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RESEARCH

Port-Hamiltonian modeling of rigid multibody systems

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Abstract We employ a port-Hamiltonian approach to model nonlinear rigid multibody systems subject to both position and velocity constraints. Our formulation accommodates Cartesian and redundant coordinates, respectively, and captures kinematic as well as gyroscopic effects. The resulting equations take the form of nonlinear differential-algebraic equations that inherently preserve an energy balance. We show that the proposed class is closed under interconnection, and we provide several examples to illustrate the theory.

Keywords Port-Hamiltonian systems · Multibody systems · Position and velocity constraints · Dirac structures · Lagrangian submanifolds · Resistive relations · Differential-algebraic equations

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1 Introduction

Port-Hamiltonian system models encompass a broad class of nonlinear physical systems [1,2] and originate from port-based network modeling of complex dynamical systems in various physical domains, such as mechanical and electrical systems [1,3]. Modeling with port-Hamiltonian systems has garnered significant attention, and considerable progress has recently been made in port-Hamiltonian modeling of constrained dynamical systems, leading to differential-algebraic equations (DAEs) [3–10]. In particular, the extension of the port-Hamiltonian framework to implicit energy storage [7,8,11,12] enables the modeling of a significantly broader range of physical systems with constraints, such as electrical circuits [13].

An inherent feature of port-Hamiltonian systems, in addition to their ability to maintain an energy balance, is the incorporation of a modular principle. This principle is built upon an elegant theory of power-preserving coupling, resulting once more in a port-Hamiltonian system [1,14–16]. Historically, the theory of port-Hamiltonian systems has been developed based on the observation that principles from rational mechanics (see e.g. [17]) apply to various physical domains. Over time, however, the theory has advanced well beyond its mechanical origins and has been further

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developed as an independent framework. In this article, we take a step back and analyze rigid multibody systems through the lens of port-Hamiltonian theory.

Multibody systems [18–20] constitute a modern computer-oriented modeling approach for mechanical systems that undergo large rigid body translational and rotational motion. This approach is widely used in fields such as machine dynamics, vehicle dynamics, robotics or biomechanics. Multibody systems consist of rigid bodies which are connected by joints and bearings as well as coupling elements such as springs and dampers. Due to occurring finite rotational motions, the systems are inherently nonlinear. As a special case, multibody systems also include vibration systems of rigid bodies, which typically undergo only small linear motions.

The Hamiltonian approach is typically formulated in terms of state variables consisting of position and momentum of each body. However, in the literature on mechanical systems, it is more common to use velocity – a so-called co-energy variable – instead of momentum. To bridge this gap, we employ the theory presented in [8], which addresses implicit energy storage. This approach also facilitates the incorporation of both position and velocity constraints. Another advantage of the presented modeling approach for mechanical systems is that numerous time discretization methods have been developed for general port-Hamiltonian systems, which specifically aim at preserving the energy balance [21–25]. Using the framework presented here, these methods can then also be applied to mechanical systems.

We begin by formulating translational motion in the port-Hamiltonian framework, considering systems of point masses with positions and velocities expressed in Cartesian coordinates. While this approach does not require much conceptual effort, it is limited from a practical perspective: when spatially expanded (in particular rotating) rigid bodies are involved, their orientation must also be incorporated into the model. In this case, the nonlinear relationship between angular velocity and the rotational parameters that determine the body's orientation must be taken into account. This introduces an additional layer of complexity compared to purely translational motion, which is also explored in this article within the port-Hamiltonian framework.

This article is structured as follows: After establishing basic notations, we provide a concise introduction to port-Hamiltonian systems in Sect. 2. Subsequently, we consider multibody systems within this framework: point masses in Cartesian coordinates are discussed in Sect. 3. Multibody systems in redundant coordinates are considered in Sect. 4 while Sect. 5 expands on the use of rotation matrices. The interconnection of two systems via their ports is introduced in Sect. 6 and it is shown that the class of port-Hamiltonian systems is closed under interconnection. A selection of illustrative examples are discussed in Sect. 7 and Conclusions are given in Sect. 8. Some additional proofs as well as auxiliary results on Dirac structures and Lagrangian submanifolds are collected in Appendices A and B.

1.1 Notation

Throughout this article, the space $\mathbb{R}^n$ will be identified with matrices of size $n \times 1$. In particular, the Euclidean inner product of $z_1, z_2 \in \mathbb{R}^n$ is given by $z_1^\top z_2$.

The vector space framework is often insufficient for describing physical systems, particularly those involving ideal constraints. Instead, the state evolution is represented within a *manifold*, a topological space that, at least locally, resembles the structure of a real vector space. This concept is further discussed in [26] in conjunction with port-Hamiltonian systems. For our purposes, it is adequate to consider the slightly simpler notion of submanifolds of a finite dimensional space $\mathbb{R}^n$.

Definition 1 (Submanifold of $\mathbb{R}^n$) Let $\mathcal{M} \subset \mathbb{R}^n$ be a subset. Then $\mathcal{M}$ is called *differentiable submanifold of* $\mathbb{R}^n$ (in this article just *submanifold of* $\mathbb{R}^n$ for sake of brevity), if for all $x \in \mathcal{M}$, there exists a neighborhood $U_x \subset \mathbb{R}^n$ of x, a number $k \in \mathbb{N}$, and a continuously differentiable mapping $f_x : U_x \to \mathbb{R}^k$, such that $f_x'(x)$ is surjective, and

$$\mathcal{M} \cap U_x = \{\, y \in U_x \mid f_x(y) = 0 \,\}.$$

For a submanifold $\mathcal{M} \subset \mathbb{R}^n$ and $x \in \mathcal{M}$, the implicit function theorem implies the existence of a number $k \in \mathbb{N}$, a neighborhood $V_x \subset \mathbb{R}^k$ of zero, and a continuously differentiable mapping $g_x : V_x \to \mathbb{R}^n$ satisfying the conditions $g_x(0) = x$, $g_x(V_x) \subset \mathcal{M}$ and $g_x'(0)$ has a trivial kernel. Such a mapping g_x possessing these properties is termed a *chart of* $\mathcal{M}$ *at* x.

Next we introduce the concept of the tangent space. To this end, we utilize continuously differentiable curves on $\mathcal{M}$, i.e., continuously differentiable mappings $x : (-1, 1) \to \mathbb{R}^n$ with $x(t) \in \mathcal{M}$ for all

$t \in (-1, 1)$. Defining the concept of the tangent space can be quite abstract in a general differential geometric context. However, the situation becomes significantly simpler when dealing with submanifolds of $\mathbb{R}^n$. Loosely speaking, the tangent space $T_{x_0}\mathcal{M}$ at $x_0 \in \mathcal{M}$ consists of all vectors that can be obtained by taking derivatives of smooth curves within $\mathcal{M}$ that pass through x_0.

Definition 2 (Tangent space of a submanifold) Let $\mathcal{M} \subset \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$. The *tangent space* of $\mathcal{M}$ at $x_0 \in \mathcal{M}$ is defined by

$$T_{x_0}\mathcal{M} = \left\{ v \in \mathcal{W} \,\middle|\, \begin{array}{l} \text{there exists a continuously differentiable curve } x : (-1, 1) \to \mathcal{M} \\ \text{with } x(0) = x_0 \text{ and } \dot{x}(0) = v \end{array} \right\}.$$

The elements of $T_{x_0}\mathcal{M}$ are called *tangent vectors of $\mathcal{M}$ at x_0*.

It can be observed from the definition of the tangent space that, for a chart $g_x : V_x \to \mathbb{R}^n$ of $\mathcal{M}$ at x, we obtain $\operatorname{im} g'_x(0) = T_x\mathcal{M}$. In particular, $T_x\mathcal{M}$ is a subspace.

2 Port-Hamiltonian systems

We recall some fundamental concepts in port-Hamiltonian systems from [7,8]. An essential concept to grasp is the notion of a *Dirac structure*, which describes the power-preserving energy routing within the system.

Definition 3 (Dirac structure) A subspace $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^n$ is called a *Dirac structure*, if for all $f, e \in \mathbb{R}^n$, we have

$$(f, e) \in \mathcal{D} \iff \forall (\hat{f}, \hat{e}) \in \mathcal{D} : \ \hat{f}^\top e + f^\top \hat{e} = 0.$$

For $(f, e) \in \mathcal{D}$, we call e an *effort* and f is termed a *flow*. By equipping $\mathbb{R}^n \times \mathbb{R}^n$ with the indefinite inner product

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle : \left(\mathbb{R}^n \times \mathbb{R}^n\right) \times \left(\mathbb{R}^n \times \mathbb{R}^n\right) \to \mathbb{R},$$
$$\left((f_1, e_1), (f_2, e_2)\right) \mapsto \langle f_1, e_2 \rangle + \langle f_2, e_1 \rangle,$$

we can conclude that $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^n$ is a Dirac structure if, and only if, $\mathcal{D} = \mathcal{D}^{\perp\!\!\perp}$, where the latter denotes the orthogonal complement of $\mathcal{D}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Any Dirac structure $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^n$ is n-dimensional. Furthermore, for matrices $K, L \in \mathbb{R}^{n \times n}$, $\mathcal{D} = \operatorname{im} \begin{bmatrix} K & L \end{bmatrix}$ is a Dirac structure if, and only if, $\operatorname{rank} \begin{bmatrix} K & L \end{bmatrix} = n$ and $KL^\top + LK^\top = 0$, see [27, Prop. 1.1.5].

Dirac structures are oftentimes insuffcent for being a component in port-Hamiltonian modeling of rigid multibody systems. Therefore, we require the more comprehensive concept of a *modulated Dirac structure* which is, loosely speaking, a family of Dirac structures depending on a parameter.

In a more general setting, a modulated Dirac structure on a manifold $\mathcal{M}$ is defined as a certain subbundle of $T\mathcal{M} \oplus T'\mathcal{M}$ (representing the sum of the tangent bundle and cotangent bundle of $\mathcal{M}$) [27, Def. 2.2.1], as discussed in [1]. This general manifold setup is not necessary for the class of multibody systems discussed in the present article.

Definition 4 (Modulated Dirac structure) Let $U \subset \mathbb{R}^k$ be open. A family $(\mathcal{D}_x)_{x \in U}$ of subspaces of $\mathbb{R}^n \times \mathbb{R}^n$ is called a *modulated Dirac structure*, if the following holds for all $x \in U$:

(a) $\mathcal{D}_x$ is a Dirac structure.
(b) There exists a neighborhood $U_x \subset U$ of x and a family $(T_y)_{y \in U_x}$ of linear and bijective mappings $T_y : \mathbb{R}^n \to \mathcal{D}_y$, such that, for all $z \in \mathbb{R}^n$, the mapping $y \mapsto T_y z$ is continuous from U_x to $\mathbb{R}^n \times \mathbb{R}^n$.

The conditions (a) and (b) outlined above imply that a modulated Dirac structure aligns with the definition of a vector bundle, as described in [28, Chap. III, § 1], where (b) is referred to as *local trivialization*.

Next, we introduce a relation that describes the energy storage of the system, known as a *Lagrangian submanifold*. We note that the general definition of this concept, as found in [29, p. 568], is not required for the systems considered in this work. Instead of dealing with submanifolds of general manifolds, it suffices to consider submanifolds of $\mathbb{R}^n \times \mathbb{R}^n$. Typically, these manifolds are assumed to be smooth in the sense that the charts are infinitely often differentiable. However, for our purposes, we can relax this assumption and consider less smooth submanifolds.

Definition 5 (Lagrangian submanifold) A submanifold $\mathcal{L} \subset \mathbb{R}^n \times \mathbb{R}^n$ is called *Lagrangian submanifold* of $\mathbb{R}^n \times \mathbb{R}^n$, if for all $z \in \mathcal{L}$ and $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$(v_1, v_2) \in T_z\mathcal{L} \iff \forall (w_1, w_2) \in T_z\mathcal{L}: \; v_1^\top w_2 - w_1^\top v_2 = 0. \tag{1}$$

In [29, Prop. 22.12], it is shown that for a smooth function $H : U \to \mathbb{R}^n$ defined on a simply connected domain $U \subset \mathbb{R}^n$, the set defined as

$$\mathcal{L} := \{ (x, H(x)) \mid x \in U \} \subset \mathbb{R}^n \times \mathbb{R}^n$$

is a Lagrangian submanifold if, and only if, H is a gradient field. That is, $\nabla \mathcal{H} = H$ for some smooth function $\mathcal{H} : U \to \mathbb{R}$. In Appendix B, we present a generalization of this result, which will be used in the context of position constraints in mechanical systems. Additionally, Appendix B includes some results on special sets that form a modulated Dirac structure.

Another essential concept for port-Hamiltonian systems is that of a *(modulated) resistive relation*, which characterizes the internal energy dissipation within the system. This relation is defined on $\mathbb{R}^n \times \mathbb{R}^n$. The components of $(f_\mathcal{R}, e_\mathcal{R}) \in \mathcal{R}$ are called *resistive flows* $f_\mathcal{R}$ and *resistive efforts* $e_\mathcal{R}$, resp. [1, Sec. 2.4].

Definition 6 ((Modulated) resistive relation) A relation $\mathcal{R} \subset \mathbb{R}^n \times \mathbb{R}^n$ is called *resistive*, if

$$\forall \, (f_\mathcal{R}, e_\mathcal{R}) \in \mathcal{R} : \; f_\mathcal{R}^\top e_\mathcal{R} \geq 0.$$

Let $U \subset \mathbb{R}^k$ be open. A family $(\mathcal{R}_x)_{x \in U}$ is called *modulated resistive relation*, if $\mathcal{R}_x \subset \mathbb{R}^n \times \mathbb{R}^n$ is a resistive relation for all $x \in U$.

Having established the definitions of (modulated) Dirac structures, Lagrangian submanifolds, and (modulated) resistive relations, we are now prepared to introduce port-Hamiltonian systems, cf. also Fig. 1. Again, we note that this class can be defined in a more general setting involving manifolds [1,8]. However, we simplify this to the specific setup required for our class of rigid multibody systems.

Definition 7 (Port-Hamiltonian system) A *port-Hamiltonian (pH) system* is a differential inclusion

$$\left(\begin{pmatrix} \dot{x}(t) \\ f_\mathcal{R}(t) \\ f_\mathcal{P}(t) \end{pmatrix}, \begin{pmatrix} e_\mathcal{L}(t) \\ e_\mathcal{R}(t) \\ e_\mathcal{P}(t) \end{pmatrix} \right) \in \mathcal{D}_{x(t)},$$

$$\big(x(t), e_\mathcal{L}(t)\big) \in \mathcal{L}, \quad (f_\mathcal{R}(t), e_\mathcal{R}(t)) \in \mathcal{R}_{x(t)},$$

where, for $U \subset \mathbb{R}^{n_\mathcal{L}}$ being open,

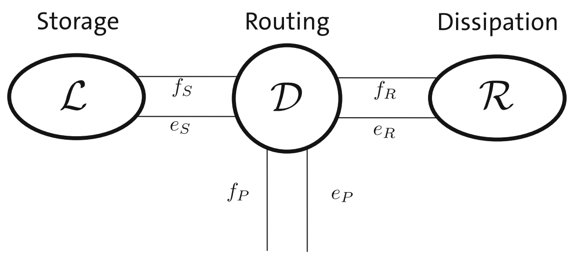

Fig. 1 Visual representation of a port-Hamiltonian system

- $\mathcal{D} = (\mathcal{D}_x)_{x \in U}$ with $\mathcal{D}_x \subset (\mathbb{R}^{n_\mathcal{L}} \times \mathbb{R}^{n_\mathcal{R}} \times \mathbb{R}^{n_\mathcal{P}}) \times (\mathbb{R}^{n_\mathcal{L}} \times \mathbb{R}^{n_\mathcal{R}} \times \mathbb{R}^{n_\mathcal{P}})$ is a modulated Dirac structure (see Definition 3),
- $\mathcal{L} \subset \mathbb{R}^{n_\mathcal{L}} \times \mathbb{R}^{n_\mathcal{L}}$ is a Lagrangian submanifold (see Definition 5), and
- $\mathcal{R} = (\mathcal{R}_x)_{x \in U}$ with $\mathcal{R}_x \subset \mathbb{R}^{n_\mathcal{R}} \times \mathbb{R}^{n_\mathcal{R}}$ is a modulated resistive relation (see Definition 6).

The elements of $\mathbb{R}^{n_\mathcal{L}}$, $\mathbb{R}^{n_\mathcal{R}}$, $\mathbb{R}^{n_\mathcal{P}}$ are, accordingly, called the *energy-storing flows/efforts*, *resistive flows/ efforts* and *external flows/efforts*.

