Lower variance bounds and normal approximation of Poisson functionals in stochastic geometry

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Summary

In this thesis, we study the asymptotic behaviour of sequences of Poisson functionals, i.e. random variables that depend on a Poisson process. An essential step in establishing normal approximation results for Poisson functionals is in many cases the determination of lower variance bounds. To address this, a generalised reverse Poincaré inequality is introduced in this thesis, which provides a lower bound for the variance of Poisson functionals that depends on the difference operator of order n of Malliavin calculus. As demonstrated throughout the thesis, by combining this generalised reverse Poincaré inequality with already existing normal approximation results such as those of G. Last, G. Peccati and M. Schulte from 2016 or those of M. Schulte and J. Yukich from 2023, one can derive qualitative and quantitative central limit theorems. With the help of the latter, it is even possible to obtain multivariate quantitative normal approximation results. Some of these results require the positive definiteness of the covariance matrices of the considered vectors of Poisson functionals. As illustrated in some of our applications, the generalised reverse Poincaré inequality can be used to prove this property as well. In contrast to previous results for lower variance bounds, the generalised reverse Poincaré inequality mainly simplifies the problem of deriving a lower variance bound for Poisson functionals to constructing configurations that arise with a positive probability and for which the difference operator of a fixed order is bounded away from zero.

Since Poisson processes can be interpreted as collections of random points, Poisson functionals have many applications in stochastic geometry. Hence, to illustrate how to derive lower variance bounds by constructing specific point configurations, we explore different applications from stochastic geometry: random graphs, random polytopes and excursion sets of Poisson shot noise processes.

We start by showing the positive definiteness of the covariance matrix for different vectors of functionals of spatial random graphs in Euclidean space, beginning with vectors of degree counts or component counts of random geometric graphs and proceeding to vectors of degree counts or total edge length functionals of k nearest neighbour graphs. To highlight the influence of the geometry of the underlying space, we further consider radial spanning trees in hyperbolic space, analyse the behaviour of their total edge length functionals and compare them to their counterpart in Euclidean space. Next, we consider random polytopes in the Euclidean unit ball and use the reverse Poincaré inequality to show positive definiteness of the covariance matrix and a two-dimensional central limit theorem for the vector of two L^p surface areas. Finally, we consider a special class of random sets, namely excursion sets of Poisson shot noise processes. For different families of kernel functions, we analyse the asymptotic behaviour of the expectation and the variance of several geometric functionals of excursion sets of Poisson shot noise processes and derive corresponding central limit theorems.

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Contents

Chapter 1

Introduction

The classical central limit theorem states that for a sequence $(X_i)_{i\in\mathbb{N}}$ of i.i.d. random variables with $\mathbb{E}[X_1] = \mu$ and $\mathrm{Var}[X_1] = \sigma^2 \in (0,\infty)$ it holds for $S_n = \frac{1}{\sqrt{n\sigma^2}} \sum_{i=1}^n (X_i - \mu)$ with $n \in \mathbb{N}$ that

$$S_n \xrightarrow{d} N \quad \text{as} \quad n \to \infty,$$

where N is a standard normally distributed random variable and $\xrightarrow{d}$ denotes convergence in distribution (see e.g. [61, Theorem 15.37]). Although it is a powerful statement in the sense that it is true independently of the original distribution of the considered random variables, it does not answer the question of how close the distribution of the standardised sum S_n is to a standard normal distribution for fixed $n \in \mathbb{N}$. This question was answered by the Berry-Esseen theorem, which was first shown in 1941 and 1942 independently by A. C. Berry in [19] and C.-G. Esseen in [45]. It states that if $\mathbb{E}[|X_1|^3] < \infty$, there exists a constant $C > 0$ such that

$$\sup_{x\in\mathbb{R}} |\mathbb{P}(S_n \le x) - \mathbb{P}(N \le x)| \le \frac{C\mathbb{E}[|X_1|^3]}{\sigma^3\sqrt{n}}$$

for all $n \in \mathbb{N}$ (see e.g. [61, Theorem 15.51]).

Naturally, the question arises of what happens if we relax the conditions on the random variables $(X_i)_{i\in\mathbb{N}}$ and allow random variables that are not identically distributed or weakly dependent. On the one hand, if the considered random variables are not identically distributed, one can show a central limit theorem, for example, under the so-called Lindeberg condition (see for instance [61, Theorem 15.43]). On the other hand, for random variables with specific dependency structures one can also derive central limit theorems in some cases (see e.g. [37]). The aim of this thesis is to show qualitative and quantitative central limit theorems for sequences of Poisson functionals arising in stochastic geometry, which can be interpreted in a certain sense as sums of weakly dependent random variables.

1.1 Normal approximation and lower variance bounds of Poisson functionals

A Poisson functional is a random variable that depends on a Poisson process. In this thesis, we consider sequences of Poisson functionals $(F_s)_{s\ge1}$. Our aim is to analyse the asymptotic behaviour of their distributions as $s \to \infty$ and, specifically, show that their distributions approach a normal distribution. In order to compare two probability distributions, we need to be able to measure the distance between them. To this end, for two random variables X and Y we typically use distances of the form

$$\mathrm{d}(X,Y) = \sup_{h\in\mathcal{H}} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|,$$

where $\mathcal{H}$ is a suitable class of test functions. Note that the distance used in the Berry-Esseen theorem, also known as Kolmogorov distance, fits into this framework by choosing $\mathcal{H}$ as the family of indicator functions of the form $\mathbb{1}\{\cdot \leq x\}$ for $x \in \mathbb{R}$. Besides the Kolmogorov distance, we are also interested in quantitative normal approximation results in Wasserstein distance and, in the multivariate case, in d_2- and d_{convex}-distance, which are also of this form (see Subsection 2.3.3).

In order to derive qualitative and quantitative central limit theorems for $(F_s)_{s\geq 1}$, we mainly use the results from [69], [70] and [103], which are based on the Malliavin-Stein method. Malliavin calculus is a variational calculus for random variables, which was first introduced for the Gaussian setting in [77] and later transferred to the Poisson case in [81]. Stein's method is a collection of techniques to derive qualitative central limit theorems, which started with the paper [106] by C. Stein and was further developed in several works. For an overview, see for example [37]. The Malliavin-Stein method is a combination of Stein's method and Malliavin calculus, which was first introduced in the Gaussian setting in [79] and applied to Poisson functionals in [82]. Over time, using the Malliavin-Stein method, normal approximation results for Poisson functionals were shown for the Wasserstein distance in [82] and the Kolmogorov distance in [44] and [99]. Additionally, [67] and [70] derived bounds for both distances. In contrast to the previous results, in which the normal approximation bounds also depend on the inverse Ornstein-Uhlenbeck generator, the results in [67] and [70] only depend on the first order or the first and second order difference operator, respectively. Since, by [67, Remark 1.1], in case that the results from [67] and [70] are both applicable, one can expect the results from [70] to be tighter, we mainly use the results from [70] for the Kolmogorov and Wasserstein distance, which provide second order Poincaré inequalities.

The concept of stabilisation is repeatedly used in the context of normal approximation. In the literature the exact definitions of stabilisation differ but, roughly speaking, the idea of stabilisation is to consider functionals that are in some sense locally defined. In this thesis we stick to the definition of stabilising functionals from [69] (see also Subsection 2.3.4). The concept of stabilisation was first introduced in the context of minimal spanning trees in [74] and [75] and over time further extended and improved in, for example, [14, 15, 67, 69, 85, 87, 88] for the univariate case and in [84, 86, 103] for the multivariate case. To handle those of our functionals that are stabilising, we mainly use the results from [69]. For multivariate normal approximation of stabilising functionals we employ the results from [103], which provide bounds for the d_2-, d_3- and d_{convex}-distance.

With the described normal approximation results one can show in many applications estimates of the form

$$\mathrm{d}\left(\frac{F_s - \mathbb{E}[F_s]}{\mathrm{Var}[F_s]}, N\right) \leq \sum_{i=1}^{k} \frac{C_i(s)}{\mathrm{Var}[F_s]^{\alpha_i}}$$

for some distance d, some $k \in \mathbb{N}$, expressions $C_i(s) > 0$, which might depend on s and powers $\alpha_i > 0$ for $i \in \{1, \dots, k\}$, if s is sufficiently large. This leads to the question of how

$\mathrm{Var}[F_s]$ can be bounded from below and how it can be controlled for $s \to \infty$. Investigating the behaviour of the variance usually requires a lot of effort. Even if one has an explicit expression for the asymptotic variance, it can be hard to show positivity because positive and negative terms could cancel out. Therefore, proving the non-degeneracy of the asymptotic variance can be a different problem than computing an explicit representation for it. In this case, it can be helpful to employ lower bounds for variances to deduce positivity of the asymptotic variance constant. This is the reason why this thesis treats the problem of deriving a lower variance bound as a separate issue from establishing central limit theorems. To this end in our main theoretical result in Chapter 3, we introduce a lower variance bound for Poisson functionals, which can be seen as a generalised reverse Poincaré inequality. This lower bound depends on the difference operator of Malliavin calculus and mainly reduces the problem of deriving a lower variance bound to constructing configurations, which occur with a positive probability and for which the difference operator is bounded away from zero.

In the literature, lower variance bounds were already derived for several applications (see e.g. [5, 11, 15, 38, 70, 86, 87, 90]). There exist several strategies for deriving lower variance bounds. The approach that is most comparable to ours is the strategy from [70, Theorem 5.2] although there the lower variance bound contains the expectation of a difference of random variables, which can be both positive or negative and can thus neutralise itself in the expectation. This cannot happen in our main theorem as the random variable in the expectation in our theorem is non-negative. For a more detailed comparison between these two results and an overview of the other strategies used in the literature, see also Section 3.3.

Lower variance bounds can also be used for multivariate normal approximation since the covariance matrix $\Sigma_s \in \mathbb{R}^{m\times m}$ of Poisson functionals $F_s^{(1)}, \dots, F_s^{(m)}$, $s \geq 1$, satisfies

$$\mathrm{Var}\left[\sum_{i=1}^{m} \alpha_i F_s^{(i)}\right] = \alpha^T \Sigma_s \alpha$$

for all $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$. This means that we can check whether Σ_s is positive definite by verifying whether $\mathrm{Var}[\sum_{i=1}^{m} \alpha_i F_s^{(i)}] > 0$ for all $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m \setminus \{(0, \dots, 0)\}$. If the asymptotic covariance matrix $\Sigma = \lim_{s\to\infty} \Sigma_s$ exists, this relation can also be used to establish the positive definiteness of Σ, which is required in some bounds for the quantitative multivariate normal approximation (see e.g. [103]).

1.2 Poisson functionals in stochastic geometry

Since Poisson processes can be interpreted as collections of random points in some space, Poisson functionals have a wide range of applications in stochastic geometry. In this thesis, we are mostly interested in the asymptotic behaviour of Poisson functionals in two frameworks, namely the one of increasing intensity and the one of increasing observation window. More precisely, we study for $s \to \infty$ a family of Poisson functionals $(F_s)_{s\geq 1}$, where F_s is either a Poisson functional on a homogeneous Poisson process with intensity s or a functional of a fixed Poisson process depending on an observation window that extends to the full space for $s \to \infty$. Geometrically, these two frameworks coincide with the idea of having either a fixed observation window, in which we add more and more points randomly or the idea of keeping the expected number of points in a fixed area constant but considering larger and larger observation windows. In Euclidean space one can often use scaling properties to switch from one framework to the other one but for other underlying spaces (such as, e.g., the hyperbolic space, which we consider in Section 4.3), the asymptotic behaviour of Poisson functionals can differ substantially when switching between these frameworks.

Many results already exist for central limit theorems and the asymptotic behaviour of the variance of functionals from stochastic geometry. One can usually differentiate between three different scaling types of the variance. The best studied case are functionals whose variance is of order s in the framework of increasing intensity. This is referred to as volume order scaling, as this usually corresponds to the case where in the other framework of growing observation windows the variance scales like the volume of the observation windows. In the case of increasing intensity it was shown for several functionals of random graphs that the variance is of order s (see for instance [85, 87, 88]). The corresponding volume order was also shown for some functionals of random sets in the framework of growing observation windows (see e.g. [55, 66, 67]). In line with this, in Chapters 4 and 6 we show that both the variance of the statistics of random graphs and the variance of the functionals of excursion sets of Poisson shot noise processes that we consider are of volume order. For $d \in \mathbb{N}$ the surface order scaling case in $\mathbb{R}^d$ contains all functionals, whose variance is of order $s^{(d-1)/d}$ for increasing intensity. This occurs for example in the case of the number of maximal points (see e.g. [114]) or, with the right scaling of the considered functionals such that a specific moment condition is of constant order (cf. (2.24)), also for the Poisson-Voronoi volume estimator (see [114]). Finally, the variance of some functionals of the convex hull of random points in a convex subset of $\mathbb{R}^d$ with smooth boundary scales like $s^{(d-1)/(d+1)}$ for increasing intensity (see e.g. [34]). In Chapter 5, we show that the L^p surface area of random polytopes in a Euclidean ball also fits into this category if we consider, as before, the functional with the right scaling.

In this thesis, we especially analyse functionals of spatial random graphs, random polytopes and excursion sets of Poisson shot noise processes. Central limit theorems for such functionals were, for example, established in [5, 15, 34, 66, 67, 69, 70, 84, 85, 86, 87, 88, 90, 103]. For more details on previous results for these functionals, we refer the reader to the respective chapters of this thesis. In the following, an overview of these three applications and the results derived in this thesis is given.

Spatial random graphs

Random graphs are constructed by interpreting the points of the Poisson process as vertices and connecting two vertices according to deterministic rules. We consider three different types of spatial random graphs: random geometric graphs, k-nearest neighbour graphs and radial spanning trees. To construct a random geometric graph, also known as Gilbert graph, we connect two vertices if their distance is smaller than or equal to some given threshold. A k-nearest neighbour graph is constructed by connecting each point to its k nearest neighbours. The radial spanning tree is a modification of the 1-nearest neighbour graph, where each vertex is connected to its nearest neighbour that is closer to the origin or, if no such point exists, to the origin itself. Note that in contrast to the usual k-nearest neighbour graph, in this case, we add the origin to the set of vertices. In this thesis, we analyse random geometric graphs and k-nearest neighbour graphs in Euclidean space, while we consider radial spanning trees in hyperbolic space to give an idea of how the geometry of the underlying space can influence the considered functionals.

There is a variety of functionals that can be studied for these graphs. For random geometric graphs on a compact convex observation window we consider the degree count and component count, i.e. the number of vertices with a specific degree or the number of components of a specific size. In both cases, we show that with the right scaling, the asymptotic covariance matrices of vectors of degree or component counts are positive definite for increasing intensity. The degree count is also considered for the k-nearest neighbour graph, as well as total edge length functionals, i.e. sums of specific powers of the length of all edges. As for the random geometric graph, we show that the asymptotic covariance matrices of the corresponding vectors supplied with the right scaling are positive definite in both cases for increasing intensity. Together with [103, Section 3] this provides multivariate central limit theorems in d_2- and d_{convex}-distance for all these applications. Finally, we study the asymptotic behaviour of total edge length functionals of the radial spanning tree in hyperbolic space, analyse its expectation and variance and show a central limit theorem for increasing observation windows. To illustrate the influence of the geometry of the underlying space on the graph, we further compare the degree of the origin of a radial spanning tree in hyperbolic space to that of a radial spanning tree in Euclidean space.

Random polytopes

Random polytopes are constructed by taking the convex hull of the points of a Poisson process. As functional we study a generalisation of the surface area and the volume, namely the L^p surface area for $p \in [0, 1]$, which corresponds to a multiple of the volume for $p = 0$ and the surface area for $p = 1$. The notion of L^p surface areas originates from the Minkowski problem, which asks for conditions under which a Borel measure on the sphere is the L^p surface area of a convex body (see also [76]). In this thesis, we study the vector of two L^p surface areas of random polytopes of homogeneous Poisson processes in a Euclidean d-dimensional ball for $p \in [0, 1]$. We start with embedding these functionals in the framework of stabilising functionals and show that the asymptotic covariance matrix of the vector of two L^p surface areas is positive definite. Together with [103, Section 3] this provides a two-dimensional central limit theorem. Note that, in particular, this proves a central limit theorem for the vector of volume and surface area.

Excursion sets of Poisson shot noise processes

Poisson shot noise processes are random fields that arise from adding up the contributions of all points of a Poisson process. Thereby, how a point from the Poisson process contributes to the field depends on the family of kernel functions that specifies the Poisson shot noise process. Excursion sets of Poisson shot noise processes are random sets that are obtained by considering the regions where a Poisson shot noise process is larger than or equal to some given threshold. In this thesis, we first derive lower variance bounds for the volume of excursion sets of Poisson shot noise processes for different families of kernel functions. Compared to [32], [66] or [67] we consider slightly modified conditions on the kernel functions. In a second step, we study the asymptotic behaviour of more general geometric functionals such as intrinsic volumes of excursion sets of Poisson shot noise processes. To this end, we restrict ourselves to a smaller class of kernel functions to ensure that the excursion sets are almost surely polyconvex and the geometric functionals are thus well defined. A prominent example of random sets is the Boolean model, which is constructed by considering the union of the compact convex sets of an underlying Poisson process on the space of compact convex sets. This model arises as a special case of our model for a specific choice of kernel functions. In Section 6.3, we particularly generalise the results from [55] for the variance asymptotics and the central limit theorem of geometric functionals of the Boolean model to the case of excursion sets of Poisson shot noise processes.

To summarise, the main achievements of this thesis are the introduction of a new lower variance bound for Poisson functionals, its application in three models from stochastic geometry, namely spatial random graphs, random polytopes and excursion sets of Poisson shot noise processes, and the derivation of corresponding central limit theorems.

1.3 Outline

The following work is mainly based on and partially extracted from the following papers, which are jointly written with Daniel Rosen, Matthias Schulte and Christoph Thäle:

[96] Rosen, Schulte, Thäle, Trapp (2024+). The radial spanning tree in hyperbolic space

[101] Schulte, Trapp (2024). Lower bounds for variances of Poisson functionals

[109] Trapp (2024+). Geometric functionals of polyconvex excursion sets of Poisson shot noise processes

This thesis is structured as follows. In Chapter 2, we establish some basic notation and summarise the geometric and stochastic preliminaries. The section about geometric preliminaries is divided into two parts. In the first one, basic facts from convex geometry are presented, while the second one introduces the hyperbolic space. The last section of this chapter contains an introduction to Poisson processes and functionals, summarises the normal approximation results for Poisson processes that we use throughout the thesis and introduces the concept of stabilisation.

Chapter 3 presents the theoretical main theorem of this thesis, the generalised reverse Poincaré inequality, which provides a lower variance bound for Poisson functionals and gives an idea of how this lower variance bound can be applied.

The remaining chapters are devoted to the applications from stochastic geometry introduced in the previous section. In Chapter 4, we consider statistics of spatial random graphs. The L^p surface area of random polytopes is analysed in Chapter 5 and geometric functionals of excursion sets of Poisson shot noise processes are studied in Chapter 6. For better readability, we only note at the beginning of each chapter its main source on which the respective chapter is based if not stated otherwise.

Chapter 2

Preliminaries

2.1 Basic notation

We start with introducing some basic notation. Throughout this thesis we denote by $\mathbb{N}$ the natural numbers and define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $\mathbb{R}$ we denote the real numbers, $\mathbb{R}_{\geq 0} = [0, \infty)$ and $\mathbb{R}_+ = (0, \infty)$.

For a set A we use the notation $|A|$ for the number of elements in A and, if A is a subset of a topological space, $\mathrm{int}(A)$ for the interior of A, ∂A for the boundary of A and $\bar{A}$ for its closure. The indicator function of A is defined by

$$\mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{else.} \end{cases}$$

For better readability we also use the notation $\mathbb{1}\{\cdot\}$ if the condition that the indicator is testing is more complex.

For $d \in \mathbb{N}$, the Euclidean space $\mathbb{R}^d$ is equipped with the usual topology. By $\mathcal{B}(\mathbb{R}^d)$ we denote the Borel σ-algebra on $\mathbb{R}^d$ and we write $\|\cdot\|$ for the Euclidean norm. For a set $A \subseteq \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, we define

$$d(x, A) = \inf_{y \in A} \|x - y\|,$$

where inf denotes the infimum. Analogously, we use the abbreviations sup, min and max for the supremum, the minimum and the maximum, respectively.

The d-dimensional closed ball in $\mathbb{R}^d$ around x with radius $r \geq 0$ is denoted by $B^d(x, r)$. By κ_d we denote the volume of the d-dimensional Euclidean unit ball. For the surface content of the corresponding $(d-1)$-dimensional Euclidean unit sphere we write ω_d. Note that $\omega_d = d\kappa_d$.

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space with σ-finite measure μ and let $f\colon \mathbb{X} \to \mathbb{R}$ be a measurable function on $\mathbb{X}$. The restriction of f to some set $A \subseteq \mathbb{X}$ is denoted by $f|_A$ and, if $\mathbb{X}$ is a topological space, the support of f is defined by $\mathrm{supp}(f) = \overline{\{x \in \mathbb{X}\colon f(x) > 0\}}$. Moreover, we say that f is square integrable (or $f \in L^2(\mu)$) if

$$\|f\|^2 = \int_{\mathbb{X}} |f(x)|^2 \, \mu(\mathrm{d}x) < \infty.$$

For $d \in \mathbb{N}$ we denote by μ^d the product measure of μ on $\mathbb{X}^d$. In many examples we consider measures, which are multiples of the d-dimensional Lebesgue measure, which is denoted by λ_d. Integrating with respect to the Lebesgue measure is usually abbreviated by $\mathrm{d}x$. Moreover, in the context of Poisson processes we often use the Dirac measure on a point x, which we denote by δ_x.

2.2 Background material from geometry

In this section we recap some definitions and basic facts from geometry. In the first part of this section, results about convex bodies in $\mathbb{R}^d$ are presented, which are mainly applied in Chapter 6 and partially also in Chapter 5. The second part of this chapter introduces the hyperbolic space $\mathbb{H}^d$ of constant sectional curvature -1, which will be used in Chapter 4 to illustrate the influence of the geometry of the underlying space on the behaviour of a specific spatial random graph.

2.2.1 Convex geometry

We start with summarising some basic facts from convex geometry. The following notation, definitions and results are mainly based on [58] by D. Hug and W. Weil and [98] by R. Schneider and W. Weil.

By $\mathcal{F}^d, \mathcal{U}^d$ and $\mathcal{C}^d$ we denote the set of closed, open and compact sets in $\mathbb{R}^d$, respectively. The space of closed sets $\mathcal{F}^d$ is equipped in this thesis with the Fell topology, also known as topology of closed convergence, whose σ-algebra $\mathcal{B}(\mathcal{F}^d)$ can be generated by $\{\{F \in \mathcal{F}^d\colon F \cap U \neq \emptyset\}\colon U \in \mathcal{U}^d\}$ (see e.g. [98, Lemma 2.1.1]). Note that on $\mathcal{C}^d$ the topology of the Hausdorff metric is strictly finer than the trace topology induced by $\mathcal{F}^d$. Anyway, since by [98, Theorem 12.3.2] both topologies coincide on the set $\{C \in \mathcal{C}^d\colon C \subset K\}$ for $K \in \mathcal{C}^d$, it does not matter for our purposes, which topology we use.

Intrinsic volumes

For a set $M \subseteq \mathbb{R}^d$, some constant $c > 0$ and a point $x \in \mathbb{R}^d$ we define $M + x = \{y + x\colon y \in M\}$ and $cM = \{cx\colon x \in M\}$. For $M_1, M_2 \subseteq \mathbb{R}^d$ the *Minkowski sum* of M_1 and M_2 is given by

$$M_1 + M_2 = \{x + y\colon x \in M_1, y \in M_2\}.$$

Let $\mathcal{K}^d$ denote the set of non-empty compact convex sets. Then, for $K \in \mathcal{K}^d$ the parallel body K^r is defined by

$$K^r = K + B^d(\mathbf{0}, r) = \{a + b\colon a \in K, b \in B^d(\mathbf{0}, r)\} = \{x \in \mathbb{R}^d\colon d(K, x) \leq r\}$$

for $r \geq 0$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$. With this definition we can introduce intrinsic volumes. By Steiner's formula we know that the volume of a parallel body is a polynomial of degree at most d in the radius of the ball. The intrinsic volumes $V_0, \dots, V_d$ are then defined with the help of the coefficients of this polynomial. By e.g. [98, p. 2] it holds that

$$V_d(K^r) = \sum_{k=0}^{d} r^{d-k} \kappa_{d-k} V_k(K). \tag{2.1}$$

The intrinsic volume $V_i(K)$ of a compact convex set K provides geometric information about K. From e.g. [97, Chapter 4] we know that for a d-dimensional compact convex set K, $V_d(K)$ is the volume, $V_{d-1}(K)$ half the surface area and $V_0(K)$ the Euler characteristic of K. Steiner's formula especially provides the existence of a constant $C_1 > 0$, which might depend on d and r such that

$$V_d(K^r) \leq C_1 \sum_{k=0}^{d} V_k(K). \tag{2.2}$$

The following basic properties of intrinsic volumes are summarised in [58, Remarks 3.21 and 3.22]. For $K_1, K_2 \in \mathcal{K}^d$ one can show that for $i \in \{1, \dots, d\}$, intrinsic volumes are non-negative, i.e. $V_i(K_1) \geq 0$, monotone in the sense that $V_i(K_1) \leq V_i(K_2)$ if $K_1 \subseteq K_2$ and homogenous of order i, i.e. $V_i(rK_1) = r^i V_i(K_1)$ for all $r > 0$. Moreover, one can show that the map $K \mapsto V_i(K)$ is continuous with respect to the Hausdorff metric for all non-empty compact convex sets $K \in \mathcal{K}^d$.

In the remaining part of this section we summarise some more advanced properties of intrinsic volumes, which are used in Chapter 6. We start with an estimate of $\int_{\mathbb{R}^d} V_i((K_1 + x) \cap K_2) \, \mathrm{d}x$ for $K_1, K_2 \in \mathcal{K}^d$. For $i = d$ this integral can be computed exactly by the translative integral formula, which can be found in [98, Theorem 5.2.1]. A bound of such an integral for intrinsic volumes was derived in [55, Lemma 3.4]. Both statements are summarised in the following lemma.

Lemma 2.1. *For $K_1, K_2 \in \mathcal{K}^d$ it holds*

$$\int_{\mathbb{R}^d} V_d((K_1 + x) \cap K_2) \, \mathrm{d}x = V_d(K_1) V_d(K_2) \tag{2.3}$$

and for $i \in \{1, \dots, d\}$ there exists a constant $c > 0$, which might depend on d, such that for all $K_1, K_2 \in \mathcal{K}^d$,

$$\int_{\mathbb{R}^d} V_i((K_1 + x) \cap K_2) \, \mathrm{d}x \leq c \sum_{j=i}^{d} V_j(K_1) \sum_{r=0}^{d} V_r(K_2). \tag{2.4}$$

Moreover, the following lemma is taken from [55, Lemma 3.6].

Lemma 2.2. *Let $K_1, K_2 \in \mathcal{K}^d$. Then it holds for a suitable constant $C_2 > 0$, which might depend on d,*

$$V_d(\{x \in \mathbb{R}^d \colon (K_1 + x) \cap \partial K_2 \neq \emptyset\}) \leq C_2 \sum_{k=0}^{d-1} V_k(K_2) \sum_{\ell=0}^{d} V_\ell(K_1). \tag{2.5}$$

As shown in [98, p. 613], for mixed volumes, which can be interpreted as generalisations of intrinsic volumes, the Alexandrov-Fenchel inequality provides the following relation between $V_i(K)$ and $V_j(K)$ for $i < j$.

