$ appear further up in the tree T than nodes in $\mathsf{low}(Y_j')$).

[10] The function f is defined in this way for convenience during the proof. We remind the reader about the specific subfunctions throughout the proof.

Let $\mathcal{Z} \subseteq \{X'_1, \ldots, X'_{g(t)}, Y'_1, \ldots, Y'_{g(t)}\}$ be a set of parts. We say that a (directed) path P in T *covers* $\mathcal{Z}$ if P contains some node from P_z for every $z \in \bigcup_{Z\in\mathcal{Z}} Z$. If P covers $\mathcal{Z}$ then for every $v \in \bigcup_{Z\in\mathcal{Z}} Z$ we denote by $\mathsf{last}_P(v)$ the last node of P_v which is contained on P. We extend this notation to sets $W \subseteq \bigcup_{Z\in\mathcal{Z}} Z$ by writing $\mathsf{last}_P(W)$ to denote the set $\{\mathsf{last}_P(w) : w \in W\}$. Note that $\leq_T$ defines a canonical linear quasi-order $\leq_P$ on $\bigcup_{Z\in\mathcal{Z}} Z$. Namely, for $v, v' \in \bigcup_{Z\in\mathcal{Z}} Z$ we have $v \leq_P v'$ if and only if $\mathsf{last}_P(v) \leq_T \mathsf{last}_P(v')$. Clearly it is a quasi-order (reflexive and transitive). We fix *representative* vertices $x_i \in X'_i$ and $y_j \in Y'_j$ for every $X'_i \in \mathcal{Z}$ and every $Y'_j \in \mathcal{Z}$ such that $x_i \leq_P x$ for every $x \in X'_i$ and $y_j \leq_P y$ for every $y \in Y'_j$. We define $\mathsf{REP}^P_{\mathcal{Z}} = \{x_i, y_j : X'_i \in \mathcal{Z}, Y'_j \in \mathcal{Z}\}$ to be the set of representatives. We say that $\mathcal{Z}$ is *k-diverse* on P if $\mathsf{REP}^P_{\mathcal{Z}}$ contains a chain of size at least k with respect to the order $\leq_P$. Specifically, we ask that for any two elements z, z' in the chain either $z \leq_P z'$ or $z' \leq_P z$, but not both.

Claim 23. *Let $\mathcal{Z} = \{X'_1, \ldots, X'_{g(t)}, Y'_1, \ldots, Y'_{g(t)}\}$. There is a root-to-leaf path P of T which covers $\mathcal{Z}$.*

Proof of Claim: We first consider the case when $T' \neq T''$. By our assumption that $Y'_j \prec X'_i$ we know that $p \leq_T q$ for nodes $p \in V(T')$, $q \in V(T'')$. Since $\mathsf{low}(x) \in V(T')$ for every $x \in \bigcup_{i\in[g(t)]} X'_i$, we know that P_x cannot contain any node in T'' by Observation 19. Since every $y \in \bigcup_{j\in[g(t)]} Y'_j$ is adjacent to some $x \in \bigcup_{i\in[g(t)]} X'_i$, we conclude that P_y must have its start node in T'. Let P be any path starting in the root of T' and ending in some leaf of T'' (which must exist by the preceding argument). Let p be the last node on P which is contained in T'. Note that because T is a tree and T', T'' are subtrees of T, the definition of p is independent of the choice we made for P. In particular, it follows that $p \in V(P_y)$ for every $y \in \bigcup_{j\in[g(t)]} Y'_j$. In particular, $V(P_y) \cap V(T') \subseteq V(P)$ for every $y \in \bigcup_{j\in[g(t)]} Y'_j$ and hence every P_x for $x \in \bigcup_{i\in[g(t)]} X'_i$ must contain some node on P as x must be adjacent to some $y \in \bigcup_{j\in[g(t)]} Y'_j$. The claim holds in this case.

Hence, assume $T' = T''$. Let P be a directed root-to-leaf path of T' maximizing $|\{v \in \bigcup_{Z\in\mathcal{Z}} Z : V(P_v) \cap V(P) \neq \emptyset\}|$. By contradiction, assume that there is a $v \in \bigcup_{Z\in\mathcal{Z}} Z$ such that $V(P_v) \cap V(P) = \emptyset$. By our assumption $\mathsf{low}(v) \in V(T')$ and hence $\mathsf{low}(v)$ is a descendant of the root of T'. Let $q \in V(P)$ be the (proper) ancestor of $\mathsf{low}(v)$ that is closest to $\mathsf{low}(v)$. If q has no children in $V(P)$, then we have a contradiction since q is not a leaf, and thus we assume that this is not the case. Let q' be the child of q that is contained in P and q'' the child of q which is an ancestor of $\mathsf{low}(v)$. Note that by assumption $q'' \notin V(P)$. Now let P' be the subpath of P from q' to the last vertex of P. Notice that there must be $u \in \bigcup_{Z\in\mathcal{Z}} Z$ with $q' \leq_T \mathsf{high}(u)$ since otherwise we can extend P to contain $\mathsf{high}(v)$, contradicting our choice of P.

If $v \in X'_i$ for some $i \in [g(t)]$ then, by construction, v has a neighbor $w \in Y'_j$ for some $j \in [g(t)]$. Similarly, if $v \in Y'_j$ then v has a neighbor w in some X'_i. In both cases, $q'' \leq_T \mathsf{low}(w)$ since $q'' \leq_T \mathsf{low}(v)$ and $\mathsf{low}(w) \in V(T')$, and thus there are $x \in X'_i$ and $y \in Y'_j$ with $q'' <_T \mathsf{low}(x)$ and $q'' <_T \mathsf{low}(y)$ (with $v = x$ or $v = y$). Repeating this analysis with relation to u and q', we conclude that there are $x' \in X'_{i'}$ and $y' \in Y'_{j'}$ satisfying $q' \leq_T \mathsf{low}(x')$ and $q' \leq_T \mathsf{low}(y')$ (with $u = x'$ or $u = y'$). This contradicts Observation 20, we conclude that no $v \in \bigcup_{Z\in\mathcal{Z}} Z$ with $V(P_v) \cap V(P) = \emptyset$ can exist, and the claim follows. $\diamond$

For the remainder of the proof, fix $P_\forall$ to be a root-to-leaf path of T covering $\{X'_1, \ldots, X'_{g(t)}, Y'_1, \ldots, Y'_{g(t)}\}$. We first argue that either $\{X'_1, \ldots, X'_{g(t)}\}$ or $\{Y'_1, \ldots, Y'_{g(t)}\}$ must be $(8t^2 \cdot \rho_{\max} + 1)^2 + 1$-diverse on $P_\forall$. For this we utilize the following behavior of parts with respect to their representatives.

Claim 24. *Let $\mathcal{Z} \subseteq \{X'_1, \ldots, X'_{g(t)}, Y'_1, \ldots, Y'_{g(t)}\}$ and P a root-to-leaf path covering $\mathcal{Z}$. Let $i_1, i_2, i_3 \in [g(t)]$ with $i_1 < i_2 < i_3$ or $i_3 < i_2 < i_1$. The following holds:*

(i) if $x_{i_1} <_P x_{i_2} <_P x_{i_3}$ for $x_{i_j} \in X'_{i_j} \cap \mathsf{REP}^P_{\mathcal{Z}}$, $j \in [3]$ then $x <_P x_{i_3}$ for every $x \in X'_{i_1}$.

(ii) if $y_{i_1} <_P y_{i_2} <_P y_{i_3}$ for $y_{i_j} \in Y'_{i_j} \cap \mathsf{REP}^P_{\mathcal{Z}}$, $j \in [3]$ then $y <_P y_{i_3}$ for every $y \in Y'_{i_1}$.

(iii) if $\mathsf{last}_P(x_{i_j}) = p$ for every $x_{i_j} \in X'_{i_j} \cap \mathsf{REP}^P_{\mathcal{Z}}$, $j \in [3]$ then either $\mathsf{last}_P(x) = p$ for every $x \in X'_{i_1}$ or $\mathsf{last}_P(x) = p$ for every $x \in X'_{i_3}$.

(iv) if $\mathsf{last}_P(y_{i_j}) = p$ *for every* $y_{i_j} \in Y'_{i_j} \cap \mathsf{REP}^P_{\mathcal{Z}}$, $j \in [3]$ *then either* $\mathsf{last}_P(y) = p$ *for every* $y \in Y'_{i_1}$ *or* $\mathsf{last}_P(y) = p$ *for every* $y \in Y'_{i_3}$.