Remark 1 It should be noted that our definition of port-Hamiltonian systems slightly differs from the one e.g. in [1], where the negative of $\dot{x}$ enters the Dirac structure, and the resistive relation fulfills $f_\mathcal{R}^\top e_\mathcal{R} \leq 0$ for all $(f_\mathcal{R}, e_\mathcal{R}) \in \mathcal{R}_x$, $x \in U$. By substituting the flow with its negative, it is straightforward to establish a one-to-one correspondence between these two definitions. In practical terms, one approach views the system from the perspective of a consumer, while the other adopts a producer-centric viewpoint.

Remark 2 (Incorporation of more general spaces) Instead of $\mathbb{R}^{n_\mathcal{L}}$, $\mathbb{R}^{n_\mathcal{R}}$, and $\mathbb{R}^{n_\mathcal{P}}$, one can also consider general finite-dimensional real inner product spaces. This is simply due to the fact that any real inner product space $\mathcal{W}$ with $n := \dim \mathcal{W} < \infty$ is isometrically isomorphic to $\mathbb{R}^n$. This becomes particularly evident when the flow and effort spaces are given, for example, by certain matrix spaces or Cartesian products of matrix spaces with $\mathbb{R}^n$. Indeed, we will encounter such spaces later in this paper.

Note that throughout our considerations in this work, we restrict ourselves to finite-dimensional spaces. Nevertheless, it is worth emphasizing at this point that the inclusion of infinite-dimensional spaces is particularly important in the context of mechanical systems, as it allows for the modeling of flexible structures (such as beams).

3 Newtownian mechanical systems: point masses in Cartesian coordinates

In this section, we consider mechanical systems consisting of point masses, with their positions expressed in conventional Cartesian coordinates, such as in $\mathbb{R}^2$ or $\mathbb{R}^3$. Loosely speaking, we consider multibody systems in which each rigid body has a negligible spatial expansion and possesses both potential and kinetic energy. The position and velocity of all masses are given by the vector-valued functions $\boldsymbol{r}, \boldsymbol{v} : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ denotes the considered time interval. The functions $\boldsymbol{r}$ and $\boldsymbol{v}$ collectively encode the position coordinates and velocities, respectively, of all point masses in the system. Specifically, for a system with p point masses in three-dimensional space, we have $n = 3p$, meaning each mass contributes three coordinates to the overall position vector.

We first present the modulated Dirac structure for this kind of systems. It contains velocity and force balances, as well as constraints formulated on velocity level. These velocity constraints are described by an equation of the form

$$A(\boldsymbol{r}(t))\boldsymbol{v}(t) = 0,$$

where A is a continuous matrix-valued function with constant rank. Note that all or some of these constraints may be *holonomic* – that is, they represent position constraints differentiated with respect to time – if they satisfy an integrability condition. Otherwise, they are referred to as *nonholonomic*.

Let $U_{\text{pos}} \subset \mathbb{R}^n$ be an open set, representing the admissible positions of the point masses. We consider the family that is modulated by the positions, namely $(\mathcal{D}_{\boldsymbol{r}})_{\boldsymbol{r}\in U_{\text{pos}}}$ with

$$\mathcal{D}_{\boldsymbol{r}} := \left\{ \begin{pmatrix} \boldsymbol{v}_{\mathcal{L},f} \\ \boldsymbol{F}_{\mathcal{L},f} \\ \boldsymbol{v}_{\mathcal{R}} \\ \boldsymbol{v}_{\text{ext}} \\ \boldsymbol{F}_{\mathcal{L},e} \\ \boldsymbol{v}_{\mathcal{L},e} \\ \boldsymbol{F}_{\mathcal{R}} \\ \boldsymbol{F}_{\text{ext}} \end{pmatrix} \in \mathbb{R}^{6n+2m} \;\middle|\; \begin{array}{l} \boldsymbol{v}_{\mathcal{L},f} = \boldsymbol{v}_{\mathcal{L},e} = \boldsymbol{v}_{\mathcal{R}},\ A(\boldsymbol{r})\boldsymbol{v}_{\mathcal{L},e} = 0, \\ \boldsymbol{v}_{\text{ext}} = B(\boldsymbol{r})^\top \boldsymbol{v}_{\mathcal{L},e}, \\ \exists\, \boldsymbol{\mu} \in \mathbb{R}^\ell : \\ \boldsymbol{F}_{\mathcal{L},f} + \boldsymbol{F}_{\mathcal{L},e} + \boldsymbol{F}_{\mathcal{R}} + B(\boldsymbol{r})\boldsymbol{F}_{\text{ext}} + A(\boldsymbol{r})^\top \boldsymbol{\mu} = 0 \end{array} \right\}. \tag{2}$$

Before proving that this constitutes a modulated Dirac structure, let us first discuss the practical meaning of the involved vectors and matrices. The identical vectors $\boldsymbol{v}_{\mathcal{L},f}, \boldsymbol{v}_{\mathcal{L},e}, \boldsymbol{v}_{\mathcal{R},e} \in \mathbb{R}^n$ represent the velocities of the point masses. One might wonder why three copies of the velocity appear in the Dirac structure. The reason is that velocity is needed in conjunction with potential energy, kinetic energy, and damping. Furthermore, $\boldsymbol{F}_{\mathcal{L},f} \in \mathbb{R}^n$ represents the inertial force, $\boldsymbol{F}_{\mathcal{R}} \in \mathbb{R}^n$ corresponds to the damping force, and $\boldsymbol{F}_{\mathcal{L},e} \in \mathbb{R}^n$ denotes a restoring force. The vectors $\boldsymbol{F}_{\text{ext}}, \boldsymbol{v}_{\text{ext}} \in \mathbb{R}^m$ collect the forces and velocities at the external ports, respectively. Here, the matrix $B : U_{\text{pos}} \to \mathbb{R}^{n\times m}$ describes the direction of the external forces. Moreover, as mentioned above, $A : U_{\text{pos}} \to \mathbb{R}^{\ell\times n}$ encodes the velocity constraints. The term $A(\boldsymbol{r})^\top \boldsymbol{\mu}$ represents the forces that ensure the fulfillment of the velocity constraints.

The following result states that $(\mathcal{D}_{\boldsymbol{r}})_{\boldsymbol{r}\in U_{\text{pos}}}$ is a modulated Dirac structure. We do not present a proof at this point, but instead refer to Proposition 2, which establishes a more general statement that encompasses the one below; we emphasize that Proposition 2 does not rely on any results from this section.

Proposition 1 *Let $U_{\text{pos}} \subset \mathbb{R}^n$ be open, let $A : U_{\text{pos}} \to \mathbb{R}^{\ell\times n}$, $B : U_{\text{pos}} \to \mathbb{R}^{n\times m}$ be continuous, and assume that A has constant rank on U_{pos}. Then the family $(\mathcal{D}_{\boldsymbol{r}})_{\boldsymbol{r}\in U_{\text{pos}}}$ with $\mathcal{D}_{\boldsymbol{r}}$ as in (2) is a modulated Dirac structure.*

The damping elements are modeled using a modulated resistive structure. In the most general case, damping is described by a dissipative relation between the damping force and velocity. A typical application scenario is the consideration of kinetic friction (see [30,31]). A dependence of this relation on the position vector $\boldsymbol{r}$ may arise in non-homogeneous setups. For example, this can occur when a car moves on a road with varying surface properties.

Let us assume a damping model that is slightly simpler than the relational setup described above: Specifically, for $U_{\text{d}} \subset U_{\text{pos}} \times \mathbb{R}^n$, we consider a function $\boldsymbol{F}_{\text{d}} : U_{\text{d}} \to \mathbb{R}^n$ with

$$\forall\,(\boldsymbol{r},\boldsymbol{v})\in U_{\mathrm{d}}:\quad \boldsymbol{v}^\top \boldsymbol{F}_{\mathrm{d}}(\boldsymbol{r},\boldsymbol{v})\geq 0.$$

Then it follows immediately that $\mathcal{R}=(\mathcal{R}_{\boldsymbol{r}})_{\boldsymbol{r}\in U_{\mathrm{pos}}}$ with

$$\mathcal{R}_{\boldsymbol{r}} := \left\{ \begin{pmatrix}\boldsymbol{F}\\ \boldsymbol{v}\end{pmatrix}\in\mathbb{R}^{2n} \;\middle|\; \boldsymbol{F}=\boldsymbol{F}_{\mathrm{d}}(\boldsymbol{r},\boldsymbol{v}) \right\} \tag{3}$$

is a modulated resistive relation.

The remaining key component is the Lagrangian submanifold, which is, in the context of mechanical systems, responsible for storage of kinetic and potential energy. In particular, it encodes the restoring and inertial forces. Let $M\in\mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix (representing the mass matrix), $\mathcal{V}_{\mathrm{pot}}: U_{\mathrm{pos}}\to\mathbb{R}$ a twice continuously differentiable mapping (representing the potential energy), and $c: U_{\mathrm{pos}}\to\mathbb{R}^k$ a twice continuously differentiable function (representing the holonomic constraints on the position r), which vanishes on a non-empty subset of U_{pos} and whose Jacobian satisfies $\operatorname{rank} c'(\boldsymbol{r})=k$ for all $\boldsymbol{r}\in U_{\mathrm{pos}}$. We then define

$$\mathcal{L} := \left\{ \begin{pmatrix}\boldsymbol{r}\\ \boldsymbol{p}\\ \boldsymbol{F}\\ \boldsymbol{v}\end{pmatrix}\in U_{\mathrm{pos}}\times\mathbb{R}^{3n} \;\middle|\; \begin{array}{l} M\boldsymbol{v}=\boldsymbol{p},\; c(\boldsymbol{r})=0,\\ \exists\,\boldsymbol{\lambda}\in\mathbb{R}^k:\;\boldsymbol{F}=\nabla\mathcal{V}_{\mathrm{pot}}(\boldsymbol{r})+c'(\boldsymbol{r})^\top\boldsymbol{\lambda}\end{array} \right\}. \tag{4}$$

Here, $\boldsymbol{p}$ stands for the momenta and $\nabla\mathcal{V}_{\mathrm{pot}}(\boldsymbol{r})$ is the restoring force caused by potential energy storage. The term $c'(\boldsymbol{r})^\top\boldsymbol{\lambda}$ represents the forces that ensure the fulfillment of the position constraints.

We can conclude from Proposition 5 in Appendix B (by setting $\mathcal{H}(\boldsymbol{r},\boldsymbol{p})=\frac{1}{2}\boldsymbol{p}^\top M^{-1}\boldsymbol{p}+\mathcal{V}_{\mathrm{pot}}(\boldsymbol{r})$ and $d(\boldsymbol{r},\boldsymbol{p})=c(\boldsymbol{r})$) that $\mathcal{L}$ as in (4) is a Lagrangian submanifold. Indeed, it is even a Lagrangian submanifold, if M is only positive semi-definite. This follows by an application of Proposition 5 with $\mathcal{H}(\boldsymbol{r},\boldsymbol{p})=\frac{1}{2}\boldsymbol{p}^\top M^{+}\boldsymbol{p}+\mathcal{V}_{\mathrm{pot}}(\boldsymbol{r})$, where $M^{+}\in\mathbb{R}^{n\times n}$ is the Moore-Penrose-inverse of M, and

$$d(\boldsymbol{r},\boldsymbol{p})=\begin{pmatrix}c(\boldsymbol{r})\\ W\boldsymbol{p}\end{pmatrix},$$

where, for $r=\operatorname{rank} M$, $W\in\mathbb{R}^{(n-r)\times n}$ is a matrix with $\operatorname{im} W^\top=\ker M$.

The overall port-Hamiltonian model according to Definition 7 is, for $(\mathcal{D}_{\boldsymbol{r}})_{\boldsymbol{r}\in U_{\mathrm{pos}}}$ as in (2), $\mathcal{L}$ as in (4), and $(\mathcal{R}_{\boldsymbol{r}})_{\boldsymbol{r}\in U_{\mathrm{pos}}}$ as in (3), given by

$$\begin{pmatrix}\dot{\boldsymbol{r}}(t)\\ \dot{\boldsymbol{p}}(t)\\ \boldsymbol{v}_{\mathcal{R}}(t)\\ \boldsymbol{v}_{\mathrm{ext}}(t)\\ \boldsymbol{F}(t)\\ \boldsymbol{v}(t)\\ \boldsymbol{F}_{\mathcal{R}}(t)\\ \boldsymbol{F}_{\mathrm{ext}}(t)\end{pmatrix}\in\mathcal{D}_{\boldsymbol{r}(t)},\quad \begin{pmatrix}\boldsymbol{F}_{\mathcal{R}}(t)\\ \boldsymbol{v}_{\mathcal{R}}(t)\end{pmatrix}\in\mathcal{R}_{\boldsymbol{r}(t)},\quad \begin{pmatrix}\boldsymbol{r}(t)\\ \boldsymbol{p}(t)\\ \boldsymbol{F}(t)\\ \boldsymbol{v}(t)\end{pmatrix}\in\mathcal{L}.$$

The first relation gives

$$\begin{aligned}
\dot{\boldsymbol{r}}(t)&=\boldsymbol{v}(t),\\
\dot{\boldsymbol{p}}(t)&=-\boldsymbol{F}(t)-\boldsymbol{F}_{\mathcal{R}}(t)-B(\boldsymbol{r}(t))\boldsymbol{F}_{\mathrm{ext}}(t)\\
&\quad -A(\boldsymbol{r}(t))^\top\boldsymbol{\mu}(t),\\
0&=A(\boldsymbol{r}(t))\dot{\boldsymbol{r}}(t),\\
\boldsymbol{v}(t)&=\boldsymbol{v}_{\mathcal{R}}(t),\\
\boldsymbol{v}_{\mathrm{ext}}(t)&=B(\boldsymbol{r}(t))^\top\boldsymbol{v}(t).
\end{aligned}\tag{5}$$

Now incorporating the Lagrangian submanifold and the modulated resistive relation, we have

$$\begin{aligned}
M\boldsymbol{v}(t)&=\boldsymbol{p}(t),\\
\boldsymbol{F}(t)&=\nabla\mathcal{V}_{\mathrm{pot}}(\boldsymbol{r}(t))+c'(\boldsymbol{r}(t))^\top\boldsymbol{\lambda}(t),\\
0&=c(\boldsymbol{r}(t)),\\
\boldsymbol{F}_{\mathcal{R}}(t)&=\boldsymbol{F}_{\mathrm{d}}(\boldsymbol{r}(t),\boldsymbol{v}_{\mathcal{R}}(t)),
\end{aligned}$$

and inserting this into (5), we obtain the overall model

$$\begin{aligned}
\dot{\boldsymbol{r}}(t)&=\boldsymbol{v}(t),\\
\tfrac{\mathrm{d}}{\mathrm{d}t}M\boldsymbol{v}(t)&=-\nabla\mathcal{V}_{\mathrm{pot}}(\boldsymbol{r}(t))-\boldsymbol{F}_{\mathrm{d}}(\boldsymbol{r}(t),\boldsymbol{v}(t))\\
&\quad -c'(\boldsymbol{r}(t))^\top\boldsymbol{\lambda}(t)-A(\boldsymbol{r}(t))^\top\boldsymbol{\mu}(t)\\
&\quad -B(\boldsymbol{r}(t))\boldsymbol{F}_{\mathrm{ext}}(t),\\
0&=c(\boldsymbol{r}(t)),\\
0&=A(\boldsymbol{r}(t))\boldsymbol{v}(t),\\
\boldsymbol{v}_{\mathrm{ext}}(t)&=B(\boldsymbol{r}(t))^\top\boldsymbol{v}(t).
\end{aligned}\tag{6}$$

We note that the total energy

$$\mathcal{H}(\boldsymbol{r},\boldsymbol{v})=\tfrac{1}{2}\boldsymbol{v}^\top M\boldsymbol{v}+\mathcal{V}_{\mathrm{pot}}(\boldsymbol{r})$$

fulfills, along the solutions of (6),

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\boldsymbol{r}(t),\boldsymbol{v}(t)) = &-\boldsymbol{v}_{\mathrm{ext}}(t)^\top \boldsymbol{F}_{\mathrm{ext}}(t)\\ &-\boldsymbol{v}(t)^\top \boldsymbol{F}_{\mathrm{d}}(\boldsymbol{r}(t),\boldsymbol{v}(t))\\ \leq &-\boldsymbol{v}_{\mathrm{ext}}(t)^\top \boldsymbol{F}_{\mathrm{ext}}(t).\end{aligned}$$

That is, $\boldsymbol{v}_{\mathrm{ext}}(t)^\top \boldsymbol{F}_{\mathrm{ext}}(t)$ can be regarded as the power extracted from the system, whereas $\boldsymbol{v}(t)^\top \boldsymbol{F}_{\mathrm{d}}(\boldsymbol{r}(t),\boldsymbol{v}(t))$ is the dissipated power.