Lemma 2.3. *Let $K \in \mathcal{K}^d$. Then, for $i, j \in \{0, \dots, d\}$ with $i < j$ it holds*

$$V_j(K) \leq C(i,j) V_i(K)^{j/i}, \tag{2.6}$$

where

$$C(i,j) = \frac{\kappa_{d-i}^{j/i} \binom{d}{j}}{\kappa_d^{(j-i)/i} \kappa_{d-j} \binom{d}{i}^{j/i}}.$$

For $j \in \{1, \dots, d\}$ and a given line $N \subset \mathbb{R}^d$ through the origin, let $G(N, j)$ be the set of j-dimensional subsets of $\mathbb{R}^d$ containing N and let ν_j^N denote the Haar measure on $G(N, j)$. Then, for $K_1, K_2 \in \mathcal{K}^d$ with $K_1 \subseteq K_2$ the *projection avoidance function* $\vartheta_j^{K_1,K_2} \colon \mathbb{R}^d \backslash \{\mathbf{0}\} \mapsto [0, 1]$ is given by

$$\vartheta_j^{K_1,K_2}(x) = \int_{G(L(x),j)} \mathbb{1}\{x \in (K_2|L) \backslash (K_1|L)\}\, \nu_j^{L(x)}(\mathrm{d}L),$$

where $K_i|L$ denotes the orthogonal projection of K_i to L for $i \in \{1, 2\}$ and $L(x)$ is the line spanned by x for $x \in \mathbb{R}^d \backslash \{\mathbf{0}\}$. With the help of the projection avoidance function the following lemma shows a representation of $V_i(K_2) - V_i(K_1)$ for $i \in \{1, \dots, d\}$, which is taken from [69, Lemma 3.8].

Lemma 2.4. *Let $i \in \{1, \dots, d\}$. Then, there exists a constant $\kappa_{d,i} > 0$ such that for all $K_1, K_2 \in \mathcal{K}^d$ with $K_1 \subseteq K_2$,*

$$V_i(K_2) - V_i(K_1) = \kappa_{d,i} \int_{K_2 \backslash K_1} \vartheta_i^{K_1,K_2}(x) \|x\|^{i-d}\, \mathrm{d}x.$$

Geometric functionals

Let $\mathcal{R}^d$ be the convex ring which contains all polyconvex subsets of $\mathbb{R}^d$, i.e. all subsets of $\mathbb{R}^d$ that can be written as a finite union of compact convex sets. In this thesis we aim to analyse geometric properties of polyconvex sets. To this end, we consider geometric functionals.

Definition 2.5 (Geometric functional)**.** A geometric functional $\varphi \colon \mathcal{R}^d \to \mathbb{R}$ is a $\mathcal{B}(\mathcal{F}^d)$-$\mathcal{B}(\mathbb{R})$-measurable function with the following three properties.

i) φ is translation invariant, i.e. $\varphi(A + x) = \varphi(A)$ for $A \in \mathcal{R}^d$ and $x \in \mathbb{R}^d$.

ii) φ is additive, i.e. $\varphi(\emptyset) = 0$ and

$$\varphi(A_1 \cup A_2) = \varphi(A_1) + \varphi(A_2) - \varphi(A_1 \cap A_2)$$

for $A_1, A_2 \in \mathcal{R}^d$.

iii) φ is locally bounded, i.e.

$$|\varphi(K)| \le M_\varphi \tag{2.7}$$

for all $K \in \mathcal{K}^d$ with $K \subseteq [0, 1]^d$, where M_φ only depends on φ.

Together with the translation invariance, the bound in iii) also holds for each translation of $[0, 1]^d$. For a polyconvex set $A = \bigcup_{j=1}^n K_j \in \mathcal{R}^d$ for $K_j \in \mathcal{K}^d$, $j \in \{1, \dots, n\}$ and $n \in \mathbb{N}$, additivity opens up the possibility of tracing the definition of $\varphi(A)$ back to $\varphi(K_j)$, $j \in \{1, \dots, n\}$, so that functionals such as the intrinsic volumes, which we originally introduced only on $\mathcal{K}^d$ can be defined for all $A \in \mathcal{R}^d$. The inclusion exclusion formula provides

$$\varphi(A) = \sum_{\emptyset \neq J \subseteq \{1,\dots,n\}} (-1)^{|J|+1} \varphi\Big(\bigcap_{j \in J} K_j\Big). \tag{2.8}$$

In harmony with this we define $\varphi(A \cap C_0^d)$ for $A \in \mathcal{R}^d$ and the half-open unit cube $C_0^d = [0,1)^d$ as

$$\varphi(A \cap C_0^d) = \varphi(A \cap C^d) - \varphi(A \cap \partial^+ C^d), \tag{2.9}$$

where $C^d = [0,1]^d$ and $\partial^+ C^d = C^d \backslash C_0^d$ (see also [98, p. 394]).

The most prominent example of geometric functionals are intrinsic volumes but there are also other types of geometric functionals like the more general example of mixed volumes $V(K[j], K_1, \ldots, K_{d-j})$, where $K[j]$ means that K is repeated j times, $j \in \{1, \ldots, d\}$ and $K_1, \ldots, K_{d-j}$ are fixed compact convex sets (see [97, Section 5.1]), or total measures from translative integral geometry (see [98, Section 6.4]). For further details and more examples of geometric functionals see also [55, p. 79] and the references therein. The following lemma is taken from [98, Lemma 9.2.2].

Lemma 2.6. *Let $\varphi : \mathcal{R}^d \to \mathbb{R}$ be a geometric functional. Then, it holds*

$$\lim_{s \to \infty} \frac{\varphi(sW)}{V_d(sW)} = \varphi(C_0^d)$$

for all $W \in \mathcal{K}^d$ with $V_d(W) > 0$.

2.2.2 Hyperbolic geometry

The d-dimensional *hyperbolic space* $\mathbb{H}^d$ of curvature -1 is the unique, complete and simply connected Riemannian manifold of constant sectional curvature -1. We usually work with a model that represents the hyperbolic space. Nevertheless, to illustrate the concepts behind the formal definition above, we give in the following a short informal description of Riemannian manifolds and curvatures. A more detailed introduction to and exact definitions of these concepts can be found in [73]. The notation, definitions and results from this subsection are mainly based on [36] by J. W. Cannon, W. J. Floyd, R. Kenyon and W. R. Parry, [73] by J. M. Lee, [89] by J. G. Ratcliffe and [96] by D. Rosen, M. Schulte, C. Thäle and V. Trapp.

A *Riemannian manifold* is a pair (M, g), where M is a smooth manifold and g a Riemannian metric. By a smooth manifold, we mean a second-countable Hausdorff space, which is locally Euclidean of dimension d and which is equipped with a smooth structure that allows to differentiate functions and maps. For more details on smooth structures see [73, Appendix A]. The Riemannian metric assigns an inner product on the tangent space to each point $p \in M$.

To understand the concept of curvatures of Riemannian manifolds, let us start with defining the *principle curvatures* at a point $p \in S$ for a surface S in $\mathbb{R}^3$, i.e. a two-dimensional embedded submanifold of $\mathbb{R}^3$. To this end, we fix a normal vector n and consider the intersections of all possible normal planes of S at p, i.e. planes through p that contain n, with S. For a specific normal plane, in a neighbourhood of p this intersection is a plane curve τ, where we can choose τ to be parametrised such that $\|\tau'\| = 1$. Then, its signed curvature at $p = \tau(t)$ is defined by $\pm\|\tau''(t)\|$, where the sign of the curvature depends on the orientations of n and the curve, determined by its parametrisation. The principle curvatures are now the maximum and minimum curvatures at p, which can be derived this way among all normal planes and give a first impression of the curvature of the surface. Anyway, one problem arises when talking about principle curvatures because one can show that principle curvatures are not intrinsic properties of the considered surface S, i.e. they are not properties that a 2-dimensional creature living in this surface could compute. Therefore, we also consider the *Gaussian curvature*, which is defined as the product of the principle curvatures and which turns out to be an intrinsic property of a surface. Note that the concept of Gaussian curvature can be generalised in such a way that it is well defined for any 2-dimensional Riemannian manifold and can be computed directly from the Riemannian metric. To transfer this idea of curvature now to Riemannian manifolds in higher dimension we use a similar procedure. The main difficulty here is that in higher dimensions the manifold may curve in many different directions. To this end we need to compute the Gaussian curvature for all possible 2-dimensional submanifolds, which correspond to a 2-dimensional subspace of the tangent space of the manifold at point p. Then, the *sectional curvature* at a point p is defined as the map, which maps each 2-dimensional subspace of the tangent space at p to the Gaussian curvature of the corresponding 2-dimensional submanifold. For more details see e.g. [73, Chapter 1]. Finally, we say that a Riemannian manifold has constant sectional curvature $\varkappa$, if the sectional curvature of all two-dimensional subspaces of the tangent space at point p is $\varkappa$ for any point $p \in M$.

Let us now return from this general description of Riemannian manifolds and curvatures to the special case of the hyperbolic space $\mathbb{H}^d$, which is, as mentioned above, the unique, complete and simply connected Riemannian manifold of constant sectional curvature -1. In the following, rather than relying on this formal definition, we work with the upper half-space model to represent the hyperbolic space. Nevertheless, it is important for the results of this thesis to remember that, based on this formal definition, we expect the hyperbolic space to behave locally similar to the Euclidean space as it is a Riemannian manifold while we expect the curvature of the hyperbolic space to have more influence on larger scales.

The upper half-space model

As already mentioned, the hyperbolic space can be represented in different models, which are all equivalent. The most common models are the hyperboloid model, the Klein model, the Poincaré disk model and the upper half-space model. For an overview of all these models we refer the reader to [36, Section 7]. In this thesis we use the upper half-space model to define and represent the hyperbolic space. As the name suggests, the upper half-space model represents the hyperbolic space in the upper half-plane

$$\{x = (x_1, \dots, x_d) \in \mathbb{R}^d \colon x_d > 0\}$$

equipped with the Riemannian metric

$$g = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2}{x_d^2}.$$

Then, the length $L(\tau)$ of a piecewise regular parametric curve $\tau = (\tau_1, \dots, \tau_d)$ with $\tau \colon [a, b] \to \mathbb{R}^{d-1} \times \mathbb{R}_+$ for $a, b \in \mathbb{R}$ with $a < b$ is given by

$$L(\tau) = \int_a^b \sqrt{\frac{\|\tau'(t)\|^2}{\tau_d(t)^2}}\, \mathrm{d}t,$$

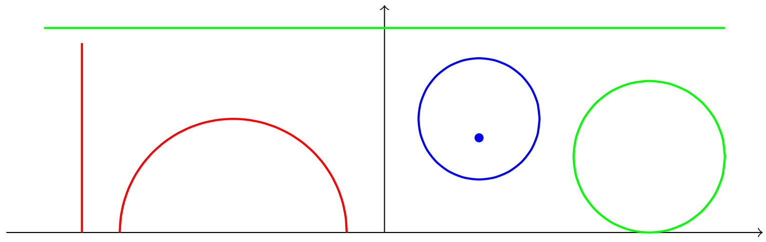

Figure 2.1: Hyperbolic lines (red), balls (blue) and horoballs (green) in the upper half-plane model for $\mathbb{H}^2$ ([96, Figure 2]).

where τ' denotes the derivative of τ. Here, piecewise regular means that there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ for some $n \in \mathbb{N}$ such that $\|\tau'(t)\| \neq 0$ for all $t \in (t_{i-1}, t_i)$ and $i \in \{1, \ldots, n\}$. Note that the length of such a curve is invariant under reparametrisation (see e.g. [73, Proposition 2.47 (b)]). We call the curve τ *parametrised by its arclength* if $L(\tau|_{[s,t]}) = t - s$ for $a \leq s \leq t \leq b$. By [73, Proposition 2.49] one can show that for each piecewise regular parametric curve $\tau\colon [a, b] \to \mathbb{R}^{d-1} \times \mathbb{R}_+$ there exists a reparametrisation by arc length. Note that this idea can be generalised to curves whose domain is $[t_0, \infty)$ for some $t_0 > 0$ or $\mathbb{R}$.

By [89, Theorem 4.6.1] the induced distance function in the upper half-space model, i.e. the minimal length of all piecewise regular parametric curves, which start in $x = (x_1, \ldots, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ and end in $y = (y_1, \ldots, y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, is given by

$$d_h(x, y) = \operatorname{arcosh}\left(1 + \frac{\|x - y\|}{2x_d y_d}\right).$$

Similarly to the Euclidean case, the distance in the upper half-space model between a point $x \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ and a set $M \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$ is defined by $d_h(x, M) = \inf_{y \in M} d_h(x, y)$.

A *geodesic* between the points x and y is a piecewise regular parametric curve, which minimises the length of curves between these two points. Note that by [73, Theorem 4.27] this geodesic is unique up to reparametrisation of the curve in hyperbolic space. As before, we can generalise this concept to piecewise regular parametric curves with unbounded domains and call τ a *geodesic ray* or a *complete geodesic* if the domain of τ is $[t_0, \infty)$ for some $t_0 > 0$ or $(-\infty, \infty)$, respectively, and any segment of the curve is the geodesic between both endpoints. In the upper half-plane model, geodesics are given by (segments of) vertical lines and (segments of) half-circles, which are orthogonal to the plane $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \colon x_d = 0\}$ (see [36, Theorem 9.3]). For an illustration for $d = 2$ see also the red line and the red semicircle in Figure 2.1. The boundary $\partial\mathbb{H}^d$ of the hyperbolic space is in this model given by the points of the boundary hyperplane $\{x_d = 0\}$ compactified by adding the point at ∞. In particular, topologically it is a $(d-1)$-dimensional sphere. We call points of $\partial\mathbb{H}^d$ *ideal points*. Observe that every geodesic ray $\tau\colon [t_0, \infty) \to \mathbb{H}^d$ tends to a unique ideal point $\tau(\infty) \in \partial\mathbb{H}^d$ and analogously, every complete geodesic tends to a unique ideal point for $t \to -\infty$ and to another unique ideal point for $t \to \infty$.

By [36, Fact 1], in the upper half-space model, hyperbolic spheres or balls are also Euclidean spheres or balls but their centres do not coincide with the centres of Euclidean balls (see e.g. the blue circle in Figure 2.1 whose centre is given by the blue dot). Of particular importance to us are *horospheres* in hyperbolic space, which are, informally

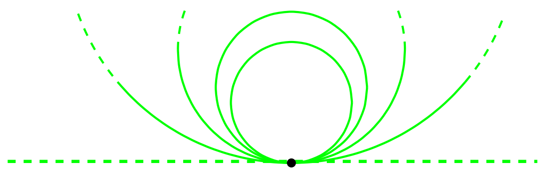

Figure 2.2: Construction of a horosphere (cf. [96, Figure 3]).

speaking, spheres of infinite radius. These can be constructed as follows. Fix a geodesic ray $\tau(t)$ in $\mathbb{H}^d$ and a point p on τ. For fixed t, consider the hyperbolic sphere S_t centered at $\tau(t)$ and passing through p. As $t \to \infty$ the spheres S_t converge to an unbounded hypersurface, called the horosphere passing through p around the ideal point $\tau(\infty)$ (see Figure 2.2). For concreteness, in the upper half-space model, horospheres are realised as Euclidean spheres tangent to the boundary, or as Euclidean hyperplanes parallel to the boundary, see Figure 2.1 for the case $d = 2$. Any horosphere H is the boundary of two unbounded domains, only one of which is convex. We call this the *horoball* bounded by H. More specifically, we call the corresponding horoball of the horosphere that passes through p around the ideal point $\tau(\infty)$, the horoball around the ideal point $\tau(\infty)$ passing through p. It can be seen as the limit of the balls bounded by the spheres converging to H. In the upper half-space model, a horoball is realised as the Euclidean ball bounded by a Euclidean sphere tangent to the boundary, or as the Euclidean half-space lying above a Euclidean hyperplane parallel to the boundary. Note that in contrast to the Euclidean case, horoballs do not coincide with half-spaces. While, as described, the boundary of a horoball is given by a horosphere, the boundary of a half-space is given by a complete geodesic (cf. Figure 2.1).

Hyperbolic trigonometry

In this section we consider hyperbolic triangles (see also Figure 2.3) whose vertices are three noncollinear points A, B, C, i.e. points which do not lie on a geodesic and whose sides are given by the geodesics from A to B, B to C and C to A.

In contrast to Euclidean triangles, the angle sum of a hyperbolic triangle is always smaller than π (see e.g. [89, Corollary 1]). Indeed, in hyperbolic space one can construct triangles with an arbitrary angle sum, which is smaller than π. If we allow the vertices of the triangle to be ideal points on the boundary of $\mathbb{H}^d$ we can even construct triangles with angle sum 0. The following law of sines can be shown in hyperbolic space (see e.g. [89, Theorem 3.5.2]).

Lemma 2.7. *Let α, β, γ be the angles of a hyperbolic triangle and let a, b, c be the lengths of the opposite sides. Then, it holds*

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}.$$

Note that for a triangle with $\gamma = \frac{\pi}{2}$ the law of sines provides

$$\sin\alpha = \frac{\sinh(a)}{\sinh(c)}, \tag{2.10}$$

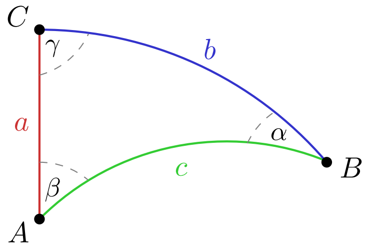

Figure 2.3: A hyperbolic triangle

i.e. compared to the Euclidean set-up, the side length a and c are replaced by $\sinh(a)$ and $\sinh(c)$. The hyperbolic law of cosine in hyperbolic space is given in the following lemma (see also [89, Theorem 3.5.3]).

Lemma 2.8. *Let α, β, γ be the angles of a hyperbolic triangle and let a, b, c be the lengths of the opposite sides. Then, it holds*

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma).$$

Note that this lemma implies for a triangle with $\gamma = \frac{\pi}{2}$ that

$$\cosh(c) = \cosh(a)\cosh(b), \tag{2.11}$$

which can be interpreted as the *hyperbolic Pythagorean theorem.*

Integration formulas

Let $\mathcal{H}^d$ denote the d-dimensional Hausdorff measure on $\mathbb{H}^d$. This subsection introduces several integration formulas in $\mathbb{H}^d$, which may be viewed as disintegration formulas for hyperbolic volume along spheres, hyperplanes, or horospheres and are taken from [96, Section 2.2].

The first one is the standard formula for polar integration in $\mathbb{H}^d$. Fix an arbitrary origin $o \in \mathbb{H}^d$, and associate to each point $x \in \mathbb{H}^d \setminus \{o\}$ the pair $(s, u) \in (0, \infty) \times S^{d-1}$ such that $s = d_h(o, x)$ and u is the unit vector tangent to the (unique) geodesic starting at o and passing through x. Then, we have for every measurable function $f\colon \mathbb{H}^d \to [0, \infty)$ the formula

$$\int_{\mathbb{H}^d} f(x)\,\mathcal{H}^d(\mathrm{d}x) = \int_0^\infty \int_{S^{d-1}} f(s, u) \sinh^{d-1}(s)\,\mathrm{d}u\,\mathrm{d}s, \tag{2.12}$$

where we denote by $\mathrm{d}u$ the volume element of the standard rotationally-invariant volume form on the unit sphere S^{d-1} in the tangent space of $\mathbb{H}^d$ at o. In particular, we get the following formula for the volume of a hyperbolic ball for $R > 0$:

$$V_h(R) = \mathcal{H}^d(B_h^d(x, R)) = \omega_d \int_0^R \sinh^{d-1}(u)\,\mathrm{d}u. \tag{2.13}$$

The next formula is a disintegration of the hyperbolic volume along hyperplanes (see also [96, Lemma 2.1]).

Lemma 2.9. *Let $L \subset \mathbb{H}^d$ be a complete geodesic with an arclength parametrisation $\tau(t)$. For $t \in \mathbb{R}$ let H_t be the hyperbolic hyperplane meeting L at $\tau(t)$ orthogonally. Then, for any measurable function $f\colon \mathbb{H}^d \to [0,\infty)$ one has that*

$$\int_{\mathbb{H}^d} f(x)\,\mathcal{H}^d(\mathrm{d}x) = \int_{\mathbb{R}} \int_{H_t} f(x) \cosh(d_h(x,L))\,\mathcal{H}^{d-1}(\mathrm{d}x)\,\mathrm{d}t.$$

We call horospheres centered at the same ideal point *parallel*. The family of all parallel horospheres corresponding to some ideal point cover the entire hyperbolic space, and it is convenient to parametrise the family in terms of the signed distance from a given horosphere H to some fixed point $o \in \mathbb{H}^d$, where we take the distance to be *negative* if o lies inside the horoball bounded by H, and positive otherwise. Then, one can show the following disintegration of the hyperbolic volume with respect to horospheres (see also [96, Lemma 2.2]).

Lemma 2.10. *Let $(H_t)_{t\in\mathbb{R}}$ be a family of parallel horospheres in $\mathbb{H}^d$, parametrised such that H_t has signed distance t from some fixed point $o \in \mathbb{H}^d$. Then, for any measurable function $f\colon \mathbb{H}^d \to [0,\infty)$ it holds*

$$\int_{\mathbb{H}^d} f(x)\,\mathcal{H}^d(\mathrm{d}x) = \int_{\mathbb{R}} \int_{H_t} f(x)\,\mathcal{H}^{d-1}(\mathrm{d}x)\,\mathrm{d}t.$$

In order to apply this lemma in Chapter 4, we use the following result from [59, Proposition 4.1.(1)].

Lemma 2.11. *Let $B_h^d(o,r)$ denote the hyperbolic ball around some fixed point $o \in \mathbb{H}^d$ and radius $r > 0$ and let H_t be a horosphere in $\mathbb{H}^d$ with signed distance t from o. Then, $H_t \cap B_h^d(o,r)$ is a $(d-1)$-dimensional ball in H_t for $|t| < r$ and it holds*

$$\mathcal{H}^{d-1}(H_t \cap B_h^d(o,r)) = \kappa_{d-1}[2e^{-t}(\cosh(r) - \cosh(t))]^{(d-1)/2}.$$

2.3 Background material from probability theory

Throughout this thesis let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the underlying probability space and $\mathbb{E}[\cdot]$, $\mathrm{Var}[\cdot]$ and $\mathrm{Cov}(\cdot,\cdot)$ the expectation, variance and covariance, respectively. This section provides a brief overview of the field of Poisson processes and gives some normal approximation results for Poisson functionals. For a more detailed introduction to Poisson processes and functionals we refer the reader to [71].

2.3.1 Poisson processes

We start with introducing Poisson processes and summarise their most important properties. The results from this subsection and the following notation are based on [71] by G. Last and M. Penrose. Let $(\mathbb{X}, \mathcal{X})$ be a measurable space and denote by $\mathbf{N}$ the set of all measures, which can be written as a countable sum of integer valued measures, equipped with the σ-field $\mathcal{N}$ generated by mappings of the form $\nu \mapsto \nu(B)$ for $B \in \mathcal{X}$. Recall that a measure ν on $\mathbb{X}$ is called σ-finite if there exists a sequence $(B_m)_{m\in\mathbb{N}}$ with $B_m \in \mathbb{X}$, $\nu(B_m) < \infty$ for $m \in \mathbb{N}$ and $\bigcup_{m\in\mathbb{N}} B_m = \mathbb{X}$. Note that any σ-finite $\overline{\mathbb{N}}_0$-valued measure is contained in $\mathbf{N}$, where $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$.

A *point process* is an $\mathcal{F}$-$\mathcal{N}$-measurable map $\eta\colon \Omega \to \mathbf{N}$. In the following we denote the mapping $\omega \mapsto \eta(\omega, B)$ by $\eta(B)$ for all $B \in \mathcal{X}$. By the definition of a point process, $\eta(B)$ is a random variable taking values in $\overline{\mathbb{N}}_0$, which can be interpreted as the number of points in B. We call η a *proper point process* if there exist random elements $X_1, X_2, \dots$ in $\mathbb{X}$ and a $\overline{\mathbb{N}}_0$-valued random variable τ such that almost surely

$$\eta = \sum_{i=1}^{\tau} \delta_{X_i}. \tag{2.14}$$

Note that for $\tau = 0$, this is interpreted as zero measure. This representation justifies the idea of interpreting a proper point process as a countable collection of random points (where the same point can occur more than once) instead of interpreting it as a measure. We call the measure λ, which is defined by $\lambda(B) = \mathbb{E}[\eta(B)]$ for $B \in \mathcal{X}$, the *intensity measure* of η.

Definition 2.12. Let λ be a σ-finite measure on $\mathbb{X}$. A *Poisson process* η on $\mathbb{X}$ with intensity measure λ is a point process which satisfies

i) $\eta(B)$ is Poisson distributed with parameter $\lambda(B)$ for all $B \in \mathcal{X}$, i.e.

$$\mathbb{P}(\eta(B) = k) = \frac{\lambda(B)^k}{k!} e^{-\lambda(B)}$$

for all $k \in \mathbb{N}_0$ and $B \in \mathcal{X}$,

ii) $\eta(B_1), \dots, \eta(B_n)$ are independent for pairwise disjoint sets $B_1, \dots, B_n \in \mathcal{X}$ and $n \in \mathbb{N}$.

Note that by [71, Theorem 3.6] for each σ-finite measure λ on $\mathbb{X}$ there exists a Poisson process on $\mathbb{X}$. Moreover, by [71, Corollary 3.7], if η is a Poisson process, there exists a proper point process with the same distribution as η. Thus, without loss of generality, we always assume that Poisson processes are proper throughout this thesis. This is why one can think of a Poisson process η as a random collection of points, where $\eta(B)$ stands for the number of points of η in B for all $B \in \mathcal{X}$, which satisfies i) and ii). If $\{x\} \in \mathcal{X}$ for all $x \in \mathbb{X}$ and λ does not have atoms, i.e. $\lambda(\{x\}) = 0$ for all $x \in \mathbb{X}$, it holds $\eta(\{x\}) \leq 1$ for all $x \in \mathbb{X}$ almost surely, i.e. no multiple points occur almost surely and we can identify the Poisson process directly with a set of points. If λ has atoms, we would lose the multiplicity of points by considering just the set of points. To avoid this we can provide each point with a mark so that we can distinguish two points at the same location and identify the random measure again with a set of random points. This justifies the notation $x \in \eta$ that we use throughout the thesis.

To equip each point of a Poisson process with a mark from some measurable space $(\mathbb{Y}, \mathcal{Y})$, which we call the *mark space*, we use a probability kernel $K\colon \mathbb{X} \times \mathcal{Y} \to [0,1]$, where $K(x, \cdot)$ is a probability measure for all $x \in \mathbb{X}$ and $K(\cdot, C)$ is measurable for all $C \in \mathcal{Y}$. Then, if η is represented as in (2.14) and $Y_1, Y_2, \dots$ are random variables in $\mathbb{Y}$ such that for given $\tau = n \in \overline{\mathbb{N}}$ and $(X_m)_{m \leq n}$ the conditional distribution of $(Y_m)_{m \leq n}$ is the distribution of independent random variables with distributions $K(X_m, \cdot)$ for $m \leq n$, the process

$$\xi = \sum_{i=1}^{\tau} \delta_{(X_i, Y_i)}$$

is a *marked Poisson process* on $\mathbb{X} \times \mathbb{Y}$. By the marking theorem (see [71, Theorem 5.5]) such a marked Poisson process is a Poisson process on $\mathbb{X} \times \mathbb{Y}$ with intensity measure $\lambda \otimes K$. Note that if $\mathbb{Y}$ is equipped with a probability measure $\mathbb{Q}$ a possible choice of the probability kernel is $K(x, \cdot) = \mathbb{Q}$ for all $x \in \mathbb{X}$, which is the type of marked Poisson processes we consider in Chapter 6.

Throughout this thesis we denote for $k \in \mathbb{N}$ by $\eta^k_{\neq}$ the set of all k-tuples of distinct points of the support of η, where multiple points are taken into account. With this notation we can introduce the multivariate Mecke equation (see for example [71, Theorem 4.4]).

Theorem 2.13 (multivariate Mecke equation)**.** *Let η be a Poisson process on $\mathbb{X}$ with σ-finite intensity measure λ. Then it holds for any measurable function $f\colon \mathbb{X}^n \times \mathbf{N} \to [0, \infty)$,*

$$\mathbb{E}\Big[\sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} f(x_1, \dots, x_k, \eta)\Big] = \int_{\mathbb{X}^k} \mathbb{E}[f(x_1, \dots, x_k, \eta + \delta_{x_1} + \dots + \delta_{x_k})] \, \lambda^k(\mathrm{d}(x_1, \dots, x_k)). \tag{2.15}$$

Note that the multivariate Mecke equation continues to hold for any function $f\colon \mathbb{X}^k \times \mathbf{N} \to \mathbb{R}$ if

$$\int_{\mathbb{X}^k} \mathbb{E}[|f(x_1, \dots, x_k, \eta + \delta_{x_1} + \dots + \delta_{x_k})|] \, \lambda^k(\mathrm{d}(x_1, \dots, x_k)) < \infty.$$

Instead of working with Poisson processes one could also think about working with a fixed number of random points. This means that for some probability distribution $\mathbb{Q}$ one would take independent random elements $X_1, \dots, X_n$, which are distributed according to $\mathbb{Q}$ for fixed $n \in \mathbb{N}$, and consider the point process $\eta_b = \sum_{i=1}^n \delta_{X_i}$. This process is called *binomial point process* with sample size n and sampling distribution $\mathbb{Q}$ since $\eta_b(B)$ is binomially distributed with parameters n and $\mathbb{Q}(B)$ for all $B \in \mathcal{X}$. Compared to binomial point processes, working with Poisson processes has the advantage that we do not only know the distribution of the number of points in a region but the numbers of points in disjoint regions are also independent in the case of Poisson processes (see Definition 2.12 ii)). Roughly speaking, this means that the numbers of points in disjoint regions do not influence each other, which is usually very helpful for the analysis of the process and is not fulfilled for binomial point processes since the total number of points is fixed in this case. Nevertheless, Poisson and binomial point processes are linked by the following proposition, which is taken from [71, Proposition 3.8].