Proof of Claim: Suppose item (i) does not hold, i.e., $x_{i_3} \le_P x$ for some $x \in X'_{i_1}$. Observe that in this case the least common ancestor of $\mathsf{low}(x)$ and $\mathsf{low}(x_{i_3})$ is $\mathsf{last}_P(x_{i_3})$ while the least common ancestor of $\mathsf{low}(x_{i_2})$ and $\mathsf{low}(x_{i_3})$ is $\mathsf{last}_P(x_{i_2})$. Therefore, applying Observation 21 with either $x \prec x_{i_2} \prec x_{i_3}$ or $x_{i_3} \prec x_{i_2} \prec x$ (depending on whether $i_1 < i_2 < i_3$ or $i_3 < i_2 < i_1$) yields a contradiction to our assumption. Part (ii) can be shown with an analogous proof.

Suppose item (iii) does not hold and let $x \in X'_{i_1}$ and $x' \in X'_{i_3}$ be two vertices with $\mathsf{last}_P(x) \neq p$ and $\mathsf{last}_P(x') \neq p$. In particular, $p <_T \mathsf{last}_P(x)$ and $p <_T \mathsf{last}_P(x')$ as $x_{i_1} \le_P x$ and $x_{i_3} \le_P x'$ by choice of the representatives x_{i_1} and x_{i_3}. Note that in this setup the least common ancestor of $\mathsf{low}(x_{i_2})$ and $\mathsf{low}(x')$ is p while the least common ancestor of $\mathsf{low}(x)$ and $\mathsf{low}(x')$ is a proper descendant of p. As $x \prec x_{i_2} \prec x'$ or $x' \prec x_{i_2} \prec x$ this yields a contradiction to Observation 21. The proof of part (iv) is similar. $\diamond$

Claim 25. *Let* $\mathcal{X} \subseteq \{X'_1, \dots, X'_{g(t)}\}$ *and* $\mathcal{Y} \subseteq \{Y'_1, \dots, Y'_{g(t)}\}$ *and* P *a path covering* $\mathcal{X} \cup \mathcal{Y}$. *If there is a node* $p \in V(P)$ *satisfying* $|\{y \in \mathsf{REP}^P_{\mathcal{Y}} : \mathsf{last}_P(y) = p\}| \ge 3$, *then* $\mathsf{last}_P(x) \le_T p$ *for every* $x \in \mathsf{REP}^P_{\mathcal{X}}$.

Proof of Claim: Assume that the statement of the claim does not hold and let p be a node, $i_1 < i_2 < i_3$ with $\mathsf{last}_P(y_{i_j}) = p$ for $y_{i_j} \in \mathsf{REP}^P_{\mathcal{Y}} \cap Y'_{i_j}$, $j \in [3]$ and $x_i \in \mathsf{REP}^P_{\mathcal{X}} \cap X'_i$ be a vertex for which $\mathsf{last}_P(x) >_P p$. By Claim 24 (iv) this implies that either $\mathsf{last}_P(y) = p$ for every $y \in Y'_{i_1}$ or $\mathsf{last}_P(y) = p$ for every $y \in Y'_{i_3}$. Without loss of generality, assume that the former is true. As $x_i \le_P x$ for every $x \in X'_i$, this implies that $x \in X'_i$ is adjacent to any $y \in Y'_{i_1}$ if and only if the path P_x contains the node p. Hence, the zone $X'_i \cap Y'_{i_1}$ can have at most two distinct row types (constant 0 and constant 1) and therefore the rank of $X'_i \cap Y'_{i_1}$ is at most 2, a contradiction. $\diamond$

It is not hard to see that we can swap the roles of x_i's and y_j's in the statement of Claim 25. Hence we have the following.

Claim 26. *Let* $\mathcal{X} \subseteq \{X'_1, \dots, X'_{g(t)}\}$ *and* $\mathcal{Y} \subseteq \{Y'_1, \dots, Y'_{g(t)}\}$ *and* P *a path covering* $\mathcal{X} \cup \mathcal{Y}$. *If there is a node* $p \in V(P)$ *satisfying* $|\{x \in \mathsf{REP}^P_{\mathcal{X}} : \mathsf{last}_P(x) = p\}| \ge 3$, *then* $\mathsf{last}_P(y) \le_P p$ *for every* $y \in \mathcal{Y}^P$.

Now let $\mathcal{X} = \{X'_1, \dots, X'_{g(t)}\}$ and $\mathcal{Y} \subseteq \{Y'_1, \dots, Y'_{g(t)}\}$ and assume that neither $\mathcal{X}$ nor $\mathcal{Y}$ is $(8t^2 \cdot \rho_{\max} + 1)^2 + 1$-diverse on $P_\forall$. In this case there must be nodes p_X and p_Y on $P_\forall$ satisfying $|\{x \in \mathsf{REP}^{P_\forall}_{\mathcal{X}} : \mathsf{last}_{P_\forall}(x) = p_X\}| \ge 3$ and $|\{y \in \mathsf{REP}^{P_\forall}_{\mathcal{Y}} : \mathsf{last}_{P_\forall}(y) = p_Y\}| \ge 3$, respectively. If $p_X = p_Y$ then by Claim 24 (iii) and (iv) there are indices $i, j \in [g(t)]$ such that $\mathsf{last}_{P_\forall}(x) = p_X$ for every $x \in X'_i$ and $\mathsf{last}_{P_\forall}(y) = p_Y$ for every $y \in Y'_j$. Hence, the set $X'_i \cup Y'_j$ forms a clique in G, implying that the zone $X'_i \cap Y'_j$ cannot have large rank, a contradiction. If $p_X \neq p_Y$ then we obtain a contradiction with either Claim 25 or Claim 26, depending on whether $p_X <_T p_Y$ or $p_Y <_T p_X$. Hence, either $\mathcal{X}$ or $\mathcal{Y}$ must be $(8t^2 \cdot \rho_{\max} + 1)^2 + 1$-diverse on $P_\forall$. Without loss of generality, we assume for the remainder of the proof that $\mathcal{X}$ is $(8t^2 \cdot \rho_{\max} + 1)^2 + 1$-diverse on $P_\forall$ (note that in case $T' \neq T''$ this must be the case whereas in the case $T' = T''$ we cannot deduce that $p_X <_T p_Y$ from the assumption that $Y_j \prec X_i$). In case we obtain diversity, we obtain the following structure.

Claim 27. *Let* $\mathcal{Z} \subseteq \{X'_1, \dots, X'_{g(t)}\}$ *or* $\mathcal{Z} \subseteq \{Y'_1, \dots, Y'_{g(t)}\}$ *and* P *a path that covers* $\mathcal{Z}$. *If* $\mathcal{Z}$ *is* $(2k-1)^2+1$-*diverse on* P *then there is a subset* $\{Z_1, \dots, Z_k\} \subseteq \mathcal{Z}$ *with* $Z_1 \prec \dots \prec Z_k$ *and pairwise vertex disjoint (directed) subpaths* $I_1 <_T \dots <_T I_k$ *of* P *such that for every* $i \in [k]$ *we have* $\mathsf{last}_P(z) \in I_i$ *for every* $z \in Z_i$.

Proof of Claim: Let $S \subseteq \mathsf{REP}^P_{\mathcal{Z}}$ be a chain of size at least $(2k-1)^2 + 1$. Let $S = \{s_1, \dots, s_{(2k-1)^2+1}\}$ where $s_i \prec s_j$ for $i < j$. We can now apply the Erdős–Szekeres theorem to the sequence $s_1, \dots, s_{(2k-1)^2+1}$ to obtain a subsequence $s_{i_1}, \dots, s_{i_{2k}}$ of length at least $2k$ which is monotone, i.e., either $s_{i_1} <_P \dots <_P s_{i_{2k}}$ or $s_{i_{2k}} <_P \dots <_P s_{i_1}$ holds.

Since $\le_P$ is monotone on $s_{i_1}, \dots, s_{i_{2k}}$ we can apply Claim 24(i) or (ii). We define Z_i to be the part in $\mathcal{Z}$ which contains $s_{i_{2i-1}}$ for every $i \in [k]$, i.e., we discard every other s_{i_j}. Claim 24 now guarantees that $\mathsf{last}_P(z) <_P \mathsf{last}_P(z')$ for $z \in Z_i$, $z' \in Z_j$ and $i < j$. Hence, we can find k pairwise node disjoint subpaths $I_1 <_T \cdots <_T I_k$ of P such that for every $i \in [k]$ we have that $\mathsf{last}_P(z)$ is in I_i for every $z \in Z_i$. $\diamond$

As $\mathcal{X}$ is $(8t^2 \cdot \rho_{\max}+1)^2+1$-diverse on $P_\forall$ we can apply Claim 27 to obtain a set of $4t^2 \cdot \rho_{\max}+1$ parts $\{\overline{X}_1, \ldots, \overline{X}_{4t^2\cdot\rho_{\max}+1}\}$ and pairwise vertex disjoint subpaths $I_1 <_T \cdots <_T I_{4t^2\cdot\rho_{\max}+1}$ of $P_\forall$ such that $\mathsf{last}_P(x)$ is in I_i for every $x \in X_i'$ (note that we need $4t^2 \cdot \rho_{\max}+1$ such parts later in the argument but require a smaller number for Claim 28). We now show that if we find a set of parts in $\{Y_1', \ldots, Y_{g(t)}'\}$ of size at least $2t^2$ for which there are vertex disjoint path $J_1, \ldots, J_{2t^2}$ containing the appropriate nodes of T then we satisfy all conditions for applying Lemma 16.