4 Multibody systems: redundant coordinates

In this section, we consider a much more general class of multibody systems involving rigid bodies. Unlike point masses, the mass of a rigid body is distributed over its volume. Since the mass distribution is constant within a body-fixed coordinate frame, a full description of a rigid body requires the position of that frame's origin (three parameters per body in space, oftentimes the center of mass) along with the orientation of its basis vectors (at least three parameters per body in space, of which only three are independent).

The most general representation for the orientation is the special orthogonal group $\mathbb{SO}_3$. It utilizes nine parameters along with six unique constraint equations to describe the body's orientation yielding a total of 12 spacial coordinates (six in planar motion). It is noted, however, that these rotation matrices can also be expressed in terms of a different parameterization such as unit-quaternions or Euler angles to reduce the number of parameters and thus also reduce or eliminate the number of constraints. These parameterizations also fit within the port-Hamiltonian framework and have different advantages [32,33]. The approach presented in this section contains all these different ways to describe rotations. In Sect. 5 some comments about the use of $\mathbb{SO}_3$ are given.

In addition to the aforementioned constraints resulting from the orientation parametrization, the bodies of the multibody system are usually subject to geometric constraints. Since there are more coordinates available than necessary to describe the system's kinematics uniquely, these coordinates are called *redundant coordinates*. In contrast, generalized coordinates are the minimal number (corresponding to the system's degrees of freedom) of coordinates to uniquely describe the kinematics [18]. Generalized coordinates are used e.g. in the Newton-Euler formalism or the Lagrangian equation of 2nd order, yielding the equations of motion as an ordinary differential equation. In contrast, the use of redundant coordinates yields the equation of motion as a differential algebraic equation. Both approaches have case-dependent advantages and disadvantages.

In the following, redundant coordinates are used to describe rigid multibody systems as port-Hamiltonian systems. With this approach, the kinematic relations between angular velocity and rotational parameters and the appearance of gyroscopic forces must be considered. The latter appear when the translational and angular velocities are defined within a body-fixed coordinate frame. This approach is used to maintain a constant mass matrix.

Subsequently, we use the following notation: Note

$\boldsymbol{\zeta}$:	Global positions, redundant coordinates
$\boldsymbol{\Gamma}$:	Global momenta,
$\boldsymbol{\varpi}$:	Global velocities, velocity coordinates,
$\boldsymbol{\tau}$:	Global forces.

that there is no universally agreed-upon standard notation for this in the literature. The global positions $\boldsymbol{\zeta}$, i.e. the redundant coordinates of all bodies, collect all positions (as discussed in the previous section) and orientation parameters of all rigid bodies. Consequently, $\boldsymbol{\Gamma}$ may encompass both linear and angular momenta of all bodies, $\boldsymbol{\varpi}$ may consist of both linear and angular velocities of all bodies, and $\boldsymbol{\tau}$ may include both forces and torques. The collection of the quantities of all bodies is reflected by the term *global*.

Again, we start with a function $\mathcal{V}_{\mathrm{pot}} : U_{\mathrm{pos}} \to \mathbb{R}$ representing the potential energy, where $U_{\mathrm{pos}} \subset \mathbb{R}^{n_{\mathrm{pot}}}$ is open. It is reasonable to assume that, in general, the number n_{pot} of variables determining the potential energy may differ from the number n_{kin} of variables governing the kinetic energy. This is evident in spring-mass-damper systems [34], where each spring serves as a potential energy storage element, while each mass acts as a kinetic energy storage element. The kinetic energy is given by the term $\frac{1}{2}\boldsymbol{\varpi}^\top M\boldsymbol{\varpi}$, again with $M \in \mathbb{R}^{n_{\mathrm{kin}}\times n_{\mathrm{kin}}}$ being symmetric. This matrix incorporates inertia effects of all bodies, which is why it is referred to as the *global mass matrix*. For instance, it includes masses but may also contain inertia tensors. As in the case of systems in Cartesian coordinates, assuming positive semi-definiteness is reasonable from a practical perspective, although it does not mathematically contribute to the findings of this article. Note that positive semi-definiteness becomes relevant when studying the existence of global solutions, which is not addressed here.

For now, we set aside considerations of energy and focus on the structural relationships between the involved global velocities and global forces. These relationships are governed by a Dirac structure, which generalizes the one in (2) in several ways: First, a kinematic relation between the velocities associated with kinetic and potential energies is incorporated. This is represented by the position-dependent matrix $Z : U_{\text{pos}} \to \mathbb{R}^{n_{\text{pot}} \times n_{\text{kin}}}$. Additionally, gyroscopic forces are introduced, which are captured by the pointwise skew-symmetric matrix $G : \mathbb{R}^{n_{\text{kin}}} \to \mathbb{R}^{n_{\text{kin}} \times n_{\text{kin}}}$, whose argument consists of the globale momenta. We introduce the family $(\mathcal{D}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})})_{(\boldsymbol{\zeta},\boldsymbol{\Gamma}) \in U_{\text{pos}} \times \mathbb{R}^{n_{\text{kin}}}}$, modulated by the global positions and momenta, as

$$\mathcal{D}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})} := \left\{ \begin{pmatrix} \boldsymbol{\varpi}_{\mathcal{L},f} \\ \boldsymbol{\tau}_{\mathcal{L},f} \\ \boldsymbol{\varpi}_{\mathcal{R}} \\ \boldsymbol{\varpi}_{\text{ext}} \\ \boldsymbol{\tau}_{\mathcal{L},e} \\ \boldsymbol{\varpi}_{\mathcal{L},e} \\ \boldsymbol{\tau}_{\mathcal{R}} \\ \boldsymbol{\tau}_{\text{ext}} \end{pmatrix} \in \mathbb{R}^{2n_{\text{pot}}+4n_{\text{kin}}+2m} \;\middle|\; \begin{array}{l} \boldsymbol{\varpi}_{\mathcal{L},f} = Z(\boldsymbol{\zeta})\boldsymbol{\varpi}_{\mathcal{L},e}, \ \boldsymbol{\varpi}_{\mathcal{R}} = \boldsymbol{\varpi}_{\mathcal{L},e}, \\ A(\boldsymbol{\zeta})\boldsymbol{\varpi}_{\mathcal{L},e} = 0, \\ \boldsymbol{\varpi}_{\text{ext}} = B(\boldsymbol{\zeta})^\top \boldsymbol{\varpi}_{\mathcal{L},e}, \\ \exists\, \boldsymbol{\mu} \in \mathbb{R}^{\ell} : \\ \boldsymbol{\tau}_{\mathcal{L},f} + Z(\boldsymbol{\zeta})^\top \boldsymbol{\tau}_{\mathcal{L},e} + G(\boldsymbol{\Gamma})\boldsymbol{\varpi}_{\mathcal{L},e} \\ +\boldsymbol{\tau}_{\mathcal{R}} + B(\boldsymbol{\zeta})\boldsymbol{\tau}_{\text{ext}} + A(\boldsymbol{\zeta})^\top \boldsymbol{\mu} = 0 \end{array} \right\} \tag{7}$$

Next we show that this family is a modulated Dirac structure.

Proposition 2 *Let $U_{\text{pos}} \subset \mathbb{R}^{n_{\text{pot}}}$ be open, let $A : U_{\text{pos}} \to \mathbb{R}^{\ell \times n_{\text{kin}}}$, $B : U_{\text{pos}} \to \mathbb{R}^{n_{\text{kin}} \times m}$, $G : \mathbb{R}^{n_{\text{kin}}} \to \mathbb{R}^{n_{\text{kin}} \times n_{\text{kin}}}$, $Z : U_{\text{pos}} \to \mathbb{R}^{n_{\text{pot}} \times n_{\text{kin}}}$ be continuous, and assume that A has constant rank on U_{pos}, and $G(\boldsymbol{\Gamma})$ is skew-symmetric for all $\boldsymbol{\Gamma} \in \mathbb{R}^{n_{\text{kin}}}$. Then the family $(\mathcal{D}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})})_{(\boldsymbol{\zeta},\boldsymbol{\Gamma}) \in U_{\text{pos}} \times \mathbb{R}^{n_{\text{kin}}}}$ with $\mathcal{D}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})}$ as in (7) is a modulated Dirac structure.*

The proof of Proposition 2 is relegated to Appendix A.

Remark 3 Recall that in Sect. 3 we mentioned that Proposition 1 would be a consequence of Proposition 2. Indeed, this is the case for $n_{\text{kin}} = n_{\text{pot}}$, $Z \equiv I_{n_{\text{kin}}}$ and $G \equiv 0_{n_{\text{kin}} \times n_{\text{kin}}}$.

As in the previous section, we consider resistive relations defined by a function. That is, for $U_{\text{d}} \subset U_{\text{pos}} \times \mathbb{R}^n$, we consider a function $\boldsymbol{\tau}_{\text{d}} : U_{\text{d}} \to \mathbb{R}^{n_{\text{kin}}}$ with

$$\forall\, (\boldsymbol{\zeta}, \boldsymbol{\varpi}) \in U_{\text{d}} : \quad \boldsymbol{\varpi}^\top \boldsymbol{\tau}_{\text{d}}(\boldsymbol{\zeta}, \boldsymbol{\varpi}) \geq 0.$$

This defines a modulated resistive relation $\mathcal{R} = (\mathcal{R}_{\boldsymbol{\zeta}})_{\boldsymbol{\zeta} \in U_{\text{pos}}}$ via

$$\mathcal{R}_{\boldsymbol{\zeta}} := \left\{ \begin{pmatrix} \boldsymbol{\tau} \\ \boldsymbol{\varpi} \end{pmatrix} \in \mathbb{R}^{2n_{\text{kin}}} \;\middle|\; \boldsymbol{\tau} = \boldsymbol{\tau}_{\text{d}}(\boldsymbol{\zeta}, \boldsymbol{\varpi}) \right\}. \tag{8}$$

Basically, we consider the same Lagrangian submanifold as in (4). The only difference is that we might have different dimensions for the position and momentum. As before, let $c : U_{\text{pos}} \to \mathbb{R}^k$ denote the position constraints in the form $c(\boldsymbol{\zeta}) = 0$, and let $M \in \mathbb{R}^{n_{\text{kin}} \times n_{\text{kin}}}$ be the mass matrix, which is assumed to be symmetric. We assume that c as well as $\mathcal{V}_{\text{pot}} : U_{\text{pos}} \to \mathbb{R}$ are twice continuously differentiable and that c vanishes in a non-empty subset of U_{pos}, and $c'(\boldsymbol{\zeta})$ has full row rank for all $\boldsymbol{\zeta} \in U_{\text{pos}}$. The same argumentation as in the previous section then yields that

$$\mathcal{L} := \left\{ \begin{pmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\Gamma} \\ \boldsymbol{\tau} \\ \boldsymbol{\varpi} \end{pmatrix} \in \mathbb{R}^{2n_{\text{pot}}+2n_{\text{kin}}} \;\middle|\; \begin{array}{l} M\boldsymbol{\varpi} = \boldsymbol{\Gamma}, \ c(\boldsymbol{\zeta}) = 0, \\ \exists\, \boldsymbol{\lambda} \in \mathbb{R}^k : \ \boldsymbol{\tau} = \nabla \mathcal{V}_{\text{pot}}(\boldsymbol{\zeta}) + c'(\boldsymbol{\zeta})^\top \boldsymbol{\lambda} \end{array} \right\} \tag{9}$$

is a Lagrangian submanifold.

Let us now derive the equations for the port-Hamiltonian system governed by the modulated Dirac structure, modulated resistive relation, and Lagrangian submanifold introduced thus far. This is given by

$$\begin{pmatrix} \dot{\boldsymbol{\zeta}}(t) \\ \dot{\boldsymbol{\Gamma}}(t) \\ \boldsymbol{\varpi}_{\mathcal{R}}(t) \\ \boldsymbol{\varpi}_{\mathrm{ext}}(t) \\ \boldsymbol{\tau}(t) \\ \boldsymbol{\varpi}(t) \\ \boldsymbol{\tau}_{\mathcal{R}}(t) \\ \boldsymbol{\tau}_{\mathrm{ext}}(t) \end{pmatrix} \in \mathcal{D}_{(\boldsymbol{\zeta}(t),\boldsymbol{\Gamma}(t))},$$

$$\begin{pmatrix} \boldsymbol{\tau}_{\mathcal{R}}(t) \\ \boldsymbol{\varpi}_{\mathcal{R}}(t) \end{pmatrix} \in \mathcal{R}_{\boldsymbol{\zeta}(t)}, \quad \begin{pmatrix} \boldsymbol{\zeta}(t) \\ \boldsymbol{\Gamma}(t) \\ \boldsymbol{\tau}(t) \\ \boldsymbol{\varpi}(t) \end{pmatrix} \in \mathcal{L}.$$

The first relation gives

$$\begin{aligned} \dot{\boldsymbol{\zeta}}(t) &= Z(\boldsymbol{\zeta}(t))\boldsymbol{\varpi}(t), \\ \dot{\boldsymbol{\Gamma}}(t) &= -Z(\boldsymbol{\zeta}(t))^\top \boldsymbol{\tau}(t) - \boldsymbol{\tau}_{\mathcal{R}}(t) - G(\boldsymbol{\Gamma}(t))\boldsymbol{\varpi}(t) \\ &\quad - B(\boldsymbol{\zeta}(t))\boldsymbol{\tau}_{\mathrm{ext}}(t) - A(\boldsymbol{\zeta}(t))^\top \boldsymbol{\mu}(t), \\ 0 &= A(\boldsymbol{\zeta}(t))\boldsymbol{\varpi}(t), \\ \boldsymbol{\varpi}(t) &= \boldsymbol{\varpi}_{\mathcal{R}}(t), \\ \boldsymbol{\varpi}_{\mathrm{ext}}(t) &= B(\boldsymbol{\zeta}(t))^\top \boldsymbol{\varpi}(t). \end{aligned}$$

Then, by the relations from the Lagrangian submanifold and the modulated resistive relation, we have

$$\begin{aligned} M\boldsymbol{\varpi}(t) &= \boldsymbol{\Gamma}(t), \\ \boldsymbol{\tau}(t) &= \nabla\mathcal{V}_{\mathrm{pot}}(\boldsymbol{\zeta}(t)) + c'(\boldsymbol{\zeta}(t))^\top \boldsymbol{\lambda}(t), \\ 0 &= c(\boldsymbol{\zeta}(t)), \\ \boldsymbol{\tau}_{\mathcal{R}}(t) &= \boldsymbol{\tau}_{\mathrm{d}}(\boldsymbol{\zeta}(t), \boldsymbol{\varpi}_{\mathcal{R}}(t)). \end{aligned}$$

Overall, this leads to the differential-algebraic system

$$\begin{aligned} \dot{\boldsymbol{\zeta}}(t) &= Z(\boldsymbol{\zeta}(t))\boldsymbol{\varpi}(t), \\ M\dot{\boldsymbol{\varpi}}(t) &= -Z(\boldsymbol{\zeta}(t))^\top \nabla\mathcal{V}_{\mathrm{pot}}(\boldsymbol{\zeta}(t)) - Z(\boldsymbol{\zeta}(t))^\top \\ &\quad \times c'(\boldsymbol{\zeta}(t))^\top \boldsymbol{\lambda}(t) - \boldsymbol{\tau}_{\mathrm{d}}(\boldsymbol{\zeta}(t), \boldsymbol{\varpi}(t)) \\ &\quad - G(M\boldsymbol{\varpi}(t))\boldsymbol{\varpi}(t) - A(\boldsymbol{\zeta}(t))^\top \boldsymbol{\mu}(t) \\ &\quad - B(\boldsymbol{\zeta}(t))\boldsymbol{\tau}_{\mathrm{ext}}(t), \\ 0 &= c(\boldsymbol{\zeta}(t)), \\ 0 &= A(\boldsymbol{\zeta}(t))\boldsymbol{\varpi}(t), \\ \boldsymbol{\varpi}_{\mathrm{ext}}(t) &= B(\boldsymbol{\zeta}(t))^\top \boldsymbol{\varpi}(t). \end{aligned} \tag{10}$$

Let us consider the total energy of the system, which is the sum of the kinetic energy $\frac{1}{2}\boldsymbol{\varpi}^\top M\boldsymbol{\varpi}$ and the potential energy $\mathcal{V}_{\mathrm{pot}}(\boldsymbol{\zeta})$. Along the solutions of (10), the total energy satisfies