Proposition 2.14. *Let η be a Poisson process on $\mathbb{X}$ with intensity measure λ and $0 < \lambda(\mathbb{X}) < \infty$. Then, the conditional distribution $\mathbb{P}(\eta \in \cdot | \eta(\mathbb{X}) = n)$ for fixed $n \in \mathbb{N}$ is the distribution of a binomial point process with sample size n and sampling distribution $\frac{\lambda}{\lambda(\mathbb{X})}$.*

2.3.2 Poisson functionals and the Fock space representation

A *Poisson functional* F is a measurable function that depends on the Poisson process η. This means that there exists a measurable function $f\colon \mathbf{N} \to \mathbb{R}$ such that $F = f(\eta)$ $\mathbb{P}$-almost surely. This function f is called representative of F. For simplicity we use the notation $F = F(\eta)$ throughout this thesis. If F is square-integrable, we write $F \in L^2_\eta$. Again, the results from this subsection and the following notation are based on [71] by G. Last and M. Penrose.

The *difference operator* of a Poisson functional F is defined by

$$D_x F = F(\eta + \delta_x) - F(\eta)$$

for $x \in \mathbb{X}$. If one interprets η as a collection of random points, this means that the difference operator, also known as add-one cost operator, measures how adding an additional point x changes the functional. Iteratively, higher order difference operators are defined as

$$D^n_{x_1,\dots,x_n} F = D_{x_1}(D^{n-1}_{x_2,\dots,x_n} F)$$

for $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{X}$. This yields the explicit representation

$$D^n_{x_1,\dots,x_n} F = \sum_{J \subseteq \{1,\dots,n\}} (-1)^{n-|J|} F\Big(\eta + \sum_{j \in J} \delta_{x_j}\Big) \tag{2.16}$$

for the difference operator of order n for $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{X}$. Note that for any permutation $\pi \colon \{1, \dots, n\} \to \{1, \dots, n\}$ of $\{1, \dots, n\}$ this provides

$$D^n_{x_1,\dots,x_n} F = D^n_{x_{\pi(1)},\dots,x_{\pi(n)}} F, \tag{2.17}$$

i.e. the difference operator of order n is symmetric in $x_1, \dots, x_n$.

For $n \in \mathbb{N}$ let $\mathbf{H}_n$ be the space of symmetric functions in $L^2(\lambda^n)$ and define $\mathbf{H}_0 = \mathbb{R}$. Then, the *Fock space* $\mathbf{H}$ is the set of all sequences $(u_n)_{n \geq 0} \in \times_{n=0}^{\infty} \mathbf{H}_n$ satisfying $\langle (u_n)_{n\geq 0}, (u_n)_{n \geq 0} \rangle_{\mathbf{H}} < \infty$, where for sequences $(u_n)_{n\geq 0}, (v_n)_{n\geq 0}$,

$$\langle (u_n)_{n\geq 0}, (v_n)_{n\geq 0} \rangle_{\mathbf{H}} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle u_n, v_n \rangle_n.$$

Here, $\langle \cdot, \cdot \rangle_n$ denotes the inner product on $L^2(\lambda^n)$ for $n \in \mathbb{N}$ and $\langle a, b \rangle_0 = ab$ for $a, b \in \mathbb{R}$. Note that $\mathbf{H}$ is a Hilbert space, i.e. $\mathbf{H}$ is a complete metric space with respect to the distance function induced by $\langle \cdot, \cdot \rangle_{\mathbf{H}}$.

For any $F \in L^2_\eta$ there exists a sequence of expected difference operators of order n for $n \geq 0$, where $D^0 F = F$. The following theorem will show that the second moment of F coincides with the squared norm of this sequence of expected difference operators in the Fock space. To this end we abbreviate $f_n(x_1, \dots, x_n) = \frac{1}{n!}\mathbb{E}[D^n_{x_1,\dots,x_n} F]$ for $x_1, \dots, x_n \in \mathbb{X}$, $n \in \mathbb{N}$ and $f_0 = \mathbb{E}[F]$. With this, we can introduce the Fock space representation, which can be found for example in [71, Theorem 18.6] and [72, Theorem 1.1].

Theorem 2.15 (Fock space representation)**.** *Let η be a Poisson process on $\mathbb{X}$ with intensity measure λ and let $F \in L^2_\eta$. Then, it holds*

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|_n^2, \tag{2.18}$$

where $\|\cdot\|_n$ denotes the norm on $L^2(\lambda^n)$ for $n \in \mathbb{N}$ and $\|\cdot\|_0 = |\cdot|$.

Note that the Fock space representation yields a representation of the variance in terms of difference operators since subtracting $\mathbb{E}[F]^2$ on both sides yields

$$\mathrm{Var}[F] = \sum_{n=1}^{\infty} n! \|f_n\|_n^2. \tag{2.19}$$

For any square-integrable Poisson functional there also exists an orthogonal decomposition of F into a series of Wiener-Itô integrals, which is known as chaos expansion and converges in $L^2(\mathbb{P})$ to F. The integrands of this chaos expansion are given by f_n for $n \in \mathbb{N}_0$ (see [71, Theorem 18.10] and [72, Theorem 1.3]). Note that the integrands are unique in the sense that if F has such a representation as a series of Wiener-Itô integrals with integrands g_n for $n \in \mathbb{N}_0$, it holds $g_0 = f_0$ and g_n is λ^n-a.e. equal to f_n for $n \in \mathbb{N}$.

A consequence of the Fock space representation is the following upper bound for the variance known as Poincaré inequality (see [71, Theorem 18.7]).

Theorem 2.16 (Poincaré inequality). *Let η be a Poisson process on $\mathbb{X}$ with intensity measure λ and let $F \in L^2_\eta$. Then it holds*

$$\operatorname{Var}[F] \leq \int_{\mathbb{X}} \mathbb{E}[(D_x F)^2]\, \lambda(\mathrm{d}x). \tag{2.20}$$

2.3.3 Normal approximation of Poisson functionals

Throughout this thesis we often aim to show convergence of a sequence of Poisson functionals to a standard Gaussian random variable in distribution. To quantify these normal approximation results, we need to be able to measure the distance between probability distributions. To this end we introduce the Wasserstein and the Kolmogorov distance.

Let X, Y be random variables with $\mathbb{E}[|X|], \mathbb{E}[|Y|] < \infty$. Then, the *Wasserstein distance* between X and Y is given by

$$\mathrm{d}_W(X, Y) = \sup_{h \in \mathrm{Lip}_1} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|,$$

where Lip_1 denotes the set of Lipschitz functions with Lipschitz constant less than or equal to one. The *Kolmogorov distance* between X and Y is defined by

$$\mathrm{d}_K(X, Y) = \sup_{t \in \mathbb{R}} |\mathbb{P}(X \leq t) - \mathbb{P}(Y \leq t)|,$$

i.e. it is given by the supremum norm of the distance of the distribution functions of X and Y. To bound the Wasserstein distance between a Poisson functional and a standard normally distributed random variable, we use the following bounds from [16, Theorem 3.1] and [70, Theorem 1.1].

Theorem 2.17. *Let η be a Poisson process on $\mathbb{X}$ with intensity measure λ and let $F \in L^2_\eta$ be a Poisson functional satisfying $\mathbb{E}[F] = 0$, $\operatorname{Var}[F] = 1$ and $\mathbb{E}[\int (D_x F)^2\, \lambda(\mathrm{d}x)] < \infty$. By N we denote a standard Gaussian random variable. Then,*

$$\mathrm{d}_W(F, N) \leq \gamma_1 + \gamma_2 + \gamma_3 \quad \text{and} \quad \mathrm{d}_W(F, N) \leq \gamma_1 + \gamma_2 + \widetilde{\gamma}_3,$$

where

$$\gamma_1 = 2\Big[\int_{\mathbb{X}^3} (\mathbb{E}[(D_{x_1}F)^2 (D_{x_2}F)^2])^{1/2} (\mathbb{E}[(D^2_{x_1,x_3}F)^2 (D^2_{x_2,x_3}F)^2])^{1/2}\, \lambda^3(\mathrm{d}(x_1, x_2, x_3))\Big]^{1/2},$$

$$\gamma_2 = \Big[\int_{\mathbb{X}^3} \mathbb{E}[(D^2_{x_1,x_3}F)^2 (D^2_{x_2,x_3}F)^2]\, \lambda^3(\mathrm{d}(x_1, x_2, x_3))\Big]^{1/2},$$

$$\gamma_3 = \int_{\mathbb{X}} \mathbb{E}[|D_x F|^3] \, \lambda(\mathrm{d}x),$$
$$\widetilde{\gamma}_3 = \int_{\mathbb{X}} (\mathbb{E}[|D_x F|^3])^{1/3} (\mathbb{E}[\min\{\sqrt{8}|D_x F|^{3/2}, |D_x F|^3\}])^{2/3} \, \lambda(\mathrm{d}x).$$

The following similar result for the Kolmogorov distance can be found in [70, Theorem 1.2].

Theorem 2.18. *Let η be a Poisson process on $\mathbb{X}$ with intensity measure λ and let $F \in L^2_\eta$ be a Poisson functional satisfying $\mathbb{E}[F] = 0$, $\mathrm{Var}[F] = 1$ and $\mathbb{E}[\int (D_x F)^2 \, \lambda(\mathrm{d}x)] < \infty$. By N we denote a standard Gaussian random variable. Then,*

$$\mathrm{d}_K(F, N) \leq \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6,$$

where $\gamma_1, \gamma_2, \gamma_3$ are defined as in Theorem 2.17 and

$$\gamma_4 = \frac{1}{2} (\mathbb{E}[F^4])^{1/4} \int_{\mathbb{X}} [(\mathbb{E}[(D_x F)^4])^{3/4}] \, \lambda(\mathrm{d}x),$$
$$\gamma_5 = \Big(\int_{\mathbb{X}} \mathbb{E}[(D_x F)^4] \, \lambda(\mathrm{d}x) \Big)^{1/2},$$
$$\gamma_6 = \Big(\int_{\mathbb{X}^2} 6 (\mathbb{E}[(D_{x_1} F)^4])^{1/2} (\mathbb{E}[(D^2_{x_1,x_2} F)^4])^{1/2} + 3\mathbb{E}[(D^2_{x_1,x_2} F)^4] \, \lambda^2(\mathrm{d}(x_1, x_2)) \Big)^{1/2}.$$

As a consequence of Theorems 2.17 and 2.18, the additional following bounds for the Kolmogorov and Wasserstein distance were shown in [70, Theorem 6.1].

Theorem 2.19. *Let η be a Poisson process on $\mathbb{X}$ with intensity measure λ and let F be a Poisson functional satisfying $\mathbb{E}[\int_{\mathbb{X}} (D_x F)^2 \, \lambda(\mathrm{d}x)] < \infty$ and $\mathrm{Var}[F] > 0$, which fulfils the moment conditions*

$$\mathbb{E}[|D_x F|^5] \leq \bar{c} \quad \text{for} \quad \lambda\text{-a.e. } x \in \mathbb{X},$$
$$\mathbb{E}[|D^2_{x_1,x_2} F|^5] \leq \bar{c} \quad \text{for} \quad \lambda\text{-a.e. } (x_1, x_2) \in \mathbb{X}^2$$

for some constant $\bar{c} \geq 1$.

Denote by N a standard Gaussian random variable. Then it holds

$$d_W\left(\frac{F - \mathbb{E}[F]}{\sqrt{\mathrm{Var}[F]}}, N \right) \leq \frac{5\bar{c}}{\mathrm{Var}[F]} \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \mathbb{P}(D^2_{x_1,x_2} F \neq 0)^{1/20} \, \lambda(\mathrm{d}x_2) \right)^2 \lambda(\mathrm{d}x_1) \right)^{1/2} + \frac{\bar{c}}{\mathrm{Var}[F]^{3/2}} \int_{\mathbb{X}} \mathbb{P}(D_x F \neq 0)^{2/5} \, \lambda(\mathrm{d}x)$$

and

$$d_K\left(\frac{F - \mathbb{E}[F]}{\sqrt{\mathrm{Var}[F]}}, N \right) \leq \frac{5\bar{c}}{\mathrm{Var}[F]} \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}} \mathbb{P}(D^2_{x_1,x_2} F \neq 0)^{1/20} \, \lambda(\mathrm{d}x_2) \right)^2 \lambda(\mathrm{d}x_1) \right)^{1/2} + \frac{\bar{c} I_F^{1/2}}{\mathrm{Var}[F]} + \frac{2\bar{c} I_F}{\mathrm{Var}[F]^{3/2}} + \frac{\bar{c} I_F^{5/4} + 2\bar{c} I_F^{3/2}}{\mathrm{Var}[F]^2}$$

$$+ \frac{\sqrt{6}\bar{c} + \sqrt{3}\bar{c}}{\operatorname{Var}[F]} \Big(\int_{\mathbb{X}^2} \mathbb{P}(D^2_{x_1,x_2} F \neq 0)^{1/10} \, \lambda^2(\mathrm{d}(x_1, x_2)) \Big)^{1/2},$$

where

$$I_F := \int_{\mathbb{X}} \mathbb{P}(D_x F \neq 0)^{1/10} \, \lambda(\mathrm{d}x).$$

Note that in the previous theorem, the bound for the Kolmogorov distance is a bound for the Wasserstein distance, too. This can also be seen from comparing Theorems 2.17 and 2.18, since Theorem 2.19 can be proven directly by estimating $\gamma_1, \dots, \gamma_6$ from Theorems 2.17 and 2.18.

Since we aim to show multivariate central limit theorems in Chapters 4 and 5, we introduce in the following two distances for random vectors. Let $\mathbf{X}, \mathbf{Y}$ be random vectors in $\mathbb{R}^m$ for some $m \in \mathbb{N}$ with $\mathbb{E}[\|\mathbf{X}\|], \mathbb{E}[\|\mathbf{Y}\|] < \infty$. By $\mathcal{H}^2_m$ we denote the set of all C^2-functions $h\colon \mathbb{R}^m \to \mathbb{R}$, which satisfy

$$|h(x) - h(y)| \leq \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^m \quad \text{and} \quad \sup_{x \in \mathbb{R}^m} \|\operatorname{Hess} h(x)\|_{\mathrm{op}} \leq 1,$$

where Hess h denotes the Hessian of h and $\|\cdot\|_{\mathrm{op}}$ stands for the operator norm of a matrix. Then, the d_2-distance is defined by

$$d_2(\mathbf{X}, \mathbf{Y}) = \sup_{h \in \mathcal{H}^2_m} |\mathbb{E}[h(\mathbf{X})] - \mathbb{E}[h(\mathbf{Y})]|.$$

Moreover, we use the d_{convex}-distance, which is given by

$$d_{\mathrm{convex}}(\mathbf{X}, \mathbf{Y}) = \sup_{h \in \mathcal{I}_m} |\mathbb{E}[h(\mathbf{X})] - \mathbb{E}[h(\mathbf{Y})]|,$$

where $\mathcal{I}_m$ denotes the set of indicators of measurable convex sets in $\mathbb{R}^m$. As in the one-dimensional case one can use the Kolmogorov distance to measure the distance of the distributions of two random vectors, which was defined as the supremum norm of the difference of the distribution functions. Note that by the definition of the d_{convex}-distance, the d_{convex}-distance is for $m \geq 2$ stronger than the Kolmogorov distance. Thus, all results that we show for the d_{convex}-distance also hold for the Kolmogorov distance.

In [102, Theorem 1.1, Theorem 1.2] multivariate second order Poincaré inequalities were shown for the d_2- and d_{convex}-distance, which are similar to the one-dimensional versions for the Wasserstein and Kolmogorov distance in Theorems 2.17 and 2.18. Since we do not use these expressions explicitly in this thesis we refrain from stating them here. Just note that similarly to Theorem 2.17 and Theorem 2.18, the multivariate second order Poincaré inequalities mainly depend on the first and second order difference operator. Moreover, just as we need the positivity of the variance for the normal approximation results in d_W- and d_K-distance, the bounds for the d_2- and d_{convex}-distance are only valid if the corresponding covariance matrix is positive definite. Finally, there is also an analogous version of Theorem 2.19 for vectors of random variables, which can be found in [102, Theorem 4.5].

2.3.4 Stabilising functionals

In this section we assume that the measurable space $(\mathbb{X}, \mathcal{X})$ is equipped with a σ-finite intensity measure λ and a semi-metric d. We denote by $B(x, r)$ the ball of radius r with respect to d around $x \in \mathbb{X}$ and assume that there exist constants $\beta, \varkappa > 0$ such that

$$\limsup_{\varepsilon \to 0} \frac{\lambda(B(x, r+\varepsilon)) - \lambda(B(x, r))}{\varepsilon} \leq \beta \varkappa r^{\varkappa - 1} \tag{2.21}$$

for all $r \geq 0$ and $x \in \mathbb{X}$. Clearly, this assumption is satisfied if $\mathbb{X}$ is $\mathbb{R}^d$ or a full-dimensional subset of $\mathbb{R}^d$ equipped with the usual Euclidean norm and λ has a bounded density with respect to the Lebesgue measure. The following definitions follow the notation from [69] by R. Lachièze-Rey, M. Schulte and J. E. Yukich, [101] by M. Schulte and V. Trapp and [103] by M. Schulte and J. E. Yukich.

For $s \geq 1$ let η_s be a Poisson process with intensity measure $s\lambda$. We consider a Poisson functional F_s for $s \geq 1$. In many applications F_s can be written as a sum of scores, i.e.

$$F_s = F_s(\eta_s) = \sum_{x \in \eta_s} \xi_s(x, \eta_s) \tag{2.22}$$

for some measurable function $\xi_s \colon \mathbb{X} \times \mathbf{N} \to \mathbb{R}$. One can think of F_s as the sum of contributions associated with the points of η_s. The idea of stabilising functionals is that the score of a point only depends on the points of the Poisson process η_s in a random neighbourhood of x.

Let $s \geq 1$. A measurable map $R_s \colon \mathbb{X} \times \mathbf{N} \to \mathbb{R}$ is called *radius of stabilisation* for ξ_s if

$$\xi_s\Big(x, \Big(\nu + \delta_x + \sum_{a \in A} \delta_a\Big)\Big|_{B(x, R_s(x, \nu+\delta_x))}\Big) = \xi_s\Big(x, \nu + \delta_x + \sum_{a \in A} \delta_a\Big)$$

for all $x \in \mathbb{X}$, $\nu \in \mathbf{N}$ and $A \subset \mathbb{X}$ with $|A| \leq 9$. Broadly speaking, this means that the value of the score only depends on the points of the underlying point process in a ball with radius $R_s(x, \nu + \delta_x)$ around x. One can also consider sums of scores, which fulfil the described condition for more general random neighbourhoods as done e.g. in [21]. Here, so-called stabilisation regions are introduced, which are not necessarily balls. This approach can be useful to handle applications, where balls are not suitable. However, in this thesis, we stick to the concept of stabilisation radii, as they are sufficient for our purposes.

The scores $(\xi_s)_{s \geq 1}$ are called *exponentially stabilising* if there exist radii of stabilisation $(R_s)_{s \geq 1}$ and constants $C_{stab}, c_{stab}, \alpha_{stab} > 0$ such that

$$\mathbb{P}(R_s(x, \eta_s + \delta_x) \geq r) \leq C_{stab} \exp[-c_{stab}(s^{1/\varkappa} r)^{\alpha_{stab}}] \tag{2.23}$$

for $x \in \mathbb{X}$, $r \geq 0$, $s \geq 1$ and $\varkappa$ from (2.21). For $q > 0$, the scores $(\xi_s)_{s \geq 1}$ fulfil a *q-th moment condition* if there exists a constant $C_q > 0$ satisfying

$$\sup_{s \geq 1} \sup_{x \in \mathbb{X}} \mathbb{E}\Big[\Big|\xi_s\Big(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a\Big)\Big|^q\Big] \leq C_q \tag{2.24}$$

for $A \subset \mathbb{X}$ with $|A| \leq 9$. Finally, the scores $(\xi_s)_{s \geq 1}$ *decay exponentially fast with distance to a measurable set* $K \subseteq \mathbb{X}$ if there are constants $C_K, c_K, \alpha_K > 0$ such that for $x \in \mathbb{X}$, $s \geq 1$ and $A \subset \mathbb{X}$ with $|A| \leq 9$,

$$\mathbb{P}\Big(\xi_s\Big(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a\Big) \neq 0\Big) \leq C_K \exp[-c_K s^{\alpha_K/\varkappa} \mathrm{d}(x, K)^{\alpha_K}], \tag{2.25}$$

where $\mathrm{d}(x, K)$ denotes the distance from x to K with respect to the semi-metric d. In contrast to the definitions in [69], those in [103] and in this section require that one can add up to nine additional points instead of seven, but this difference is not essential and all results from [69] we refer to throughout this thesis are still valid. The additional points come from considering difference operators and applying the multivariate Mecke formula since the k-th power of a sum of scores can be rewritten as sums over up to k different points so that the multivariate Mecke formula leads to adding up to $k-1$ additional points. Thus, the different numbers of points in the works [69] and [103] are caused by different moment conditions. For more details on stabilising functionals we refer to [69] or [103] and the references therein.

In Chapter 4 and Chapter 5 we represent the considered functionals in the framework of stabilising functionals. In order to apply the normal approximation results from the previous section to stabilising functionals the following properties from [69, Lemma 5.5], [69, (5.8) in Lemma 5.10] and [69, Lemma 5.9] can be helpful. (Note that the statement of [69, Lemma 5.9] contains a typo since the exponent α of $\mathrm{d}_s(x_1, K) = s^{1/\varkappa}\mathrm{d}(x_1, K)$ is missing in the upper bound.)

Lemma 2.20. *Let $(F_s)_{s\geq 1}$ be a family of stabilising functionals of the form (2.22), whose corresponding scores fulfil (2.23), (2.24) for $q = 4 + p$ for some $p \in (0, 1]$ and (2.25) and let $\alpha = \min\{\alpha_{stab}, \alpha_K\}$.*

a) For any $\varepsilon \in (4, 4 + p)$ there exists a constant $C > 0$ such that

$$\mathbb{E}\Big|D_x F_s\Big(\eta_s + \sum_{u\in U} \delta_u\Big)\Big|^{\varepsilon} \leq C \tag{2.26}$$

for all $x \in \mathbb{X}$, $U \subset \mathbb{X}$ with $|U| \leq 1$ and $s \geq 1$.

b) For any $\beta > 0$,

$$s\int \mathbb{P}(D_x F_s \neq 0)^{\beta}\, \lambda(\mathrm{d}x) \leq \tilde{C}_\beta s \int \exp[-\tilde{c}_\beta s^{\alpha/\varkappa}\mathrm{d}(x, K)^{\alpha}]\, \lambda(\mathrm{d}x)$$

for some constants $\tilde{C}_\beta, \tilde{c}_\beta > 0$ and all $s \geq 1$.

c) For any $\beta > 0$ there exists constants $C_\beta, c_\beta > 0$ such that

$$s\int \mathbb{P}(D^2_{x,y} F_s \neq 0)^{\beta}\, \lambda(\mathrm{d}y) \leq C_\beta \exp[-c_\beta s^{\alpha/\varkappa}\mathrm{d}(x, K)^{\alpha}] \tag{2.27}$$

for all $x \in \mathbb{X}$ and $s \geq 1$.

The following multivariate normal approximation result in d_{convex}-distance for a vector of centred stabilising functionals was shown in [103, Theorem 4.1 c)].

Theorem 2.21. *For $m \in \mathbb{N}$ and $s \geq 1$ let $F_s^{(1)}, \dots, F_s^{(m)}$ be functionals of the form (2.22) such that the corresponding scores fulfil (2.23), (2.24) and (2.25). Let $\alpha = \min\{\alpha_{stab}, \alpha_K\}$ and $\tau > 0$. Then, there exists a constant $C > 0$ such that for any positive definite matrix $\Theta = (\theta_{i,j})_{i,j\in\{1,\dots,m\}}$,*

$$d_{\mathrm{convex}}(s^{-\tau}(F_s^{(1)} - \mathbb{E}[F_s^{(1)}], \dots, F_s^{(m)} - \mathbb{E}[F_s^{(m)}]), N_\Theta)$$

$$
\begin{aligned}
&\leq C m^{13/2} \max\{\|\Theta^{-1}\|_{\mathrm{op}}^{1/2}, \|\Theta^{-1}\|_{\mathrm{op}}^{3/2}\} \\
&\quad \times \max\Big\{ \sum_{i,j=1}^{m} \Big|\theta_{i,j} - \frac{\mathrm{Cov}(F_s^{(i)}, F_s^{(j)})}{s^{2\tau}}\Big|, s^{-\tau} \max\{s^{-2\tau} I_{K,s}, (s^{-2\tau} I_{K,s})^{1/4}\}\Big\},
\end{aligned}
$$

where N_Θ *is a centred Gaussian random vector in* $\mathbb{R}^{m\times m}$ *with covariance matrix* Θ *and*

$$
I_{K,s} = s \int_{\mathbb{X}} \exp\Big(- \frac{\min\{c_{\mathrm{stab}}, c_K\} \min\{p, 1\} s^{\alpha/\gamma} \mathrm{d}(x, K)^\alpha}{39 \cdot 4^{\alpha+1}} \Big) \, \lambda(\mathrm{d}x)
$$

for $s \geq 1$.

Note that we refrain from introducing the analogous bound for the d_2-distance here since we do not use this bound throughout the thesis. Nevertheless, there is a similar result for the d_2-distance, which can be found in [103, Theorem 4.1 b)].

To conclude, it should be noted that all normal approximation results mentioned throughout this and the previous section require either the positivity of the variance in the one-dimensional setting or the positive definiteness of the covariance matrix in the multivariate setting. This is the reason why we introduce in the following chapter a lower variance bound, which can also be used to show the positive definiteness of a covariance matrix, as discussed in Section 1.1.

Chapter 3

Lower bounds for variances of Poisson functionals

In this chapter we derive the main theoretical result of this thesis: a lower variance bound for Poisson functionals, which can be interpreted as a generalised reverse Poincaré inequality. This chapter is based on [101] by M. Schulte and V. Trapp and [109] by V. Trapp. While [101, Theorem 1.1] introduced the reverse Poincaré inequality, [109, Theorem 2.3] generalised this result to higher order difference operators.

3.1 Main theorem

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space and let η be a Poisson process on $\mathbb{X}$ with σ-finite intensity measure λ. To establish lower variance bounds for a Poisson functional $F \in L^2_\eta$ that depend on difference operators we can use the Fock space representation. In a first approach, with (2.19), one can directly derive for fixed $n \in \mathbb{N}$,

$$\operatorname{Var}[F] = \sum_{k=1}^{\infty} k! \|f_k\|_k^2 \geq n! \|f_n\|_n^2 = \frac{1}{n!} \int_{\mathbb{X}^n} \mathbb{E}[D^n_{x_1,\dots,x_n} F]^2 \, \lambda^n(\mathrm{d}(x_1, \dots, x_n)). \tag{3.1}$$

The problem with this lower variance bound is that difference operators can in general be positive or negative and, thus, can have expectation zero. To overcome this issue, we provide in this chapter a counterpart to the well-known Poincaré inequality

$$\operatorname{Var}[F] \leq \int_{\mathbb{X}} \mathbb{E}[(D_x F)^2] \, \lambda(\mathrm{d}x)$$

for $F \in L^2_\eta$ (see Theorem 2.16). More precisely, we give conditions under which the variance of F can be bounded from below by a constant times the right-hand side of the Poincaré inequality or its analogue for higher order difference operators. Thus, we can think of it as a generalised reverse Poincaré inequality.