Claim 28. *Let $\overline{\mathcal{X}} = \{\overline{X}_1, \ldots, \overline{X}_{2t^2}\} \subseteq \{X_1', \ldots, X_{g(t)}'\}$ and $\overline{\mathcal{Y}} = \{\overline{Y}_1, \ldots, \overline{Y}_{2t^2}\} \subseteq \{Y_1', \ldots, Y_{g(t)}'\}$ and P a path that covers $\overline{\mathcal{X}} \cup \overline{\mathcal{Y}}$. If there are two sets of pairwise vertex disjoint subpaths $I_1 <_T \cdots <_T I_{2t^2}$ and $J_1 <_T \cdots <_T J_{2t^2}$ of P such that for every $i \in [2t^2]$ we have $\mathsf{last}_P(x) \in I_i$ for every $x \in \overline{X}_i$ and $\mathsf{last}_P(y) \in J_i$ for every $y \in \overline{Y}_i$ then G contains a semi-induced T_t.*

Proof of Claim: We first discard half of each of the sets $\overline{X}_1, \ldots, \overline{X}_{2t^2}$ and half of $\overline{Y}_1, \ldots, \overline{Y}_{2t^2}$ so that either $I_1 <_T \ldots I_{t^2} <_T J_{t^2+1} <_T \cdots <_T J_{2t^2}$ or $J_1 <_T \cdots <_T J_{t^2} <_T I_{t^2+1} <_T \cdots <_T I_{2t^2}$. This can be done as follows. Choose an edge e_X of T which divides P into two subpaths so that the first subpath contains $I_1, \ldots, I_{k^2}$ and the second subpath contains $I_{k^2+1}, \ldots, I_{2k^2}$. Choose an edge e_Y similarly. If e_X comes before e_Y on P, then we have $I_1 <_T \ldots I_{t^2} <_T J_{t^2+1} <_T \cdots <_T J_{2t^2}$ and otherwise we get $J_1 <_T \cdots <_T J_{t^2} <_T I_{t^2+1} <_T \cdots <_T I_{2t^2}$. Without loss of generality assume that we are in the former case.

Note that the subgraph G_I of G induced by vertices in $\bigcup_{i\in[t^2]} \overline{X}_i \cup \overline{Y}_{t^2+i}$ is an interval graph because the vertices in $\bigcup_{i\in[t^2]} \overline{X}_i \cup \overline{Y}_{t^2+i}$ are covered by a path (this is generally true for any set of vertices that is covered by a path). Further note that the interval representation of G_I that can be obtained by looking at P in reverse satisfies that $\mathsf{start}(\overline{X}_1) < \cdots \mathsf{start}(\overline{X}_{t^2}) < \mathsf{start}(\overline{Y}_{t^2+1}) < \mathsf{start}(\overline{Y}_{2t^2})$. Finally the zone $\overline{X}_i \cup \overline{Y}_j$ in the adjacency matrix of G_I has rank at least 3 since it also is a zone in M' for which $(\{X_1', \ldots, X_{g(t)}'\}, \{Y_1', \ldots, Y_{g(t)}'\})$ is a high rank division. Therefore Lemma 16 applies to G_I which implies that G contains a semi-induced transversal pair T_t. $\diamond$

In case $\{Y_1', \ldots, Y_{g(t)}'\}$ is $(4t^2-1)^2+1$-diverse on $P_\forall$ we can apply first Claim 27 on $\{Y_1', \ldots, Y_{g(t)}'\}$ to obtain parts $\{\overline{Y}_1, \ldots, \overline{Y}_{2t^2}\}$ with corresponding paths $J_1 <_T \cdots <_T J_{2t^2}$. We then apply Claim 28 to $\{\overline{X}_1, \ldots, \overline{X}_{2t^2}\}$, $\{\overline{Y}_1, \ldots, \overline{Y}_{2t^2}\}$, $I_1 <_T \cdots <_T I_{2t^2}$ and $J_1 <_T \cdots <_T J_{2t^2}$. Hence, we can find T_t and conclude that in this case the statement is true. Therefore, we will assume for the remainder of the proof that $\{Y_1', \ldots, Y_{g(t)}'\}$ is not $(4t^2-1)^2+1$-diverse on $P_\forall$. Hence, there is a node p_Y on $P_\forall$ such that $|\{y \in \mathsf{REP}^{P_\forall}_{\mathcal{Y}} : \mathsf{last}_{P_\forall}(y) = p_y\}| \geq g_0(t) + \rho_{\max} + 4$. By Claim 25 we know that $\mathsf{last}_{P_\forall}(x) \leq_T p_Y$ for every $x \in \mathsf{REP}^{P_\forall}_{\{\overline{X}_1,\ldots,\overline{X}_{4t^2\cdot\rho_{\max}+1}\}}$. As $I_1 <_T \cdots <_T I_{4t^2\cdot\rho_{\max}+1}$, this implies that $\mathsf{last}_{P_\forall}(x) <_T p_Y$ for every $x \in \bigcup_{i\in[4t^2\cdot\rho_{\max}]} \overline{X}_i$. Hence, we discard $\overline{X}_{4t^2\cdot\rho_{\max}+1}$ and set $\overline{\mathcal{X}} = \{\overline{X}_1, \ldots, \overline{X}_{4t^2\cdot\rho_{\max}}\}$.

The next claim allows us to obtain slightly cleaner picture, i.e., it yields the property, that there is a node p_0 such that for all considered column parts, for every vertex v the path P_v representing v diverges from $P_\forall$ at p_0 and ends below p_0. We define this formally. Let $\mathcal{Z} \subseteq \{Y_1', \ldots, Y_{g(t)}'\}$ be a set of parts covered by some path P. We say that a node p of P *concentrates* $\mathcal{Z}$ if $\mathsf{last}_P(z) = p$ and $\mathsf{low}(z) >_T p$ for every $z \in \bigcup_{Z\in\mathcal{Z}} Z$.

Claim 29. *Let $\mathcal{Z} \subseteq \{Y_1', \ldots, Y_{g(t)}'\}$, P a path covering $\mathcal{Z}$ and p a node on P appearing after $I_{4t^2\cdot\rho_{\max}}$ on $P_\forall$ such that $\mathsf{last}_P(y_j) = p$ for every $y_j \in \mathsf{REP}^P_{\mathcal{Z}}$. There is a subset $\mathcal{Z}' \subseteq \mathcal{Z}$ containing all but at most $\rho_{\max}+4$ parts of $\mathcal{Z}$ such that p concentrates $\mathcal{Z}'$.*

Proof of Claim: We first argue that there are at most $\rho_{\max}+2$ distinct parts $Z \in \mathcal{Z}$ such that there is $z \in Z$ with $\mathsf{low}(z) = p$. Assume that this is not the case and there are indices $i_1 < \cdots < i_{\rho_{\max}+3}$ such that $Y_{i_j}' \in \mathcal{Z}$ and $\mathsf{low}(z_{i_j}) = p$ for some $z_{i_j} \in Y_{i_j}'$ for every $j \in [\rho_{\max}+3]$. By construction of $\prec$, we get that $\mathsf{low}(y) = p$ for every $z_{i_1} \prec y \prec z_{i_{\rho_{\max}+3}}$ which in particular includes every $y \in Y_{i_2}' \cup \cdots \cup Y_{i_{\rho_{\max}+2}}'$. Again using the properties of $\prec$, it must be the case that $\mathsf{high}(y) \leq_T \mathsf{high}(y')$

for every pair $y \in Y'_{i_j}, y' \in Y'_{i_k}$, $2 \le j \le k \le \rho_{\max}+2$ (as $y \prec y'$). Recall that $I_1 <_T \cdots <_T I_{4t^2\cdot\rho_{\max}}$ and p appears after $I_{4t^2\cdot\rho_{\max}}$. As the rank of the zone $\overline{X}_1 \cap Y'_{i_{\rho_{\max}+3}}$ is at least 3, there must be $y_1 \in Y'_{i_{\rho_{\max}+3}}$ which is adjacent to some $x \in \overline{X}_1$ and non-adjacent to some $x' \in \overline{X}_1$. Hence, there is $p_1 \in I_1$ such $p_1 \in V(P_{y_1})$ and $q \notin V(P_{y_1})$ for every ancestor $q \in V(T')$ of p_1. Similarly, for every $j \in [2, \rho_{\max}+2]$ there must be $y_j \in Y'_{i_{\rho_{\max}+3-j}}$ which is adjacent to some $x \in \overline{X}_j$ and non-adjacent to some $x' \in \overline{X}_j$. Hence, there is $p_j \in I_j$ such $p_j \in V(P_{y_j})$ and $q \notin V(P_{y_j})$ for every ancestor $q \in V(T')$ of p_j. In total we have obtained $y_{\rho_{\max}+1} \prec \cdots \prec y_1$ and nodes $p_1 <_T \cdots <_T p_{\rho_{\max}+1}$ on P such that p_j is the first node of P contained in P_{y_j}.