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\tfrac{1}{2}\boldsymbol{\varpi}(t)^\top M\boldsymbol{\varpi}(t) + \mathcal{V}_{\mathrm{pot}}(\boldsymbol{\zeta}(t))\big) \\ &= \boldsymbol{\varpi}(t)^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(M\boldsymbol{\varpi}(t)\big) + \dot{\boldsymbol{\zeta}}(t)^\top \nabla\mathcal{V}_{\mathrm{pot}}(\boldsymbol{\zeta}(t)) \\ &= \boldsymbol{\varpi}(t)^\top \Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(M\boldsymbol{\varpi}(t)\big) + Z(\boldsymbol{\zeta}(t))^\top \nabla\mathcal{V}_{\mathrm{pot}}(\boldsymbol{\zeta}(t))\Big) \\ &\overset{(10)}{=} -\boldsymbol{\varpi}(t)^\top \Big(Z(\boldsymbol{\zeta}(t))^\top c'(\boldsymbol{\zeta}(t))^\top \boldsymbol{\lambda}(t) \\ &\quad + \boldsymbol{\tau}_{\mathrm{d}}(\boldsymbol{\zeta}(t), \boldsymbol{\varpi}(t)) + G(M\boldsymbol{\varpi}(t))\boldsymbol{\varpi}(t) \\ &\qquad + A(\boldsymbol{\zeta}(t))^\top \boldsymbol{\mu}(t) + B(\boldsymbol{\zeta}(t))\boldsymbol{\tau}_{\mathrm{ext}}(t)\Big). \end{aligned}$$

The velocity constraint gives $\boldsymbol{\varpi}(t)^\top A(\boldsymbol{\zeta}(t))^\top = 0$, skew-symmetry of $G(M\boldsymbol{\varpi}(t))$ implies $\boldsymbol{\varpi}(t)^\top G(M\boldsymbol{\varpi}(t))\boldsymbol{\varpi}(t) = 0$. Moreover,

$$\begin{aligned} &\boldsymbol{\varpi}(t)^\top Z(\boldsymbol{\zeta}(t))^\top c'(\boldsymbol{\zeta}(t))^\top \\ &\quad = \dot{\boldsymbol{\zeta}}(t)^\top c'(\boldsymbol{\zeta}(t))^\top = \tfrac{\mathrm{d}}{\mathrm{d}t} c(\boldsymbol{\zeta}(t)) = 0. \end{aligned}$$

By further invoking $\boldsymbol{\varpi}_{\mathrm{ext}}(t) = B(\boldsymbol{\zeta}(t))^\top \boldsymbol{\varpi}(t)$, we see that the above energy balance reduces to

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\tfrac{1}{2}\boldsymbol{\varpi}(t)^\top M\boldsymbol{\varpi}(t) + \mathcal{V}_{\mathrm{pot}}(\boldsymbol{\zeta}(t))\big) \\ &\quad = -\boldsymbol{\varpi}(t)^\top \boldsymbol{\tau}_{\mathrm{d}}(\boldsymbol{\zeta}(t), \boldsymbol{\varpi}(t)) - \boldsymbol{\varpi}_{\mathrm{ext}}(t)^\top \boldsymbol{\tau}_{\mathrm{ext}}(t), \end{aligned}$$

again with the interpretation that $\boldsymbol{\varpi}_{\mathrm{ext}}(t)^\top \boldsymbol{\tau}_{\mathrm{ext}}(t)$ is the power extracted from the system, whereas $\boldsymbol{\varpi}(t)^\top \boldsymbol{\tau}_{\mathrm{d}}(\boldsymbol{\zeta}(t), \boldsymbol{\varpi}(t))$ represents the dissipated power.

5 Incorporation of rotation matrices

As mentioned in Sect. 4, rotation matrices can be used to describe orientation. In the following, we provide a more detailed discussion of this approach.

The rotational configuration of a rigid body is represented by a rotation matrix, that is, an element of the *special orthogonal group*

$$\mathbb{SO}_d = \left\{ \boldsymbol{R} \in \mathbb{R}^{d\times d} \;\middle|\; \boldsymbol{R}^\top \boldsymbol{R} = I_d \wedge \det \boldsymbol{R} = 1 \right\}, \quad d = 2, 3,$$

where $d = 2$ corresponds to the planar case, and $d = 3$ to the spatial case. However, it should be noted that in the planar case the orientation is easily described by one rotation angle per body. Thus, $\mathbb{SO}_2$ is rarely used in practice.

Roughly speaking, our approach proceeds as follows: we include a matrix-valued component both in the space of flows and efforts, and introduce an additional state variable (represented by a matrix) that describes the global orientation of the body. Since this matrix is meant to represent a rotation, we must impose global position constraints of the form $\boldsymbol{R}^\top \boldsymbol{R} - I_d = 0$, which are incorporated into the Lagrangian submanifold.

Note that we do not explicitly enforce any constraint on the determinant. However, any matrix satisfying $\boldsymbol{R}^\top \boldsymbol{R} = I_d$ necessarily has a determinant of either 1 or -1, where the latter corresponds to a reflection. If the system is initialized with a proper rotation matrix (i.e., an element of $\mathbb{SO}_d$), then by continuity the determinant will remain 1 and cannot jump to -1.

For the sake of brevity, we restrict ourselves to the case involving a single rotation in the three-dimensional space; the generalization to multiple rotating bodies (or two-dimensional rotations) follows analogously. First, we note that the space of matrices must be embedded into a Euclidean space in order to describe such systems within the framework of Sect. 4; see also Remark 2. This is achieved by stacking the columns $\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3 \in \mathbb{R}^3$ of the rotation matrix $\boldsymbol{R} = [\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3]$ on top of each other to obtain a vector $(\boldsymbol{R}_1^\top, \boldsymbol{R}_2^\top, \boldsymbol{R}_3^\top)^\top \in \mathbb{R}^9$ – a so-called vectorization.

Let $\tilde{\boldsymbol{\zeta}} \in \mathbb{R}^{\tilde{n}_{\text{pot}}}$ denote the vector of global positions of a given multibody system. We augment this by the rotation matrix $\boldsymbol{R}$ corresponding to the considered single rotation, which is thus interpreted – also in an abstract sense – as part of the global position. To ensure that $\boldsymbol{R}$ is indeed a rotation matrix, we impose a position constraint of the form $\boldsymbol{R}^\top \boldsymbol{R} - I_3 = 0$. Note that this constraint represents six scalar equations, since $\boldsymbol{R}^\top \boldsymbol{R}$ is symmetric and hence three of the nine scalar conditions are redundant. The above explained incorporation leads, for $n_{\text{pot}} = \tilde{n}_{\text{pot}} + 9$ to the Lagrangian submanifold

$$\mathcal{L} := \left\{ \begin{pmatrix} \boldsymbol{R}_1 \\ \boldsymbol{R}_2 \\ \boldsymbol{R}_3 \\ \tilde{\boldsymbol{\zeta}} \\ \boldsymbol{\Gamma} \\ \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \\ \boldsymbol{\tau}_3 \\ \tilde{\boldsymbol{\tau}} \\ \boldsymbol{\varpi} \end{pmatrix} \in \mathbb{R}^{2n_{\text{pot}}+2n_{\text{kin}}} \;\middle|\; \begin{array}{l} M\boldsymbol{\varpi} = \boldsymbol{\Gamma},\ c(\tilde{\boldsymbol{\zeta}}) = 0,\ \boldsymbol{R}_i^\top \boldsymbol{R}_i - 1 = 0, i = 1, 2, 3, \\ \boldsymbol{R}_1^\top \boldsymbol{R}_2 = 0,\ \boldsymbol{R}^\top \boldsymbol{R}_3 = 0,\ \boldsymbol{R}_2^\top \boldsymbol{R}_3 = 0, \\ \exists\, \boldsymbol{\lambda} \in \mathbb{R}^k :\ \tilde{\boldsymbol{\tau}} = \nabla \mathcal{V}_{\text{pot}}(\tilde{\boldsymbol{\zeta}}) + c'(\tilde{\boldsymbol{\zeta}})^\top \boldsymbol{\lambda}, \\ \exists\, \boldsymbol{\lambda}_R \in \mathbb{R}^6 : \\ \begin{pmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \\ \boldsymbol{\tau}_3 \end{pmatrix} = \begin{bmatrix} \boldsymbol{R}_1 & 0 & 0 & \boldsymbol{R}_2 & \boldsymbol{R}_3 & 0 \\ 0 & \boldsymbol{R}_2 & 0 & \boldsymbol{R}_1 & 0 & \boldsymbol{R}_3 \\ 0 & 0 & \boldsymbol{R}_3 & 0 & \boldsymbol{R}_1 & \boldsymbol{R}_2 \end{bmatrix} \boldsymbol{\lambda}_R. \end{array} \right\}$$

In the above submanifold, the orthogonality condition $\boldsymbol{R}^\top \boldsymbol{R} = I_3$ is enforced by the constraints $\boldsymbol{R}_i^\top \boldsymbol{R}_i - 1 = 0$ for $i = 1, 2, 3$, and $\boldsymbol{R}_1^\top \boldsymbol{R}_2 = 0$, $\boldsymbol{R}_1^\top \boldsymbol{R}_3 = 0$, $\boldsymbol{R}_2^\top \boldsymbol{R}_3 = 0$. The corresponding *constraint forces* are given by $\boldsymbol{\tau}_i \in \mathbb{R}^3$, for $i = 1, 2, 3$. Note that the matrix multiplying $\boldsymbol{\lambda}_R$ is the transpose of the Jacobian of the function

$$\begin{pmatrix} \boldsymbol{R}_1 \\ \boldsymbol{R}_2 \\ \boldsymbol{R}_3 \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{R}_1^\top \boldsymbol{R}_1 - 1 \\ \boldsymbol{R}_2^\top \boldsymbol{R}_2 - 1 \\ \boldsymbol{R}_3^\top \boldsymbol{R}_3 - 1 \\ \boldsymbol{R}_1^\top \boldsymbol{R}_2 \\ \boldsymbol{R}_1^\top \boldsymbol{R}_3 \\ \boldsymbol{R}_2^\top \boldsymbol{R}_3 \end{pmatrix}.$$

The Jacobian of this function clearly has full row rank at any $(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3)$ satisfying the above orthogonality constraints (and thus, by a continuity argument, also in an open neighborhood of such points). Hence, the set $\mathcal{L}$ introduced above is, in principle, of the same structure as in (9), and therefore defines a Lagrangian submanifold.

The overall model is completed by a modulated resistive relation (8) and a modulated Dirac structure $(\mathcal{D}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})})_{(\boldsymbol{\zeta},\boldsymbol{\Gamma}) \in U_{\text{pos}} \times \mathbb{R}^{n_{\text{kin}}}}$ as in (7). Note that, here, the tuple of global positions and corresponding forces are respectively given by

$$\boldsymbol{\zeta} = \begin{pmatrix} \boldsymbol{R}_1 \\ \boldsymbol{R}_2 \\ \boldsymbol{R}_3 \\ \tilde{\boldsymbol{\zeta}} \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \\ \boldsymbol{\tau}_3 \\ \tilde{\boldsymbol{\tau}} \end{pmatrix}.$$

This means that the rotation matrix is incorporated into the system through kinematic relations. Typically, the vector of global velocities includes angular velocities, which are then related to the rotation matrix via the kinematic relation

$$\begin{bmatrix}\dot{\boldsymbol{R}}_1(t) & \dot{\boldsymbol{R}}_2(t) & \dot{\boldsymbol{R}}_3(t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{R}_1(t) & \boldsymbol{R}_2(t) & \boldsymbol{R}_3(t)\end{bmatrix} \cdot \begin{bmatrix} 0 & -\boldsymbol{\omega}_{z'}(t) & \boldsymbol{\omega}_{y'}(t) \\ \boldsymbol{\omega}_{z'}(t) & 0 & -\boldsymbol{\omega}_{x'}(t) \\ -\boldsymbol{\omega}_{y'}(t) & \boldsymbol{\omega}_{x'}(t) & 0 \end{bmatrix}$$

if the angular velocities are expressed in a body-fixed coordinate frame [18,19]. This can be rewritten as

$$\begin{pmatrix}\dot{\boldsymbol{R}}_1(t)\\ \dot{\boldsymbol{R}}_2(t)\\ \dot{\boldsymbol{R}}_3(t)\end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\boldsymbol{R}_3(t) & \boldsymbol{R}_2(t) \\ \boldsymbol{R}_3(t) & 0 & -\boldsymbol{R}_1(t) \\ -\boldsymbol{R}_2(t) & \boldsymbol{R}_1(t) & 0 \end{bmatrix}}_{Z_{\mathrm{rot}}(\boldsymbol{R}_1(t),\boldsymbol{R}_2(t),\boldsymbol{R}_3(t))} \begin{pmatrix}\boldsymbol{\omega}_{x'}(t)\\ \boldsymbol{\omega}_{y'}(t)\\ \boldsymbol{\omega}_{z'}(t)\end{pmatrix}, \tag{11}$$

with the kinematic matrix $Z_{\mathrm{rot}}(\boldsymbol{R}_1(t), \boldsymbol{R}_2(t), \boldsymbol{R}_3(t)) \in \mathbb{R}^{9\times 3}$. The quantities $\boldsymbol{\omega}_{x'}, \boldsymbol{\omega}_{y'}, \boldsymbol{\omega}_{z'}$ are part of the global velocity vector $\boldsymbol{\varpi}$, and the upper left corner of the full kinematic matrix consists of Z_{rot}, with zeros to its right. It can be further seen that

$$Z_{\mathrm{rot}}(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3)^\top \begin{bmatrix} \boldsymbol{R}_1 & 0 & 0 & \boldsymbol{R}_2 & \boldsymbol{R}_3 & 0 \\ 0 & \boldsymbol{R}_2 & 0 & \boldsymbol{R}_1 & 0 & \boldsymbol{R}_3 \\ 0 & 0 & \boldsymbol{R}_3 & 0 & \boldsymbol{R}_1 & \boldsymbol{R}_2 \end{bmatrix} = 0.$$

The fact that, in the kinetic equation of the full model, the expression involving the constraint forces with respect to global positions is multiplied from the left by the transposed kinematic matrix (see (10)) implies that the Lagrangian multiplier $\boldsymbol{\lambda}_R$ corresponding to the constraints on $\boldsymbol{R}$ does not enter the resulting full model at all. This is due to the fact that (11) ensures that the time evolution of $\boldsymbol{R}$ always lies in the tangent space of $\mathbb{SO}_3$. The kinematic constraints given by (11) therefore automatically imply the abstract position constraint $\boldsymbol{R}^\top \boldsymbol{R} - I_3 = 0$.

It is furthermore noted that any rotation matrix can also be expressed in terms of successive elementary rotations such as

$$\boldsymbol{R} = \begin{bmatrix} \cos(\boldsymbol{\gamma}) & -\sin(\boldsymbol{\gamma}) & 0 \\ \sin(\boldsymbol{\gamma}) & \cos(\boldsymbol{\gamma}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\boldsymbol{\beta}) & 0 & \sin(\boldsymbol{\beta}) \\ 0 & 1 & 0 \\ -\sin(\boldsymbol{\beta}) & 0 & \cos(\boldsymbol{\beta}) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\boldsymbol{\alpha}) & -\sin(\boldsymbol{\alpha}) \\ 0 & \sin(\boldsymbol{\alpha}) & \cos(\boldsymbol{\alpha}) \end{bmatrix}$$

with the Euler angles $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$. Note that the three Euler angles can be uniquely reconstructed from $\boldsymbol{R}$, unless $\boldsymbol{\beta} = \frac{\pi}{2}$; see [19, Sec. 3.6].