Theorem 3.1. *For $n \in \mathbb{N}$ let $F \in L^2_\eta$ with*

$$\mathbb{E}\Big[\int_{\mathbb{X}^n} (D^n_{x_1,\dots,x_n} F)^2 \, \lambda^n(\mathrm{d}(x_1, \dots, x_n)) \Big] < \infty$$

satisfy

$$\mathbb{E}\Big[\int_{\mathbb{X}^{n+1}}(D^{n+1}_{x_1,\dots,x_{n+1}}F)^2\lambda^{n+1}(\mathrm{d}(x_1,\dots,x_{n+1}))\Big] \le \alpha\mathbb{E}\Big[\int_{\mathbb{X}^n}(D^n_{x_1,\dots,x_n}F)^2\,\lambda^n(\mathrm{d}(x_1,\dots,x_n))\Big] \tag{3.2}$$

for some $\alpha \ge 0$ *and* $n \in \mathbb{N}$*. Then,*

$$\mathrm{Var}[F] \ge \frac{1}{c(\alpha,n)}\mathbb{E}\Big[\int_{\mathbb{X}^n}(D^n_{x_1,\dots,x_n}F)^2\,\lambda^n(\mathrm{d}(x_1,\dots,x_n))\Big]$$

for some constant $c(\alpha,n) > 0$*, which depends only on* n *and* α*.*

For $n = 1$ this theorem was proven in [101, Section 2]. The following proof for general n is analogue to the one in [109, Appendix A].

Proof. For $m \in \mathbb{N}$ and $x_1,\dots,x_m \in \mathbb{X}$ let $f_m(x_1,\dots,x_m) = \frac{1}{m!}\mathbb{E}[D^m_{x_1,\dots,x_m}F]$ and $f_0 = \mathbb{E}[F]$. Then, the Fock space representation (2.18) provides

$$\mathbb{E}[F^2] = \sum_{m=0}^{\infty} m!\|f_m\|_m^2.$$

For $j \in \{n, n+1\}$ this yields together with Fubini's theorem and the monotone convergence theorem,

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{X}^j}(D^j_{x_1,\dots,x_j}F)^2\,\lambda^j(\mathrm{d}(x_1,\dots,x_j))\Big]\\
&= \int_{\mathbb{X}^j}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{X}^m}\mathbb{E}[D^m_{y_1,\dots,y_m}(D^j_{x_1,\dots,x_j}F)]^2\,\lambda^m(\mathrm{d}(y_1,\dots,y_m))\,\lambda^j(\mathrm{d}(x_1,\dots,x_j))\\
&= \sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mathbb{X}^{m+j}}\mathbb{E}[D^{m+j}_{x_1,\dots,x_{m+j}}F]^2\,\lambda^{m+j}(\mathrm{d}(x_1,\dots,x_{m+j}))\\
&= \sum_{m=j}^{\infty}\frac{\prod_{i=0}^{j-1}(m-i)}{m!}\int_{\mathbb{X}^m}\mathbb{E}[D^m_{x_1,\dots,x_m}F]^2\,\lambda^m(\mathrm{d}(x_1,\dots,x_m))\\
&= \sum_{m=1}^{\infty}\prod_{i=0}^{j-1}(m-i)m!\|f_m\|_m^2.
\end{aligned}$$

This means, assumption (3.2) is equivalent to

$$\sum_{m=1}^{\infty}\prod_{i=0}^{n-1}(m-i)m!\|f_m\|_m^2(\alpha - m + n) \ge 0. \tag{3.3}$$

Now, choose $c(\alpha,n) \ge \prod_{i=0}^{n-1}(m-i)(\alpha - m + n + 1) =: g(m)$ for all $m \in \mathbb{N}_0$, which is possible since $g\colon \mathbb{N}_0 \to \mathbb{R}$ is uniformly bounded from above in m as the leading monomial occurs with a negative sign. Then,

$$c(\alpha,n)\mathrm{Var}[F] - \mathbb{E}\Big[\int(D^n_{x_1,\dots,x_n}F)^2\,\lambda^n(\mathrm{d}(x_1,\dots,x_n))\Big]$$

$$= \sum_{m=1}^{\infty} m! \|f_m\|_m^2 \Big(c(\alpha, n) - \prod_{i=0}^{n-1} (m-i) \Big)$$
$$\geq \sum_{m=1}^{\infty} m! \|f_m\|_m^2 \prod_{i=0}^{n-1} (m-i)(\alpha - m + n) \geq 0$$

by (3.3), which completes the proof. □

We are also able to give an explicit constant $c(\alpha, n)$. Remember that by the proof of Theorem 3.1, the theorem holds for any $c(\alpha, n)$ with

$$c(\alpha, n) \geq \max_{m \in \mathbb{N}_0} \prod_{i=0}^{n-1} (m-i)(\alpha - m + n + 1).$$

For $n = 1$ it was shown in [101, Theorem 1.1] that $c(\alpha, 1)$ can be chosen as

$$c(\alpha, 1) = \frac{(\alpha+2)^2}{4}.$$

This is due to the fact that $\Big(m - \frac{(\alpha+2)}{2}\Big)^2 \geq 0$ implies $\frac{(\alpha+2)^2}{4} \geq m(\alpha - m + 2)$ for all $m \in \mathbb{N}_0$. Similarly, one can show that the function $h \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $h(m) = m^n(\alpha - m + n + 1)$ is maximised for $m = \frac{n(\alpha+n+1)}{n+1}$. Hence,

$$\prod_{i=0}^{n-1} (m-i)(\alpha - m + n + 1) \leq m^n(\alpha - m + n + 1) \leq \frac{n^n(\alpha+n+1)^{n+1}}{(n+1)^{n+1}}$$

for all $m \in \mathbb{R}_{\geq 0}$ and thus also for all $m \in \mathbb{N}_0$ so that we can choose

$$c(\alpha, n) = \frac{n^n(\alpha+n+1)^{n+1}}{(n+1)^{n+1}}$$

for $n \in \mathbb{N}$. This means that we also have an explicit bound for $c(\alpha, n)$. Anyway, since we are usually only interested in the order of the variance and not in explicit constants in this thesis, we refrain from going deeper into the topic of optimising this constant.

For $n = 1$ we refer to Theorem 3.1 as reverse Poincaré inequality. Since it is sufficient to work with the case $n = 1$ in many applications, we recapitulate this version from [101, Theorem 1.1] in the following corollary.

Corollary 3.2. *Let $F \in L^2_\eta$ be a Poisson functional satisfying*

$$\mathbb{E}\Big[\int_{\mathbb{X}^2} (D^2_{x,y}F)^2 \, \lambda^2(\mathrm{d}(x,y))\Big] \leq \alpha \mathbb{E}\Big[\int_{\mathbb{X}} (D_x F)^2 \, \lambda(\mathrm{d}x)\Big] < \infty$$

for some constant $\alpha \geq 0$. Then,

$$\mathrm{Var}[F] \geq \frac{4}{(\alpha+2)^2} \mathbb{E}\Big[\int_{\mathbb{X}} (D_x F)^2 \, \lambda(\mathrm{d}x)\Big].$$

3.2 Application of the generalised reverse Poincaré inequality

The generalised reverse Poincaré inequality can be applied to show that the variance is of a specific order. Typically, we consider a family of Poisson functionals $(F_s)_{s\geq 1}$ and aim to show that $\mathrm{Var}[F_s] \geq c_1 f(s)$ for some constant $c_1 > 0$ and some function $f\colon \mathbb{R} \to \mathbb{R}_+$. Establishing such a lower variance bound requires the following two steps for some fixed $n \in \mathbb{N}$.

(1) Showing an upper bound for the left-hand side of (3.2), i.e. proving that

$$\mathbb{E}\Big[\int_{\mathbb{X}^{n+1}} (D^{n+1}_{x_1,\dots,x_{n+1}} F_s)^2 \, \lambda^{n+1}(\mathrm{d}(x_1,\dots,x_{n+1}))\Big] \leq c_2 f(s) \tag{3.4}$$

for some constant $c_2 > 0$.

(2) Deriving a lower bound for the right-hand side of (3.2), i.e. showing that

$$\mathbb{E}\Big[\int_{\mathbb{X}^{n}} (D^{n}_{x_1,\dots,x_{n}} F_s)^2 \, \lambda^{n}(\mathrm{d}(x_1,\dots,x_{n}))\Big] \geq c_3 f(s) \tag{3.5}$$

for some constant $c_3 > 0$.

Then, condition (3.2) is fulfilled for $\alpha = \frac{c_2}{c_3}$ since in this case

$$\begin{aligned}&\mathbb{E}\Big[\int_{\mathbb{X}^{n+1}} (D^{n+1}_{x_1,\dots,x_{n+1}} F_s)^2 \lambda^{n+1}(\mathrm{d}(x_1,\dots,x_{n+1}))\Big] \leq c_2 f(s) = \alpha c_3 f(s)\\ &\leq \alpha\mathbb{E}\Big[\int_{\mathbb{X}^{n}} (D^{n}_{x_1,\dots,x_{n}} F_s)^2 \, \lambda^{n}(\mathrm{d}(x_1,\dots,x_{n}))\Big]\end{aligned}$$

and we derive as lower bound for the variance

$$\mathrm{Var}[F_s] \geq \frac{1}{c(\alpha,n)}\mathbb{E}\Big[\int_{\mathbb{X}^{n}} (D^{n}_{x_1,\dots,x_{n}} F_s)^2 \, \lambda^{n}(\mathrm{d}(x_1,\dots,x_{n}))\Big] \geq \frac{c_3}{c(\alpha,n)} f(s).$$

In the following chapters we consider applications from stochastic geometry. Here, we typically derive bounds of variances scaling like the volume of the observation window or like a specific power of s. For most applications it is sufficient to use the reverse Poincaré inequality (i.e. Theorem 3.1 for $n = 1$). Only in Section 6.3 we will use the more general version and show how this version can help to reduce our problem to a deterministic geometric problem.

For stabilising functionals of the form (2.22), one can often use properties of the score functions for step (1). For random graphs this will be illustrated for example in Lemma 4.2. The more involved case of random polytopes is treated in Section 5.2. Anyway, functionals do not have to be stabilising to fulfil (3.4) as can be seen in the chapter about excursion sets of Poisson shot noise processes. The functionals, which are considered in this chapter are not stabilising. Nevertheless, it is possible to show (3.4).

Note that upper bounds as in (3.4) are often also required to derive quantitative central limit theorems in Wasserstein and Kolmogorov distance (cf. Theorems 2.17 and 2.18). Thus, the main additional task that occurs when not only deriving a quantitative central limit theorem but also a lower variance bound is to find a lower bound as in (3.5). To show (3.5) one usually constructs special point configurations that lead to a non-zero difference operator and occur with positive probability. This is often much easier than to verify that the expectation of the difference operator is non-zero as required in (3.1).

3.3 Comparison to other lower variance bounds

Let us discuss some alternative approaches to derive lower variance bounds for Poisson functionals or statistics arising in stochastic geometry. The following theorem can be found in [70, Theorem 5.2].

Theorem 3.3. *Let $F \in L^2_\eta$ and assume that there exist $n \in \mathbb{N}$, $I_1, I_2 \subseteq \{1, \dots, n\}$ with $I_1 \cup I_2 = \{1, \dots, n\}$, a measurable set $U \subseteq \mathbb{X}^n$ and a constant $c > 0$ such that*

$$\left| \mathbb{E}\left[F\Big(\eta + \sum_{i \in I_1} \delta_{x_i}\Big) - F\Big(\eta + \sum_{i \in I_2} \delta_{x_i}\Big) \right] \right| \geq c$$

for λ^n-a.e. $(x_1, \dots, x_n) \in U$. Then,

$$\mathrm{Var}[F] \geq \frac{c^2}{4^{n+1} n!} \min_{\emptyset \neq J \subseteq \{1,\dots,n\}} \inf_{\substack{V \subseteq U \\ \lambda^n(V) \geq \lambda^n(U)/2^{n+1}}} \lambda^{|J|}(\Pi_J(V)),$$

where Π_J denotes the projection onto the components whose indices belong to J.

The approach from Theorem 3.3 is the one from the literature, which is most comparable to ours since, to apply this theorem, one also needs to control how the functional changes, if points are added to the underlying Poisson processes, to derive a lower bound for

$$\left| \mathbb{E}\left[F\Big(\eta + \sum_{i \in I_1} \delta_{x_i}\Big) - F\Big(\eta + \sum_{i \in I_2} \delta_{x_i}\Big) \right] \right|$$

for fixed $n \in \mathbb{N}$, $I_1, I_2 \subseteq \{1, \dots, n\}$ and $x_1, \dots, x_n \in \mathbb{X}$. Note that adding more than one point allows to enforce particular point configurations, which is why this expression is often easier to control than the expectation of the first difference operator in (3.1). Nevertheless, there is still the problem that the difference within the expectation can be both positive and negative. In contrast to that, for an application of the generalised reverse Poincaré inequality, we need a lower bound for $\mathbb{E}[(D^n_{x_1,\dots,x_n} F)^2]$ for $x_1, \dots, x_n \in \mathbb{X}$. Due to the square, the expression in the expectation is non-negative, which provides

$$\mathbb{E}[(D^n_{x_1,\dots,x_n} F)^2] \geq c^2 \mathbb{P}(|D^n_{x_1,\dots,x_n} F| \geq c)$$

for any $c > 0$. Thus, in contrast to the strategy from Theorem 3.3, it is sufficient for our approach to construct configurations of points $x_1, \dots, x_n \in \mathbb{X}$ for which $|D^n_{x_1,\dots,x_n} F| \geq c$ for some constant $c > 0$ and $n \in \mathbb{N}$ to obtain a lower bound for $\mathbb{P}(|D^n_{x_1,\dots,x_n} F| \geq c)$.

In the literature there are of course several further results for lower variance bounds. In [15, 86, 87], lower bounds for variances of stabilising functionals of Poisson processes and sometimes also binomial point processes were deduced. These results have all in common that generalised difference or add-one-cost operators are required to be non-degenerate.

This is similar to our work, but the random variable that has to be non-degenerate is more involved than the difference operator and, moreover, the results apply only to stabilising functionals and not to general Poisson functionals.

A further approach is to condition on some σ-field and to bound the variance from below by the expectation of the conditional variance with respect to this σ-field. In the context of stochastic geometry this was used, for example, in [5, 11, 90]. By conditioning on the σ-field it is sufficient to consider some particular point configurations similarly as in Theorem 3.1. In [38], a condition requiring that some conditional expectations are not degenerate is used to establish lower variance bounds for stabilising functionals.

As already mentioned, we consider functionals of spatial random graphs, random polytopes and excursion sets of Poisson shot noise processes in the following chapters. Since the considered statistics of spatial random graphs in Euclidean space fit into the framework of stabilising functionals of Poisson processes, the results for the non-degeneracy of the asymptotic variance of stabilising functionals discussed above might be applicable, while this is not the case for the considered statistics in hyperbolic space. The L^p surface area is again stabilising, but here the variance does not scale like the intensity of the underlying Poisson process, whence the previously mentioned results are not available anymore. Finally, in case of general Poisson shot noise processes we do not have stabilisation at all. Thus, the previously mentioned results cannot be used either.

Remark 3.4. Note that for functionals of isonormal Gaussian processes and for functionals of Rademacher sequences (i.e. sequences of independent random variables with values ± 1) one can also define an operator D^n for $n \in \mathbb{N}$. For isonormal Gaussian processes, this operator is usually referred to as the derivative operator. In the case of Rademacher sequences it is often called discrete gradient. This operator fulfils a similar role for its respective functionals as the difference operator has for Poisson functionals. As in the Poisson case there exist Fock space representations and chaos decompositions of functionals of isonormal Gaussian processes and functionals of Rademacher sequences.

Since our proof of Theorem 3.1 only requires the Fock space representations of $F, D^n F$ and $D^{n+1}F$ for some fixed $n \in \mathbb{N}$, the statement of Theorem 3.1 continues to hold for functionals of isonormal Gaussian processes and for functionals of Rademacher sequences if we rewrite the integral with respect to λ^j, which is the squared norm of $D^j F$ in $L^2(\lambda^j)$ for $j \in \{n, n+1\}$, in a proper way. This means that for isonormal Gaussian processes we take the squared norm of $D^j F$ in $H^{\otimes j}$, where H is the underlying Hilbert space and in the case of Rademacher sequences one uses the squared norm of $D^j F$ in $L^2(\mu^{\otimes j})$, where μ is the counting measure on $\mathbb{N}$ for $j \in \{n, n+1\}$. For more details on the Fock space representations and the operators D^k for $k \in \mathbb{N}$ see, for example, [80] for the Gaussian case and [65] for the Rademacher case.

Chapter 4

Spatial random graphs

In this chapter we analyse functionals of three different spatial random graphs. Sections 4.1 and 4.2 consider random graphs in Euclidean space while Section 4.3 analyses the radial spanning tree in hyperbolic space to illustrate the influence of the curvature of the underlying space to the behaviour of the graph. If not stated otherwise, Sections 4.1 and 4.2 are based on [101, Section 3] by M. Schulte and V. Trapp whereas Section 4.3 is mainly taken from [96] by D. Rosen, M. Schulte, C. Thäle and V. Trapp.

4.1 Random geometric graphs

Throughout this section let $W \subset \mathbb{R}^d$ be a non-empty compact convex set with $\lambda_d(W) > 0$. For $s \geq 1$ let η_s be a homogeneous Poisson process on W with intensity s, i.e. a Poisson process on $\mathbb{R}^d$ with intensity measure $\lambda = s\lambda_d|_W$. This section analyses the asymptotic behaviour of different functionals of random geometric graphs as $s \to \infty$.

A *random geometric graph* G_{r_s} with radius $r_s = \varrho s^{-1/d}$ for a fixed $\varrho > 0$ generated by η_s is a graph with vertex set η_s, whose vertices $v_1, v_2 \in \eta_s$ with $v_1 \neq v_2$ are connected by an edge if $\|v_1 - v_2\| \leq r_s$. A simulation of a random geometric graph can be found in Figure 4.1.

The random geometric graph, also known as Gilbert graph, was originally introduced by E. N. Gilbert in [48]. There are already many results in the literature of how functionals of random geometric graphs behave. For an overview of results until 2003 see for example [83]. In the following years more properties of random geometric graphs were studied. For high dimensions different functionals like the clique count, the number of edges or more general geometric functionals were analysed for instance in [4, 28, 42, 51]. Results for several functionals concerning the edges, degrees or components in the setting were the intensity parameter goes to infinity while the radius goes to zero can be found in [41, 68, 91, 92, 103], among others.

In this section we consider the vector of degree counts and the vector of component counts of a random geometric graph and want to analyse these functionals as $s \to \infty$. For $j \in \mathbb{N}_0$ let $V_j^{r_s}$ be the number of vertices of degree j in G_{r_s}, i.e.

$$V_j^{r_s} = \sum_{y \in \eta_s} \mathbb{1}\{\deg(y, \eta_s) = j\},$$

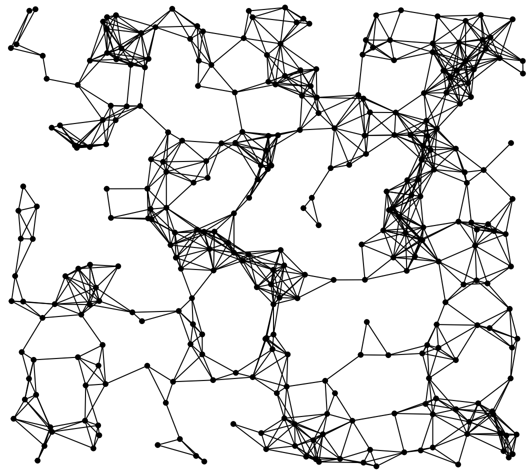

Figure 4.1: Simulation of a random geometric graph in $[0,1]^2$

where $\deg(y,\eta_s)$ stands for the degree of y in G_{r_s}. Moreover, let $C_j^{r_s}$ denote the number of components of size j in G_{r_s}, i.e.

$$C_j^{r_s} = \frac{1}{j} \sum_{y \in \eta_s} \mathbb{1}\{|\mathrm{C}(y,\eta_s)| = j\},$$

where $|\mathrm{C}(y,\eta_s)|$ is the number of vertices of the component $\mathrm{C}(y,\eta_s)$ of y in G_{r_s}. By the component $\mathrm{C}(y,\eta_s)$ we mean the set of all vertices that can be reached from y via edges. In the following we study the asymptotic behaviour of these functionals as $s \to \infty$.

Theorem 4.1. *a) For $s \to \infty$ the asymptotic covariance matrix of the rescaled vector of degree counts $\frac{1}{\sqrt{s}}(V_{j_1}^{r_s}, \dots, V_{j_n}^{r_s})$ for distinct $j_i \in \mathbb{N}_0$, $i \in \{1,\dots,n\}$, is positive definite, i.e. for any $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ there exists a constant $c > 0$ such that for s sufficiently large*

$$\operatorname{Var}\left[\sum_{i=1}^n \alpha_i V_{j_i}^{r_s}\right] \geq cs.$$

b) For $s \to \infty$ the asymptotic covariance matrix of the rescaled vector of component counts $\frac{1}{\sqrt{s}}(C_{j_1}^{r_s}, \dots, C_{j_n}^{r_s})$ for distinct $j_i \in \mathbb{N}_0$, $i \in \{1,\dots,n\}$, is positive definite, i.e. for any $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ there exists a constant $c > 0$ such that for s sufficiently large

$$\operatorname{Var}\left[\sum_{i=1}^n \alpha_i C_{j_i}^{r_s}\right] \geq cs.$$

One can show that the vectors $\frac{1}{\sqrt{s}}(V_{j_1}^{r_s}, \dots, V_{j_n}^{r_s})$ and $\frac{1}{\sqrt{s}}(C_{j_1}^{r_s}, \dots, C_{j_n}^{r_s})$ for distinct $j_i \in \mathbb{N}_0$, $i \in \{1,\dots,n\}$, fulfil the assumptions of Theorem 2.21. With this and an analogous theorem for the d_2-distance it was shown in [103, Section 3.2] that, after centering, both vectors fulfil a quantitative central limit theorem in d_2- and d_{convex}- distance if the corresponding asymptotic covariance matrix is positive definite. This means that Theorem 4.1 completes the proof of these multivariate quantitative central limit theorems from [103, Section 3.2].

For the proof of Theorem 4.1 we introduce the following lemma, which is helpful for verifying condition (3.2). More specifically, this means that we will derive lower bounds of the form (3.5) on a case by case basis while (3.4) is controlled by the lemma below as the considered functionals can be embedded in the framework of stabilising functionals.

Lemma 4.2. *Let $F_s^{(1)},\dots,F_s^{(n)}$ be Poisson functionals on η_s of the form* (2.22) *whose corresponding scores $\xi_s^{(1)},\dots,\xi_s^{(n)}$ satisfy a $(4+p)$-th moment condition as described in* (2.24) *for $p\in(0,1]$ and are exponentially stabilising, i.e. fulfil* (2.23). *Then, for any $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n\backslash\{\mathbf{0}\}$ there exists a constant $c>0$ such that for $s\geq 1$,*

$$\mathbb{E}\Big[\int_W\int_W\Big(\sum_{i=1}^n \alpha_i D^2_{x,y}F_s^{(i)}\Big)^2\,\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y)\Big]\leq cs.$$

Proof. Note at first that the scores $\xi_s^{(1)},\dots,\xi_s^{(n)}$ clearly fulfil (2.25) for a choice of $K=W$ and $C_K=1$. Hence, we can apply Lemma 2.20 a) and c). Fix $\varepsilon\in(4,4+p)$. Then, using Lemma 2.20 a), Hölder's inequality for $\frac{\varepsilon}{2}$ and $q=(1-\frac{2}{\varepsilon})^{-1}$ and Jensen's inequality provides

$$\begin{aligned}
\mathbb{E}[|D^2_{x,y}F_s^{(i)}|^2]&=\mathbb{E}\left[|D^2_{x,y}F_s^{(i)}|^2\mathbb{1}\{D^2_{x,y}F_s^{(i)}\neq 0\}\right]\\
&\leq(\mathbb{E}[|D^2_{x,y}F_s^{(i)}|^\varepsilon])^{2/\varepsilon}\mathbb{P}(D^2_{x,y}F_s^{(i)}\neq 0)^{1/q}\\
&=(\mathbb{E}[|D_xF_s^{(i)}(\eta_s+\delta_y)-D_xF_s^{(i)}(\eta_s)|^\varepsilon])^{2/\varepsilon}\mathbb{P}(D^2_{x,y}F_s^{(i)}\neq 0)^{1/q}\\
&\leq\left(2^{\varepsilon-1}\left(\mathbb{E}[|D_xF_s^{(i)}(\eta_s+\delta_y)|^\varepsilon]+\mathbb{E}[|D_xF_s^{(i)}(\eta_s)|^\varepsilon]\right)\right)^{2/\varepsilon}\mathbb{P}(D^2_{x,y}F_s^{(i)}\neq 0)^{1/q}\\
&\leq 4C^{2/\varepsilon}\mathbb{P}(D^2_{x,y}F_s^{(i)}\neq 0)^{1/q}
\end{aligned}$$

for $i\in\{1,\dots,n\}$, where $C>0$ is the constant from Lemma 2.20 a). Therefore, using Jensen's inequality and Lemma 2.20 c), it follows

$$\begin{aligned}
&\mathbb{E}\Big[\int_W\int_W\Big(D^2_{x,y}\sum_{i=1}^n\alpha_iF_s^{(i)}\Big)^2\,\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y)\Big]\\
&\leq\int_W\int_W\mathbb{E}\Big[n\sum_{i=1}^n\alpha_i^2(D^2_{x,y}F_s^{(i)})^2\Big]\,\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y)\\
&=n\sum_{i=1}^n\alpha_i^2\int_W\int_W\mathbb{E}[|D^2_{x,y}F_s^{(i)}|^2]\,\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y)\\
&\leq n\sum_{i=1}^n\alpha_i^2 s\int_W s\int_W 4C^{2/\varepsilon}\mathbb{P}(D^2_{x,y}F_s^{(i)}\neq 0)^{1/q}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq n\sum_{i=1}^n\alpha_i^2 s\int_W 4C^{2/\varepsilon}C_{1/q}\,\mathrm{d}x\leq cs
\end{aligned}$$

for some constant $c>0$, which completes the proof. □

Proof of Theorem 4.1. For $x \in W$ and $j \in \mathbb{N}_0$ the difference operators are given by

$$D_x V_j^{r_s} = \mathbb{1}\{\deg(x, \eta_s + \delta_x) = j\} + \sum_{y \in \eta_s} (\mathbb{1}\{\deg(y, \eta_s + \delta_x) = j\} - \mathbb{1}\{\deg(y, \eta_s) = j\})$$

and

$$D_x C_j^{r_s} = \frac{1}{j}\mathbb{1}\{|\mathrm{C}(x, \eta_s + \delta_x)| = j\} + \frac{1}{j}\sum_{y \in \eta_s} (\mathbb{1}\{|\mathrm{C}(y, \eta_s + \delta_x)| = j\} - \mathbb{1}\{|\mathrm{C}(y, \eta_s)| = j\}).$$

Let $m = \operatorname{argmax}_{i \in \{1,\dots,n\}\colon \alpha_i \neq 0} j_i$ and $x \in W$. For a) we consider configurations where

$$\eta_s\left(B^d\left(x, \frac{r_s}{2}\right)\right) = j_m + 1 \quad \text{ and } \quad \eta_s\Big(B^d\Big(x, \frac{3}{2}r_s\Big) \Big\backslash B^d\Big(x, \frac{r_s}{2}\Big)\Big) = 0.$$

Then, it follows for any $y \in \eta_s$ with $y \in B^d(x, \frac{r_s}{2})$ that

$$\deg(y, \vartheta) = \begin{cases} j_m, & \text{for } \vartheta = \eta_s, \\ j_m + 1, & \text{for } \vartheta = \eta_s + \delta_x. \end{cases}$$

The degrees of all the other points are not affected by adding x. Thus, in this situation only the numbers of points with degree j_m and $j_m + 1$ change. Due to the choice of m, we have

$$\left| D_x \left(\sum_{i=1}^n \alpha_i V_{j_i}^{r_s} \right) \right| = \left| \sum_{i=1}^n \alpha_i D_x V_{j_i}^{r_s} \right| = |\alpha_m D_x V_{j_m}^{r_s}| = |\alpha_m(-(j_m + 1))| \geq |\alpha_m|.$$

For b) we consider configurations where

$$\eta_s\Big(B^d\Big(x, \frac{r_s}{2}\Big)\Big) = j_m \quad \text{ and } \quad \eta_s\Big(B^d\Big(x, \frac{3}{2}r_s\Big) \Big\backslash B^d\Big(x, \frac{r_s}{2}\Big)\Big) = 0.$$

It follows that $C_{j_m}^{r_s}$ decreases by 1 by adding x and $C_{j_m+1}^{r_s}$ increases by 1. The other component counts are not affected. Because of the choice of m, it holds

$$\left| D_x \left(\sum_{i=1}^n \alpha_i C_{j_i}^{r_s} \right) \right| = \left| \sum_{i=1}^n \alpha_i D_x C_{j_i}^{r_s} \right| = |\alpha_m D_x C_{j_m}^{r_s}| = |\alpha_m|.$$

Let $A_s = \{x \in W\colon B^d(x, \frac{r_s}{2}) \subset W\}$. Then, for $F_{j_i}^{r_s} = V_{j_i}^{r_s}$ or $F_{j_i}^{r_s} = C_{j_i}^{r_s}$ for $i \in \{1, \dots, n\}$ and

$$k = \begin{cases} j_m + 1, & \text{for } F_{j_i}^{r_s} = V_{j_i}^{r_s}, \\ j_m, & \text{for } F_{j_i}^{r_s} = C_{j_i}^{r_s}, \end{cases}$$

it follows for s sufficiently large such that $\lambda_d(A_s) \geq \frac{\lambda_d(W)}{2}$ that

$$\begin{aligned} \mathbb{E}\Big[\int_W \Big(\sum_{i=1}^n \alpha_i D_x F_{j_i}^{r_s} \Big)^2 \lambda(\mathrm{d}x) \Big] &\geq s\alpha_m^2 \int_W \mathbb{P}\Big(\Big| \sum_{i=1}^n \alpha_i D_x F_{j_i}^{r_s} \Big| \geq |\alpha_m| \Big) \,\mathrm{d}x \\ &\geq s\alpha_m^2 \int_{A_s} \mathbb{P}\Big(\eta_s\Big(B^d\Big(x, \frac{r_s}{2}\Big)\Big) = k, \eta_s\Big(B^d\Big(x, \frac{3}{2}r_s\Big) \Big\backslash B^d\Big(x, \frac{r_s}{2}\Big)\Big) = 0 \Big) \,\mathrm{d}x \end{aligned}$$

$$\geq s\alpha_m^2 \int_{A_s} \frac{(s\kappa_d r_s^d)^k}{2^{dk}k!} e^{-s\kappa_d r_s^d/2^d} e^{-s\kappa_d(3^d-1)r_s^d/2^d} \,\mathrm{d}x$$
$$\geq s\alpha_m^2 \frac{\lambda_d(W)}{2} \frac{(\kappa_d \varrho^d)^k}{2^{dk}k!} e^{-\kappa_d 3^d \varrho^d/2^d} =: c \cdot s,$$

where $c > 0$ depends on W, α, ϱ, k and d.