Let $\widehat{T}^1 \in \{T^1, \dots, T^\rho\}$ be the tree containing $\mathsf{high}(y_1)$. Since, $\mathsf{high}(y_2) <_T \mathsf{high}(y_1)$ but $p_1 <_T p_2$ we get that $\mathsf{high}(y_2)$ must be contained in a subtree $\widehat{T}^2 \in \{T^1, \dots, T^\rho\}$ of T which appears before $\widehat{T}^1$ in $\le_T$. Inductively, assume that we have shown for every $j < k$ that $\mathsf{high}(y_j)$ is contained in the subtree $\widehat{T}^j \in \{T^1, \dots, T^\rho\}$ and that $\widehat{T}^{k-1} <_T \cdots <_T \widehat{T}^1$. Since $p_k <_T p_{k_1} <_T \cdots <_T p_1$ we know that $\mathsf{high}(y_k)$ cannot be contained in any tree of $\widehat{T}^2, \dots, \widehat{T}^{k-1}$. Let $\overline{T}^k \in \{T^1, \dots, T^\rho\}$ be the tree containing $\mathsf{high}(y_k)$. Since $y_k \prec y_{k-1}$ it must be the case that $\widehat{T}^k <_T \widehat{T}^{k-1}$. Hence, after repeating the argument we get that there must be pairwise different trees $\widehat{T}^1, \dots, \widehat{T}^{\rho_{\max}+1} \in \{T^1, \dots, T^\rho\}$, a contradiction since $\{T^1, \dots, T^\rho\}$ contains only ρ elements. Hence, there are at most $\rho_{\max}+2$ distinct parts $Z \in \mathcal{Z}$ such that there is $z \in Z$ with $\mathsf{low}(z) = p$.

Next, we consider parts $Z \in \mathcal{Z}$ for which there is $z \in Z$ with $\mathsf{last}_P(z) \neq p$. By Claim 24, we have that for all but at most two parts $Z \in \mathcal{Z}$ it holds that $\mathsf{last}_P(z) = p$ for all $z \in Z$ (if there are three we can choose i_1, i_2, i_3 to contradict Claim 24 (iv)). Therefore, in total for all but at most $\rho_{\max}+4$ parts $Z \in \mathcal{Z}$ we have that $\mathsf{low}(z) >_T p$ and $\mathsf{last}_P(z) = p$ for every $z \in Z$. Therefore, we can define $\mathcal{Z}'$ accordingly to prove the claim. $\diamond$

Next we analyze how having a node p which concentrates many column parts can be used to find a transversal pair $\mathcal{T}_{t,2}$. In order for this to work, we need that the endpoints of paths representing vertices in these parts are distributed nicely among the subtrees rooted at children of p. To formalize this, we need the following definitions.

Let $\mathcal{Z} \subseteq \{Y'_1, \dots, Y'_{g(t)}\}$, P a path covering $\mathcal{Z}$ and p a node which concentrates $\mathcal{Z}$. We say that a node q of T *hosts* a vertex $y \in V(G)$ if $q \le_T \mathsf{low}(y)$. For every $Z \in \mathcal{Z}$ we define Q^p_Z to be the set of children of p which host some vertex $z \in Z$.

Claim 30. *Let $\mathcal{Z} \subseteq \{Y'_1, \dots, Y'_{g(t)}\}$ with $|\mathcal{Z}| \ge g_i(t)$ for some $i \in [(4t^2+1)^2]$, P a path covering $\mathcal{Z}$ and p a node concentrating $\mathcal{Z}$. If every child q of p hosts at most $g_{i+1}(t) + \rho_{\max} + 4$ vertices from $\mathsf{REP}^P_{\mathcal{Z}}$, then we can find a set $\mathcal{Z}' \subseteq \mathcal{Z}$ of size at least $(2t^2 \cdot \rho_{\max})^{2t^2\cdot\rho_{\max}}$ such that the sets Q^p_Z with $Z \in \mathcal{Z}'$ are pairwise disjoint.*

Proof of Claim: Assume that every child q of p host at most $g_{i+1}(t) + \rho_{\max} + 4$ vertices from $\mathsf{REP}^P_{\mathcal{Z}}$ and recall that $g_i(t) = (g_{i+1}(t) + \rho_{\max} + 4) \cdot 3(2t^2 \cdot \rho_{\max})^{2t^2\cdot\rho_{\max}}$. Let $Q_{\ge 3}$ be the set of all children q of p that host vertices from at least three different parts of $\mathcal{Z}$ and $Q_{<3}$ be the set of all children of p that host vertices from one or two parts in $\mathcal{Z}$. By our assumption every node in $Q_{\ge 3}$ hosts at most $g_{i+1}(t) + \rho_{\max} + 4$ vertices from $\mathsf{REP}^P_{\mathcal{Z}}$. Hence, there are at least $(g_{i+1}(t) + \rho_{\max} + 4)(3(2t^2 \cdot \rho_{\max})^{2t^2\cdot\rho_{\max}} - |Q_{\ge 3}|)$ vertices from $\mathsf{REP}^P_{\mathcal{Z}}$ which are hosted by nodes in $Q_{<3}$. Since each node in $Q_{<3}$ hosts at most two vertices from $\mathsf{REP}^P_{\mathcal{Z}}$, there must be at least $2(3(2t^2 \cdot \rho_{\max}))^{2t^2\cdot\rho_{\max}} - |Q_{\ge 3}|$ nodes in $Q_{<3}$.

We first argue that for any $q \in Q_{\ge 3}$ there are at most two parts $Z \in \mathcal{Z}$ of which some vertex but not all vertices are hosted by q. Assume this is not the case and there are $Z_1, Z_2, Z_3 \in \mathcal{Z}$ and $u_i, v_i \in Z_i$, $i \in [3]$ such that q hosts u_1, u_2, u_3 but q does not host v_1, v_2, v_3. Without loss of generality assume that $u_1 \prec u_2 \prec u_3$. Since $u_1 \prec z \prec u_3$ for any $z \in Z_2$, applying Observation 21 to $u_1 \prec v_2 \prec u_3$ yields a contradiction and hence the statement holds. We let $\mathcal{Z}_{\ge 3} \subseteq \mathcal{Z}$ be a set of parts consisting of one part Z fully hosted by q for every $q \in Q_{\ge 3}$. Additionally, we let $\mathcal{Z}_{<3}$ be the set of parts containing a $z \in \mathsf{REP}^P_{\mathcal{Z}}$ which is hosted by some $q \in Q_{<3}$. Observe that by the above statement each $q \in Q_{\ge 3}$ hosts vertices from at most two different parts in $\mathcal{Z}_{<3}$.

In case $Q_{\ge 3}$ contains at least $(2t^2 \cdot \rho_{\max})^{2t^2\cdot\rho_{\max}}$ parts the statement follows as each Q^P_Z consists of one distinct node from $Q_{\ge 3}$ for the parts $Z \in \mathcal{Z}_{\ge 3}$. Therefore, assume that $|Q_{\ge 3}| < (2t^2 \cdot \rho_{\max})^{2t^2\cdot\rho_{\max}}$.

We let $Q'_{<3} \subseteq Q_{<3}$ be the set of nodes hosting some $z \in \mathsf{REP}^P_{\mathcal{Z}}$ for which no vertex of the part Z containing z is hosted by any $q \in Q_{\geq 3}$. Since $|Q_{<3}| \geq 2(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}} + 3((2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}} - |Q_{\geq 3}|)$ and each of the at most $(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}$ nodes in $Q_{\geq 3}$ hosts vertices from at most two different parts in $\mathcal{Z}_{<3}$, we know that $|Q'_{<3}| \geq 3((2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}} - |Q_{\geq 3}|)$. We let $\mathcal{Z}'_{<3} \subseteq \mathcal{Z}_{<3}$ be the set of parts containing some $z \in \mathsf{REP}^P_{\mathcal{Z}}$ which is hosted by some $q \in Q'_{<3}$.