6 Interconnection

One of the biggest advantages of port-Hamiltonian models is their modular structure. That is, the class of port-Hamiltonian systems is closed under a certain type of interconnection [14]. This type of interconnection is based on a splitting of the external ports of two systems into those to be linked and those that are 'truly external', that is

$$f_{\mathcal{P}_i}(t) = \begin{pmatrix} f_{\mathcal{P}\mathrm{c},i}(t) \\ f_{\mathcal{P}\mathrm{ext},i}(t)\end{pmatrix},\ e_{\mathcal{P}_i}(t) = \begin{pmatrix} e_{\mathcal{P}\mathrm{c},i}(t) \\ e_{\mathcal{P}\mathrm{ext},i}(t)\end{pmatrix} \in \mathbb{R}^{m_{\mathrm{c}}} \times \mathbb{R}^{m_{\mathrm{ext},i}}, \quad i = 1, 2.$$

The coupling relations are

$$f_{\mathcal{P}\mathrm{c},1}(t) = f_{\mathcal{P}\mathrm{c},2}(t), \quad e_{\mathcal{P}\mathrm{c},1}(t) = -e_{\mathcal{P}\mathrm{c},2}(t), \tag{12}$$

and the resulting external ports of the interconnected system are given by

$$f_{\mathcal{P}}(t) = \begin{pmatrix} f_{\mathcal{P}\,\mathrm{ext},1}(t) \\ f_{\mathcal{P}\mathrm{ext},2}(t)\end{pmatrix},\ e_{\mathcal{P}}(t) = \begin{pmatrix} e_{\mathcal{P}\mathrm{ext},1}(t) \\ e_{\mathcal{P}\mathrm{ext},2}(t)\end{pmatrix} \in \mathbb{R}^{m_{\mathrm{ext},1}} \times \mathbb{R}^{m_{\mathrm{ext},2}}.$$

For completeness, note that [14] uses the interconnection rules $f_{\mathcal{P}\mathrm{c},1} = -f_{\mathcal{P}\mathrm{c},2}$ and $e_{\mathcal{P}\mathrm{c},1} = e_{\mathcal{P}\mathrm{c},2}$, which are equivalent to ours. However, we prefer the coupling relations in (12), as they are more natural in the context of the class of multibody systems studied in this article.

For constant (i.e., unmodulated) Dirac structures, the coupling (12) again results in a Dirac structure, as shown in [14]. For modulated Dirac structures, it remains unknown whether this type of coupling still results in a modulated Dirac structure, as it is unclear whether the local trivialization property holds for the interconnected system. In [35,36], additional conditions on the involved modulated Dirac structures have been imposed to ensure that the interconnection preserves the modulated Dirac structure property.

Our situation is somewhat more specific, as we focus on a subclass of port-Hamiltonian systems – namely, multibody systems as described in Sect. 4. We demonstrate that, under a rather mild additional assumption on the matrices A and B, this class remains closed under interconnection via (12).

More precisely, we consider two multibody systems of type (10), where all involved matrices, functions, and variables are indexed by $i = 1, 2$, depending on the respective system. We further partition the matrices

B_i by

$$B_i(\boldsymbol{\zeta}_i) = [B_{\mathrm{c},i}(\boldsymbol{\zeta}_i), B_{\mathrm{ext},i}(\boldsymbol{\zeta}_i)], \quad B_{\mathrm{c},i} \in \mathbb{R}^{n_{\mathrm{kin},i} \times m_{\mathrm{c}}}, \quad B_{\mathrm{ext},i} \in \mathbb{R}^{n_{\mathrm{kin},i} \times m_{\mathrm{ext},i}}, \quad i = 1, 2,$$

and the external forces and velocities by

$$\boldsymbol{\tau}_{\mathrm{ext},i}(t) = \begin{pmatrix} \boldsymbol{\tau}_{\mathrm{c},i}(t) \\ \boldsymbol{\tau}_{\widetilde{\mathrm{ext}},i}(t) \end{pmatrix}, \ \boldsymbol{\varpi}_{\mathrm{ext},i}(t) = \begin{pmatrix} \boldsymbol{\varpi}_{\mathrm{c},i}(t) \\ \boldsymbol{\varpi}_{\widetilde{\mathrm{ext}},i}(t) \end{pmatrix}, \quad \boldsymbol{\tau}_{\mathrm{c},i}(t), \boldsymbol{\varpi}_{\mathrm{c},i}(t) \in \mathbb{R}^{m_{\mathrm{c}}}, \ \boldsymbol{\tau}_{\widetilde{\mathrm{ext}},i}(t), \boldsymbol{\varpi}_{\widetilde{\mathrm{ext}},i}(t) \in \mathbb{R}^{m_{\mathrm{ext},i}}.$$

We show that under the assumption that

$$\operatorname{rank} \begin{bmatrix} A_1(\boldsymbol{\zeta}_1) & 0 \\ 0 & A_2(\boldsymbol{\zeta}_2) \\ B_{\mathrm{c},1}(\boldsymbol{\zeta}_1)^\top & -B_{\mathrm{c},2}(\boldsymbol{\zeta}_2)^\top \end{bmatrix} \equiv \mathrm{const} \tag{13}$$

on $U_{\mathrm{pos},1} \times U_{\mathrm{pos},2}$, the coupling

$$\boldsymbol{\tau}_{\mathrm{c},1}(t) = -\boldsymbol{\tau}_{\mathrm{c},2}(t), \quad \boldsymbol{\varpi}_{\mathrm{c},1}(t) = \boldsymbol{\varpi}_{\mathrm{c},2}(t),$$

which is induced by condition (12), leads to an interconnected system which is again port-Hamiltonian. Note that the coupling corresponds to a connection between the two systems. More precisely, the velocities of the connected ports are equal, while the force on one side of the link acts as the counterforce on the other side. Altogether, this gives rise to the system

$$\begin{aligned}
\dot{\boldsymbol{\zeta}}_1(t) &= Z_1(\boldsymbol{\zeta}_1(t))\boldsymbol{\varpi}_1(t), \\
\dot{\boldsymbol{\zeta}}_2(t) &= Z_2(\boldsymbol{\zeta}_2(t))\boldsymbol{\varpi}_2(t), \\
M_1\dot{\boldsymbol{\varpi}}_1(t) &= -Z_1(\boldsymbol{\zeta}_1(t))^\top \nabla \mathcal{V}_{\mathrm{pot},1}(\boldsymbol{\zeta}_1(t)) \\
&\quad - Z_1(\boldsymbol{\zeta}_1(t))^\top c_1'(\boldsymbol{\zeta}_1(t))^\top \boldsymbol{\lambda}_1(t) \\
&\quad - \boldsymbol{\tau}_{\mathrm{d},1}(\boldsymbol{\zeta}_1(t), \boldsymbol{\varpi}_1(t)) \\
&\quad - G_1(M_1\boldsymbol{\varpi}_1(t))\boldsymbol{\varpi}_1(t) \\
&\quad - A_1(\boldsymbol{\zeta}_1(t))^\top \boldsymbol{\mu}_1(t) \\
&\quad - B_{\mathrm{c},1}(\boldsymbol{\zeta}_1(t))\boldsymbol{\tau}_{\mathrm{c},1}(t) \\
&\quad - B_{\mathrm{ext},1}(\boldsymbol{\zeta}_1(t))\boldsymbol{\tau}_{\widetilde{\mathrm{ext}},1}(t), \\
M_2\dot{\boldsymbol{\varpi}}_2(t) &= -Z_2(\boldsymbol{\zeta}_2(t))^\top \nabla \mathcal{V}_{\mathrm{pot},2}(\boldsymbol{\zeta}_2(t)) \\
&\quad - Z_2(\boldsymbol{\zeta}_2(t))^\top c_2'(\boldsymbol{\zeta}_2(t))^\top \boldsymbol{\lambda}_2(t) \\
&\quad + \boldsymbol{\tau}_{\mathrm{d},2}(\boldsymbol{\zeta}_2(t), \boldsymbol{\varpi}_2(t)) \\
&\quad - G_2(M_1(\boldsymbol{\zeta}_2(t))\boldsymbol{\varpi}_2(t))\boldsymbol{\varpi}_2(t) \\
&\quad - A_2(\boldsymbol{\zeta}_2(t))^\top \boldsymbol{\mu}_2(t) \\
&\quad + B_{\mathrm{c},2}(\boldsymbol{\zeta}_2(t))\boldsymbol{\tau}_{\mathrm{c},1}(t) \\
&\quad - B_{\mathrm{ext},2}(\boldsymbol{\zeta}_2(t))\boldsymbol{\tau}_{\widetilde{\mathrm{ext}},2}(t), \\
0 &= c_1(\boldsymbol{\zeta}_1(t)), \\
0 &= c_2(\boldsymbol{\zeta}_2(t)), \\
0 &= A_1(\boldsymbol{\zeta}_1(t))\boldsymbol{\varpi}_1(t), \\
0 &= A_2(\boldsymbol{\zeta}_2(t))\boldsymbol{\varpi}_2(t), \\
0 &= B_{\mathrm{c},1}(\boldsymbol{\zeta}_1(t))^\top \boldsymbol{\varpi}_1(t) - B_{\mathrm{c},2}(\boldsymbol{\zeta}_2(t))^\top \boldsymbol{\varpi}_2(t), \\
\boldsymbol{\varpi}_{\widetilde{\mathrm{ext}},1}(t) &= B_{\mathrm{ext},1}(\boldsymbol{\zeta}_1(t))^\top \boldsymbol{\varpi}_1(t), \\
\boldsymbol{\varpi}_{\widetilde{\mathrm{ext}},2}(t) &= B_{\mathrm{ext},2}(\boldsymbol{\zeta}_2(t))^\top \boldsymbol{\varpi}_2(t).
\end{aligned}$$

It may appear quite complex at first glance, yet the system has an inherent structure. Specifically, it is of the form of (10) with

$$\begin{aligned}
n_{\mathrm{pot}} &= n_{\mathrm{pot},1} + n_{\mathrm{pot},2}, \quad n_{\mathrm{kin}} = n_{\mathrm{kin},1} + n_{\mathrm{kin},2}, \\
m &= m_{\mathrm{ext},1} + m_{\mathrm{ext},2}, \\
\boldsymbol{\zeta}(t) &= \begin{pmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{pmatrix}, \ \boldsymbol{\varpi} = \begin{pmatrix} \boldsymbol{\varpi}_1 \\ \boldsymbol{\varpi}_2 \end{pmatrix}, \ \boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Gamma}_1 \\ \boldsymbol{\Gamma}_2 \end{pmatrix}, \\
\boldsymbol{\varpi}_{\mathrm{ext}} &= \begin{pmatrix} \boldsymbol{\varpi}_{\widetilde{\mathrm{ext}},1} \\ \boldsymbol{\varpi}_{\widetilde{\mathrm{ext}},2} \end{pmatrix}, \ \boldsymbol{\tau}_{\mathrm{ext}}(t) = \begin{pmatrix} \boldsymbol{\tau}_{\widetilde{\mathrm{ext}},1} \\ \boldsymbol{\tau}_{\widetilde{\mathrm{ext}},2} \end{pmatrix}, \\
\mathcal{V}_{\mathrm{pot}}(\boldsymbol{\zeta}) &= \mathcal{V}_{\mathrm{pot},1}(\boldsymbol{\zeta}_1) + \mathcal{V}_{\mathrm{pot},2}(\boldsymbol{\zeta}_2), \\
Z(\boldsymbol{\zeta}) &= \begin{bmatrix} Z_1(\boldsymbol{\zeta}_1) & 0 \\ 0 & Z_2(\boldsymbol{\zeta}_2) \end{bmatrix}, \ M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \\
G(\boldsymbol{\Gamma}) &= \begin{bmatrix} G_1(\boldsymbol{\Gamma}_1) & 0 \\ 0 & G_2(\boldsymbol{\Gamma}_2) \end{bmatrix}, \\
\boldsymbol{\tau}_{\mathrm{d}}(\boldsymbol{\zeta}, \boldsymbol{\varpi}) &= \begin{pmatrix} \boldsymbol{\tau}_{\mathrm{d},1}(\boldsymbol{\zeta}_1, \boldsymbol{\varpi}_1) \\ \boldsymbol{\tau}_{\mathrm{d},2}(\boldsymbol{\zeta}_2, \boldsymbol{\varpi}_2) \end{pmatrix}, \\
c(\boldsymbol{\zeta}) &= \begin{pmatrix} c_1(\boldsymbol{\zeta}_1) \\ c_2(\boldsymbol{\zeta}_2) \end{pmatrix}, \\
A(\boldsymbol{\zeta}) &= \begin{bmatrix} A_1(\boldsymbol{\zeta}_1) & 0 \\ 0 & A_2(\boldsymbol{\zeta}_2) \\ B_{\mathrm{c},1}(\boldsymbol{\zeta}_1)^\top & -B_{\mathrm{c},2}(\boldsymbol{\zeta}_2)^\top \end{bmatrix}, \\
B(\boldsymbol{\zeta}) &= \begin{bmatrix} B_{\mathrm{ext},1}(\boldsymbol{\zeta}_1) & 0 \\ 0 & B_{\mathrm{ext},2}(\boldsymbol{\zeta}_2) \end{bmatrix}.
\end{aligned}$$

This means that the interconnection rules cause for an additional velocity constraint $B_{\mathrm{c},1}(\boldsymbol{\zeta}_1(t))^\top \boldsymbol{\varpi}_1(t) - B_{\mathrm{c},2}(\boldsymbol{\zeta}_2(t))^\top \boldsymbol{\varpi}_2(t) = 0$ and the corresponding Lagrange multiplier $\boldsymbol{\tau}_{c,1}(t)$ also appears in the force balance.

By condition (13) the matrix A as above has constant rank on $U_{\mathrm{pos},1} \times U_{\mathrm{pos},2}$, hence it follows from the results in Sect. 4 that the interconnected system is again port-Hamiltonian.

Finally, we note that the interconnection of more than two multibody systems can be reduced to the interconnection of two systems by an inductive approach.

7 Examples

Here, we present three exemplary mechanical systems that are (partially) subject to constraints, incorporate gyroscopic effects, and include kinematic relationships. In particular the first example is a planar system subjected to nonholonomic constraints. The second example considers the spatial rotational motion of a gyroscope, using both $\mathbb{SO}_3$ and Euler angles for description of the rotation. In the final example, we illustrate for a multibody system the interconnection of individual mechanical systems in port-Hamiltonian form.

7.1 A differential drive robot in the plane

Consider the differential drive robot depicted in Fig. 2. Its body-fixed coordinate system is defined such that its y'-axis coincides with the axis of its wheels, while its x'-axis passes through its center of mass, whose y'-coordinate is $\ell \in \mathbb{R}$. The robot is symmetric about its x'-axis and has a mass $m > 0$ and a moment of inertia $\mathrm{I_O}$ about the z'-axis at its origin. The identity

$$\mathrm{I_O} = \mathrm{I_S} + m\ell^2 \tag{14}$$

holds, where $\mathrm{I_S}$ is the moment of inertia about the center of mass. We assume that $\mathrm{I_S} > 0$, which means that the robot's mass is not entirely concentrated at the center of mass. The system has three geometric degrees of freedom, described by the redundant coordinates

$$\boldsymbol{\zeta}(t) = \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{y}(t) \\ \boldsymbol{\varphi}(t) \end{pmatrix},$$

where $\boldsymbol{r}(t) = \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{y}(t) \end{pmatrix}$ represents the position of the point O′ in the inertial frame, and $\boldsymbol{\varphi}(t)$ denotes the orientation of the body-fixed frame relative to the inertial frame. Moreover, the velocity vector of the robot at point O′, expressed in the body-fixed frame, is defined as

$$\boldsymbol{\varpi}(t) = \begin{pmatrix} \boldsymbol{v}_x(t) \\ \boldsymbol{v}_y(t) \\ \boldsymbol{\omega}(t) \end{pmatrix}.$$

The left and right wheels generate propulsion forces $\boldsymbol{F}_\mathrm{l}(t)$ and $\boldsymbol{F}_\mathrm{r}(t)$, respectively, acting in the x'-direction. The wheels are spaced by a distance b. As wheel slip is neglected, the robot cannot move along its y'-axis, leading to the nonholonomic velocity constraint $\boldsymbol{v}_y(t) = 0$.