Both functionals can be written as sums of scores as in (2.22). For $j \in \mathbb{N}_0$, $y \in \eta_s$ and $s \geq 1$ the score for the degree count of degree j is given by

$$\xi_s(y, \eta_s) = \mathbb{1}\{\deg(y, \eta_s) = j\}$$

and for $j \in \mathbb{N}$, $y \in \eta_s$ and $s \geq 1$ the score for the number of components of size j is

$$\xi_s(y, \eta_s) = \frac{1}{j}\mathbb{1}\{|\mathrm{C}(y, \eta_s)| = j\}.$$

These scores clearly fulfil a $(4+p)$-th moment condition and are by [103, proofs of Theorem 3.5 (b) and Theorem 3.6 (b)] exponentially stabilising. Therefore, we can apply Lemma 4.2. Together with Theorem 3.1 for $n = 1$ the proof is complete. □

4.2 k-nearest neighbour graphs

Let $W \subset \mathbb{R}^d$ be a non-empty compact convex set with $\lambda_d(W) > 0$. For $s \geq 1$ let η_s be a homogeneous Poisson process on W with intensity s, i.e. a Poisson process on $\mathbb{R}^d$ with intensity measure $\lambda = s\lambda_d|_W$. In the following we study the asymptotic behaviour of a k-nearest neighbour graph as $s \to \infty$.

A *k-nearest neighbour graph* for $k \in \mathbb{N}$ generated by the Poisson process η_s is the undirected graph with vertex set η_s, where each vertex is connected with its k-nearest neighbours. The set of all k-nearest neighbours of $v_1 \in \eta_s$ contains all $v_2 \in \eta_s$ with $v_2 \neq v_1$ and for which $\|v_1 - v_2\| \geq \|v_1 - x\|$ for at most k vertices $v_1 \neq x \in \eta_s$ or, in other words, $\eta_s(B^d(v_1, \|v_1 - v_2\|)\setminus\{v_1\}) \leq k$. Note that the set of all k-nearest neighbours of a point $x \in \eta_s$, denoted by $N(x, \eta_s)$, contains k points almost surely since the distances of two distinct points to some vertex is almost surely not equal. A simulation of a k-nearest neighbour graph can be seen in Figure 4.2. The asymptotic behaviour of functionals of k-nearest neighbour graphs are for example analysed in e.g. [5, 15, 69, 70, 87, 88, 103, 110], to name just a few. In particular, the total edge lengths or edge length functionals of k-nearest neighbour graphs are studied in a variety of papers. These are also the first type of functionals that we consider.

For $q \in [0, \infty)$ let L_q denote the edge length functional of power q of the k-nearest neighbour graph generated by η_s, which is defined by

$$L_q = \frac{1}{2} \sum_{(y,z)\in\eta_{s,\neq}^2} \mathbb{1}\{z \in N(y, \eta_s) \text{ or } y \in N(z, \eta_s)\}\|y - z\|^q.$$

Let $F_q = s^{q/d}L_q$ be its scaled version.

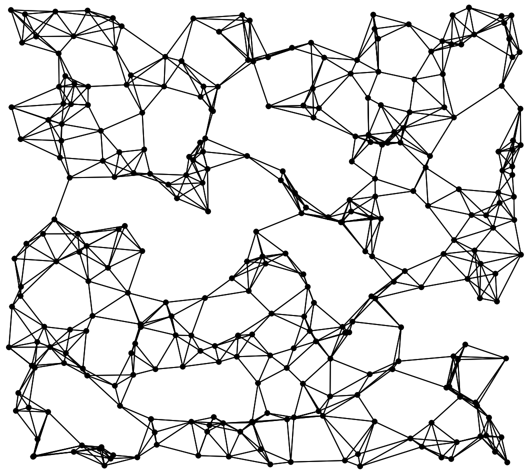

Figure 4.2: Simulation of a k-nearest neighbour graph in $[0,1]^2$ for $k = 5$

Theorem 4.3. *For $s \to \infty$ the asymptotic covariance matrix of $\frac{1}{\sqrt{s}}(F_{q_1}, \ldots, F_{q_n})$ for distinct $q_i \geq 0$, $i \in \{1, \ldots, n\}$, is positive definite, i.e. for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ there exists a constant $c > 0$ such that for s sufficiently large*

$$\operatorname{Var}\left[\sum_{i=1}^{n} \alpha_i F_{q_i}\right] \geq cs.$$

The first quantitative result for the total edge length of a k-nearest neighbour graph can be found in [5]. This convergence rate was further improved in [88] before in [70] the presumably optimal rate was shown. In [103] this result was transferred to the multivariate case of a vector of edge length functionals with the help of results as those from Theorem 2.21 but it was left open to show in general that its asymptotic covariance matrix is positive definite. As before for the functionals of random geometric graphs, Theorem 4.3 closes this gap and completes the proofs of the multivariate central limit theorems in d_2- and d_{convex}-distance of the centred versions of vectors of the form $\frac{1}{\sqrt{s}}(F_{q_1}, \ldots, F_{q_n})$ for distinct $q_i \geq 0$, $i \in \{1, \ldots, n\}$.

In order to apply the reverse Poincaré inequality in the proof of Theorem 4.3, we need to construct configurations for which we know how the difference operator behaves. To this end, we aim to construct specific point configurations in some ball $B^d(x,r)$ around the added point x for some $r > 0$. Additionally, to guarantee that the configuration in the ball around x is not influenced by the points in $\mathbb{R}^d \backslash B^d(x,r)$, we also need to control the behaviour of the points of η in $\mathbb{R}^d \backslash B^d(x,r)$ and have to make sure that no point of η from $\mathbb{R}^d \backslash B^d(x,r)$ destroys the configuration in $B^d(x,r)$. For the random geometric graph, we could construct such configurations rather easy by only considering configurations without any points in a large enough annulus around $B^d(x,r)$ so that there were no edges between point of η in $B^d(x,r)$ and points of η in $\mathbb{R}^d \backslash B^d(x,r)$. To guarantee the same for the k-nearest neighbour graph we need to make sure that the k nearest neighbours of all points of η, which are in $\mathbb{R}^d \backslash B^d(x,r)$, also lie in $\mathbb{R}^d \backslash B^d(x,r)$. The following lemma helps bounding the probability of such point configurations. Since we will not only use this lemma to prove Theorem 4.3 but also employ it in a slightly more general version in a further proof, we present the lemma here directly in its generalised version.

Lemma 4.4. *Let $k, j \in \mathbb{N}$ be fixed. Then there exist constants $c_1, c_2 > 0$ depending on k, j, d and W such that for all $\varepsilon > 0$ and $x \in W$ with $B^d(x, 2(j+1)\varepsilon) \subset W$,*

$$\mathbb{P}(\exists y \in \eta_s|_{\mathbb{R}^d \setminus B^d(x,j\varepsilon)} \colon \eta_s(A_{j,\varepsilon}(x,y)) \leq k-1) \leq c_1 e^{-sc_2\varepsilon^d},$$

where $A_{j,\varepsilon}(x,y) = (B^d(y, \|x-y\| - (j-1)\varepsilon) \cap W) \setminus (B^d(x, j\varepsilon) \cup \{y\})$.

Proof. Let $x \in W$ with $B^d(x, 2(j+1)\varepsilon) \subset W$. Then, for $y \in W$ with $j\varepsilon < \|x-y\| \leq (j+1)\varepsilon$ we have that $B^d(y, \|x-y\|) \subset W$. Therefore, since $y \notin B^d(x, j\varepsilon)$,

$$\lambda_d(A_{j,\varepsilon}(x,y)) \geq \frac{1}{2}\kappa_d(\|x-y\| - (j-1)\varepsilon)^d.$$

For $y \in W$ with $\|x-y\| > (j+1)\varepsilon$ it holds that $\|x-y\| - j\varepsilon \geq \frac{1}{2}(\|x-y\| - (j-1)\varepsilon)$. Moreover, $(B^d(y, \|x-y\| - j\varepsilon) \cap W) \setminus \{y\} \subset A_{j,\varepsilon}(x,y)$. Hence, with [70, Lemma 7.4] there is a constant $c_W > 0$ only depending on W such that

$$\begin{aligned}\lambda_d(A_{j,\varepsilon}(x,y)) &\geq \lambda_d(B^d(y, \|x-y\| - j\varepsilon) \cap W) \geq c_W(\|x-y\| - j\varepsilon)^d \\ &\geq c_W\Big(\frac{1}{2}(\|x-y\| - (j-1)\varepsilon)\Big)^d.\end{aligned}$$

Altogether, for $y \in W \setminus B^d(x, j\varepsilon)$ it follows

$$\lambda_d(A_{j,\varepsilon}(x,y)) \geq c(\|x-y\| - (j-1)\varepsilon)^d$$

for some constant $c > 0$. For $t \in \mathbb{N}_0$ there exist constants $\tilde{c}_1, \tilde{c}_2 > 0$ such that $z^t e^{-z} \leq \tilde{c}_1 e^{-\tilde{c}_2 z}$ for all $z > 0$. Hence, using the Mecke formula (2.15) and spherical coordinates, we get

$$\begin{aligned}&\mathbb{P}(\exists y \in \eta_s|_{\mathbb{R}^d \setminus B^d(x,j\varepsilon)} \colon \eta_s(A_{j,\varepsilon}(x,y)) \leq k-1) \\ &\leq \mathbb{E}\Big[\sum_{y \in \eta_s|_{\mathbb{R}^d \setminus B^d(x,j\varepsilon)}} \mathbb{1}\{\eta_s(A_{j,\varepsilon}(x,y)) \leq k-1\}\Big] \\ &\leq s\int_{\mathbb{R}^d \setminus B^d(x,j\varepsilon)} \mathbb{P}(\eta_s(A_{j,\varepsilon}(x,y)) \leq k-1)\,\mathrm{d}y \\ &= s\int_{\mathbb{R}^d \setminus B^d(x,j\varepsilon)} \sum_{i=0}^{k-1} \frac{\lambda(A_{j,\varepsilon}(x,y))^i}{i!} e^{-\lambda(A_{j,\varepsilon}(x,y))}\,\mathrm{d}y \\ &\leq s\int_{\mathbb{R}^d \setminus B^d(x,j\varepsilon)} \hat{c}_1 e^{-\hat{c}_2\lambda(A_{j,\varepsilon}(x,y))}\,\mathrm{d}y \\ &\leq \int_{\mathbb{R}^d \setminus B^d(x,j\varepsilon)} \hat{c}_1 s e^{-\hat{c}_2 sc(\|x-y\| - (j-1)\varepsilon)^d}\,\mathrm{d}y \\ &= d\kappa_d \int_\varepsilon^\infty \hat{c}_1 s(r + (j-1)\varepsilon)^{d-1} e^{-\hat{c}_2 scr^d}\,\mathrm{d}r \leq c_1 e^{-sc_2\varepsilon^d}\end{aligned}$$

for suitable constants $\hat{c}_1, \hat{c}_2, c_1, c_2 > 0$. $\square$

Proof of Theorem 4.3. Let e_i denote the d-dimensional standard unit vector in the i-th direction. For $\varepsilon > 0$, $x \in W$ with $B^d(x, 4\varepsilon) \subset W$ and $\hat{x} = x + \frac{3}{4}\varepsilon e_1$, we consider configurations where $\eta_s(B^d(\hat{x}, \varepsilon/4)) = k + 1$, $\eta_s(B^d(x, \varepsilon)\backslash B^d(\hat{x}, \varepsilon/4)) = 0$ and $\eta_s(A_{1,\varepsilon}(x, y)) \geq k$ for all $y \in \eta_s|_{\mathbb{R}^d\backslash B^d(x,\varepsilon)}$, where $A_{1,\varepsilon}(x, y)$ is defined as in Lemma 4.4. Then, for $q \geq 0$ the difference operator of F_q is given by

$$D_x F_q = s^{q/d} \sum_{y \in N(x, \eta_s + \delta_x)} \|x - y\|^q.$$

Inserting $j = 1$ in Lemma 4.4 provides

$$\mathbb{P}(\exists y \in \eta_s|_{\mathbb{R}^d\backslash B^d(x,\varepsilon)} \colon \eta_s(A_{1,\varepsilon}(x, y)) \leq k - 1) \leq c_1 e^{-sc_2\varepsilon^d}$$

for some constants $c_1, c_2 > 0$.

Now, let $m = \mathrm{argmax}_{i \in \{1,\dots,n\} \colon \alpha_i \neq 0} q_i$ and assume without loss of generality $\alpha_m > 0$. If $\alpha_i \geq 0$ for all $i \in \{1, \dots, n\}$, we choose $\varepsilon = \bar{c}s^{-1/d}$ with $\bar{c} \geq 1$ large enough such that we have for the configurations mentioned above

$$D_x \sum_{i=1}^n \alpha_i F_{q_i} \geq \alpha_m s^{q_m/d} \sum_{y \in N(x, \eta_s + \delta_x)} \|x - y\|^{q_m} \geq \alpha_m k \left(\frac{s^{1/d}\varepsilon}{2} \right)^{q_m} \geq 1$$

and $c_1 e^{-sc_2\varepsilon^d} < \frac{1}{2}$. Otherwise, let $\ell = \mathrm{argmax}_{i \in \{1,\dots,n\} \colon \alpha_i < 0} q_i$. Then, $q_m > q_\ell$ and it follows for the configurations introduced above for $s^{1/d}\varepsilon \geq 1$,

$$\begin{aligned}
D_x \sum_{i=1}^n \alpha_i F_{q_i} &= \sum_{i=1}^n \alpha_i s^{q_i/d} \sum_{y \in N(x, \eta_s + \delta_x)} \|x - y\|^{q_i} \\
&\geq \alpha_m s^{q_m/d} \sum_{y \in N(x, \eta_s + \delta_x)} \|x - y\|^{q_m} - \sum_{\substack{i \in \{1,\dots,n\} \colon \\ \alpha_i < 0}} (-\alpha_i) s^{q_i/d} \sum_{y \in N(x, \eta_s + \delta_x)} \|x - y\|^{q_i} \\
&\geq \alpha_m \sum_{y \in N(x, \eta_s + \delta_x)} \left(s^{1/d} \|x - y\| \right)^{q_m} - \sum_{\substack{i \in \{1,\dots,n\} \colon \\ \alpha_i < 0}} (-\alpha_i) \sum_{y \in N(x, \eta_s + \delta_x)} (s^{1/d}\varepsilon)^{q_i} \\
&\geq \alpha_m k \left(\frac{s^{1/d}\varepsilon}{2} \right)^{q_m} - \sum_{\substack{i \in \{1,\dots,n\} \colon \\ \alpha_i < 0}} (-\alpha_i) k (s^{1/d}\varepsilon)^{q_\ell} \\
&\geq k (s^{1/d}\varepsilon)^{q_\ell} \left(\alpha_m \frac{1}{2^{q_m}} \left(s^{1/d}\varepsilon \right)^{q_m - q_\ell} - \sum_{\substack{i \in \{1,\dots,n\} \colon \\ \alpha_i < 0}} (-\alpha_i) \right).
\end{aligned}$$

In this case, choose $\varepsilon = s^{-1/d}\bar{c} > 0$ with $\bar{c} \geq 1$ large enough such that $c_1 e^{-sc_2\varepsilon^d} < \frac{1}{2}$ and

$$D_x \sum_{i=1}^n \alpha_i F_{q_i} \geq \alpha_m \frac{1}{2^{q_m}} \left(s^{1/d}\varepsilon \right)^{q_m - q_\ell} - \sum_{\substack{i \in \{1,\dots,n\} \colon \\ \alpha_i < 0}} (-\alpha_i) \geq 1.$$

To conclude, let $A_s = \{x \in W\colon B^d(x,4\varepsilon) \subset W\}$. Due to the independence of $\eta_s(B^d(\hat{x},\varepsilon/4))$, $\eta_s(B^d(x,\varepsilon)\backslash B^d(\hat{x},\varepsilon/4))$ and $\eta_s(A_{1,\varepsilon}(x,y))$ for $y \in \eta_s|_{\mathbb{R}^d\backslash B^d(x,\varepsilon)}$ and $x \in A_s$ and by Lemma 4.4 we have for s large enough such that $\lambda_d(A_s) \geq \frac{\lambda_d(W)}{2}$,

$$\begin{aligned}
\mathbb{E}\Big[\int_W \Big(D_x \sum_{i=1}^n \alpha_i F_{q_i}\Big)^2 \lambda(\mathrm{d}x)\Big] &\geq s\int_W \mathbb{P}\Big(D_x \sum_{i=1}^n \alpha_i F_{q_i} \geq 1\Big)\,\mathrm{d}x \\
&\geq s\int_W \mathbb{P}\big(\eta_s(B^d(\hat{x},\varepsilon/4)) = k+1, \eta_s(B^d(x,\varepsilon)\backslash B^d(\hat{x},\varepsilon/4)) = 0, \\
&\qquad \eta_s(A_{1,\varepsilon}(x,y)) \geq k\ \forall y \in \eta_s|_{\mathbb{R}^d\backslash B^d(x,\varepsilon)}\big)\,\mathrm{d}x \\
&\geq s\int_{A_s} \frac{(s\kappa_d\varepsilon^d)^{k+1}}{4^{d(k+1)}(k+1)!} e^{-s\kappa_d\varepsilon^d/4^d} e^{-s\kappa_d\varepsilon^d(1-1/4^d)}(1-c_1 e^{-sc_2\varepsilon^d})\,\mathrm{d}x \\
&\geq s\frac{\lambda_d(W)}{2}\frac{(\kappa_d\bar{c}^d)^{k+1}}{4^{d(k+1)}(k+1)!} e^{-\kappa_d\bar{c}^d}\cdot\frac{1}{2} =: c_{q,\alpha,k,W,d}\, s.
\end{aligned}$$

Our functionals can be written as sums of scores as in (2.22). For $y \in \eta_s$, $q \geq 0$ and $s \geq 1$ the corresponding score of F_q is given by

$$\xi_s(y,\eta_s) = \sum_{z\in N(y,\eta_s)} \mathbb{1}\{y \in N(z,\eta_s)\}\frac{\|y-z\|^q}{2} + \mathbb{1}\{y \notin N(z,\eta_s)\}\|y-z\|^q.$$

The scores $(\xi_s)_{s\geq 1}$ fulfil a $(4+p)$-th moment condition (see the proof of [69, Theorem 3.1]) and are by [103, proof of Theorem 3.1] exponentially stabilising. Therefore, we can apply Lemma 4.2, which completes the proof together with Theorem 3.1. □

In the following we consider a second statistic of k-nearest neighbour graphs, namely the number of vertices with a given degree. For $j \in \mathbb{N}_0$ let V_j^k denote the number of vertices of degree j in the k-nearest neighbour graph generated by η_s, i.e.

$$V_j^k = \sum_{y\in\eta_s} \mathbb{1}\{\deg(y,\eta_s) = j\}.$$

We want to study the vector $(V_{j_1}^k,\dots,V_{j_n}^k)$ for distinct $j_i \geq k$, $i \in \{1,\dots,n\}$. The following property of the degrees of k-nearest neighbour graphs can be found in [113, Lemma 8.4].

Lemma 4.5. *The vertices of a k-nearest neighbour graph in $\mathbb{R}^d$ have bounded degree.*

Proof. The following arguments are based on the proof ideas of [23, Lemma S.] and [113, Lemma 8.4]. Without loss of generality we assume that the origin $\mathbf{0} \in \mathbb{R}^d$ is a vertex of the k-nearest neighbour graph and show that its degree is bounded. Then, by translation invariance, all vertices have bounded degree.

Since the unit sphere is compact, there exists a partition of the unit sphere into disjoint spherically convex sets $C_1,\dots,C_n$ for some $n \in \mathbb{N}$, which might depend on d, such that for any $i \in \{1,\dots,n\}$, $\|x-y\| < 1$ for $x,y \in C_i$. Then, $\mathbb{R}^d = \bigcup_{i=1}^n \mathcal{C}_i$, where for $i \in \{1,\dots,n\}$,

$$\mathcal{C}_i = \{\lambda x\colon x \in C_i, \lambda \geq 0\}.$$

Let $\hat{x}, \hat{y} \in \mathcal{C}_i$ with $\hat{x} = \lambda_1 x$ and $\hat{y} = \lambda_2 y$ with $0 < \lambda_1 \leq \lambda_2$ and $x, y \in C_i$. Then, since $\|y\| = 1$ and $\|x - y\| < 1$,

$$\begin{aligned} \|\hat{x} - \hat{y}\| &= \|\lambda_1 x - \lambda_2 y\| = \|\lambda_1(x - y) - (\lambda_2 - \lambda_1) y\| \\ &\leq \lambda_1 \|x - y\| + (\lambda_2 - \lambda_1)\|y\| < \lambda_2 = \|\hat{y}\|, \end{aligned}$$

i.e. for $\hat{x}, \hat{y} \in \mathcal{C}_i$ it holds

$$\|\hat{x} - \hat{y}\| < \max\{\|\hat{x}\|, \|\hat{y}\|\}. \tag{4.1}$$

Now, the degree of $\mathbf{0}$ is bounded by nk for the following reason. Assume the degree of $\mathbf{0}$ is larger than nk. Then, there exists $i \in \{1, \dots, n\}$ such that $\mathcal{C}_i$ contains more than k points, which are connected to $\mathbf{0}$. Denote $k+1$ of these points by $x_1, \dots, x_{k+1}$, where x_{k+1} is such that $\|x_{k+1}\| \geq \|x_i\|$ for $i \in \{1, \dots, k\}$. Then, x_{k+1} is not one of the k nearest neighbours of $\mathbf{0}$ and $\mathbf{0}$ is not one of the k nearest neighbours of x_{k+1} since $\|x_i - x_{k+1}\| < \|x_{k+1}\|$ for $i \in \{1, \dots, k\}$ by (4.1), which is a contradiction and completes the proof. □

By Lemma 4.5 there exists $k_{\max} \in \mathbb{N}$, the maximal possible degree that occurs with a positive probability in a k-nearest neighbour graph. The smallest possible degree that appears with positive probability in a k-nearest neighbour graph is clearly k. It arises when a point is only connected to its k nearest neighbours but is not one of the k nearest neighbours of any other point. Therefore, we consider in the following theorem only degree counts of degree j with $j \in \{k, k+1, \dots, k_{\max}\}$ for $i \in \{1, \dots, n\}$.

Theorem 4.6. *For $d \geq 2$, $n \in \mathbb{N}$ with $n \leq k_{\max} - k + 1$ and $s \to \infty$ the asymptotic covariance matrix of $\frac{1}{\sqrt{s}}(V_{j_1}^k, \dots, V_{j_n}^k)$ for distinct $j_i \in \{k, k+1, \dots, k_{\max}\}$, $i \in \{1, \dots, n\}$, is positive definite, i.e. for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ there exists a constant $c > 0$ such that for s sufficiently large*

$$\operatorname{Var}\left[\sum_{i=1}^{n} \alpha_i V_{j_i}^k\right] \geq cs.$$

Similarly to the example of edge length functionals, it was shown in [103, Theorem 3.3] that the scaled vector of degree counts $\frac{1}{\sqrt{s}}(V_{j_1}^k, \dots, V_{j_n}^k)$ for distinct $j_i \in \{k, k+1, \dots, k_{\max}\}$, $i \in \{1, \dots, n\}$ fulfils a quantitative multivariate central limit theorem in d_2- and d_{convex}-distance if its asymptotic covariance matrix is positive definite. Thus, with Theorem 4.6 the proofs of these multivariate central limit theorems are complete.

Proof of Theorem 4.6. First note that the degrees $j_1, \dots, j_n$ are chosen in such a way that they can occur in a k-nearest neighbour graph. A vertex can have k neighbours if it is only connected to its k nearest neighbours and can have up to $k_{\max}$ neighbours by the definition of $k_{\max}$. All degrees in between can occur as well as can be seen from the following construction. Assume we have a configuration where x has $k_{\max}$ neighbours. Then we delete $1 \leq t \leq k_{\max} - k$ vertices which are connected to x but are not one of the k nearest neighbours of x and all other vertices that are not connected to x. Consequently, we obtain a configuration where x has degree $k_{\max} - t$. This means that $\mathbb{P}(\deg(x, \beta_{j_i} + \delta_x) = j_i) > 0$ for $i \in \{1, \dots, n\}$, where β_{j_i} denotes a binomial point process of j_i independent random points uniformly distributed in $B^d(\mathbf{0},1)$ as defined in Subsection 2.3.1. Obviously, these probabilities do not change if we take a binomial point process on any other ball.

The difference operator of V_j^k is given by

$$D_x V_j^k = \mathbb{1}\{\deg(x,\eta_s+\delta_x)=j\} + \sum_{y\in\eta_s}\left(\mathbb{1}\{\deg(y,\eta_s+\delta_x)=j\} - \mathbb{1}\{\deg(y,\eta_s)=j\}\right)$$

for $x \in W$. Denote $I = \{i \in \{1,\dots,n\}\colon \alpha_i \neq 0\}$ and $m = \operatorname{argmin}_{i\in I} j_i$. We can assume $\alpha_m > 0$ without loss of generality. In the following we distinguish several cases that are illustrated in Figure 4.3.