Next, we argue that for every $Z \in \mathcal{Z}'_{<3}$ there are at most two other parts $Z' \in \mathcal{Z}'_{<3}$ such that Q^P_Z and $Q^P_{Z'}$ are not disjoint. Assume this is not the case and there are three distinct parts $Z_1, Z_2, Z_3 \in \mathcal{Z}'_{<3}$ and $u_i \in Z_i, i \in [3]$ and $v_1, v_2, v_3 \in Z$ such that u_i and v_i are hosted by the same child q_i of p for every $i \in [3]$. As each q_i can host nodes from at most two distinct parts of $\mathcal{Z}'_{<3}$ we know that the q_i are pairwise different. Assume without loss of generality that $v_1 \prec v_2 \prec v_3$. As v_1, v_2, v_3 are from the same part, either $u_2 \prec v_1 \prec v_2 \prec v_3$ or $v_1 \prec v_2 \prec v_3 \prec u_2$. Hence, applying Observation 21 to either $u_2 \prec v_1 \prec v_2$ or $v_2 \prec v_3 \prec u_2$ yields a contradiction. Since for every $Z \in \mathcal{Z}'_{<3}$ there are at most two parts $Z' \in \mathcal{Z}'_{<3}$ such that Q^P_Z and $Q^P_{Z'}$ are not disjoint, we can find a subset $\mathcal{Z}''_{<3} \subseteq \mathcal{Z}'_{<3}$ containing at least a third of the parts of $\mathcal{Z}'_{<3}$ such that the sets Q^P_Z are pairwise disjoint for the $Z \in \mathcal{Z}''_{<3}$. By our previous count we have that $|Q''_{<3}| \geq (2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}} - |Q_{\geq 3}|$ and hence $\mathcal{Z}_{\geq 3} \cup \mathcal{Z}''_{<3}$ by construction is the desired set of size $(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}$. ◇

Claim 31. *Let $\mathcal{X} = \{\overline{X}_1, \ldots, \overline{X}_{4t^2 \cdot \rho_{\max}}\} \subseteq \{X'_1, \ldots, X'_{g(t)}\}$ with $\overline{X}_1 \prec \cdots \prec \overline{X}_{4t^2 \cdot \rho_{\max}}$ and additionally let $\mathcal{Y} = \{\overline{Y}_1, \ldots, \overline{Y}_{(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}}\} \subseteq \{Y'_1, \ldots, Y'_{g(t)}\}$, P a path that covers $\mathcal{X} \cup \mathcal{Y}$, $I_1, \ldots, I_{4t^2 \cdot \rho_{\max}}$ pairwise disjoint subpaths of P such that $\mathsf{last}_P(x) \in I_i$ for every $x \in \overline{X}_i$ for every $i \in [4t^2 \cdot \rho_{\max}]$ and p a node which concentrates $\mathcal{Y}$. If the sets Q^p_Y with $Y \in \mathcal{Y}$ are pairwise disjoint, then G contains a semi-induced $T_{t,2}$.*

Proof of Claim: In this proof we aim to find a semi-induced $T_{k,2}$ (A, B, C, D, E). Similar to the interval graph case, we find A among vertices in pairwise distinct parts from $\mathcal{X}$ and B among vertices in pairwise distinct parts from $\mathcal{Y}$ using the large rank of M' to force the adjacency as required. But as all vertices in parts in $\mathcal{Y}$ diverge from P in the same point, we cannot choose a set C of vertices which has the required adjacency to B in order to obtain T_t consisting out of (A, B, C). To overcome this obstacle we use the property that the sets of children hosting vertices from $\overline{Y}_i$ are pairwise disjoint. We show that for every $b \in B$ we can find $d \in V(G)$ which is hosted by the same child of p as b, such that the high endpoints of d's are distinct, appear in the same tree $\overline{T} \in \{T^1, \ldots, T^\rho\}$ and are ordered (by $<_T$) in the desired way. The vertices in C and E can now be chosen using minimality of the chosen tree model. For an illustration see Fig. 4. A formal argument follows.

Without loss of generality assume $\overline{Y}_1 \prec \cdots \prec \overline{Y}_{(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}}$. We first argue that there exist a set $\tilde{S} \subseteq [(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}]$ of size $2t^2$ and a set of vertices $\{\tilde{s}_i : i \in \tilde{S}\} \subseteq V(G)$ such that the following properties hold:

(i) $\tilde{s}_i$ is hosted by some $q \in Q^p_{\overline{Y}_i}$ for every $i \in \tilde{S}$,

(ii) $\mathsf{high}(\tilde{s}_i) <_T \mathsf{high}(\tilde{s}_j)$ for $i < j$,

(iii) every $q \in Q^p_{\overline{Y}_i}$ hosts some vertex s with $\mathsf{high}(\tilde{s}_i) \leq_T \mathsf{high}(s) \leq_T \mathsf{high}(\tilde{s}_{i'})$ for $i, i' \in \tilde{S}$, $i < i'$ and

(iv) $\mathsf{high}(\tilde{s}_i) \leq_T \mathsf{last}_P(x)$ for some $x \in \overline{X}_{2t^2}$.

(v) there is $\overline{T} \in \{T^1, \ldots, T^\rho\}$ and a path $\overline{P}$ in $\overline{T}$ such that $\mathsf{high}(\tilde{s}_i) \in V(\overline{P})$ for every $i \in \tilde{S}$.

First, we let s^1_i be the vertex such that $\mathsf{high}(s^1_i)$ is the minimum of the set $\{\mathsf{high}(v) : \mathsf{low}(v) \in T_q, q \in Q^p_{\overline{Y}_i}\}$ with respect to $\leq_T$. Since the set $Q^p_{\overline{Y}_i}$ hosts all vertices from $\overline{Y}_i$ and for $i < i'$ the vertices in $\overline{Y}_i$ get discovered before the vertices in $\overline{Y}_{i'}$, the tie breaking rule for obtaining $\prec$ implies that $\mathsf{high}(s^1_1) \leq_T \cdots \leq_T \mathsf{high}\left(s^1_{(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}}\right)$. There are two cases to consider. Either there is a subset $S \subseteq [(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}]$ of size $2t^2 \cdot \rho_{\max}$ such that $\mathsf{high}(s^1_i) <_T \mathsf{high}(s^1_j)$ for $i, j \in S$, $i < j$ or there is a node p^1 on P such that there is a subset $S^1 \subseteq [(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}]$ of size at least $(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max} - 1}$ such that $\mathsf{high}(s^1_i) = p^1$ for every $i \in S^1$. In the first case we terminate. In the second case we define

s_i^2 such that $\mathsf{high}(s_i^2)$ is the minimum of the set $\{\mathsf{high}(v) : \mathsf{low}(v) \in T_q, q \in Q^p_{\overline{Y}_i}\} \setminus \{p^1\}$ with respect to $\leq_T$ and repeat the argument iteratively no more than $2t^2 \cdot \rho_{\max}$ times.

That is in the kth iteration for every $i \in S^{k-1}$ we select s_i^k such that $\mathsf{high}(s_i^k)$ is the minimum of the set $\{\mathsf{high}(v) : \mathsf{low}(v) \in T_q, q \in Q^p_{\overline{Y}_i}\} \setminus \{p^j : j < k\}$ with respect to $\leq_T$. Note that each set of nodes $Q^p_{\overline{Y}_i}$ hosts at least $2t^2 \cdot \rho_{\max}$ vertices with different high endpoints. To see this, observe that for every $i \in [4t^2 \cdot \rho_{\max}]$ and $j \in [(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}]$ there must be $y \in \overline{Y}_j$ such that P_y contains some node in I_i but not in I_{i-1} which follows from the condition that M' has high rank and the assumption on the layout of the parts in $\mathcal{X}$. Hence, the set $\{\mathsf{high}(v) : \mathsf{low}(v) \in T_q, q \in Q_{\overline{Y}_i}\} \setminus \{p^k : j < k\}$ from which we choose s_i^k is not empty. By the tie breaking rule we used to obtain $\prec$ we infer that $\mathsf{high}(s_i^k) \leq_T \mathsf{high}(s_{i'}^k)$ for every $i, i' \in S^{k-1}$ for which $i < i'$. Again we either find a subset $S \subset [(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}]$ of size $2t^2 \cdot \rho_{\max}$ such that $\mathsf{high}(s_i^k) <_T \mathsf{high}(s_j^k)$ for $i, j \in S$, $i < j$ or a node p^k and a set $S^k \subseteq S^{k-1}$ of size $(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max} - k}$ such that $\mathsf{high}(s_i^k) = p^k$ for every $i \in S^k$. We repeat this procedure until we obtain either

(C1) a set $S \subseteq [(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}]$ of size $2t^2 \cdot \rho_{\max}$ and vertices $\tilde{s}_i$, $i \in S$ such that $\tilde{s}_i$ is hosted by some $q \in Q_{\overline{Y}_i^p}$ and $\mathsf{high}(\tilde{s}_i) <_T \mathsf{high}(\tilde{s}_j)$ for $i, j \in S$, $i < j$ or

(C2) $2t^2 \cdot \rho_{\max}$ points $p^1, \ldots, p^{2t^2 \cdot \rho_{\max}}$, a set $S^{2t^2 \cdot \rho_{\max}}$ and for every $i \in S^{2t^2 \cdot \rho_{\max}}$, $j \in [t]$ a vertex s_i^j such that $\mathsf{high}(s_i^j) = p_j$ for every $i \in S^{2t^2 \cdot \rho_{\max}}$.