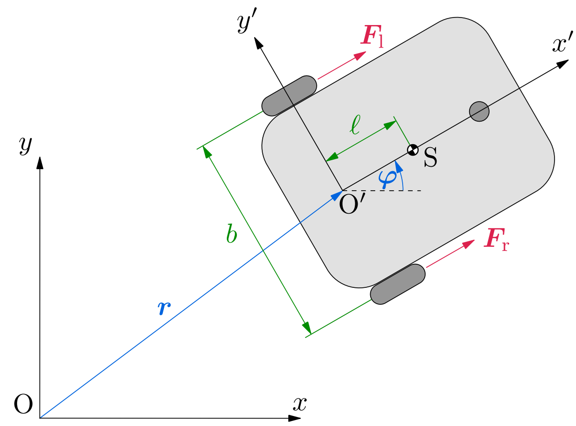

Fig. 2 Differential drive robot

This constraint is enforced by the reaction force $\boldsymbol{\mu}(t) \in \mathbb{R}$. Furthermore, we introduce the wheel velocities as $\boldsymbol{v}_\mathrm{l}(t) = \boldsymbol{v}_x(t) - \frac{b}{2}\boldsymbol{\omega}(t)$ and $\boldsymbol{v}_\mathrm{r}(t) = \boldsymbol{v}_x(t) + \frac{b}{2}\boldsymbol{\omega}(t)$ of the left and right wheel, respectively. The equations of motion in implicit form are then given by

$$\begin{pmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{y}}(t) \\ \dot{\boldsymbol{\varphi}}(t) \end{pmatrix} = \begin{bmatrix} \cos(\boldsymbol{\varphi}(t)) & -\sin(\boldsymbol{\varphi}(t)) & 0 \\ \sin(\boldsymbol{\varphi}(t)) & \cos(\boldsymbol{\varphi}(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_x(t) \\ \boldsymbol{v}_y(t) \\ \boldsymbol{\omega}(t) \end{pmatrix},$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & m\,\ell \\ 0 & m\,\ell & \mathrm{I_O} \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}_x(t) \\ \dot{\boldsymbol{v}}_y(t) \\ \dot{\boldsymbol{\omega}}(t) \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 0 & m(\boldsymbol{v}_y + \ell\boldsymbol{\omega}) \\ 0 & 0 & -m\boldsymbol{v}_x \\ -m(\boldsymbol{v}_y + \ell\boldsymbol{\omega}) & m\boldsymbol{v}_x & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_x(t) \\ \boldsymbol{v}_y(t) \\ \boldsymbol{\omega}(t) \end{pmatrix}$$

$$+ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{\mu}(t) + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -\frac{b}{2} & \frac{b}{2} \end{bmatrix} \begin{pmatrix} \boldsymbol{F}_\mathrm{l}(t) \\ \boldsymbol{F}_\mathrm{r}(t) \end{pmatrix},$$

$$0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_x(t) \\ \boldsymbol{v}_y(t) \\ \boldsymbol{\omega}(t) \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{v}_\mathrm{l}(t) \\ \boldsymbol{v}_\mathrm{r}(t) \end{pmatrix} = \begin{bmatrix} 1 & 0 & -\frac{b}{2} \\ 1 & 0 & \frac{b}{2} \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_x(t) \\ \boldsymbol{v}_y(t) \\ \boldsymbol{\omega}(t) \end{pmatrix}.$$

We neither have any position constraints nor damping, and the potential energy is constant. That is, we have a system (10) with $k = 0$, $\mathcal{V}_\mathrm{pot} \equiv 0$, $\boldsymbol{\tau}_\mathrm{d} \equiv 0$ and

$$\boldsymbol{\zeta} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{\varphi} \end{pmatrix}, \quad \boldsymbol{Z}(\boldsymbol{\zeta}) = \begin{bmatrix} \cos(\boldsymbol{\varphi}) & -\sin(\boldsymbol{\varphi}) & 0 \\ \sin(\boldsymbol{\varphi}) & \cos(\boldsymbol{\varphi}) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

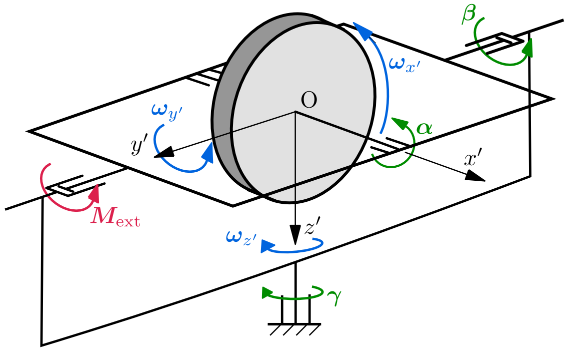

Fig. 3 Gyroscope

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & m\,\ell \\ 0 & m\,\ell & \mathrm{I_O} \end{bmatrix}, \quad B(\zeta) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -\frac{b}{2} & \frac{b}{2} \end{bmatrix},$$
$$A(\zeta) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Using (14) together with $m > 0$ and $\mathrm{I_S} > 0$, we obtain that $\boldsymbol{M}$ is positive definite. Note that, in the above model, the nonholonomic constraint can be resolved, which leads to the simplified system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\varphi}(t) \end{pmatrix} = \begin{bmatrix} \cos(\varphi(t)) & 0 \\ \sin(\varphi(t)) & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v_x(t) \\ \omega(t) \end{pmatrix},$$
$$\begin{bmatrix} m & 0 \\ 0 & \mathrm{I_O} \end{bmatrix} \begin{pmatrix} \dot{v}_x(t) \\ \dot{\omega}(t) \end{pmatrix} = \begin{bmatrix} 0 & m\omega(t)\ell \\ -m\omega(t)\ell & 0 \end{bmatrix} \begin{pmatrix} v_x(t) \\ \omega(t) \end{pmatrix}$$
$$+ \begin{bmatrix} 1 & 1 \\ -\frac{b}{2} & \frac{b}{2} \end{bmatrix} \begin{pmatrix} F_{\mathrm{l}}(t) \\ F_{\mathrm{r}}(t) \end{pmatrix},$$
$$\begin{pmatrix} v_{\mathrm{l}}(t) \\ v_{\mathrm{r}}(t) \end{pmatrix} = \begin{bmatrix} 1 & -\frac{b}{2} \\ 1 & \frac{b}{2} \end{bmatrix} \begin{pmatrix} v_x(t) \\ \omega(t) \end{pmatrix},$$

which is again of the form (10), now without any explicitly stated constraints.

7.2 Gyroscope

Consider the gimbal-mounted symmetrical gyroscope as depicted in Fig. 3.

The frames of the gimbal suspension are assumed to be massless. The gyroscope's disk has a uniform mass density, with total mass $m > 0$, radius $r > 0$, and width $w > 0$. The three axes of rotation intersect at the center of gravity O of the gyroscope. Therefore, its inertia tensor is given by

$$\mathrm{I_O} = \begin{bmatrix} \mathrm{I}_{\mathrm{O},x'} & & \\ & \mathrm{I}_{\mathrm{O},y'} & \\ & & \mathrm{I}_{\mathrm{O},z'} \end{bmatrix} = \frac{m}{12} \begin{bmatrix} 6\,r^2 & & \\ & 3\,r^2 + w^2 & \\ & & 3\,r^2 + w^2 \end{bmatrix}. \tag{15}$$

We further assume that an external torque $\boldsymbol{M}_{\mathrm{ext}}(t)$ acts on the second axis causing gyroscopic precession [37]. Let us first describe the gyroscope's orientation in terms of successive rotations about the suspension axes with the Euler angles $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ [19]. The kinematic relationships between the angular velocities along the body-fixed axes x', y', and z' and the time derivatives of the Euler angles are then given by

$$\begin{pmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \\ \dot{\gamma}(t) \end{pmatrix} = \frac{1}{\cos(\beta(t))} \begin{bmatrix} \cos(\beta(t)) & \sin(\alpha(t))\,\sin(\beta(t)) & \cos(\alpha(t))\,\sin(\beta(t)) \\ 0 & \cos(\alpha(t))\,\cos(\beta(t)) & -\sin(\alpha(t))\,\cos(\beta(t)) \\ 0 & \sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix} \begin{pmatrix} \omega_{x'}(t) \\ \omega_{y'}(t) \\ \omega_{z'}(t) \end{pmatrix}. \tag{16a}$$

Note that the singularity at $\beta = \pm\frac{\pi}{2}$ describes the state in which physical gimbal lock appears [37]. The kinetic equations of motion of the gyroscope read

$$\begin{bmatrix} \mathrm{I}_{\mathrm{O},x'} & & \\ & \mathrm{I}_{\mathrm{O},y'} & \\ & & \mathrm{I}_{\mathrm{O},z'} \end{bmatrix} \begin{pmatrix} \dot{\omega}_{x'}(t) \\ \dot{\omega}_{y'}(t) \\ \dot{\omega}_{z'}(t) \end{pmatrix} = \begin{bmatrix} 0 & -\mathrm{I}_{\mathrm{O},z'}\omega_{z'}(t) & \mathrm{I}_{\mathrm{O},y'}\omega_{y'}(t) \\ \mathrm{I}_{\mathrm{O},z'}\omega_{z'}(t) & 0 & -\mathrm{I}_{\mathrm{O},x'}\omega_{x'}(t) \\ -\mathrm{I}_{\mathrm{O},y'}\omega_{y'}(t) & \mathrm{I}_{\mathrm{O},x'}\omega_{x'}(t) & 0 \end{bmatrix} \cdot \begin{pmatrix} \omega_{x'}(t) \\ \omega_{y'}(t) \\ \omega_{z'}(t) \end{pmatrix} + \begin{bmatrix} 0 \\ \cos(\alpha(t)) \\ -\sin(\alpha(t)) \end{bmatrix} \boldsymbol{M}_{\mathrm{ext}}(t). \tag{16b}$$

Moreover, from (16a) we obtain that the angular velocity at the second axis is given by

$$\omega_{\mathrm{ext}}(t) = \dot{\beta}(t) = \begin{bmatrix} 0 & \cos(\alpha(t)) & -\sin(\alpha(t)) \end{bmatrix} \begin{pmatrix} \omega_{x'}(t) \\ \omega_{y'}(t) \\ \omega_{z'}(t) \end{pmatrix}. \tag{16c}$$

The gyroscope can therefore be written as a system of the form (10) without constraints or damping, and with trivial potential energy (i.e., $\tau_{\mathrm{d}} \equiv 0$ and $\mathcal{V}_{\mathrm{pot}} \equiv$

0). The redundant coordinates, velocities and external force are given by

$$\zeta(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \gamma(t) \end{pmatrix}, \quad \varpi(t) = \begin{pmatrix} \omega_{x'}(t) \\ \omega_{y'}(t) \\ \omega_{z'}(t) \end{pmatrix},$$
$$\tau_{\text{ext}}(t) = M_{\text{ext}}(t), \quad \varpi_{\text{ext}}(t) = \omega_{\text{ext}}(t). \tag{17}$$

Further, the mass matrix is $M = \mathrm{I_O}$ with $\mathrm{I_O} \in \mathbb{R}^{3\times 3}$ as in (15), and, for $\Gamma = (L_{x'}, L_{y'}, L_{z'})$, the kinematic, gyrator and external force matrix, respectively, read

$$G(\Gamma) = \begin{bmatrix} 0 & -L_{z'} & L_{y'} \\ L_{z'} & 0 & -L_{x'} \\ -L_{y'} & L_{x'} & 0 \end{bmatrix}, \quad B(\zeta) = \begin{bmatrix} 0 \\ \cos(\alpha) \\ -\sin(\alpha) \end{bmatrix},$$
$$Z(\zeta) = \frac{1}{\cos(\beta)} \begin{bmatrix} \cos(\beta) & \sin(\alpha)\sin(\beta) & \cos(\alpha)\sin(\beta) \\ 0 & \cos(\alpha)\cos(\beta) & -\sin(\alpha)\cos(\beta) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}. \tag{18}$$

Alternatively, instead of using Euler angles, we can describe the orientation of the gyroscope by a rotation matrix $R \in \mathbb{SO}_3$. By following the argumentation as in Sect. 5, we denote the columns of R by $R_1, R_2, R_3 \in \mathbb{R}^3$, and the kinematic relation is given by (11). The redundant velocities and external forces and velocities are now given as in (17), and the redundant coordinates read

$$\zeta(t) = \begin{pmatrix} R_1(t) \\ R_2(t) \\ R_3(t) \end{pmatrix}.$$

Further, as remarked at the end of Sect. 5, the Euler angles can be reconstructed from R unless $\beta = \frac{\pi}{2}$. Hence there exist according functions $\alpha(\zeta)$, $\beta(\zeta)$. The kinetic equation of motion then reads as in (16b), with external force matrix replaced by

$$B(\zeta) = \begin{bmatrix} 0 \\ \cos(\alpha(\zeta)) \\ -\sin(\alpha(\zeta)) \end{bmatrix},$$

Moreover, the kinematic matrix is given by $Z(\zeta) = Z_{\text{rot}}(R_1, R_2, R_3)$, where Z_{rot} is as in (11). The mass matrix and gyrator matrix now read as in (18).

7.3 Slider crank

To illustrate port-Hamiltonian interconnection, we consider a planar slider-crank mechanism as shown in Fig. 4.

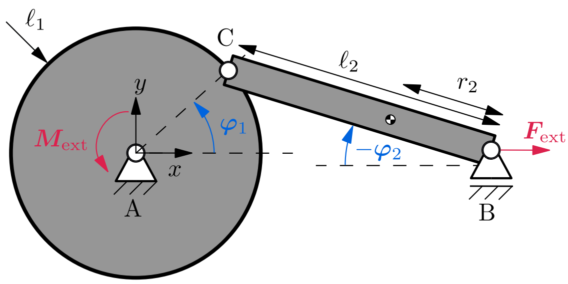

Fig. 4 Slider crank

The configuration consists of a crank (on the left), a slider and a connecting rod (on the right). For simplicity, the slider and the rod are modeled as one rigid body. We neither incorporate damping nor gravitational forces. The components are first modeled separately and then interconnected in the port-Hamiltonian framework.

We describe the dynamics of the crank about the fixed point A, around which it pivots. Consequently, there is no need to consider any translational coordinates. The crank's radius is ℓ_1, and its orientation is described by the angle $\varphi_1(t) \in \mathbb{R}$, and its angular velocity is denoted by $\omega_1(t) \in \mathbb{R}$. The moment of inertia of the crank about point A is given by $\mathrm{I}_{1,\mathrm{A}}$. An external torque $M_{\text{ext}}(t) \in \mathbb{R}$ is applied at point A and forms an external port together with $\omega_{\text{ext}}(t) := \omega_1(t)$. At the opposite end, the velocity $v_{1,\mathrm{C}}(t) \in \mathbb{R}^2$ and the force $F_{1,\mathrm{C}}(t) \in \mathbb{R}^2$ constitute another port, which is used to interconnect with the rod. The dynamics are described by

$$\begin{aligned} \dot{\varphi}_1(t) &= \omega_1(t), \\ \mathrm{I}_{1,\mathrm{A}}\, \dot{\omega}_1(t) &= M_{\text{ext}}(t) + \left[-\ell_1 \sin(\varphi_1(t)),\ \ell_1 \cos(\varphi_1(t))\right] \\ &\quad \times F_{1,\mathrm{C}}(t), \\ \omega_{\text{ext}}(t) &= \omega_1(t), \\ v_{1,\mathrm{C}}(t) &= \begin{bmatrix} -\ell_1 \sin(\varphi_1(t)) \\ \ell_1 \cos(\varphi_1(t)) \end{bmatrix} \omega_1(t). \end{aligned}$$

The system is of the form (10), with mass matrix $M = \mathrm{I}_{1,\mathrm{C}}$ and kinematics matrix $Z = 1$. The motion does not have any constraints. Moreover, the potential energy and the damping are trivial, and $n_{\text{kin}} = n_{\text{pot}} = 1$. Furthermore, the gyrator matrix is identically zero. The external ports are divided into two parts: on the one hand, the pair $(M_{\text{ext}}, \omega_{\text{ext}})$, and on the other hand, the pair $(F_{1,\mathrm{C}}, v_{1,\mathrm{C}})$. The latter will be used for interconnection, while the former remains an external port after the interconnection.