Case 1: $j_m > k$

Let $\varepsilon > 0$ and $x \in W$ with $B^d(x,8\varepsilon) \subset W$. We consider configurations with $\eta_s(B^d(x,\varepsilon)) = j_m$, $\eta_s(B^d(x,3\varepsilon)\backslash B^d(x,\varepsilon)) = 0$ and $\eta_s(A_{3,\varepsilon}(x,y)) \geq k$ for all $y \in \eta_s|_{\mathbb{R}^d\backslash B^d(x,3\varepsilon)}$, where $A_{3,\varepsilon}(x,y)$ is defined as in Lemma 4.4. Then, if x is connected to all $z \in \eta_s|_{B^d(x,\varepsilon)}$, we have

$$D_x \sum_{i=1}^n \alpha_i V_{j_i}^k \geq \alpha_m.$$

Applying Lemma 4.4 for $j = 3$ provides

$$\mathbb{P}(\exists y \in \eta_s|_{\mathbb{R}^d\backslash B^d(x,3\varepsilon)}\colon \eta_s(A_{3,\varepsilon}(x,y)) \leq k-1) \leq c_1 e^{-sc_2\varepsilon^d}.$$

Now, choose $\varepsilon = \bar{c}s^{-1/d} > 0$ for $\bar{c} \geq 1$ such that $c_1 e^{-sc_2\varepsilon^d} \leq \frac{1}{2}$. Let $A_s = \{x \in W\colon B^d(x,8\varepsilon) \subset W\}$ and s large enough such that $\lambda_d(A_s) > \frac{\lambda_d(W)}{2}$. Then, using the independence property of Poisson processes and Proposition 2.14, we have for $p_m = \mathbb{P}\left(\deg(x,\beta_{j_m}+\delta_x) = j_m\right) > 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\int_W \Big(D_x \sum_{i=1}^n \alpha_i V_{j_i}^k\Big)^2 \lambda(\mathrm{d}x)\Big] \geq \alpha_m^2 \int_W \mathbb{P}\Big(D_x \sum_{i=1}^n \alpha_i V_{j_i}^k \geq \alpha_m\Big)\ \lambda(\mathrm{d}x)\\
&\geq \alpha_m^2 \int_{A_s} \mathbb{P}\left(\eta_s(B^d(x,\varepsilon)) = j_m, \eta_s(B^d(x,3\varepsilon)\backslash B^d(x,\varepsilon)) = 0, \deg(x,\eta_s|_{B^d(x,\varepsilon)}+\delta_x) = j_m\right)\\
&\qquad\cdot \mathbb{P}\left(\eta_s(A_{3,\varepsilon}(x,y)) \geq k\ \forall y \in \eta_s|_{\mathbb{R}^d\backslash B^d(x,3\varepsilon)}\right)\ \lambda(\mathrm{d}x)\\
&\geq s\frac{\alpha_m^2}{2}\int_{A_s} \mathbb{P}\left(\eta_s(B^d(x,\varepsilon)) = j_m, \eta_s(B^d(x,3\varepsilon)\backslash B^d(x,\varepsilon)) = 0\right)\\
&\qquad\qquad\cdot \mathbb{P}\left(\deg(x,\eta_s|_{B^d(x,\varepsilon)}+\delta_x) = j_m | \eta_s(B^d(x,\varepsilon)) = j_m\right)\ \mathrm{d}x\\
&= s\frac{\alpha_m^2}{2}\int_{A_s} \frac{(s\kappa_d\varepsilon^d)^{j_m}}{j_m!} e^{-s\kappa_d\varepsilon^d} e^{-s\kappa_d(3^d-1)\varepsilon^d}\,\mathbb{P}\left(\deg(x,\beta_{j_m}+\delta_x) = j_m\right)\ \mathrm{d}x\\
&\geq s\frac{\alpha_m^2}{2}\frac{(s\kappa_d\varepsilon^d)^{j_m}}{j_m!} e^{-s\kappa_d 3^d\varepsilon^d} p_m \frac{\lambda_d(W)}{2} =: c_{\alpha,k,W,d}\, s.
\end{aligned}$$

Case 2: $j_m = k$**.**

If it exists, we denote by $\ell \in \{1,\dots,n\}$ the index with $j_\ell = k+1$. Then,

$$\hat{\alpha} = \begin{cases}\alpha_\ell, & \text{if } \ell \text{ exists},\\ 0, & \text{if } \ell \text{ does not exist}.\end{cases}$$

Let $\varepsilon > 0$ and let $x \in W$ be such that $B^d(x, 8\varepsilon) \subset W$. We consider four different configurations to deal with all possible vectors $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ (see Figure 4.3). Let e_i denote the d-dimensional standard unit vector in the i-th direction.

1. $k \in \mathbb{N}$ and $\alpha_m(1-k) + \hat{\alpha}k \neq 0$:
   In this case we consider the event S_1 that for $\hat{x} = x + \frac{3\varepsilon}{4} e_1$ we have $\eta_s(B^d(\hat{x}, \varepsilon/4)) = k+1$, $\eta_s(B^d(x, 3\varepsilon) \backslash B^d(\hat{x}, \varepsilon/4)) = 0$ and $\eta_s(A_{3,\varepsilon}(x, y)) \geq k$ for all $y \in \eta_s|_{\mathbb{R}^d \backslash B^d(x, 3\varepsilon)}$. Then it follows
   $$D_x \sum_{i=1}^{n} \alpha_i V_{j_i}^k = \alpha_m D_x V_k^k + \hat{\alpha} D_x V_{k+1}^k = \alpha_m(1-k) + \hat{\alpha}k \neq 0.$$

2. $k \geq 3$ and $\alpha_m(1-k) + \hat{\alpha}k = 0$:
   The condition $\alpha_m(1-k) + \hat{\alpha}k = 0$ implies
   $$\alpha_m(3-k) + \hat{\alpha}(k-2) = 2(\alpha_m - \hat{\alpha}) = 2\frac{\alpha_m}{k} \neq 0. \tag{4.2}$$
   We consider the event S_2 where $\eta_s(B^d(\hat{x}_i, \varepsilon/16)) = 1$ for $i \in \{1, \dots, 4\}$ with $\hat{x}_j = x + (-1)^j \frac{3\varepsilon}{4} e_1$ for $j \in \{1, 2\}$ and $\hat{x}_j = x + (-1)^j \frac{3\varepsilon}{4} e_2$ for $j \in \{3, 4\}$, $\eta_s(B^d(x, \varepsilon/4)) = k-3$, $\eta_s(B^d(x, 3\varepsilon) \backslash (B^d(x, \varepsilon/4) \cup \bigcup_{i=1}^{4} B^d(\hat{x}_i, \varepsilon/16))) = 0$ and $\eta_s(A_{3,\varepsilon}(x, y)) \geq k$ for all $y \in \eta_s|_{\mathbb{R}^d \backslash B^d(x, 3\varepsilon)}$. Then we have with (4.2),
   $$D_x \sum_{i=1}^{n} \alpha_i V_{j_i}^k = \alpha_m D_x V_k^k + \alpha_\ell D_x V_{k+1}^k = \alpha_m(3-k) + \hat{\alpha}(k-2) \neq 0.$$

3. $k = 2$ and $\alpha_m(1-k) + \hat{\alpha}k = 0$:
   In this case we use the event S_3 where $\eta_s(B^d(\hat{x}_i, \varepsilon/16)) = 1$ for $i \in \{1, 2, 3\}$ with $\hat{x}_j = x + \frac{7\varepsilon}{16} e_1 + (-1)^j \frac{7\varepsilon}{16} e_2$ for $j \in \{1, 2\}$ and $\hat{x}_3 = x + \frac{7\varepsilon}{8} e_1$. Additionally, we assume $\eta_s(B^d(x, 3\varepsilon) \backslash (\bigcup_{i=1}^{3} B^d(\hat{x}_i, \varepsilon/16))) = 0$ and $\eta_s(A_{3,\varepsilon}(x, y)) \geq k$ for all $y \in \eta_s|_{\mathbb{R}^d \backslash B^d(x, 3\varepsilon)}$. Hence,
   $$D_x \sum_{i=1}^{n} \alpha_i V_{j_i}^k = \alpha_m D_x V_k^k = \alpha_m \neq 0.$$

4. $k = 1$ and $\alpha_m(1-k) + \hat{\alpha}k = 0$:
   We look at the event S_4 where $\eta_s(B^d(\hat{x}_1, \varepsilon/4)) = 1$ for $\hat{x}_1 = x - \frac{\varepsilon}{4} e_1$, $\eta_s(B^d(\hat{x}_2, \varepsilon/4)) = 2$ for $\hat{x}_2 = x + \frac{3\varepsilon}{4} e_1$, $\eta_s(B^d(x, 3\varepsilon) \backslash (\bigcup_{i=1}^{2} B^d(\hat{x}_i, \varepsilon/4))) = 0$ and $\eta_s(A_{3,\varepsilon}(x, y)) \geq k$ for all $y \in \eta_s|_{\mathbb{R}^d \backslash B^d(x, 3\varepsilon)}$. Since $\hat{\alpha} = 0$, it follows
   $$D_x \sum_{i=1}^{n} \alpha_i V_{j_i}^k = 2\alpha_m \neq 0.$$

Let $\varepsilon = \bar{c}s^{-1/d}$ for $\bar{c} \geq 1$ such that $c_1 e^{sc_2\varepsilon^d} \leq \frac{1}{2}$. Then, analogously to Case 1, we get for all $x \in W$ with $B^d(x, 8\varepsilon) \subset W$, $\mathbb{P}(S_u) \geq c_{\alpha,k,d}$ for a constant $c_{\alpha,k,d} > 0$ and $u \in \{1, \dots, 4\}$.

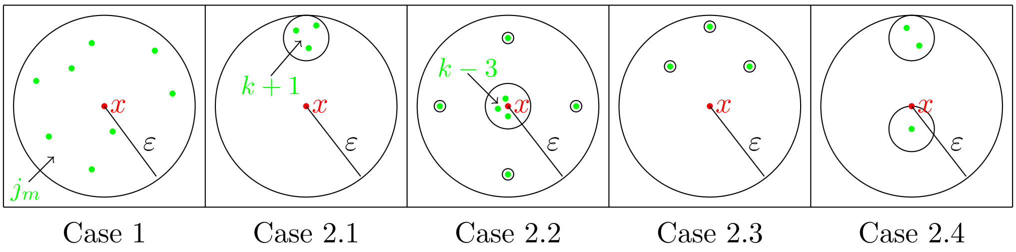

Figure 4.3: Configurations in $B^d(x,\varepsilon)$ ([101, Figure 1]).

Moreover, let

$$c_\alpha = \begin{cases} \alpha_m(1-k)+\hat{\alpha}k, & \text{for } k\in\mathbb{N} \text{ and } \alpha_m(1-k)+\hat{\alpha}k\neq 0,\\ \alpha_m(3-k)+\hat{\alpha}(k-2), & \text{for } k\geq 3 \text{ and } \alpha_m(1-k)+\hat{\alpha}k=0,\\ \alpha_m, & \text{for } k=2 \text{ and } \alpha_m(1-k)+\hat{\alpha}k=0,\\ 2\alpha_m, & \text{for } k=1 \text{ and } \alpha_m(1-k)+\hat{\alpha}k=0. \end{cases}$$

Then, for $A_s=\{x\in W\colon B^d(x,8\varepsilon)\subset W\}$ and s large enough such that $\lambda_d(A_s)>\frac{\lambda_d(W)}{2}$ it follows for $u\in\{1,\dots,4\}$,

$$\mathbb{E}\Big[\int_W\Big(D_x\sum_{i=1}^n \alpha_i V_{j_i}^k\Big)^2\,\lambda(\mathrm{d}x)\Big]\geq c_\alpha^2\int_{A_s}\mathbb{P}(S_u)\,\lambda(\mathrm{d}x)\geq c_{\alpha,k,W,d}\,s$$

for a suitable constant $c_{\alpha,k,W,d}>0$.

Our functionals can be written as sums of scores as in (2.22). For $y\in\eta_s$, $j\in\{k,\dots,k_{\max}\}$ and $s\geq 1$ the corresponding score is given by

$$\xi_s(y,\eta_s)=\mathbb{1}\{\deg(y,\eta_s)=j\}.$$

The scores $(\xi_s)_{s\geq 1}$ clearly fulfil a $(4+p)$-th moment condition and are by [103, Proof of Theorem 3.3] exponentially stabilising. Therefore, we can apply Lemma 4.2, which completes together with an application of Theorem 3.1 for $n=1$ the proof. □

Remark 4.7. Throughout this section we assume that the underlying Poisson processes have the intensity measures $s\lambda_d|_W$ for $s\geq 1$. However, we can generalise our results from these homogeneous Poisson processes to a large class of inhomogeneous Poisson processes. Let μ be a measure with a density $g\colon W\to[0,\infty)$ such that $\underline{c}\leq g(x)\leq\overline{c}$ for all $x\in W$ and constants $\underline{c},\overline{c}>0$. All results of this section continue to hold for Poisson processes with intensity measures $s\mu$ for $s\geq 1$. We only have to slightly modify the proofs by bounding the intensity measure by $s\underline{c}\lambda_d|_W$ from below or by $s\overline{c}\lambda_d|_W$ from above depending on whether a lower or an upper bound is required in our estimates. Consequently, some of the constants might change.

Remark 4.8. While we consider an underlying Poisson process on W, an alternative approach is to study a Poisson process on $\mathbb{R}^d$. In the case that the intensity measure of this Poisson process has a density $g\colon\mathbb{R}^d\to[0,\infty)$ such that $\underline{c}\leq g(x)\leq\overline{c}$ for all $x\in\mathbb{R}^d$ and constants $\underline{c},\overline{c}>0$, all arguments and, thus, also all results in this section continue to hold.

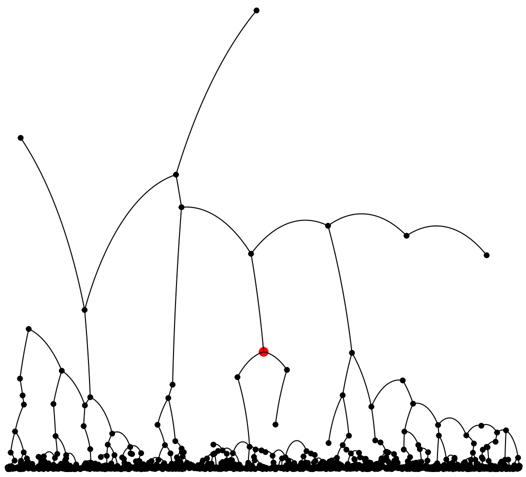

Figure 4.4: Simulation of an RST in $\mathbb{H}^2$ ([96, Figure 1]).

4.3 Hyperbolic radial spanning trees

We call an undirected graph a *rooted tree* if two vertices of the graph are connected by exactly one path and one vertex is designated as the root. To define the radial spanning tree on a metric space (X, d) we fix a base point $o \in X$ and let $\xi \in \mathbf{N}$. The *radial spanning tree* $\mathrm{RST}(\xi)$ on ξ with respect to the base point o is the tree rooted at o defined as follows. The vertices of $\mathrm{RST}(\xi)$ are the points of $\xi + \delta_o$. Any vertex $o \neq x \in \xi$ is connected by an edge to its *radial nearest neighbour* $n(x, \xi)$, which is the nearest vertex to x among all vertices which lie in the interior of the ball $B(o, \mathrm{d}(o, x))$ in X centred at o with radius $\mathrm{d}(o, x)$, that is,

$$n(x, \xi) = \underset{\substack{y \in \xi + \delta_o \\ \mathrm{d}(o,y) < \mathrm{d}(o,x)}}{\operatorname{argmin}} \ \mathrm{d}(x, y).$$

This means that y is the radial nearest neighbour of x precisely when $\mathrm{d}(o, y) < \mathrm{d}(o, x)$ and there are no points of $\xi + \delta_o$ in the interior of $B(o, \mathrm{d}(o, x)) \cap B(x, \mathrm{d}(x, y))$. Note that we assume that the radial nearest neighbour is unique in this construction, which is almost surely the case in the random set-up below, as discussed before for the k-nearest neighbour graph.

In this section we mainly study the random radial spanning tree induced by a stationary Poisson process in the d-dimensional hyperbolic space $\mathbb{H}^d$, which was introduced in Subsection 2.2.2. More precisely, let η be a stationary Poisson process on $\mathbb{H}^d$ with intensity measure $\gamma \mathcal{H}^d$ for some constant $\gamma > 0$. We fix an arbitrary point $o \in \mathbb{H}^d$, referred to as the origin, and consider the radial spanning tree $\mathrm{RST}(\eta)$ on the full Poisson process η, as well as the radial spanning tree $\mathrm{RST}(\eta_s)$ on the restriction $\eta_s = \eta|_{B_h^d(o,s)}$ of η to the closed hyperbolic ball $B_h^d(o, s)$ of radius $s > 0$, both with respect to the base point o. A simulation of such a radial spanning tree in the upper half-space model can be seen in Figure 4.4.

There are several reasons why we are interested in analysing the behaviour of stochastic models in hyperbolic space. From a mathematical point of view replacing Euclidean space by other underlying spaces is a natural problem and the hyperbolic space is the first candidate. Working with such a space is helpful in understanding which properties of a stochastic geometry model depend on the curvature of the underlying space and which do not. On the other hand, many real-world complex networks such as the internet graph seem to have an underlying hyperbolic structure (see e.g. [27, 64, 104]). In fact, the random graphs in hyperbolic space introduced in these papers share many crucial properties of complex networks.

Transferring a problem from Euclidean to hyperbolic space can lead to some new phenomena, which are not present in Euclidean space as illustrated by the following non-exhaustive list of examples. In [18] it was shown that for some parameter choices in Poisson-Voronoi-Bernoulli percolation in the hyperbolic plane, one simultaneously has infinitely many black and infinitely many white components, which is a situation that cannot occur in Euclidean space. For continuum percolation of the Boolean model in hyperbolic space it was shown in [111] that there exists a regime with infinitely many unbounded components, which is again not possible in Euclidean space. For the low intensity Boolean model in the hyperbolic plane it was shown in [17] that an observer located at a fixed non-covered point can see infinitely far in a fractal set of directions, a phenomenon in sharp contrast to the Euclidean situation, where such an observer can only see a bounded region. The same has also been shown in [17] for Poisson line processes in the hyperbolic plane. While the surface area of a Poisson hyperplane process in an increasing observation window satisfies a central limit theorem in Euclidean space, this is only true in dimensions two and three in hyperbolic space [53], whereas non-Gaussian limiting distributions arise for all higher space dimensions [59].

The radial spanning tree was originally introduced in Euclidean space in [9] motivated by applications from communication networks. In [9] and [10], in particular semi-infinite paths of the radial spanning tree in $\mathbb{R}^2$ were studied. Variance asymptotics and quantitative central limit theorems for edge-length functionals were derived in [100]. A crucial role for the analysis of the radial spanning tree plays the directed spanning forest, where each point is connected with its closest neighbour in the half-space generated by a fixed direction.

In hyperbolic space, the radial spanning tree was first considered in [40], where the same path properties as for the two-dimensional Euclidean radial spanning tree were shown for any dimension. These results were further refined in [39]. The hyperbolic directed spanning forest was first studied in [47]. It shows a radical different behaviour than its Euclidean analogue as it is a tree containing infinitely many bi-infinite branches, while the Euclidean radial spanning forest is a countable collection of trees for $d \geq 4$ and does not admit bi-infinite branches.

In the following subsection we start with analysing the degree of the origin of the radial spanning tree in hyperbolic space and compare its behaviour with that of a radial spanning tree in Euclidean space to give a first idea of how the underlying space influences the properties of a radial spanning tree. In the remaining subsections we will concentrate on the behaviour of edge length functionals and will especially derive a lower variance bound and a central limit theorem for these functionals for growing observation windows. These results are similar to those derived in [100] for the Euclidean case. The major difference is that a hyperbolic ball scales differently with its radius than a Euclidean ball and that in the asymptotic variance formula the integration over a half-space is replaced by the integration over a horoball.

4.3.1 Degree of the origin

In this subsection we consider the radial spanning tree $\mathrm{RST}(\eta)$ on $\mathbb{H}^d$, aim to analyse the behaviour of the degree of the origin and compare it with its Euclidean analogue.

Theorem 4.9.

a) The degree of the origin has finite moments of all orders, i.e. $\mathbb{E}[\deg(o)^n] < \infty$ *for all* $n \in \mathbb{N}$.

b) The expected degree of the origin is given by

$$\mathbb{E}[\deg(o)] = \gamma\omega_d \int_0^\infty \sinh^{d-1}(r) \times \exp\left(-2\gamma\kappa_{d-1} \int_0^{r/2} \left[\frac{\cosh^2(r)}{\cosh^2(r-t)} - 1\right]^{\frac{d-1}{2}} \mathrm{d}t\right) \mathrm{d}r. \tag{4.3}$$

c) The degree of the origin is not almost surely bounded, i.e. for all $n \in \mathbb{N}$ *it holds that* $\mathbb{P}(\deg(o) \geq n) > 0$.

Proof. By the definition of the radial nearest neighbour and the Mecke equation (2.15) we have that

$$\begin{aligned}
\mathbb{E}[\deg(o)] &= \mathbb{E}\Big[\sum_{x\in\eta} \mathbf{1}\{\eta(\mathrm{int}(B_h^d(x, d_h(o,x)) \cap B_h^d(o, d_h(o,x)))) = 0\}\Big] \\
&= \gamma \int_{\mathbb{H}^d} \mathbb{P}(\eta(B_h^d(x, d_h(o,x)) \cap B_h^d(o, d_h(o,x))) = 0)\, \mathcal{H}^d(\mathrm{d}x) \\
&= \gamma \int_{\mathbb{H}^d} \exp(-\gamma\mathcal{H}^d(B_h^d(x, d_h(o,x)) \cap B_h^d(o, d_h(o,x))))\, \mathcal{H}^d(\mathrm{d}x).
\end{aligned}$$

Transforming to hyperbolic spherical coordinates (see also (2.12)) leads to

$$\mathbb{E}[\deg(o)] = \gamma\omega_d \int_0^\infty \exp(-\gamma\mathcal{H}^d(B_h^d(x_r, r) \cap B_h^d(o, r)))\, \sinh^{d-1}(r)\, \mathrm{d}r, \tag{4.4}$$

where $x_r \in \mathbb{H}^d$ is an arbitrary point satisfying $d_h(o, x_r) = r$.

To compute the hyperbolic volume of the intersection $B_h^d(x_r, r) \cap B_h^d(o, r)$, first observe that the intersection is the union of two caps of height $r/2$ in a hyperbolic ball $B_h^d(o, r)$ of radius r (see Figure 4.5). The volume of such a cap can be computed using Lemma 2.9 (see Figure 4.6). We take as L the radial line of the ball passing through the apex of the cap, and set the arclength parameter at the apex to zero. Then, Lemma 2.9 provides

$$\mathcal{H}^d(B_h^d(x_r, r) \cap B_h^d(o, r)) = 2\int_0^{r/2} \int_{H_t \cap B_h^d(o,r)} \cosh(d_h(x, L))\, \mathcal{H}^{d-1}(\mathrm{d}x)\, \mathrm{d}t.$$

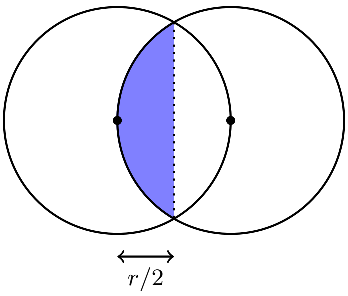

Figure 4.5: Intersection of two balls ([96, Figure 5, left])

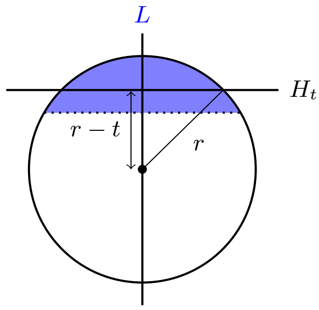

Figure 4.6: Computing the volume of a cap ([96, Figure 5, right]).

The intersection $H_t \cap B_h^d(o,r)$ of the ball with the hyperplane H_t orthogonal to L at distance $t \in (0, r/2]$ from the apex is a $(d-1)$-dimensional hyperbolic ball within the hyperplane H_t, whose radius is

$$\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(r-t)}\right)$$

by the hyperbolic Pythagorean theorem (2.11). Together with the hyperbolic spherical coordinates from (2.12) within H_t, this leads to

$$\begin{aligned}
&\mathcal{H}^d(B_h^d(x_r,r)\cap B_h^d(o,r))\\
&= 2\omega_{d-1}\int_0^{r/2}\int_0^{\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(r-t)}\right)} \cosh(u)\sinh^{d-2}(u)\,\mathrm{d}u\,\mathrm{d}t\\
&= 2\omega_{d-1}\int_0^{r/2}\frac{1}{d-1}\sinh^{d-1}\left(\operatorname{arcosh}\left(\frac{\cosh(r)}{\cosh(r-t)}\right)\right)\mathrm{d}t\\
&= 2\kappa_{d-1}\int_0^{r/2}\left[\frac{\cosh^2(r)}{\cosh^2(r-t)}-1\right]^{\frac{d-1}{2}}\mathrm{d}t,
\end{aligned}$$

where we have used that $\omega_{d-1} = (d-1)\kappa_{d-1}$ and $\sinh^2(u) = \cosh^2(u) - 1$ for $u \in \mathbb{R}$. Plugging this back into (4.4) completes the proof of b).

Next, we show a). For $n = 1$ the result is clear by the previous proof. For $n \geq 2$ consider the n-th factorial moment of $\deg(o)$. To emphasise the dependence of this random variable on the underlying intensity γ, we write $\deg_\gamma(o)$. Using the multivariate Mecke equation (2.15) we obtain

$$\begin{aligned}
&\mathbb{E}[\deg_\gamma(o)(\deg_\gamma(o)-1)\cdots(\deg_\gamma(o)-n+1)]\\
&= \mathbb{E}\Big[\sum_{(x_1,\dots,x_n)\in\eta^n_{\neq}} \mathbb{1}\Big\{n\Big(x_i,\eta+\sum_{i=1}^n\delta_{x_i}\Big) = o \text{ for all } i\in\{1,\dots,n\}\Big\}\Big]\\
&= \gamma^n\int_{\mathbb{H}^d}\cdots\int_{\mathbb{H}^d}\mathbb{P}\Big(n\Big(x_i,\eta+\sum_{i=1}^n\delta_{x_i}\Big) = o \text{ for all } i\in\{1,\dots,n\}\Big)\,\mathcal{H}^d(\mathrm{d}x_1)\dots\mathcal{H}^d(\mathrm{d}x_n)
\end{aligned}$$

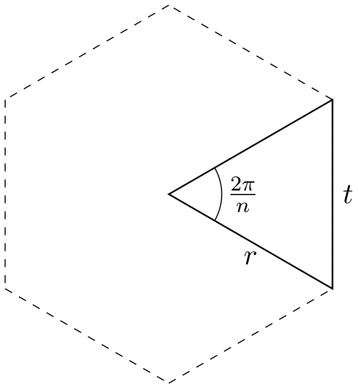

Figure 4.7: Constructing a regular n-gon ([96, Figure 6, left]).

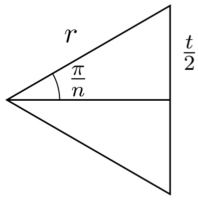

Figure 4.8: $\sin\frac{\pi}{n} = \frac{\sinh(t/2)}{\sinh r}$ by the hyperbolic law of sines (2.10) ([96, Figure 6, right])

$$\leq \gamma^n \int_{\mathbb{H}^d} \cdots \int_{\mathbb{H}^d} \exp(-H_\gamma(x_1,\dots,x_n))\, \mathcal{H}^d(\mathrm{d}x_1)\dots\mathcal{H}^d(\mathrm{d}x_n),$$

where

$$H_\gamma(x_1,\dots,x_n) = \gamma\mathcal{H}^d\Big(\bigcup_{i=1}^n (B_h^d(x_i, d_h(o,x_i)) \cap B_h^d(o, d_h(o,x_i)))\Big)$$

since $x_1,\dots,x_n$ can only have o as their radial nearest neighbour if η has no points in the union set considered in H_γ. Together with

$$\begin{aligned} H_\gamma(x_1,\dots,x_n) &\geq \gamma \max_{i\in\{1,\dots,n\}} \mathcal{H}^d(B_h^d(x_i, d_h(o,x_i)) \cap B_h^d(o, d_h(o,x_i))) \\ &\geq \frac{\gamma}{n}\sum_{i=1}^n \mathcal{H}^d(B_h^d(x_i, d_h(o,x_i)) \cap B_h^d(o, d_h(o,x_i))) \end{aligned}$$

this shows that

$$\begin{aligned} &\mathbb{E}[\deg_\gamma(o)(\deg_\gamma(o)-1)\cdots(\deg_\gamma(o)-n+1)] \\ &\leq n^n\Big(\frac{\gamma}{n}\int_{\mathbb{H}^d} \exp\Big(-\frac{\gamma}{n}\mathcal{H}^d(B_h^d(x, d_h(o,x)) \cap B_h^d(o, d_h(o,x)))\Big)\mathcal{H}^d(\mathrm{d}x)\Big)^n \\ &= n^n\big(\mathbb{E}[\deg_{\gamma/n}(o)]\big)^n. \end{aligned}$$

Since $\mathbb{E}[\deg_{\gamma/n}(o)] < \infty$ by b), $\mathbb{E}[\deg_\gamma(o)(\deg_\gamma(o)-1)\cdots(\deg_\gamma(o)-n+1)] < \infty$ and hence $\mathbb{E}[(\deg_\gamma(o))^n] < \infty$ as well. This completes the proof of a).