In case we obtain case (C1) the properties (i),(ii) are true by construction. To prove property (iii) observe that there is a k such that $\tilde{s}_i = s_i^k$ for every $i \in S$. Since every subtree T_q for $q \in Q_{\overline{Y}_i^p}$ is explored by the lex-DFS before $T_{q'}$ for $q' \in Q_{\overline{Y}_{i'}^p}$ for any $i' > i$ the tie breaking rule implies that any $q \in Q_{\overline{Y}_i^p}$ must host some vertex s with $\mathsf{high}(s) \leq_T \mathsf{high}(\tilde{s}_{i'})$. Since we further picked s_i^k in the above procedure to be the minimum of $\{\mathsf{high}(v) : \mathsf{low}(v) \in T_q, q \in Q^p_{\overline{Y}_i}\} \setminus \{p^j : j < k\}$ we obtain (iii).

In case we obtain case (C2), we set $S = S^{2t^2 \cdot \rho_{\max}}$ and $\tilde{s}_i := s_i^{2t^2 \cdot \rho_{\max} - i}$. Since $s_i^{2t^2 \cdot \rho_{\max} - i}$ is hosted by some $q \in Q^p_{\overline{Y}_i}$ and by construction $\mathsf{high}(s_1^t) = p^t <_T \cdots <_T \mathsf{high}(s_t^1) = p^1$ these choices also satisfy the conditions of the statements (i) and (ii). Note that in this case, (iii) follows from the observation that if in the kth iteration of the above procedure we find p^k such that $\mathsf{high}(s_i^k) = p^k$ for every $i \in S^k$ then the tie breaking rule implies that every $q \in \bigcup_{i \in S} Q^p_{\overline{Y}_i}$ hosts a path s with $\mathsf{high}(s) = p^k$.

In both case (C1) and (C2) condition (iv), i.e., $\mathsf{high}(\tilde{s}_i) \leq_T \mathsf{last}_P(x)$ for some $x \in \overline{X}_{2t^2}$, follows from the tie breaking rule as follows. First note that the column of $\overline{X}_{2t^2} \cap \overline{Y}_i$ corresponding to all vertices $y \in \overline{Y}_i$ for which $\mathsf{high}(y)$ coincides is the same. Since $\overline{X}_{2t^2} \cap \overline{Y}_i$ has rank larger than $2t^2 \cdot \rho_{max}$, for every $k \in [2t^2 \cdot \rho_{\max}]$ the set of nodes $Q^p_{\overline{Y}_i}$ must host an element $y \in \overline{Y}_i$ with $\mathsf{high}(y) >_T \{p^j : j < k\}$ which is adjacent to some $x \in \overline{X}_{2t^2}$. It remains to obtain property (v). Since S has size $2t^2 \cdot \rho_{\max}$ we know that there is $\overline{T} \in \{T^1, \ldots, T^\rho\}$ such that at least $2t^2$ vertices $\tilde{s}_i$, $i \in S$ satisfy that $\mathsf{high}(\tilde{s}_i) \in V(\overline{T})$. Define $\tilde{S}$ to be all indices $i \in S$ for which $\mathsf{high}(\tilde{s}_i) \in V(\overline{T})$. It remains to observe that all $\mathsf{high}(\tilde{s}_i)$, $i \in \tilde{S}$ must lie on a path $\overline{P}$ in $\overline{T}$. This follows as $P_{\tilde{s}_i}$ contains some node on P and there is only one directed path from the root of $\overline{T}$ to some node of P. Hence, we found set $\tilde{S} \subseteq [(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}]$ of size $2t^2$ and a set of vertices $\{\tilde{s}_i : i \in \tilde{S}\} \subseteq V(G)$ with properties (i), (ii), (iii), (iv) and (v).

We let $\{\overline{Y}'_1, \ldots, \overline{Y}'_{2t^2}\} \subseteq \{\overline{Y}_1, \ldots, \overline{Y}_{(2t^2 \cdot \rho_{\max})^{2t^2 \cdot \rho_{\max}}}\}$ such that $\overline{Y}'_i$ is equal to $\overline{Y}_j$ for the ith smallest index $j \in \tilde{S}$. We further let $\tilde{s}'_i$ be the vertex in $\{\tilde{s}_i : i \in \tilde{S}\}$ which is hosted by some $q \in Q^p_{\overline{Y}'_i}$.

For easier reading we define $\overline{Y}''_i := \overline{Y}'_{2(t^2-(i-1))}$, $\overline{X}'_i := \overline{X}_{2(2t^2-(i-1))+1}$, $I'_i := I_{2(2t^2-(i-1))+1}$ and $\tilde{s}'_i := \tilde{s}_{2(t^2-(i-1))}$ for every $i \in [t^2]$. Note that this is just a renaming scheme to keep the indices in the following argument small. For $i, j \in [t]$ pick $b_{i,j} \in \overline{Y}''_{(j-1)t+i}$ to be a vertex that is hosted in some $q \in Q^p_{\overline{Y}''_{(j-1)t+i}}$ and $\mathsf{high}(b_{i,j}) \in I'_{(i-1)t+j}$ which exists as the rank of $\overline{X}'_{(i-1)t+j} \cap \overline{Y}''_{(j-1)t+i}$ is at least 3 and we can therefore pick $b_{i,j}$ to be a vertex which is adjacent to some vertex in $\overline{X}'_{(i-1)t+j}$ and non-adjacent to some other vertex in $\overline{X}'_{(i-1)t+j}$. Now pick $a_{i,j} \in \overline{X}'_{(i-1)t+j}$ arbitrarily. Note that

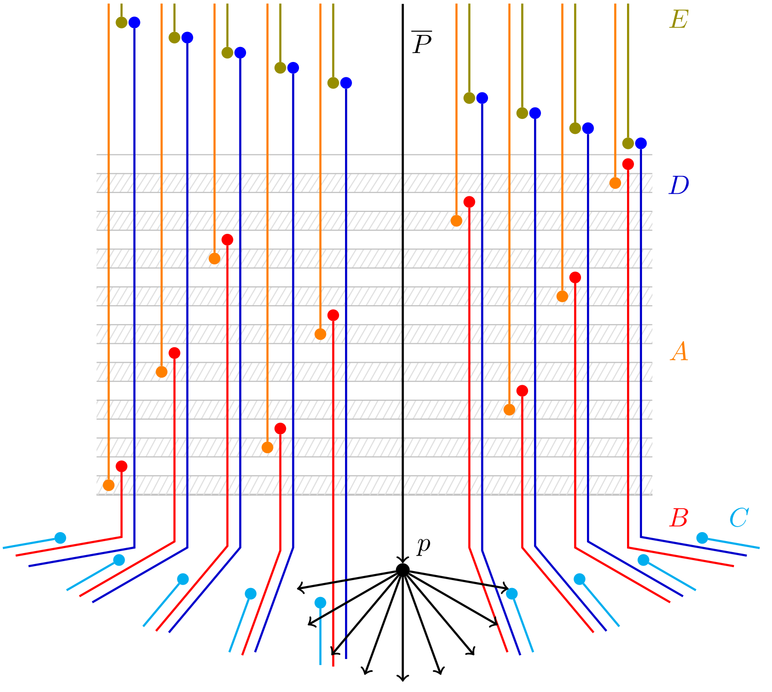


Fig. 4. Schematic illustration of the tree model of a $T_{3,2}$. Here lines depict paths of unknown length while vertices emphasize known start or end points of the paths.