Next, we consider the rod of length ℓ_2 that is connected to the slider at point B. The combined mass of the rod and the slider is given by m_2 and its center of mass is defined by r_2. The rod has length ℓ_2 and its moment of inertia about point B is denoted as $\mathrm{I}_{2,\mathrm{B}}$. The orientation of the rod is given by the angle $\boldsymbol{\varphi}_2(t) \in \mathbb{R}$, and the position of point B is described by its coordinates $\boldsymbol{r}_\mathrm{B}(t) = (\boldsymbol{x}_\mathrm{B}(t), \boldsymbol{y}_\mathrm{B}(t))^\top \in \mathbb{R}^2$. The angular velocity is denoted by $\boldsymbol{\omega}_2(t) \in \mathbb{R}$, and the linear velocity in the body-fixed frame is given by $\boldsymbol{v}_2(t) = (\boldsymbol{v}_{2,x'}(t), \boldsymbol{v}_{2,y'}(t))^\top \in \mathbb{R}^2$. The slider's bearing in point B results in a holonomic constraint in vertical direction. An external force $\boldsymbol{F}_\mathrm{ext}(t) \in \mathbb{R}$ is acting horizontally on the slider and forms an external port with the velocity $\boldsymbol{v}_\mathrm{ext}(t) = \boldsymbol{v}_{2,x'}(t)$ of the slider. In point C, the interconnecting force $\boldsymbol{F}_{2,\mathrm{C}}(t) \in \mathbb{R}^2$ is acting on the rod. The corresponding velocity is denoted as $\boldsymbol{v}_{2,\mathrm{C}}(t) \in \mathbb{R}^2$. The dynamics are then given by

$$
\begin{pmatrix} \dot{\boldsymbol{r}}_\mathrm{B}(t) \\ \dot{\boldsymbol{\varphi}}_2(t) \end{pmatrix} = \begin{bmatrix} \cos(\boldsymbol{\varphi}_2(t)) & -\sin(\boldsymbol{\varphi}_2(t)) & 0 \\ \sin(\boldsymbol{\varphi}_2(t)) & \cos(\boldsymbol{\varphi}_2(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_2(t) \\ \boldsymbol{\omega}_2(t) \end{pmatrix},
$$

$$
\begin{aligned}
&\begin{bmatrix} m_2 & 0 & 0 \\ 0 & m_2 & m_2 r_2 \\ 0 & m_2 r_2 & \mathrm{I}_{2,\mathrm{B}} \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}_2(t) \\ \dot{\boldsymbol{\omega}}_2(t) \end{pmatrix} \\
&\quad = \begin{bmatrix} \cos(\boldsymbol{\varphi}_2(t)) & \sin(\boldsymbol{\varphi}_2(t)) \\ -\sin(\boldsymbol{\varphi}_2(t)) & \cos(\boldsymbol{\varphi}_2(t)) \\ -\ell_2 \sin(\boldsymbol{\varphi}_2(t)) & \ell_2 \cos(\boldsymbol{\varphi}_2(t)) \end{bmatrix} \boldsymbol{F}_{2,\mathrm{C}}(t) \\
&\qquad + \begin{bmatrix} 0 & 0 & m_2(\boldsymbol{v}_{2,y'}(t)+r_2\boldsymbol{\omega}_2(t)) \\ 0 & 0 & -m_2\boldsymbol{v}_{2,x'}(t) \\ -m_2(\boldsymbol{v}_{2,y'}(t)+r_2\boldsymbol{\omega}_2(t)) & m_2\boldsymbol{v}_{2,x'}(t) & 0 \end{bmatrix} \\
&\qquad \times \begin{pmatrix} \boldsymbol{v}_2(t) \\ \boldsymbol{\omega}_2(t) \end{pmatrix} + \begin{bmatrix} \sin(\boldsymbol{\varphi}_2(t)) \\ \cos(\boldsymbol{\varphi}_2(t)) \\ 0 \end{bmatrix} \boldsymbol{\lambda}(t) \\
&\qquad + \begin{bmatrix} \cos(\boldsymbol{\varphi}_2(t)) \\ -\sin(\boldsymbol{\varphi}_2(t)) \\ 0 \end{bmatrix} \boldsymbol{F}_\mathrm{ext}(t),
\end{aligned}
$$

$$
0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{r}_\mathrm{B}(t) \\ \boldsymbol{\varphi}_2(t) \end{pmatrix},
$$

$$
\begin{aligned}
\boldsymbol{v}_{2,\mathrm{C}}(t) &= \begin{bmatrix} \cos(\boldsymbol{\varphi}_2(t)) & -\sin(\boldsymbol{\varphi}_2(t)) & -\ell_2 \sin(\boldsymbol{\varphi}_2(t)) \\ \sin(\boldsymbol{\varphi}_2(t)) & \cos(\boldsymbol{\varphi}_2(t)) & \ell_2 \cos(\boldsymbol{\varphi}_2(t)) \end{bmatrix} \\
&\quad \times \begin{pmatrix} \boldsymbol{v}_2(t) \\ \boldsymbol{\omega}_2(t) \end{pmatrix},
\end{aligned}
$$

$$
\boldsymbol{v}_\mathrm{ext}(t) = \begin{bmatrix} \cos(\boldsymbol{\varphi}_2(t)) & -\sin(\boldsymbol{\varphi}_2(t)) & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_2(t) \\ \boldsymbol{\omega}_2(t) \end{pmatrix}.
$$

It can be verified that this system is again of the form (10), in particular with external port consisting of $\boldsymbol{F}_\mathrm{ext}(t)$, $\boldsymbol{v}_\mathrm{ext}(t)$ and $\boldsymbol{F}_{2,\mathrm{C}}(t)$, $\boldsymbol{v}_{2,\mathrm{C}}(t)$.

To model the overall system, that is, the interconnection of rod and crank, the latter quantities can be coupled with the force-velocity pair of the crank via

$$
\boldsymbol{F}_{1,\mathrm{C}}(t) = -\boldsymbol{F}_{2,\mathrm{C}}(t), \quad \boldsymbol{v}_{1,\mathrm{C}}(t) = \boldsymbol{v}_{2,\mathrm{C}}(t).
$$

This interconnection constraint is holonomic. Moreover, condition (13) is satisfied, since the constraint matrix of the combined system

$$
\boldsymbol{A}(\boldsymbol{\zeta}) = \begin{bmatrix} -\ell_1 \sin(\boldsymbol{\varphi}_1) & -\cos(\boldsymbol{\varphi}_2) & \sin(\boldsymbol{\varphi}_2) & \ell_2 \sin(\boldsymbol{\varphi}_2) \\ \ell_1 \cos(\boldsymbol{\varphi}_1) & -\sin(\boldsymbol{\varphi}_2) & -\cos(\boldsymbol{\varphi}_2) & -\ell_2 \cos(\boldsymbol{\varphi}_2) \end{bmatrix}
$$

has always full row rank. Based on our findings in Sect. 6, the interconnection gives rise to a port-Hamiltonian system with external flows $\boldsymbol{\omega}_\mathrm{ext}(t)$ and $\boldsymbol{v}_\mathrm{ext}(t)$, and corresponding external efforts $\boldsymbol{M}_\mathrm{ext}(t)$ and $\boldsymbol{F}_\mathrm{ext}(t)$. We omit providing the equations of the overall system.

Remark 4 In the presented theory of interconnection for port-Hamiltonian systems, some velocities of the subsystems are used for interconnection. However, in the case of the slider-crank mechanism, an interconnection based on position level would be more appropriate. We note that coupling via velocities, together with a suitable choice of initial conditions, implicitly yields the desired interconnection on position level.

At the modeling level, interconnection based on positions requires a modification of the standard port concept and calls for a theoretical framework involving the interconnection of Lagrangian submanifolds rather than Dirac structures. An appropriate adaptation of the port concept has already been proposed through the notion of 'energy ports' [38].

8 Conclusion

In this article, we have presented a port-Hamiltonian formulation of a class of multibody systems with rigid components. This includes, in particular, the formulation of suitable Dirac structures and Lagrangian submanifolds that allow for the incorporation of kinematic relations, position and velocity constraints, gyroscopic forces, and restoring forces arising from potential energy. Furthermore, we have shown that the port-Hamiltonian interconnection of such systems again yields a system of the same structural form. In this context, the interconnection relations behave like velocity constraints. Our theoretical developments are illustrated by three examples.

Author contributions Thomas Berger, René Hochdahl, Timo Reis, and Robert Seifried contributed equally to this work.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

competing interest The authors declare no Conflict of interest.

Appendix A: Proof of Proposition 2

Step 1: We show the statement under the additional assumption that $\ell = 0$ (i.e., there are no constraints). Denote the vector space as in (7), with additionally $\ell = 0$, by $\hat{\mathcal{D}}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})}$. Then an image representation of this space is given by

$$\hat{\mathcal{D}}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})} = \operatorname{im} \begin{bmatrix} 0 & Z(\boldsymbol{\zeta}) & 0 & 0 \\ -Z(\boldsymbol{\zeta})^\top & -G(\boldsymbol{\Gamma}) & -I_{n_{\text{kin}}} & -B(\boldsymbol{\zeta}) \\ 0 & I_{n_{\text{kin}}} & 0 & 0 \\ 0 & B(\boldsymbol{\zeta})^\top & 0 & 0 \\ I_{n_{\text{pot}}} & 0 & 0 & 0 \\ 0 & I_{n_{\text{kin}}} & 0 & 0 \\ 0 & 0 & I_{n_{\text{kin}}} & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} = \operatorname{im} \begin{bmatrix} J(\boldsymbol{\zeta},\boldsymbol{\Gamma}) \\ I_{\hat{n}} \end{bmatrix},$$

where $\hat{n} = n_{\text{pot}} + 2n_{\text{kin}} + m$ and $J : U_{\text{pos}} \times \mathbb{R}^{n_{\text{kin}}} \to \mathbb{R}^{\hat{n}\times\hat{n}}$ is continuous and pointwise skew-symmetric. Fix $(\boldsymbol{\zeta},\boldsymbol{\Gamma}) \in U_{\text{pos}} \times \mathbb{R}^{n_{\text{kin}}}$. Let $(f_i, e_i) \in \hat{\mathcal{D}}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})}$, $i = 1, 2$, then there exist $g_1, g_2 \in \mathbb{R}^{\hat{n}}$ such that

$$\begin{pmatrix} f_i \\ e_i \end{pmatrix} = \begin{pmatrix} J(\boldsymbol{\zeta},\boldsymbol{\Gamma}) g_i \\ g_i \end{pmatrix}, \quad i = 1, 2.$$

From this it follows that

$$f_2^\top e_1 + f_1^\top e_2 = \eta_2^\top J(\boldsymbol{\zeta},\boldsymbol{\Gamma})^\top \eta_1 + \eta_1^\top J(\boldsymbol{\zeta},\boldsymbol{\Gamma})^\top \eta_2 = 0.$$

On the other hand, if for any $f, e \in \mathbb{R}^{\hat{n}}$ we have that, for all $(J(\boldsymbol{\zeta},\boldsymbol{\Gamma})g, g) \in \hat{\mathcal{D}}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})}$,

$$0 = f^\top g + g^\top J(\boldsymbol{\zeta},\boldsymbol{\Gamma})^\top e = g^\top \big(f - J(\boldsymbol{\zeta},\boldsymbol{\Gamma})e\big),$$

then $f = J(\boldsymbol{\zeta},\boldsymbol{\Gamma})e$ as $g \in \mathbb{R}^{\hat{n}}$ is arbitrary. This shows that $\hat{\mathcal{D}}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})}$ is a Dirac structure. Property (b) of Definition 4 follows from continuity of $(\boldsymbol{\zeta},\boldsymbol{\Gamma}) \mapsto J(\boldsymbol{\zeta},\boldsymbol{\Gamma})$ with the linear and bijective maps $T_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})} := \begin{bmatrix} J(\boldsymbol{\zeta},\boldsymbol{\Gamma}) \\ I_{\hat{n}} \end{bmatrix}$. Therefore, $(\hat{\mathcal{D}}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})})_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})\in U_{\text{pos}}\times\mathbb{R}^{n_{\text{kin}}}}$ is a modulated Dirac structure.

Step 2: We prove the general statement for $\ell > 0$. Since A has constant rank on U_{pos}, we find that

$$E : U_{\text{pos}} \times \mathbb{R}^{n_{\text{kin}}} \to \mathbb{R}^{\ell\times(n_{\text{pot}}+2n_{\text{kin}}+m)}, \ (\boldsymbol{\zeta},\boldsymbol{\Gamma}) \mapsto [0_{\ell\times n_{\text{pot}}}, A(\boldsymbol{\zeta}), 0_{\ell\times(n_{\text{kin}}+m)}]$$

has constant rank on $U_{\text{pos}} \times \mathbb{R}^{n_{\text{kin}}}$. Further, the structure of $\hat{\mathcal{D}}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})}$ yields

$$\forall\, (\boldsymbol{\zeta},\boldsymbol{\Gamma}) \in U_{\text{pos}} \times \mathbb{R}^{n_{\text{kin}}} : \dim\Big(\hat{\mathcal{D}}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})} \cap \big(\mathbb{R}^{n_{\text{pot}}+2n_{\text{kin}}+m} \times \ker E(\boldsymbol{\zeta},\boldsymbol{\Gamma})\big)\Big) = n_{\text{pot}} + 2n_{\text{kin}} + m - \operatorname{rank} A(\boldsymbol{\zeta}),$$

which is constant. As a consequence, the assumptions of Proposition 4 are fulfilled, and we can conclude that $(\mathcal{D}_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})})_{(\boldsymbol{\zeta},\boldsymbol{\Gamma})\in U_{\text{pos}}\times\mathbb{R}^{n_{\text{kin}}}}$ is a modulated Dirac structure. □

Appendix B Auxiliary results on Dirac structures and Lagrangian submanifolds

We present some results for special modulated Dirac structures and Lagrangian submanifolds that play a key role in this work. The necessary theoretical results for the proofs in this appendix are drawn from existing literature. The findings presented here are applied in Sections 3 and 4, which focus on mechanical systems.

We begin with a preparatory lemma on local image representations of continuous matrix-valued functions with pointwise full row rank.

Lemma 3 *Let $U \subset \mathbb{R}^k$ be an open set, and let $E : U \to \mathbb{R}^{\ell\times n}$ be a continuous function. Assume that E has constant rank $r \in \mathbb{N}_0$, i.e., $\operatorname{rank} E(x) = r$ for all $x \in U$. Then, for every $x \in U$, there exists a neighborhood $U_x \subset U$ of x and a continuous function $J : U_x \to \mathbb{R}^{n\times(n-r)}$, such that $\operatorname{im} J(y) = \ker E(y)$ for all $y \in U_x$.*

Proof Fix $x \in U$. Without loss of generality, we assume that $E(x)$ admits a partition

$$E(y) = \begin{bmatrix} E_{11}(y) & E_{12}(y) \\ E_{21}(y) & E_{22}(y) \end{bmatrix}, \quad y \in U,$$

such that $E_{11}(x) \in \mathbb{R}^{r \times r}$ is invertible (otherwise, rearrange the rows and columns of $E(x)$ accordingly). Since E is continuous and $E_{11}(x)$ is invertible, there exists a neighborhood $U_x \subset U$ of x such that $E_{11}(y)$ is invertible for all $y \in U_x$. As a consequence, since $\operatorname{rank} E(y) = r$ for all $y \in U_x$, there exists $R : U_x \to \mathbb{R}^{(\ell-r)\times r}$ such that $[E_{21}(y), E_{22}(y)] = R(y)[E_{11}(y), E_{12}(y)]$ for all $y \in U_x$. Then the function $J : U_x \to \mathbb{R}^{n\times(n-r)}$ with

$$J(y) = \begin{bmatrix} -E_{11}(y)^{-1}E_{12}(y) \\ I_{n-r} \end{bmatrix}$$

has the desired properties. □

Next we investigate a special modulated Dirac structure that is derived from another one by incorporating an additional type of constraint. This is used in our analysis of mechanical systems to account for velocity constraints.

Proposition 4 *Let $U \subset \mathbb{R}^k$ be open and $(\mathcal{D}_x)_{x\in U}$ be a modulated Dirac structure with $\mathcal{D}_x \subset \mathbb{R}^n \times \mathbb{R}^n$ for all $x \in U$. Further, let $E : U \to \mathbb{R}^{\ell\times n}$ be continuous and assume that the dimension of the space*

$$\mathcal{D}_x \cap \left(\mathbb{R}^n \times \ker E(x)\right) \tag{19}$$

is independent of $x \in U$. Then for

$$\widetilde{\mathcal{D}}_x := \left\{ (f, e) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; E(x)e = 0,\ \exists\, \mu \in \mathbb{R}^\ell : \left(f + E(x)^\top \mu, e\right) \in \mathcal{D}_x \right\}, \quad x \in U,$$

we have that $(\widetilde{\mathcal{D}}_x)_{x\in U}$ is a modulated Dirac structure.

Proof Fix $x \in U$.