Finally, we show that the degree of the origin is not bounded. Note that it is sufficient to show that $\mathbb{P}(\deg(o) \geq n) > 0$ for $n \geq 3$. Thus, we can assume that $n \geq 3$ throughout the proof without loss of generality. Now, fix an arbitrary hyperbolic 2-plane E passing through o. In a first step we show that there exists a hyperbolic regular n-gon in E whose side length t is larger than the radius r of its circumscribing circle. Such an n-gon may be constructed by gluing together n isosceles triangles with side length r, base t and apex angle $\frac{2\pi}{n}$ (see Figure 4.7).

By the hyperbolic law of sines (2.10) (see also Figure 4.8) it holds

$$t = 2\mathrm{arsinh}\Big(\sinh(r)\cdot\sin\frac{\pi}{n}\Big) \geq 2\log\Big(2\sinh(r)\cdot\sin\frac{\pi}{n}\Big),$$

which follows from $\operatorname{arsinh}(x) = \log(x + \sqrt{x^2+1}) \geq \log(2x)$ for any $x > 0$.

For $r = 3\log(n)$ we have, using in the second step the inequality $\sin x \geq \frac{2}{\pi}x$ for $x \in [0, \frac{\pi}{2}]$ (which follows from concavity of $x \mapsto \sin x$ on $[0, \frac{\pi}{2}]$) with $x = \frac{\pi}{n}$ and the definition of sinh,

$$\begin{aligned} t &\geq 2\log\left(2\sinh(3\log(n)) \cdot \sin\frac{\pi}{n}\right) \\ &\geq 2\log\left(\left(n^3 - n^{-3}\right) \cdot \frac{2}{n}\right) \\ &= 2\log\left(n^2 \cdot 2\big(1-n^{-6}\big)\right) \\ &= 2\left[2\log n + \log\left(2\big(1-n^{-6}\big)\right)\right] \\ &\geq 4\log(n) > r, \end{aligned}$$

as desired, where in the penultimate inequality we used that $2\big(1-n^{-6}\big) \geq 1$.

To complete the proof, consider a regular n-gon within the 2-plane E as above, circumscribed in the circle of radius r around the origin o. Denote its vertices by $v_1, \dots, v_n$ and choose $\varepsilon > 0$ small enough so that $r + 3\varepsilon < t$. Now, consider the event

$$A_n = \Big\{\eta(B_h^d(v_i,\varepsilon)) = 1 \text{ for } i \in \{1,\dots,n\}, \eta\Big(B_h^d(o, r+\varepsilon)\Big\backslash \bigcup_{i=1}^n B_h^d(v_i,\varepsilon)\Big) = 0\Big\}.$$

By construction it holds $d_h(o, x_i) < d_h(x_i, x_j)$ for $i \neq j$ and all $x_i \in B_h^d(v_i, \varepsilon)$ and $x_j \in B_h^d(v_j, \varepsilon)$. Hence, if the event A_n occurs it holds $\deg(o) \geq n$ and thus,

$$\begin{aligned} \mathbb{P}(\deg(o) \geq n) \geq \mathbb{P}(A_n) &= \prod_{i=1}^n \mathbb{P}(\eta(B_h^d(v_i,\varepsilon)) = 1) \cdot \mathbb{P}\Big(\eta\Big(B_h^d(o,r+\varepsilon)\Big\backslash \bigcup_{i=1}^n B_h^d(v_i,\varepsilon)\Big) = 0\Big) \\ &> 0, \end{aligned}$$

which completes the proof. □

Let us compare these results to the ones for a radial spanning tree in Euclidean space. To this end, let η^E be a Poisson process in $\mathbb{R}^d$ with intensity measure $\gamma\lambda_d$ for some $\gamma > 0$. The mean degree of the origin in this Euclidean radial spanning tree is given by

$$\begin{aligned} \mathbb{E}[\deg^{\mathrm{E}}(o)] &= \gamma\omega_d \int_0^\infty r^{d-1} \exp\left(-2\gamma\kappa_{d-1}\beta_d r^d\right) \mathrm{d}r \\ &= \frac{\omega_d}{2d\kappa_{d-1}\beta_d} = \frac{\sqrt{\pi}}{2\beta_d} \frac{\Gamma\left(\frac{d}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{d}{2}+1\right)}, \end{aligned}$$

where $2\kappa_{d-1}\beta_d$ is the volume of the intersection of two balls in $\mathbb{R}^d$ with radius one whose centres have distance one. This can be seen by adapting the proof of Theorem 4.9 b) to the Euclidean set-up and was shown for $d = 2$ in [9, Equation (7)]. The constant β_d can be written as $\beta_d = \frac{1}{2}B_{\frac{3}{4}}(\frac{d+1}{2}, \frac{1}{2})$ with the incomplete beta function $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$. For the sake of comparison, Table 4.1 shows values of the expected degree of the origin in the hyperbolic and the Euclidean case, for low dimensions d. In the table one can see that for $d \in \{2,\dots,7\}$ the expected degree of the origin in the Euclidean case is smaller than the one in the hyperbolic case.

d	$\mathbb{E}[\deg(o)]$ approx.	$\mathbb{E}[\deg^E(o)]$ exact	$\mathbb{E}[\deg^E(o)]$ approx.
2	2.931	$\frac{6\pi}{4\pi-3\sqrt{3}}$	2.558
3	3.985	$\frac{16}{5}$	3.200
4	5.397	$\frac{12\pi}{8\pi-9\sqrt{3}}$	3.950
5	7.332	$\frac{256}{53}$	4.832
6	10.010	$\frac{30\pi}{20\pi-27\sqrt{3}}$	5.866
7	13.749	$\frac{2048}{289}$	7.087

Table 4.1: Expected degree of the origin in the radial spanning tree for the hyperbolic case and the Euclidean case and $d \in \{2, \dots, 7\}$ (cf. [96, Table 1]).

The result about the expected degree of the origin also fits to the result from Theorem 4.9 c). Remember that we have shown in Lemma 4.5 that the degree of a vertex in the k-nearest neighbour graph in $\mathbb{R}^d$ is bounded. By the same argument, the degree of the origin of a radial spanning tree in Euclidean space is bounded, while Theorem 4.9 c) states that the degree of the origin is not almost surely bounded in the hyperbolic case. The reason for that lies in the geometry of the underlying space. The proof of the unboundedness in the hyperbolic setting is mainly based on the fact that in hyperbolic space, for all $n \in \mathbb{N}$ there exists a regular hyperbolic n-gon, whose side length is larger than the radius of its circumscribing circle. Due to the Euclidean Pythagorean theorem, such an n-gon does not exist in Euclidean space if n is too large, which leads to the boundedness of the degree of the origin in Euclidean space.

4.3.2 Expectation of total edge length functionals

Recall that η_s denotes the restriction of η to $B_h^d(o,s)$ for $s > 0$. We are interested in the lengths of the edges from points of η_s to their radial nearest neighbours. Therefore, for $\alpha \geq 0$ we denote by $\mathcal{L}_{s,\gamma}^{(\alpha)}$ the edge-length functional

$$\mathcal{L}_{s,\gamma}^{(\alpha)} = \sum_{x \in \eta_s} \ell(x,\eta)^\alpha = \sum_{x \in \eta_s} d_h(x, n(x,\eta))^\alpha,$$

where $\ell(x,\eta) = d_h(x, n(x,\eta))$ denotes the distance from x to its radial nearest neighbour. In the remaining part of Section 4.3 we are now interested in the behaviour of $\mathcal{L}_{s,\gamma}^{(\alpha)}$ as $s \to \infty$.

Note at first that $L_{s,\gamma}^{(0)}$ only counts the number of points of η in $B_h^d(o,s)$, which is Poisson distributed with parameter $\gamma V_h(s)$. Thus, we restrict the following theorems to the case $\alpha > 0$. Recall that, up to constants, $V_h(s)$ grows like $e^{(d-1)s}$, as $s \to \infty$ (see (2.13)).

Theorem 4.10. *Let $\alpha > 0$. Then it holds that*

$$\lim_{s\to\infty} \frac{\mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}{V_h(s)} = \gamma \int_0^\infty e^{-\gamma G(u^{1/\alpha})} \mathrm{d}u,$$

where $G(u)$ is given by

$$G(u) = \kappa_{d-1} \int_0^u \left[2e^{-t}\big(\cosh(u) - \cosh(t)\big)\right]^{\frac{d-1}{2}} \,\mathrm{d}t.$$

To show this theorem, we use the following two lemmas. Since these lemmas are also helpful for the analysis of the variance and the central limit theorem we state some of the results in a slightly more generalised version than we need for the proof of Theorem 4.10. We start with the following geometric result.

Lemma 4.11. *Let $u > 0$ be fixed and let x_s be a point with $d_h(o, x_s) = s$. Moreover, denote by $\mathrm{HB}_e(o)$ the horoball around some fixed ideal point $e \in \partial\mathbb{H}^d$ passing through o. Then it holds*

$$\lim_{s\to\infty} \mathcal{H}^d\big(B_h^d(o,s) \cap B_h^d(x_s,u)\big) = \mathcal{H}^d(\mathrm{HB}_e(o) \cap B_h^d(o,u)) = G(u)$$

with G as in Theorem 4.10.

Proof. Applying appropriate hyperbolic isometries, we may assume instead that the centre of the ball with radius u is fixed at the origin and compute the volume of the intersection of the ball $B_h^d(o,u)$ with a sequence of growing balls B_s of radii $s > 0$ passing through o with centres tending to e. As $s \to \infty$ these balls converge to a horoball $\mathrm{HB}_e(o)$. We conclude that

$$\lim_{s\to\infty} \mathcal{H}^d\big(B_h^d(o,s) \cap B_h^d(x_s,u)\big) = \mathcal{H}^d\big(B_h^d(o,u) \cap \mathrm{HB}_e(o)\big).$$

To compute the latter volume we consider the family $(H_t)_{t\in\mathbb{R}}$ of parallel horospheres corresponding to the horoball $\mathrm{HB}_e(o)$, parametrised so that H_t has signed distance t from o. With this convention, the horoball $\mathrm{HB}_e(o)$ is the union $\bigcup_{t\geq 0} H_t$. Applying Lemma 2.10, we get

$$\mathcal{H}^d\big(B_h^d(o,u) \cap \mathrm{HB}_e(o)\big) = \int_0^\infty \mathcal{H}^{d-1}(B_h^d(o,u) \cap H_t)\,\mathrm{d}t. \tag{4.5}$$

The volume appearing on the right-hand side is zero for $t \geq u$. For $t \in [0,u)$ it holds by Lemma 2.11,

$$\mathcal{H}^{d-1}(B_h^d(o,u) \cap H_t) = \kappa_{d-1}\left[2e^{-t}\big(\cosh(u) - \cosh(t)\big)\right]^{\frac{d-1}{2}}.$$

Substituting this into (4.5) yields the result. □

Lemma 4.12. *Let $\alpha > 0$.*

a) For every $x \in \mathbb{H}^d$ it holds

$$\mathbb{E}[\ell(x,\eta+\delta_x)^\alpha] = \alpha \int_0^{d_h(o,x)} u^{\alpha-1} \exp(-\gamma\mathcal{H}^d(B_h^d(x,u) \cap B_h^d(o, d_h(o,x))))\,\mathrm{d}u.$$

b) There exists a constant $c(\alpha) > 0$ only depending on α and d such that

$$\gamma^{\alpha/d}\mathbb{E}[\ell(x,\eta+\delta_x)^\alpha \mathbb{1}\{\ell(x,\eta+\delta_x) \geq t\}] \leq c(\alpha)\mathbb{1}\{t \leq d_h(o,x)\}\exp(-\gamma V_h(t/2)/2)$$

for all $x \in \mathbb{H}^d$ and $t \geq 0$.

Proof. First we compute the expectation from b), which can be written as

$$\mathbb{E}[\ell(x,\eta+\delta_x)^\alpha \mathbb{1}\{\ell(x,\eta+\delta_x)\geq t\}] = \int_0^\infty \mathbb{P}(\ell(x,\eta+\delta_x)\geq \max\{v^{1/\alpha},t\})\,\mathrm{d}v.$$

Since $\ell(x,\eta+\delta_x)\leq d_h(o,x)$ and $\ell(x,\eta+\delta_x)\geq \tilde{u}$ if and only if η has no points in the interior of $B_h^d(x,\tilde{u})\cap B_h^d(o,d_h(o,x))$ for $\tilde{u}\in(0,d_h(o,x))$, we obtain

$$\begin{aligned}
&\mathbb{E}[\ell(x,\eta+\delta_x)^\alpha \mathbb{1}\{\ell(x,\eta+\delta_x)\geq t\}]\\
&= \mathbb{1}\{t\leq d_h(o,x)\}\int_0^{d_h(o,x)^\alpha} \mathbb{P}(\ell(x,\eta+\delta_x)\geq \max\{v^{1/\alpha},t\})\,\mathrm{d}v\\
&= \mathbb{1}\{t\leq d_h(o,x)\}\int_0^{d_h(o,x)^\alpha} \mathbb{P}(\eta(B_h^d(x,\max\{v^{1/\alpha},t\})\cap B_h^d(o,d_h(o,x)))=0)\,\mathrm{d}v\\
&= \mathbb{1}\{t\leq d_h(o,x)\}\int_0^{d_h(o,x)^\alpha} \exp(-\gamma\mathcal{H}^d(B_h^d(x,\max\{v^{1/\alpha},t\})\cap B_h^d(o,d_h(o,x))))\,\mathrm{d}v\\
&= \mathbb{1}\{t\leq d_h(o,x)\}\alpha\int_0^{d_h(o,x)} u^{\alpha-1}\exp(-\gamma\mathcal{H}^d(B_h^d(x,\max\{u,t\})\cap B_h^d(o,d_h(o,x))))\,\mathrm{d}u,
\end{aligned}$$

where we used the substitution $v=u^\alpha$ in the last step. For $t=0$ this provides a). Note that for $\tilde{u}\in(0,d_h(o,x))$, the intersection $B_h^d(x,\tilde{u})\cap B_h^d(o,d_h(o,x))$ contains the hyperbolic ball of radius $\tilde{u}/2$ around the midpoint between x and the point on the boundary of $B_h^d(x,\tilde{u})$ closest to the origin. Hence, we have

$$\mathcal{H}^d(B_h^d(x,\tilde{u})\cap B_h^d(o,d_h(o,x)))\geq V_h(\tilde{u}/2). \tag{4.6}$$

This implies

$$\begin{aligned}
&\mathbb{E}[\ell(x,\eta+\delta_x)^\alpha \mathbb{1}\{\ell(x,\eta+\delta_x)\geq t\}]\\
&\leq \mathbb{1}\{t\leq d_h(o,x)\}\alpha\int_0^{d_h(o,x)} u^{\alpha-1}\exp(-\gamma V_h(\max\{u,t\}/2))\,\mathrm{d}u\\
&\leq \mathbb{1}\{t\leq d_h(o,x)\}\exp(-\gamma V_h(t/2)/2)\alpha\int_0^\infty u^{\alpha-1}\exp(-\gamma V_h(u/2)/2)\,\mathrm{d}u.
\end{aligned}$$

Since $\sinh(v)\geq v$ for all $v\geq 0$, it follows from (2.13) that

$$V_h(r)=\omega_d\int_0^r \sinh^{d-1}(v)\,\mathrm{d}v\geq d\kappa_d\int_0^r v^{d-1}\,\mathrm{d}v=\kappa_d r^d$$

for all $r\geq 0$. This leads to

$$\begin{aligned}
\gamma^{\alpha/d}\alpha\int_0^\infty u^{\alpha-1}\exp(-\gamma V_h(u/2)/2)\,\mathrm{d}u &\leq \gamma^{\alpha/d}\alpha\int_0^\infty u^{\alpha-1}\exp(-\gamma\kappa_d 2^{-d-1}u^d)\,\mathrm{d}u\\
&= \alpha\int_0^\infty v^{\alpha-1}\exp(-\kappa_d 2^{-d-1}v^d)\,\mathrm{d}v,
\end{aligned}$$

which completes the proof of b). □

Proof of Theorem 4.10. The Mecke equation (2.15) and Lemma 4.12 a) imply that

$$\mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}] = \mathbb{E}\Big[\sum_{x\in\eta_s} \ell(x,\eta)^\alpha\Big] = \gamma \int_{B_h^d(o,s)} \mathbb{E}[\ell(x,\eta+\delta_x)^\alpha]\, \mathcal{H}^d(dx)$$
$$= \gamma\alpha \int_{B_h^d(o,s)} \int_0^{d_h(o,x)} u^{\alpha-1} \exp(-\gamma\mathcal{H}^d(B_h^d(x,u)\cap B_h^d(o,d_h(o,x))))\, \mathrm{d}u\, \mathcal{H}^d(dx).$$

Using the polar integration formula (2.12) provides

$$\mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}] = \gamma\alpha\omega_d \int_0^s \int_0^t u^{\alpha-1} \exp\Big(-\gamma\mathcal{H}^d(B_h^d(x_t,u)\cap B_h^d(o,t))\Big) \sinh^{d-1}(t)\, \mathrm{d}u\, \mathrm{d}t,$$

where x_t is a point in $B_h^d(o,t)$ with $d_h(x_t,o)=t$. Since $\mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}]\to\infty$ and $V_h(s)\to\infty$ as $s\to\infty$, by l'Hospital's rule and (2.13) we obtain

$$\lim_{s\to\infty} \frac{\mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}{V_h(s)}$$
$$= \lim_{s\to\infty} \frac{\gamma\alpha\omega_d}{\omega_d \sinh^{d-1}(s)} \int_0^s u^{\alpha-1} \exp\Big(-\gamma\mathcal{H}^d(B_h^d(x_s,u)\cap B_h^d(o,s))\Big) \sinh^{d-1}(s)\, \mathrm{d}u$$
$$= \lim_{s\to\infty} \gamma\alpha \int_0^s u^{\alpha-1} \exp\Big(-\gamma\mathcal{H}^d(B_h^d(x_s,u)\cap B_h^d(o,s))\Big)\, \mathrm{d}u.$$

Since, by (4.6), the integrand is bounded by the integrable function $(0,\infty)\ni u\mapsto u^{\alpha-1}\exp(-\gamma V_h(u/2))$, the dominated convergence theorem and Lemma 4.11 lead to

$$\lim_{s\to\infty} \frac{\mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}{V_h(s)} = \gamma\alpha \int_0^\infty \lim_{s\to\infty} u^{\alpha-1} \exp\Big(-\gamma\mathcal{H}^d(B_h^d(x_s,u)\cap B_h^d(o,s))\Big)\, \mathrm{d}u$$
$$= \gamma\alpha \int_0^\infty u^{\alpha-1} \exp(-\gamma G(u))\, \mathrm{d}u.$$

Combining this with the change of variable $v=u^\alpha$ we conclude that

$$\lim_{s\to\infty} \frac{\mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}{V_h(s)} = \gamma \int_0^\infty \exp\Big(-\gamma G(v^{1/\alpha})\Big)\, \mathrm{d}v,$$

which completes the proof. □

For $\alpha>0$ the Euclidean counterpart to the edge-length functional $\mathcal{L}_{s,\gamma}^{(\alpha)}$ is

$$\mathcal{L}_{s,\gamma}^{(\alpha),\mathrm{E}} = \sum_{x\in\eta_s^{\mathrm{E}}} \ell_{\mathrm{E}}(x,\eta^{\mathrm{E}})^\alpha,$$

where $\eta_s^{\mathrm{E}} = \eta^{\mathrm{E}}|_{B^d(0,s)}$ for $s>0$ is the restriction of η^{E} to the Euclidean ball $B^d(0,s)$ and $\ell_{\mathrm{E}}(x,\eta^{\mathrm{E}})$ denotes the Euclidean distance from x to its radial nearest neighbour in the radial spanning tree on η^{E} with respect to the origin. Its asymptotic behaviour for $s\to\infty$ is given by

$$\lim_{s\to\infty} \frac{\mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha),\mathrm{E}}]}{V_d(s)} = \gamma \int_0^\infty e^{-\gamma\frac{\kappa_d}{2}u^{d/\alpha}}\, \mathrm{d}u = \gamma\Big(\frac{2}{\gamma\kappa_d}\Big)^{\alpha/d} \Gamma\Big(\frac{\alpha}{d}+1\Big) \tag{4.7}$$

d	$\lim_{s\to\infty} \frac{\mathbb{E}[\mathcal{L}_{s,1}^{(1)}]}{V_h(s)}$	$\lim_{s\to\infty} \frac{\mathbb{E}[\mathcal{L}_{s,1}^{(1),\mathrm{E}}]}{V_d(s)}$
2	0.7591	0.7071
3	0.7514	0.6979
4	0.7762	0.7232
5	0.8087	0.7566
6	0.8426	0.7920
7	0.8761	0.8273

Table 4.2: Expected asymptotic volume-normalised edge-length functional of the radial spanning tree for the hyperbolic case and the Euclidean case for $\alpha = 1$ and $d \in \{2, \dots, 7\}$ (cf. [96, Table 2]).

where $V_d(s) = \kappa_d s^d$ is the volume of a Euclidean ball of radius s. The first equality follows from rewriting [100, (1.2) in Theorem 1.1], where the $+1$ is missing in the argument of the Gamma function due to an inaccuracy in the last step of the proof. Note that the polynomial volume growth in (4.7) is in sharp contrast to the exponential growth of $V_h(s)$ in the hyperbolic case. For the sake of comparison, Table 4.2 shows values of the asymptotic expected volume-normalised Euclidean edge-length functional with $\alpha = 1$ in the hyperbolic case and the Euclidean case for low dimensions d.

Remember that in contrast to the examples from Section 4.1 and Section 4.2 we consider in this section results for fixed intensity and a sequence of growing observation windows. Naturally, the question arises of what happens for a fixed observation window and increasing intensity. In this case one can expect that the asymptotic results for edge length functionals coincide with the ones in Euclidean space (see also [96, Section 1.3]). Roughly speaking, this can be explained with the fact that the edges of the radial spanning tree become shorter for increasing intensity. Then, since the hyperbolic space can be locally approximated by its tangent space, total edge length functionals in hyperbolic space should behave similarly to edge length functionals in Euclidean space. This is the reason why we are interested in growing observation windows instead.

4.3.3 Variance of total edge length functionals

In this subsection we analyse the asymptotic behaviour of the variance of total edge length functionals. To formulate the precise variance asymptotics of the edge-length functionals, we need to consider a directed variant of the radial spanning tree, the so-called *directed spanning forest*, which was introduced in the Euclidean case in [9] and generalised to hyperbolic space in [47]. For a connection to the hyperbolic radial spanning tree see also [40]. To define it, we fix an ideal point $e \in \partial\mathbb{H}^d$. For a point $y \in \mathbb{H}^d$ denote by $\mathrm{HB}_e(y)$ the horoball around e with boundary passing through y. The directed spanning forest $\mathrm{DSF}_e(\eta)$ on the Poisson process η with respect to the ideal point e has the points of η as vertices and each vertex y is connected to its (a.s. unique) nearest neighbour in the horoball $\mathrm{HB}_e(y)$. We denote by $\ell_e(y)$ the distance between y and its nearest neighbour in $\mathrm{DSF}_e(\eta + \delta_y)$, and by $\ell_e^{(x)}(y)$ the distance from y to its nearest neighbour in $\mathrm{DSF}_e(\eta + \delta_y + \delta_x)$ for $x \in \mathbb{H}^d$.

Theorem 4.13. *Let $\alpha > 0$. Then, the limit*

$$\mathbb{V}_\gamma^{(\alpha)} = \lim_{s\to\infty} \frac{\mathrm{Var}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}{V_h(s)}$$

exists, is finite and given by

$$\mathbb{V}_\gamma^{(\alpha)} = \gamma\mathbb{E}[\ell_e(o)^{2\alpha}] + 2\gamma^2 \int_{\mathrm{HB}_e(o)} \mathbb{E}[\ell_e^{(y)}(o)^\alpha \ell_e^{(o)}(y)^\alpha] - \mathbb{E}[\ell_e(o)^\alpha]\mathbb{E}[\ell_e(y)^\alpha]\, \mathcal{H}^d(\mathrm{d}y),$$

where $e \in \partial\mathbb{H}^d$ is an arbitrary ideal point.

For the Euclidean case the asymptotic variance constant was derived in [100, Lemma 3.4]. By rewriting in that formula the integral over $\mathbb{R}^d$ as two times the integral over a half-space with the origin in its boundary, one obtains the same expression as in Theorem 4.13 with an integral over a half-space instead of over a horoball. This is due to the fact that in both the proof of Theorem 4.13 below and the proof for the Euclidean case in [100] one has to deal with increasing balls with centres tending to infinity, which converge to a horoball in the hyperbolic case and a half-space in the Euclidean case. In this context, for the variance asymptotics of geometric functionals associated with Boolean models stronger differences between Euclidean and hyperbolic space were observed in [56]. These are caused by boundary effects, which are negligible in Euclidean case but become significant in hyperbolic space. Since we do not deal with edges of the radial spanning tree crossing the boundary of the observation window, such effects are not present in the current paper. However, we expect similar phenomena to occur if we take such edges into account.

For the proof of this theorem we need some further notation. First, for a point $p \in \mathbb{H}^d$ we consider the radial spanning tree $\mathrm{RST}_p(\eta)$ build on η with base point p (i.e. p plays the role of the origin). We denote by $\ell_p(x)$ the distance from $x \in \mathbb{H}^d$ to its radial nearest neighbour (with respect to p) in $\mathrm{RST}_p(\eta + \delta_x)$ and by $\ell_p^{(y)}(x)$ the same distance in $\mathrm{RST}_p(\eta + \delta_x + \delta_y)$ for $y \in \mathbb{H}^d$. Observe that by the isometry-invariance of η, for any hyperbolic isometry ϱ and any $x, y \in \mathbb{H}^d$ one has the equalities in laws

$$\ell(x) \overset{d}{=} \ell_{\varrho(o)}(\varrho(x)) \quad \text{and} \quad \big(\ell^{(y)}(x), \ell^{(x)}(y)\big) \overset{d}{=} \big(\ell_{\varrho(o)}^{(\varrho(y))}(\varrho(x)), \ell_{\varrho(o)}^{(\varrho(x))}(\varrho(y))\big). \tag{4.8}$$

For $p, x, y \in \mathbb{H}^d$, the inequality $\ell_p^{(y)}(x) \le \ell_p(x)$ is obvious since adding the point y can only decrease the distance from x to its radial nearest neighbour with respect to p. Thus, it follows from Lemma 4.12 b) that

$$\sup_{p,x\in\mathbb{H}^d} \mathbb{E}[\ell_p(x)^\alpha] < \infty \quad \text{and} \quad \sup_{p,x,y\in\mathbb{H}^d} \mathbb{E}[\ell_p^{(y)}(x)^\alpha] < \infty \tag{4.9}$$

for all $\alpha > 0$. Moreover, if τ is a geodesic ray in $\mathbb{H}^d$ with ideal endpoint $e = \tau(\infty) \in \partial\mathbb{H}^d$, it holds for $x, y \in \mathbb{H}^d$ that

$$\ell_{\tau(t)}(x) \xrightarrow[t\to\infty]{} \ell_e(x) \quad \text{and} \quad \ell_{\tau(t)}^{(y)}(x) \xrightarrow[t\to\infty]{} \ell_e^{(y)}(x) \quad \text{a.s.} \tag{4.10}$$

For the proof of Theorem 4.13 we need the following technical lemma.