by construction, $a_{i,j}$ is adjacent to $b_{i',j'}$ if and only if $(i,j) \leqslant_{\text{lex}} (i',j')$. Set $c_{i,j}$ to be a vertex with $\mathsf{high}(c_{i,j}) = q$ where q is the node hosting $b_{i,j}$. Note that $c_{i,j}$ must exists by the assumption that the tree model is minimal. Set $d_{i,j}$ to be a vertex hosted by the node q which is hosting $b_{i,j}$ and $c_{i,j}$ with $\mathsf{high}(\tilde{s}_{(j-1)t+i}) \leq_T \mathsf{high}(d_{i,j}) \leq_T \mathsf{high}(\tilde{s}_{(j-1)t+i+1})$. The existence of $d_{i,j}$ follows from property (iii) since $b_{i,j}$ and $c_{i,j}$ are hosted by some $q \in Q^{p}_{Y''_{(j-1)t+i}}$. Furthermore, $d_{i,j}$ is distinct from $b_{i',j'}$, $i',j' \in [t]$ as $\mathsf{high}(d_{i,j}) \leq_T x <_T \mathsf{high}(b_{i',j'})$ for some element $x \in \overline{X}_{2t^2}$ by choice of $b_{i,j}$ and property (iv). Observe that if $(j,i) <_{\text{lex}} (j',i')$ then $\mathsf{high}(d_{i',j'}) \leq_T \tilde{s}_{(j'-1)t+i'+1} <_T \tilde{s}_{(j-1)t+i} \leq_T \mathsf{high}(d_{i,j})$ where the strict inequality follows from property (ii). Hence we can pick $e_{i,j}$ to be a vertex such that $\mathsf{last}_{\overline{P}}(e_{i,j}) = \mathsf{high}(d_{i,j})$ and obtain that $d_{i,j}$ is adjacent to $e_{i',j'}$ if and only if $(j,i) \leqslant_{\text{lex}} (j',i')$. Note that $e_{i,j}$ exists as a consequence of choosing a minimal tree model. Set $A := \{a_{i,j} : i,j \in [k]\}$, $B := \{b_{i,j} : i,j \in [k]\}$, $C := \{c_{i,j} : i,j \in [k]\}$, $D := \{d_{i,j} : i,j \in [k]\}$ and $E := \{e_{i,j} : i,j \in [k]\}$ and observe that indeed (A,B,C,D,E) is a semi-induced $T_{k,2}$. We refer to Fig. 4 for an illustration. $\diamond$

We are now ready to conclude the proof of the theorem by induction. More specifically, in the ith step of our inductive argument we either conclude that G contains a semi-induced $T_{t,2}$ or we find

(i) a set $\mathcal{Y}_i \subseteq \{Y'_1,\ldots,Y'_{g(t)}\}$ of at least $g_i(t)$ parts,
(ii) a path P_i which covers $\overline{\mathcal{X}} \cup \mathcal{Y}_i$ such that the last node p_i of P_i concentrates $\mathcal{Y}_i$ and no child of p_i hosts all vertices in $\mathsf{REP}^{P_i}_{\mathcal{Y}_i}$, and
(iii) a set $\mathcal{Z}_i \subseteq \{Y'_1,\ldots,Y'_{g(t)}\}$ which is covered by P_i, is i-diverse on P_i and $\mathsf{last}_{P_i}(z) <_T p_i$ for every $z \in \mathsf{REP}^{P_i}_{\mathcal{Z}_i}$.

We first consider the base case. Therefore, recall that we are in the case were we cannot apply Claim 28 which implies (as explained after Claim 28) that there is a node p_Y on $P_\forall$ and a set

$\mathcal{Y} \subseteq \{Y'_1, \ldots, Y'_{g(t)}\}$ of size more than $g_0(t) + \rho_{\max} + 4$ such that $\mathsf{last}_{P_\forall}(y_j) = p_Y$ for every $y_j \in \mathsf{REP}^{P_\forall}_{\mathcal{Y}}$. We let p_0 be the lowest descendant of p_Y which hosts the same vertices from $\mathsf{REP}^{P_\forall}_{\mathcal{Y}}$ as p_Y and hence no child of p_0 hosts all vertices of $\mathsf{REP}^{P_\forall}_{\mathcal{Y}}$. We let P_0 be the path from the start of $P_\forall$ to p_0. We let $\mathcal{Y}_0$ be the subset of $\mathcal{Y}$ of parts concentrated by p_0 and remark that $|\mathcal{Y}_0| \geq g_0(t)$ by Claim 29 and hence we satisfy (i). Note that P_0 covers $\overline{\mathcal{X}} \cup \mathcal{Y}_0$, p_0 is the last node of P_0 and no child of p_0 hosts all vertices in $\mathsf{REP}^{P_0}_{\mathcal{Y}_0}$ which implies that (ii) is satisfied. We further set $\mathcal{Z}_0 = \emptyset$ which fulfills the requirement (iii).

Now assume that in the ith step, no child of p_i hosts at least $g_{i+1}(t) + \rho_{\max} + 4$ vertices from $\mathsf{REP}^{P_i}_{\mathcal{Y}_i}$. In this case, we can apply Claims 30 and 31 to find $T_{t,2}$. Hence, assume that a child q of p_i hosts at least $g_{i+1}(t) + \rho_{\max} + 4$ vertices from $\mathsf{REP}^{P_i}_{\mathcal{Y}_i}$. We let p_{i+1} to be the lowest descendant of q which hosts the same vertices from $\mathsf{REP}^{P_i}_{\mathcal{Y}_i}$ as q, i.e., no child of p_{i+1} hosts the same vertices from $\mathsf{REP}^{P_i}_{\mathcal{Y}_i}$ as q. We let P_{i+1} be the path from the start of P_i to p_{i+1} which must cover $\overline{\mathcal{X}} \cup \mathcal{Y}_i \cup \mathcal{Z}_i$ as it contains P_i as a subpath. We define $\mathcal{Y}_{i+1}$ to be all parts from Y_i concentrated by p_{i+1} which must be at least $g_{i+1}(t)$ by Claim 29 and hence (i) and (ii) are satisfied. Finally, there must be $z \in \mathsf{REP}^{P_i}_{\mathcal{Y}_i}$ which is not hosted by q by our assumption that no child of p_i hosts all vertices in $\mathsf{REP}^{P_i}_{\mathcal{Y}_i}$. Hence, $p_i \leq_T \mathsf{last}_{P_{i+1}}(z) <_T p_{i+1}$. Since $\mathsf{last}_{P_i}(z') <_T p_i$ for every $z' \in \mathsf{REP}^{P_i}_{\mathcal{Z}_i}$, we obtain that the set $\mathcal{Z}_{i+1} = \mathcal{Z} \cup \{Z\}$ where $Z \in \{Y'_1, \ldots, Y'_{g(t)}\}$ is the part containing z is $i+1$-diverse on P_{i+1}. Since additionally $\mathcal{Z}_{i+1}$ is covered by P_{i+1} and $\mathsf{last}_{P_{i+1}}(z') <_T p_{i+1}$ for every $z' \in \mathsf{REP}^{P_{i+1}}_{\mathcal{Z}_{i+1}}$ the requirement (iii) is satisfied.

If after repeating this procedure $(4t^2+1)^2+1$ times we have not found $T_{t,2}$ we conclude as follows. Since $\mathcal{Z}_{(4t^2+1)^2+1}$ is $(4t^2+1)^2+1$-diverse on $P_{(4t^2+1)^2+1}$ we apply Claim 27 to obtain parts $\{\overline{Y}_1, \ldots, \overline{Y}_{2t^2}\} \subseteq \mathcal{Z}_{(4t^2+1)^2+1}$ with corresponding pairwise disjoint paths $J_1 <_T \cdots <_T J_{2t^2}$. Then applying Claim 28 to $\{\overline{X}_1, \ldots, \overline{X}_{2t^2}\}$, $\{\overline{Y}_1, \ldots, \overline{Y}_{2t^2}\}$, $I_1 <_T \cdots <_T I_{2t^2}$ and $J_1 <_T \cdots <_T J_{2t^2}$ allows us to find a T_t. Therefore, the statement holds. □

By Theorem 22 and Lemma 13 we conclude that the class of rooted directed path graphs is effectively delineated.

5. Delineation of other geometric graph classes

In this section we consider some further geometric graph classes and prove that they are not delineated.

5.1. Visibility graphs of simple polygons are not delineated

A *visibility graph* of a simple polygon is a graph where the vertices are the same as the polygon's vertices, and an edge exists between two vertices in the graph if the line segment connecting them lies entirely inside the polygon. We exhibit a family of simple polygons whose visibility graphs are transduction equivalent to the class of all bipartite subcubic graphs $\mathcal{B}_{\leqslant 3}$, and conclude by Lemma 14. This shows something apriori stronger than that visibility graphs of simple polygons are not delineated.

Theorem 32. *There is a class $\mathcal{P}$ of visibility graphs of simple polygons which is transduction equivalent to $\mathcal{B}_{\leqslant 3}$. In particular, the class of visibility graphs of simple polygons is not delineated.*

Proof. The class $\mathcal{P}$ consists of each image of a bipartite subcubic graph $G = (A, B, E(G))$ by the following transformation Π. Let $A = \{a_1, \ldots, a_s\}$ and $B = \{b_1, \ldots, b_t\}$. For each vertex $a_i \in A$, we create the $2d(a_i) + 1 \in \{1, 3, 5, 7\}$ first vertices of the list: $d_i, p_i, d'_i, p'_i, d''_i, p''_i, d'''_i$. For each vertex $b_j \in B$, we create a vertex q_j. Let D be the set of vertices of the form d_i, d'_i, d''_i, d'''_i, let P be the set of vertices of the form p_i, p'_i, p''_i, and $Q = \{q_j \ : \ b_j \in B\}$.