Step 1: We show that $\widetilde{\mathcal{D}}_x$ is a Dirac structure. A straighforward calculation yields that

$$\forall\, (f, e), (\hat{f}, \hat{e}) \in \widetilde{\mathcal{D}}_x : \quad \hat{f}^\top e + f^\top \hat{e} = 0.$$

Next assume that $(f, e) \in \mathbb{R}^n \times \mathbb{R}^n$ fulfills $\hat{f}^\top e + f^\top \hat{e} = 0$ for all $(\hat{f}, \hat{e}) \in \widetilde{\mathcal{D}}_x$. We use the orthogonal decomposition $\mathbb{R}^n = \operatorname{im} E(x)^\top + \ker E(x)$ (that is, $(\ker E(x))^\perp = \operatorname{im} E(x)^\top$) to obtain a decomposition $f = f_1 + f_2$, $e = e_1 + e_2$ with $f_1, e_1 \in \operatorname{im} E(x)^\top$, $f_2, e_2 \in \ker E(x)$. Then, for all $(\hat{f}_1, \hat{e}) \in \mathcal{D}_x$ with $E(x)\hat{e} = 0$, $\hat{\mu} \in \mathbb{R}^\ell$, we have that $(\hat{f}_1 + E(x)^\top \hat{\mu}, \hat{e}) \in \widetilde{\mathcal{D}}_x$ and hence

$$\begin{aligned} 0 &= (\hat{f}_1 + E(x)^\top \hat{\mu})^\top e + f^\top \hat{e} \\ &= \left(\hat{f}_1 + E(x)^\top \hat{\mu}\right)^\top (e_1 + e_2) + (f_1 + f_2)^\top \underbrace{\hat{e}}_{\in \ker E(x)} \\ &= \hat{f}_1^\top (e_1 + e_2) + \hat{\mu}^\top E(x) e_1 + f_2^\top \hat{e}. \end{aligned}$$

This gives, by setting $(\hat{f}_1, \hat{e}) = (0, 0)$ and $\hat{\mu} = 0$ one after the other, that $e_1 \in \ker E(x)$ and

$$\forall\, (\hat{f}_1, \hat{e}) \in \mathcal{D}_x : \quad 0 = \hat{f}_1^\top (e_1 + e_2) + f_2^\top \hat{e}.$$

Using $e_1 \in \ker E(x)$ together with $e_1 \in \operatorname{im} E(x)^\top$, we obtain that $e_1 = 0$, and thus

$$\forall\, (\hat{f}_1, \hat{e}) \in \mathcal{D}_x : \quad 0 = \hat{f}_1^\top e_2 + f_2^\top \hat{e}.$$

Since $\mathcal{D}_x$ is a Dirac structure, this implies that $(f_2, e_2) \in \mathcal{D}_x$. Now $e_1 = 0$ gives $e = e_2$, whence $(f_2, e) \in \mathcal{D}_x$. By $f_1 \in \operatorname{im} E(x)^\top$ there exists $\mu \in \mathbb{R}^\ell$ such that $f_1 = E(x)^\top \mu$, and thus

$$(f, e) = (f_2 + f_1, e) = (f_2 + E(x)^\top \mu, e) \in \widetilde{\mathcal{D}}_x.$$

Step 2: We show that there exists a neighborhood $U_x \subset U$ of x and a family $(T_y)_{y\in U_x}$ of linear and bijective mappings $T_y : \mathbb{R}^n \to \mathcal{D}_y$, such that, for all $z \in \mathbb{R}^n$, the mapping $y \mapsto T_y z$ is continuous from U_x to $\mathbb{R}^n \times \mathbb{R}^n$.

The definition of a modulated Dirac structure yields that there exists a neighborhood $U_{1,x} \subset U$ of x, and continuous $K, L : U_{1,x} \to \mathbb{R}^{n\times n}$, such that

$$\forall\, y \in U_{1,x} : \quad \mathcal{D}_y = \operatorname{im} \begin{bmatrix} K(y) \\ L(y) \end{bmatrix}.$$

Then

$$\begin{aligned} &\forall\, y \in U_{1,x} : \quad \mathcal{D}_y \cap \left(\mathbb{R}^n \times \ker E(y)\right) \\ &\quad = \begin{bmatrix} K(y) \\ L(y) \end{bmatrix} \ker E(y)L(y). \end{aligned} \tag{20}$$

Since $\dim \mathcal{D}_y = n$, the matrix $\begin{bmatrix} K(y) \\ L(y) \end{bmatrix}$ has full column rank. Hence, a combination of (20) with the rank-nullity theorem yields that

$$\begin{aligned} \operatorname{rank} E(y)L(y) &= n - \dim \ker E(y)L(y) \\ &= n - \dim \left(\begin{bmatrix} K(y) \\ L(y) \end{bmatrix} \times \ker E(y)L(y) \right) = n - \dim \\ &\quad \times \left(\mathcal{D}_y \cap \left(\mathbb{R}^n \times \ker E(y)\right) \right), \end{aligned}$$

which is independent of y by the assumption that the last term is constant. Consequently, $E(y)L(y)$ has constant rank, which we denote by r_1. Then, by Lemma 3, there exists a neighborhood $U_{2,x} \subset U_{1,x}$ of x and some continuous $J_1 : U_{2,x} \to \mathbb{R}^{n\times(n-r_1)}$, such that $\operatorname{im} J_1(y) = \ker E(y)L(y)$ for all $y \in U_{2,x}$. The definition of $\widetilde{\mathcal{D}}_y$ yields that

$$\forall\, y \in U_{2,x}: \quad \widetilde{\mathcal{D}}_y = \operatorname{im} \underbrace{\begin{bmatrix} K(y)J_1(y) & E(y)^\top \\ L(y)J_1(y) & 0 \end{bmatrix}}_{=:D(y)\in\mathbb{R}^{2n\times(r_1+\ell)}},$$

which can be seen as follows: If $(f,e) \in \widetilde{\mathcal{D}}_y$, then $e \in \ker E(y)$ and $(f + E(y)^\top \mu, e) \in \mathcal{D}_y$ for some $\mu \in \mathbb{R}^\ell$, that is

$$\begin{pmatrix} f + E(y)^\top \mu \\ e \end{pmatrix} \in \begin{bmatrix} K(y) \\ L(y) \end{bmatrix} \ker E(y)L(y)$$
$$\implies \quad \begin{pmatrix} f \\ e \end{pmatrix} \in \operatorname{im} \begin{bmatrix} K(y)J_1(y) & E(y)^\top \\ L(y)J_1(y) & 0 \end{bmatrix}.$$

On the other hand, let $\left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) \in \operatorname{im} \begin{bmatrix} K(y)J_1(y) & E(y)^\top \\ L(y)J_1(y) & 0 \end{bmatrix}$, then

$$\begin{pmatrix} f \\ e \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ \tilde{e} \end{pmatrix} + \begin{bmatrix} E(y)^\top \\ 0 \end{bmatrix} \mu,$$
$$\begin{pmatrix} \tilde{f} \\ \tilde{e} \end{pmatrix} \in \mathcal{D}_y \cap \left(\mathbb{R}^n \times \ker E(y)\right), \ \mu \in \mathbb{R}^\ell,$$

from which it follows that $e \in \ker E(y)$ and $(f - E(y)^\top \mu, e) \in \mathcal{D}_y$, thus $(f,e) \in \widetilde{\mathcal{D}}_y$.

Since $\dim \widetilde{\mathcal{D}}_y = n$, we have that $\operatorname{rank} D(y) = n$ for all $y \in U_{2,x}$. Consequently, again using Lemma 3, there exists a neighborhood $U_{3,x} \subset U_{2,x}$ of x, and some continuous $J_2 : U_{3,x} \to \mathbb{R}^{(r_1+\ell)\times(r_1+\ell-n)}$, such that $\operatorname{im} J_2(y) = \ker D(y)$ for all $y \in U_{3,x}$. Consequently, J_2 has pointwise full column rank, and another use of Lemma 3 yields that there exists a neighborhood $U_{4,x} \subset U_{3,x}$ of x, and some continuous $J_3 : U_{4,x} \to \mathbb{R}^{(r_1+\ell)\times n}$, such that $\operatorname{im} J_3(y) = \ker J_2(y)^\top$ for all $y \in U_{3,x}$. Using that $\operatorname{im} J_2(y)$ is the orthogonal complement of $\operatorname{im} J_3(y) = \ker J_2(y)^\top$, we have that $[J_2(y), J_3(y)]$ is an invertible matrix for all $y \in U_{4,x}$. Hence, for all $y \in U_{4,x}$,

$$\widetilde{\mathcal{D}}_y = \operatorname{im} D(y) = \operatorname{im} D(y)[J_2(y), J_3(y)]$$
$$= \operatorname{im} D(y)J_3(y).$$

Since $D(y)J_3(y) \in \mathbb{R}^{2n\times n}$ has full column rank, the desired result holds for $U_x = U_{4,x}$ and $T_y : \mathbb{R}^n \to \mathcal{D}_y$ with $z \mapsto D(y)J_3(y)z$. □

Remark 5 In Proposition 4, the assumption that the rank of the space $\mathcal{D}_x \cap \left(\mathbb{R}^n \times \ker E(x)\right)$ is constant is essential for $(\widetilde{D}_x)_{x\in U}$ to be a modulated Dirac structure. As a counterexample, consider the (constant) Dirac structure

$$\mathcal{D} = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and the matrix-valued function $E : \mathbb{R} \to \mathbb{R}^{1\times 2}$, $x \mapsto [1,\ x]$. Then the subspaces $(\widetilde{\mathcal{D}}_x)_{x\in\mathbb{R}}$, as constructed in Proposition 4, satisfy $\widetilde{\mathcal{D}}_0 = \mathcal{D}$ and

$$\forall\, x \in \mathbb{R}\setminus\{0\}: \quad \widetilde{\mathcal{D}}_x = \mathbb{R}^2 \times \{0\}.$$

In particular, for any neighborhood $U_0 \subset \mathbb{R}$ of zero and any family $(T_y)_{y\in U_0}$ of linear bijective mappings $T_y : \mathbb{R}^2 \to \widetilde{\mathcal{D}}_y$, we find that for $z \in \mathbb{R}^2$ with $T_0 z = (0,0,0,1)$, the mapping $y \mapsto T_y z$ is discontinuous at zero. Therefore, $(\widetilde{\mathcal{D}}_x)_{x\in\mathbb{R}}$ is not a modulated Dirac structure.

Indeed, the assumptions of Proposition 4 do not hold in this case, since $\mathcal{D} \cap (\mathbb{R}^2 \times \ker E(0))$ is one-dimensional, whereas $\mathcal{D} \cap (\mathbb{R}^2 \times \ker E(x)) = \{0\}$ for all $x \in \mathbb{R}\setminus\{0\}$.

It was mentioned in Section 2 that the graphs of gradient fields are Lagrangian submanifolds. In the following, we present a generalization of this result, where a gradient field is considered alongside a certain type of restriction. This generalization is used in this article to appropriately formulate position constraints. The following result can essentially be deduced from the findings in [8, Sec. 4.2] on so-called *Morse families*; see also [39,40]. However, in these references, the result is embedded within rather abstract differential geometric concepts. For this reason, we have chosen to present an elementary proof here.

Proposition 5 *Let $U \subset \mathbb{R}^n$ be open, and let $\mathcal{H} : U \to \mathbb{R}$ and $d : U \to \mathbb{R}^k$, $k, n \in \mathbb{N}_0$, be twice continuously differentiable. Further assume that $d'(x)$ has full row rank for all $x \in U$, and the set $\{\, x \in U \mid d(x) = 0 \,\}$ is non-empty. Then*

$$\mathcal{L} := \Big\{ (x, \nabla\mathcal{H}(x) + d'(x)^\top \lambda) \ \Big|\ x \in U \text{ with } d(x) = 0,\ \lambda \in \mathbb{R}^k \Big\} \subset \mathbb{R}^n \times \mathbb{R}^n$$

is a Lagrangian submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

Proof It can be seen that $\mathcal{L}$ is a differentiable submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. In the sequel we show that it is Lagrangian.

Step 1: We calculate $T_z\mathcal{L}$ for some fixed $z \in \mathcal{L}$. First observe that by the assumption that d' has full row rank everywhere it is a submersion, cf. [29, Ch. 4]. Then the submersion theorem (see e.g. [29, Cor. 5.13]) implies that the non-empty set $M := \{ x \in U \mid d(x) = 0 \}$ is a submanifold with dimension $\dim M = n - k$. Partitioning $z = (z_1, z_2)$, then $z_1 \in M$ and there exists a neighborhood $V \subseteq \mathbb{R}^{n-k}$ of zero and a chart $g_1 : V \to U$ of M at z_1, i.e., $g_1(0) = z_1$ and $\ker g_1'(0) = \{0\}$ as well as $\operatorname{im} g_1'(0) = T_{z_1}M$. Furthermore, since $d(g_1(v)) = 0$ for all $v \in V$, by differentiation we obtain that

$$d'(g_1(v))g_1'(v) = 0.$$

Since $\dim \ker d'(x) = n - k$ for all $x \in U$ by assumption, $g_1(0) = z_1$ and $\ker g_1'(0) = \{0\}$ we find that

$$\ker d'(z_1) = \operatorname{im} g_1'(0) = T_{z_1}M. \tag{21}$$

Set $W := V \times \mathbb{R}^k \subset \mathbb{R}^n$ and define

$$g : W \to \mathbb{R}^n \times \mathbb{R}^n,\ (w_1, w_2) \mapsto \big(g_1(w_1), \nabla\mathcal{H}(g_1(w_1)) - d'(g_1(w_1))^\top(\hat{z} + w_2)\big),$$

where

$$\hat{z} := \big(d'(z_1)d'(z_1)^\top\big)^{-1} d'(z_1)\big(\nabla\mathcal{H}(z_1) - z_2\big) \in \mathbb{R}^k.$$

Then g is a chart of $\mathcal{L}$ at z, hence

$$T_z\mathcal{L} = \operatorname{im} g'(0).$$

We have that

$$g'(w_1, w_2) = \begin{bmatrix} g_1'(w_1) & 0 \\ \nabla^2\mathcal{H}(g_1(w_1))g_1'(w_1) - D(g_1(w_1))(\hat{z} + w_2)g_1'(w_1) & -d'(g_1(w_1))^\top \end{bmatrix} \in \mathbb{R}^{2n \times n},$$

where $\nabla^2\mathcal{H}$ is the Hessian of $\mathcal{H}$ and $D = ((d')^\top)' : \mathbb{R}^n \to L(\mathbb{R}^k, \mathbb{R}^{n\times n})$, where the latter denotes the space of linear mappings from $\mathbb{R}^k$ to $\mathbb{R}^{n\times n}$. Then we find that

$$T_{(z_1,z_2)}\mathcal{L} = \left\{ (v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; \begin{array}{l} v_1 \in \operatorname{im} g_1'(0), \\ v_2 - \big(\nabla^2\mathcal{H}(z_1) - D(z_1)\hat{z}\big)v_1 \in \operatorname{im} d'(z_1)^\top \end{array} \right\}.$$

By the symmetry of second derivatives we have that

$$(D(z_1)\hat{z})_{ij} = \sum_{l=1}^{k} \frac{\partial^2 d_l}{\partial x_i \partial x_j}(z_1)\hat{z}_l = (D(z_1)\hat{z})_{ji}$$

for any $i, j = 1, \ldots, n$, and hence $D(z_1)\hat{z} \in \mathbb{R}^{n\times n}$ is symmetric. Furthermore, the Hessian $\nabla^2\mathcal{H}(z_1)$ is symmetric, hence $M(z_1, z_2) := \nabla^2\mathcal{H}(z_1) - D(z_1)\hat{z}$ is symmetric. Finally, by (21) we obtain that

$$T_{(z_1,z_2)}\mathcal{L} = \left\{ (v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; v_1 \in \ker d'(z_1),\ v_2 - M(z_1, z_2)v_1 \in \big(\ker d'(z_1)\big)^\perp \right\}.$$

Step 2: Fix $z = (z_1, z_2) \in \mathcal{L}$ and $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$. We show that property (1) is satisfied.

$\Rightarrow$: Assume that $(v_1, v_2) \in T_z\mathcal{L}$ and let $(w_1, w_2) \in T_z\mathcal{L}$ be arbitrary. Let $x, y \in \big(\ker d'(z_1)\big)^\perp$ be such that

$$v_2 - M(z_1, z_2)v_1 = x, \quad w_2 - M(z_1, z_2)w_1 = y$$

and observe that $v_1^\top y = w_1^\top x = 0$. Then we have

$$\begin{aligned} v_1^\top w_2 - w_1^\top v_2 &= v_1^\top M(z_1, z_2)w_1 + v_1^\top y \\ &\quad - w_1^\top M(z_1, z_2)v_1 - w_1^\top x \\ &= v_1^\top M(z_1, z_2)w_1 - w_1^\top M(z_1, z_2)v_1 \\ &= 0, \end{aligned}$$

where the last equality follows from symmetry of $M(z_1, z_2)$.

$\Leftarrow$: Let $w_1 \in \ker d'(z_1)$ and $y \in \big(\ker d'(z_1)\big)^\perp$ be arbitrary. Then $(w_1, y + M(z_1, z_2)w_1) \in T_z\mathcal{L}$ and hence we have

$$\begin{aligned} 0 &= v_1^\top w_2 - w_1^\top v_2 \\ &= v_1^\top M(z_1, z_2)w_1 + v_1^\top y - w_1^\top v_2 \\ &= \big(M(z_1, z_2)v_1 - v_2\big)^\top w_1 + v_1^\top y. \end{aligned}$$

Since w_1 and y are arbitrary we obtain that $M(z_1, z_2)v_1 - v_2 \in \big(\ker d'(z_1)\big)^\perp$ and $v_1 \in \ker d'(z_1)$, thus $(v_1, v_2) \in T_z\mathcal{L}$. This completes the proof. □

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