We make p_i (resp. p'_i, resp. p''_i) adjacent to q_j such that b_j is the neighbor of a_i with largest (resp. second largest, resp. third largest) index (if they exist). We make d_i adjacent to p_i, d'_i adjacent to p_i

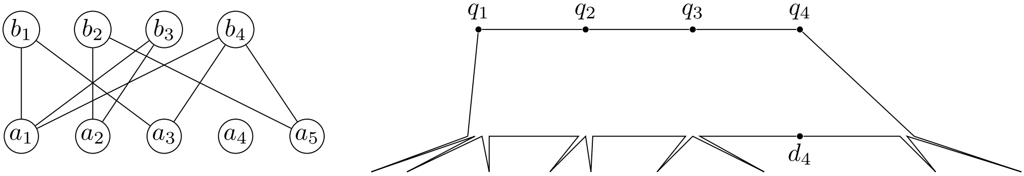


Fig. 5. The transformation Π and how its images are representable by simple polygons.

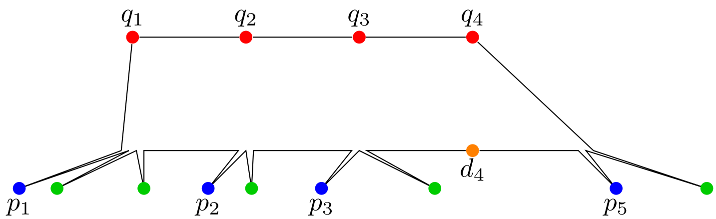


Fig. 6. The desired interpretation of unary relations A in blue, A' in green, B in red, and one vertex of C in orange while the remaining 13 vertices are also in C. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and p_i', d_i'' adjacent to p_i' and p_i'', and d_i''' adjacent to p_i'' (if they exist). Finally we turn $D \cup Q$ into a clique. This finishes the construction of $\Pi(G)$. Fig. 5 shows how to represent, for every $G \in \mathcal{B}_{\leqslant 3}$, the graph $\Pi(G)$ as the visibility graph of a simple polygon.

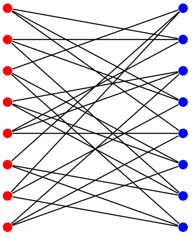
We now prove that there is a first-order transduction T_1 satisfying $\mathsf{T}_1(\mathcal{P}) \supset \mathcal{B}_{\leqslant 3}$, which in particular implies that $\mathcal{P}$ has unbounded twin-width by Theorem 8 and the second item of Theorem 9. Let G be any graph in $\mathcal{B}_{\leqslant 3}$ and $\Pi(G) \in \mathcal{P}$. We want to design an FO transduction T_1 that undoes Π.

The transduction T_1 adds four unary relations A, A', B, C to its input graph $\Pi(G)$. In the run we are interested in, they form a partition of the vertex set, where B colors the vertices q_j, A colors the vertices p_i, A' colors the vertices p_i' and p_i'', and C colors the remaining vertices (those in D); see Fig. 6. The first-order unary formula that defines the vertex set is $\nu_1(x) = A(x) \vee B(x)$. The new edge set is defined by:

$$
\begin{aligned}
\varphi_1(x,y) = A(x) \wedge B(y) \wedge \Bigg(& E(x,y) \\
& \vee \Bigg(\exists z_1 \exists x_1\ C(z_1) \wedge A'(x_1) \wedge E(x,z_1) \wedge E(z_1,x_1) \wedge \Bigg(E(x_1,y) \\
& \vee \big(\exists z_2 \exists x_2\ C(z_2) \wedge A'(x_2) \wedge E(x_1,z_2) \wedge E(z_2,x_2) \wedge E(x_2,y) \big) \Bigg) \Bigg) \Bigg).
\end{aligned}
$$

T_1 indeed creates the bipartite graph $G \in \mathcal{B}_{\leqslant 3}$ back, since vertices in C are adjacent to at most two vertices in $A \cup A'$ corresponding to the same vertex in G.

We now want to build every graph $\Pi(G) \in \mathcal{P}$ by applying a transduction T_2 to a bipartite subcubic graph. We just need to observe that $\Pi(G)$ deprived of the clique on $D \cup Q$ is itself a bipartite subcubic graph, say H. The transduction T_2 inputs H and adds one unary relation U, which in the correct run corresponds to $D \cup Q$. The unary relation ν_2 does not change the domain, while the binary relation $\varphi_2(x,y)$ adds all edges between pairs in U. $\square$



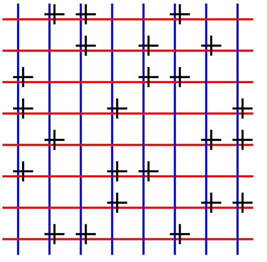


Fig. 7. A bipartite 3-regular graph on the left and a segment representation of its 2-subdivision augmented by bicliques between its two independent sets on the right.

5.2. Axis-parallel segment graphs with segments of two lengths are not delineated

Similar to the previous section, we provide a class of axis-parallel segments of two length whose intersection graphs are transduction equivalent to the class of bipartite subcubic graphs $\mathcal{B}_{\leqslant 3}$, which implies that this class is not delineated by Lemma 14.

Theorem 33. *There is a class $\mathcal{C}$ of two-lengthed axis-parallel segment graphs which is transduction equivalent to the class $\mathcal{B}_{\leqslant 3}$ of subcubic bipartite graphs. In particular, the class of two-lengthed axis-parallel segment graphs is not delineated.*

Proof. We let $\mathcal{C}$ be class of 2-subdivision of any subcubic bipartite graph, augmented with the biclique between its two partite sets. Note that this class is realizable with axis-parallel segments of two different lengths (see Fig. 7).

To see that $\mathcal{C}$ transduces $\mathcal{B}_{\leqslant 3}$ we give a transduction T defined on $\mathcal{C}$. Firstly, use three extra unary relations A, B, C which partition the vertex set of some $G \in \mathcal{C}$ into the long vertical segments, the long horizontal segments, and the short segments, respectively. Then the transduction fixes the new domain by $\nu(x) \equiv A(x) \vee B(x)$ and edge set by

$$\varphi(x, y) \equiv A(x) \ \wedge \ B(y) \ \wedge \ \exists z_1 \exists z_2 \ (C(z_1) \ \wedge \ C(z_2) \ \wedge \ E(x, z_1) \ \wedge \ E(z_1, z_2) \wedge \ E(z_2, y)).$$

We now obtain the original subcubic bipartite graph prior 2-subdivision and augmentation by a biclique and hence $\mathcal{B}_{\leqslant 3} \subseteq T(\mathcal{C})$.

Conversely, $\mathcal{B}_{\leqslant 3}$ transduces $\mathcal{C}$. Just observe that graphs G of $\mathcal{C}$ are bipartite, and admits a partition in (A_1, A_2, B_1, B_2) such that $(A_1 \cup A_2, B_1 \cup B_2)$ is their bipartition, $A_1 \cup B_1$ induces a biclique, every vertex in $A_2 \cup B_2$ has at most two neighbors, and every vertex of A_1 (resp. B_1) has at most three neighbors in B_2 (resp. A_2). Thus if one removes the biclique between A_1 and B_1, one obtains a graph $G' \in B_{\leqslant 3}$. Given G', the transduction then guesses A_1 and B_1 with two unary relations, and adds all the edges in between them. □

6. Conclusion

We have considered delineation of intersection graphs of intervals of the real line or more generally intersection graphs of paths in a tree and given a comprehensive picture of where the boundary between delineated classes and classes that are not delineated lies. For the cases considered, a linear ordering of the graph is naturally derived from the intersection model. Therefore, a good direction for future research is to consider other geometric graph classes which often can be associated with a similarly natural order derived from their geometric representation. We give some sporadic examples of classes that are not delineated, in particular, visibility graphs of simple polygons and axis-parallel segment graphs with only segments of two fixed length, but remark that this direction remains mostly unexplored.

We wonder whether the classes of visibility graphs of 1.5D terrains and unit segment graphs are delineated.

Our way of refuting delineation was to show transduction equivalence to subcubic graphs. We ask whether this method is complete for hereditary classes.

Conjecture 34. *A hereditary class is not delineated if and only if it is transduction equivalent to all subcubic graphs.*

Indeed, this conjecture was refuted [11, Corollary 20.10].

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