Lemma 4.14. *For every $\alpha > 0$ there exists a constant $C(\alpha) > 0$ only depending on α and d such that for all $p, z_1, z_2 \in \mathbb{H}^d$,*

$$\left|\mathbb{E}[\ell_p^{(z_2)}(z_1)^\alpha \ell_p^{(z_1)}(z_2)^\alpha] - \mathbb{E}[\ell_p(z_1)^\alpha]\mathbb{E}[\ell_p(z_2)^\alpha]\right| \leq C(\alpha) \exp\Big(-\frac{\gamma}{2} V_h\Big(\frac{d_h(z_1,z_2)}{4}\Big)\Big).$$

Proof. For $i \in \{1,2\}$ we define the event

$$A_i = \left\{\eta\left(B_h^d\left(\hat{z}_i, \frac{d_h(z_1,z_2)}{4}\right)\right) = 0\right\}$$

with $\hat{z}_i \in \mathbb{H}^d$ being the point on the geodesic ray in the direction of p such that $d_h(\hat{z}_i, z_i) = \frac{d_h(z_1,z_2)}{4}$. We note that in case of A_1^c, $\ell_p^{(z_2)}(z_1)$ is completely determined by the points of η within $B_h^d(z_1, \frac{d_h(z_1,z_2)}{2})$ since there are potential radial nearest neighbours of z_1 in the latter ball. Indeed, for $p \notin B_h^d(\hat{z}_1, d_h(z_1,z_2)/4)$ the points of η in $B_h^d(\hat{z}_1, d_h(z_1,z_2)/4)$ are closer to p than z_1 and, thus, potential radial nearest neighbours of z_1, while for $p \in B_h^d(\hat{z}_1, d_h(z_1,z_2)/4)$, p is a potential radial nearest neighbour of z_1. Analogously, $\ell_p^{(z_1)}(z_2)$ is in the case of A_2^c completely determined by the points of η in $B_h^d(z_2, \frac{d_h(z_1,z_2)}{2})$. Due to the independence of the events A_1 and A_2 by the defining property ii) of a Poisson process we have for $A = A_1 \cup A_2$ that

$$\mathbb{E}[\ell_p^{(z_2)}(z_1)^\alpha \ell_p^{(z_1)}(z_2)^\alpha \mathbb{1}_{A^c}] = \mathbb{E}[\ell_p(z_1)^\alpha \mathbb{1}_{A_1^c}]\mathbb{E}[\ell_p(z_2)^\alpha \mathbb{1}_{A_2^c}].$$

It follows together with the Cauchy-Schwarz inequality and (4.9) that

$$\begin{aligned}
&|\mathbb{E}[\ell_p^{(z_2)}(z_1)^\alpha \ell_p^{(z_1)}(z_2)^\alpha] - \mathbb{E}[\ell_p(z_1)^\alpha]\mathbb{E}[\ell_p(z_2)^\alpha]| \\
&= |\mathbb{E}[\ell_p^{(z_2)}(z_1)^\alpha \ell_p^{(z_1)}(z_2)^\alpha(\mathbb{1}_A + \mathbb{1}_{A^c})] - \mathbb{E}[\ell_p(z_1)^\alpha(\mathbb{1}_{A_1} + \mathbb{1}_{A_1^c})]\mathbb{E}[\ell_p(z_2)^\alpha(\mathbb{1}_{A_2} + \mathbb{1}_{A_2^c})]| \\
&\leq \mathbb{E}[\ell_p^{(z_2)}(z_1)^\alpha \ell_p^{(z_1)}(z_2)^\alpha \mathbb{1}_A] + \mathbb{E}[\ell_p(z_1)^\alpha \mathbb{1}_{A_1}]\mathbb{E}[\ell_p(z_2)^\alpha \mathbb{1}_{A_2}] \\
&\quad + \mathbb{E}[\ell_p(z_1)^\alpha \mathbb{1}_{A_1}]\mathbb{E}[\ell_p(z_2)^\alpha \mathbb{1}_{A_2^c}] + \mathbb{E}[\ell_p(z_1)^\alpha \mathbb{1}_{A_1^c}]\mathbb{E}[\ell_p(z_2)^\alpha \mathbb{1}_{A_2}] \\
&\leq \mathbb{E}[\ell_p^{(z_2)}(z_1)^{4\alpha}]^{1/4}\mathbb{E}[\ell_p^{(z_1)}(z_2)^{4\alpha}]^{1/4}\mathbb{P}(A)^{1/2} + \mathbb{E}[\ell_p(z_1)^\alpha \mathbb{1}_{A_1}]\mathbb{E}[\ell_p(z_2)^\alpha] \\
&\quad + \mathbb{E}[\ell_p(z_2)^\alpha \mathbb{1}_{A_2}]\mathbb{E}[\ell_p(z_1)^\alpha] \\
&\leq \mathbb{E}[\ell_p^{(z_2)}(z_1)^{4\alpha}]^{1/4}\mathbb{E}[\ell_p^{(z_1)}(z_2)^{4\alpha}]^{1/4}\mathbb{P}(A)^{1/2} + \mathbb{E}[\ell_p(z_1)^{2\alpha}]^{1/2}\mathbb{E}[\ell_p(z_2)^\alpha]\mathbb{P}(A_1)^{1/2} \\
&\quad + \mathbb{E}[\ell_p(z_2)^{2\alpha}]^{1/2}\mathbb{E}[\ell_p(z_1)^\alpha]\mathbb{P}(A_2)^{1/2} \\
&\leq C_1(\alpha)(\mathbb{P}(A)^{1/2} + \mathbb{P}(A_1)^{1/2} + \mathbb{P}(A_2)^{1/2}) \\
&\leq C_2(\alpha)(\mathbb{P}(A_1)^{1/2} + \mathbb{P}(A_2)^{1/2}) \\
&= 2C_2(\alpha)\exp\Big(-\frac{\gamma}{2}\mathcal{H}^d\Big(B_h^d\Big(o, \frac{d_h(z_1,z_2)}{4}\Big)\Big)\Big) = 2C_2(\alpha)\exp\Big(-\frac{\gamma}{2}V_h\Big(\frac{d_h(z_1,z_2)}{4}\Big)\Big)
\end{aligned}$$

for suitable constants $C_1(\alpha), C_2(\alpha) > 0$ only depending on α and d, which completes the proof. □

Proof of Theorem 4.13. Due to the multivariate Mecke equation (2.15), the variance is given by

$$\mathrm{Var}[\mathcal{L}_{s,\gamma}^{(\alpha)}] = \mathbb{E}\Big[\sum_{(x,y)\in\eta_{s,\neq}^2} \ell(x,\eta)^\alpha \ell(y,\eta)^\alpha\Big] - \left(\mathbb{E}\Big[\sum_{x\in\eta_s} \ell(x,\eta)^\alpha\Big]\right)^2 + \mathbb{E}\Big[\sum_{x\in\eta_s} \ell(x,\eta)^{2\alpha}\Big]$$

$$= T(s) + \mathbb{E}[\mathcal{L}_{s,\gamma}^{(2\alpha)}],$$

where

$$T(s) = \gamma^2 \int_{B_h^d(o,s)} \int_{B_h^d(o,s)} \mathbb{E}[\ell^{(y)}(x)^\alpha \ell^{(x)}(y)^\alpha] - \mathbb{E}[\ell(x)^\alpha]\mathbb{E}[\ell(y)^\alpha] \,\mathcal{H}^d(\mathrm{d}y)\,\mathcal{H}^d(\mathrm{d}x).$$

By Theorem 4.10 we have

$$\lim_{s\to\infty} \frac{\mathbb{E}[\mathcal{L}_{s,\gamma}^{(2\alpha)}]}{V_h(s)} = \gamma \int_0^\infty e^{-\gamma G(u^{1/(2\alpha)})}\,\mathrm{d}u.$$

On the other hand, a computation as in the proof of Lemma 4.12 a) shows that

$$\mathbb{E}[\ell_e(o)^{2\alpha}] = \int_0^\infty \mathbb{P}(\ell_e(o) \geq u^{1/(2\alpha)})\,\mathrm{d}u = \int_0^\infty \exp\big(-\gamma\mathcal{H}^d(\mathrm{HB}_e(o) \cap B^d(o, u^{1/(2\alpha)}))\big)\,\mathrm{d}u.$$

Lemma 4.11 implies that $\mathcal{H}^d(\mathrm{HB}_e(o) \cap B_h^d(o, u^{1/(2\alpha)})) = G(u^{1/(2\alpha)})$ and hence

$$\lim_{s\to\infty} \frac{\mathbb{E}[\mathcal{L}_{s,\gamma}^{(2\alpha)}]}{V_h(s)} = \gamma\mathbb{E}[\ell_e(o)^{2\alpha}]. \tag{4.11}$$

For the term $T(s)$ we write, by symmetry and polar integration,

$$\begin{aligned} T(s) &= 2\gamma^2 \int_{B_h^d(o,s)} \int_{B_h^d(o,d_h(o,x))} \mathbb{E}[\ell^{(y)}(x)^\alpha \ell^{(x)}(y)^\alpha] - \mathbb{E}[\ell(x)^\alpha]\mathbb{E}[\ell(y)^\alpha] \,\mathcal{H}^d(\mathrm{d}y)\,\mathcal{H}^d(\mathrm{d}x) \\ &= 2\gamma^2\omega_d \int_0^s \sinh^{d-1}(t) \int_{B_h^d(o,t)} \mathbb{E}[\ell^{(y)}(x_t)^\alpha \ell^{(x_t)}(y)^\alpha] - \mathbb{E}[\ell(x_t)^\alpha]\mathbb{E}[\ell(y)^\alpha] \,\mathcal{H}^d(\mathrm{d}y)\,\mathrm{d}t, \end{aligned}$$

where x_t is the point on the geodesic ray in $\mathbb{H}^d$ from o with ideal endpoint $e = \gamma(\infty) \in \partial\mathbb{H}^d$ such that $d_h(x_t, o) = t$ for $t \in [0, \infty)$. From (2.13) and l'Hospital's rule we deduce

$$\begin{aligned} &\lim_{s\to\infty} \frac{T(s)}{V_h(s)} \\ &= \lim_{s\to\infty} \frac{2\gamma^2\omega_d \sinh^{d-1}(s)}{\omega_d \sinh^{d-1}(s)} \int_{B_h^d(o,s)} \mathbb{E}[\ell^{(y)}(x_s)^\alpha \ell^{(x_s)}(y)^\alpha] - \mathbb{E}[\ell(x_s)^\alpha]\mathbb{E}[\ell(y)^\alpha] \,\mathcal{H}^d(\mathrm{d}y) \\ &= \lim_{s\to\infty} 2\gamma^2 \int_{B_h^d(o,s)} \mathbb{E}[\ell^{(y)}(x_s)^\alpha \ell^{(x_s)}(y)^\alpha] - \mathbb{E}[\ell(x_s)^\alpha]\mathbb{E}[\ell(y)^\alpha] \,\mathcal{H}^d(\mathrm{d}y). \end{aligned}$$

Using (4.8) to change the roles of x_s and o, this can be rewritten as

$$\begin{aligned} &\lim_{s\to\infty} \frac{T(s)}{V_h(s)} \\ &= \lim_{s\to\infty} 2\gamma^2 \int_{B_h^d(x_s,s)} \mathbb{E}[\ell_{x_s}^{(y)}(o)^\alpha \ell_{x_s}^{(o)}(y)^\alpha] - \mathbb{E}[\ell_{x_s}(o)^\alpha]\mathbb{E}[\ell_{x_s}(y)^\alpha] \,\mathcal{H}^d(\mathrm{d}y) \\ &= \lim_{s\to\infty} 2\gamma^2 \int_{\mathbb{H}^d} \mathbf{1}\{y \in B_h^d(x_s, s)\}\big(\mathbb{E}[\ell_{x_s}^{(y)}(o)^\alpha \ell_{x_s}^{(o)}(y)^\alpha] - \mathbb{E}[\ell_{x_s}(o)^\alpha]\mathbb{E}[\ell_{x_s}(y)^\alpha]\big)\,\mathcal{H}^d(\mathrm{d}y). \end{aligned} \tag{4.12}$$

The absolute value of the integrand is bounded by $C(\alpha)\exp\big(-\frac{\gamma}{2}V_h\big(\frac{d_h(o,y)}{4}\big)\big)$ for $y\in\mathbb{H}^d$ by Lemma 4.14. Since, by a further application of (2.12),

$$\int_{\mathbb{H}^d}\exp\Big(-\frac{\gamma}{2}V_h\Big(\frac{d_h(o,y)}{4}\Big)\Big)\,\mathcal{H}^d(\mathrm{d}y)=\omega_d\int_0^\infty \sinh^{d-1}(t)e^{-\gamma V_h(t/4)/2}\,\mathrm{d}t<\infty,$$

we can apply the dominated convergence theorem in (4.12). Combining (4.9) with (4.10) provides for $y\in\mathbb{H}^d$,

$$\mathbb{E}[\ell_{x_s}(o)^\alpha]\mathbb{E}[\ell_{x_s}(y)^\alpha]\xrightarrow[s\to\infty]{}\mathbb{E}[\ell_e(o)^\alpha]\mathbb{E}[\ell_e(y)^\alpha]$$

and

$$\mathbb{E}[\ell^{(y)}_{x_s}(o)^\alpha\ell^{(o)}_{x_s}(y)^\alpha]\xrightarrow[s\to\infty]{}\mathbb{E}[\ell^{(y)}_e(o)^\alpha\ell^{(o)}_e(y)^\alpha].$$

Moreover, we have

$$\mathbf{1}\{y\in B^d_h(x_s,s)\}\xrightarrow[s\to\infty]{}\mathbf{1}\{y\in \mathrm{HB}_e(o)\}$$

for $\mathcal{H}^d$-a.e. $y\in\mathbb{H}^d$. Altogether, we obtain

$$\lim_{s\to\infty}\frac{T(s)}{V_h(s)}=2\gamma^2\int_{\mathrm{HB}_e(o)}\mathbb{E}[\ell^{(y)}_e(o)^\alpha\ell^{(o)}_e(y)^\alpha]-\mathbb{E}[\ell_e(o)^\alpha]\mathbb{E}[\ell_e(y)^\alpha]\,\mathcal{H}^d(\mathrm{d}y).$$

Combining this with (4.11) completes the proof. □

Note that the explicit representation of $\mathbb{V}^{(\alpha)}_\gamma$ in Theorem 4.13 does not show its positivity due to the difference in the integral. Therefore, the following proposition answers the question of the positivity of the asymptotic variance.

Proposition 4.15. *Fix $c_0>0$. Then, there are constants $c_\ell,c_u>0$ and $s_0\geq 1$ depending on α, d and c_0, such that*

$$c_\ell\gamma^{1-\frac{2\alpha}{d}}V_h(s)\leq \mathrm{Var}[\mathcal{L}^{(\alpha)}_{s,\gamma}]\leq c_u\gamma^{1-\frac{2\alpha}{d}}V_h(s)$$

for all $\gamma\geq c_0$ and $s\geq s_0$.

For the proof of this proposition we use some additional lemmas. We start with the following geometric result.

Lemma 4.16. *There exists a constant $\overline{\varepsilon}>0$ only depending on d such that for all $z\in B^d_h(o,3)^c$ and all $x\in B^d_h(z,1)\backslash\{z\}$ the set*

$$B^d_h(z,d_h(x,z))^c\cap B^d_h(x,d_h(x,z))\cap B^d_h(o,d_h(x,o))$$

contains a ball of radius $\overline{\varepsilon}d_h(x,z)$.

Proof. For $z\in B^d_h(o,3)^c$ and $x\in B^d_h(z,1)\backslash\{z\}$ let $\tau\colon[0,1]\to\mathbb{H}^d$ be the geodesic segment from o to x, i.e. $\tau(0)=o$ and $\tau(1)=x$. Then,

$$B^d_h(\tau(u),d_h(x,\tau(u)))\subset B^d_h(\tau(v),d_h(x,\tau(v)))$$

for $u > v$, i.e. shifting the origin in the direction of x along a geodesic ray decreases the considered set. Since $d_h(x,z) \le 1$, $d_h(o,z) > 3$ and $d_h(\tau(u),z)$ is continuous in u, there exists $t \in [0,1]$ satisfying $d_h(\tau(t),z) = 3$. Hence, together with rotational invariance it is sufficient to show the desired statement for a fixed $\hat{z}$ with $d_h(\hat{z},o) = 3$.

For $x \in B_h^d(\hat{z},1)\setminus\{\hat{z}\}$ let $R(x)$ be the supremum over all radii of balls contained in the set $B_h^d(\hat{z},d_h(x,\hat{z}))^c \cap B_h^d(x,d_h(x,\hat{z})) \cap B_h^d(o,d_h(x,o))$. Note that R is continuous on $B_h^d(\hat{z},1)\setminus\{\hat{z}\}$. Assume that the function $h\colon B_h^d(\hat{z},1)\setminus\{\hat{z}\} \to \mathbb{R}, x \mapsto \frac{R(x)}{d_h(x,\hat{z})}$ is not bounded away from 0. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $B_h^d(\hat{z},1)\setminus\{\hat{z}\}$ such that

$$\lim_{n\to\infty} \frac{R(x_n)}{d_h(x_n,\hat{z})} = 0.$$

Since $B_h^d(\hat{z},1)\setminus \operatorname{int}(B_h^d(\hat{z},u))$ is compact for any $u \in (0,1)$ and $h \neq 0$ on $B_h^d(\hat{z},1)\setminus\{\hat{z}\}$, h is bounded away from zero on $B_h^d(\hat{z},1)\setminus \operatorname{int}(B_h^d(\hat{z},u))$. This implies $x_n \to \hat{z}$ as $n \to \infty$. Define $r_n = d_h(x_n,\hat{z})$ for $n \in \mathbb{N}$. We have

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{r_n^d}\mathcal{H}^d(B_h^d(\hat{z},r_n)^c \cap B_h^d(x_n,r_n) \cap B_h^d(o,d_h(x_n,o)))\\
&\ge \liminf_{n\to\infty} \frac{1}{r_n^d}\big(\mathcal{H}^d(B_h^d(x_n,r_n)\cap B_h^d(o,d_h(x_n,o))) - \mathcal{H}^d(B_h^d(x_n,r_n)\cap B_h^d(\hat{z},r_n))\big)\\
&\ge \liminf_{n\to\infty} \frac{1}{r_n^d}\big(\mathcal{H}^d(B_h^d(x_n,r_n)\cap B_h^d(y_n,d_h(x_n,y_n))) - \mathcal{H}^d(B_h^d(x_n,r_n)\cap B_h^d(\hat{z},r_n))\big)
\end{aligned}$$

where y_n is the point on the geodesic between x_n and o such that $d_h(x_n,y_n) = 2r_n$. Since hyperbolic balls of small radii can be approximated by Euclidean balls in appropriate tangent spaces, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{r_n^d}\mathcal{H}^d(B_h^d(\hat{z},r_n)^c \cap B_h^d(x_n,r_n) \cap B_h^d(o,d_h(x_n,o)))\\
&\ge V_d(B^d(0,1)\cap B^d(v_2,2)) - V_d(B^d(0,1)\cap B^d(v_1,1)) =: c_1
\end{aligned}$$

with $v_1, v_2 \in \mathbb{R}^d$ such that $\|v_1\| = 1$ and $\|v_2\| = 2$. Obviously, we have $c_1 > 0$. Next fix $\tilde{\varepsilon} \in (0,1)$. Then one has

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{r_n^d}\mathcal{H}^d(\{w \in B_h^d(\hat{z},r_n)^c \cap B_h^d(x_n,r_n)\cap B_h^d(o,d_h(x_n,o))\colon\\
&\qquad\qquad B_h^d(w,\tilde{\varepsilon}r_n) \not\subseteq B_h^d(\hat{z},r_n)^c \cap B_h^d(x_n,r_n)\cap B_h^d(o,d_h(x_n,o))\})\\
&\le \limsup_{n\to\infty} \frac{1}{r_n^d}\Big(\mathcal{H}^d(B_h^d(\hat{z},(1+\tilde{\varepsilon})r_n)\setminus B_h^d(\hat{z},r_n)) + \mathcal{H}_h^d(B_h^d(x_n,r_n)\setminus B_h^d(x_n,(1-\tilde{\varepsilon})r_n))\\
&\qquad + \mathcal{H}^d((B_h^d(o,d_h(x_n,o))\setminus B_h^d(o,d_h(x_n,o)-\tilde{\varepsilon}r_n))\cap B_h^d(x_n,r_n))\Big)\\
&= \limsup_{n\to\infty} \frac{1}{r_n^d}\Big(\mathcal{H}^d(B_h^d(o,(1+\tilde{\varepsilon})r_n)\setminus B_h^d(o,r_n)) + \mathcal{H}^d(B_h^d(o,r_n)\setminus B_h^d(o,(1-\tilde{\varepsilon})r_n))\\
&\qquad + \mathcal{H}^d(B_h^d(o,d_h(x_n,o))\cap B_h^d(x_n,r_n)) - \mathcal{H}^d(B_h^d(o,d_h(x_n,o)-\tilde{\varepsilon}r_n)\cap B_h^d(x_n,r_n))\Big)\\
&= \kappa_d\Big((1+\tilde{\varepsilon})^d - 1 + 1 - (1-\tilde{\varepsilon})^d + \frac{1}{2} - v(\tilde{\varepsilon})\Big) =: g(\tilde{\varepsilon}),
\end{aligned}$$

where $v(\tilde{\varepsilon}) = \kappa_d^{-1} V_d(\{(x_1,\dots,x_d) \in B^d(0,1)\colon x_d \geq \tilde{\varepsilon}\})$ and where we used again convergence of hyperbolic balls with small radii to Euclidean ones. Combining the previous estimates, we see that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{r_n^d} \mathcal{H}^d(\{w \in B_h^d(\hat{z},r_n)^c \cap B_h^d(x_n,r_n) \cap B_h^d(o,d_h(x_n,o))\colon \\ B_h^d(w,\tilde{\varepsilon}r_n) \subseteq B_h^d(\hat{z},r_n)^c \cap B_h^d(x_n,r_n) \cap B_h^d(o,d_h(x_n,o))\}) \\ \geq c_1 - g(\tilde{\varepsilon}).\end{aligned}$$

Since $g(t) \to 0$ as $t \to 0$, the right-hand side is positive for $\tilde{\varepsilon}$ sufficiently small. Thus, for n large enough there exists $w \in B_h^d(\hat{z},r_n)^c \cap B_h^d(x_n,r_n) \cap B_h^d(o,d_h(x_n,o))$ such that $B_h^d(w,\tilde{\varepsilon}r_n) \subseteq B_h^d(\hat{z},r_n)^c \cap B_h^d(x_n,r_n) \cap B_h^d(o,d_h(x_n,o))$ and, thus, $h(x_n) \geq \tilde{\varepsilon}$. This is a contradiction so that we have shown that h must be bounded away from zero, which proves the desired statement. □

In the following, the previous lemma is used to give a lower bound for the probability that the difference operator is bounded away from 0, which we need for showing the lower bound of the variance.

Lemma 4.17. *Let $c_0 > 0$ be fixed. Then, there exist constants $c, C > 0$, depending on α, d and c_0, such that for all $\gamma \geq c_0$ and $s > 3$ one has with $c_\gamma = \gamma^{-\alpha/d}c$ that*

$$\mathbb{P}(D_z\mathcal{L}_{s,\gamma}^{(\alpha)} \geq c_\gamma) \geq C$$

for all $z \in B_h^d(o,s) \setminus B_h^d(o,3)$.

Proof. Fix $c > 0$ such that $c_\gamma = \gamma^{-\alpha/d}c \leq 1$ for all $\gamma \geq c_0$ and let $z \in \mathbb{H}^d$ be such that $d_h(o,z) > 3$. Denote by $A_\gamma = \{x \in \mathbb{H}^d\colon c_\gamma^{1/\alpha} \leq d_h(x,z) \leq 2c_\gamma^{1/\alpha}\}$ the annulus with radii $c_\gamma^{1/\alpha} < 2c_\gamma^{1/\alpha}$ around z. For all $x \in A_\gamma$ with $d_h(x,o) > d_h(z,o)$ denote by $x^* \in A_\gamma$ the point on the geodesic from x to z satisfying $d_h(x^*,z) = c_\gamma^{1/\alpha}$. Then, by the convexity of the hyperbolic distance function along geodesics (see e.g. [3, Theorem 1.4.2]), one has $d_h(x^*,o) \leq d_h(x,o)$ and hence,

$$B_h^d(o,d_h(o,x^*)) \cap B_h^d(x^*,d_h(z,x^*)) \cap A_\gamma \subseteq B_h^d(o,d_h(o,x)) \cap B_h^d(x,d_h(z,x)) \cap A_\gamma.$$

The left-hand side can be rewritten as

$$\begin{aligned}&B_h^d(o,d_h(o,x^*)) \cap B_h^d(x^*,d_h(z,x^*)) \cap A_\gamma \\ &= B_h^d(z,d_h(x^*,z))^c \cap B_h^d(x^*,d_h(x^*,z)) \cap B_h^d(o,d_h(x^*,o))\end{aligned}$$

so that, by Lemma 4.16, $B_h^d(o,d_h(o,x^*)) \cap B_h^d(x^*,d_h(z,x)) \cap A_\gamma$ contains a ball of radius $\overline{\varepsilon} d_h(x^*,z) = \overline{\varepsilon} c_\gamma^{1/\alpha} = \overline{\varepsilon} c^{1/\alpha}\gamma^{-1/d}$. Hence, we can choose $\varepsilon > 0$ such that the interior of the intersection $B_h^d(o,d_h(o,x)) \cap B_h^d(x,d_h(z,x)) \cap A_\gamma$ contains a ball of radius $\varepsilon_\gamma = \gamma^{-1/d}\varepsilon$ for all $\gamma \geq c_0$ and $x \in A_\gamma$. Note that ε does not depend on the choice of $z \in B_h^d(o,3)^c$.

Now consider the event

$$E_z = \{\eta|_{A_\gamma} \text{ is an } \varepsilon_\gamma\text{-dense subset of } A_\gamma \text{ and } \eta(\operatorname{int}(B_h^d(z,c_\gamma^{1/\alpha}))) = 0\},$$

where a set is called ε_γ-dense in A_γ if for all $y \in A_\gamma$ there exists a point of the ε_γ-dense subset having distance smaller than or equal to ε_γ to y. We claim that assuming E_z one has $D_z\mathcal{L}_{s,\gamma}^{(\alpha)} \geq c_\gamma$. Let us first show that, in the case of E_z, z is not the radial nearest neighbour of any $x \in \eta|_{B_h^d(o,s)}$. Indeed, first let $x \in \eta|_{A_\gamma}$ be such that $d_h(o,x) > d_h(o,z)$ (so that z could potentially be the radial nearest neighbour of x). Then, by the choice of ε_γ and since $\eta|_{A_\gamma}$ is ε_γ-dense in A_γ, there exists $w \in \eta|_{A_\gamma}$ which lies in the interior of $B_h^d(o, d_h(o,x)) \cap B_h^d(x, d_h(z,x))$, which in particular shows that z is not the radial nearest neighbour of x. For $\widehat{B} = \mathbb{R}^d \setminus B_h^d(z, 2c_\gamma^{1/\alpha})$ let now $x \in \eta|_{\widehat{B}}$ with $d_h(o,x) > d_h(o,z)$. Let x_B be the intersection point of the geodesic segment $[z,x]$ with the sphere $\partial B_h^d(z, 2c_\gamma^{1/\alpha})$, and let again $w \in \eta|_{A_\gamma}$ be such that w lies in the interior of $B_h^d(o, d_h(o,x_B)) \cap B_h^d(x_B, d_h(z,x_B))$. Then, using again the convexity of the hyperbolic distance function along geodesics one has

$$d_h(o,w) < d_h(o,x_B) \leq \max\{d_h(o,x), d_h(o,z)\} = d_h(o,x).$$

Additionally, using the fact that x_B lies on the geodesic segment $[z,x]$ we see that

$$d_h(w,x) \leq d_h(w,x_B) + d_h(x_B,x) < d_h(z,x_B) + d_h(x_B,x) = d_h(z,x).$$

As above, these two conditions combined show that z is not the radial nearest neighbour of x. This proves that, assuming E_z, adding z does not delete any edges from the radial spanning tree and hence adds exactly one edge between z and its radial nearest neighbour. Since in the case of E_z there are no points of η in the interior of $B_h^d(z, c_\gamma^{1/\alpha})$, the length of this edge is at least $c_\gamma^{1/\alpha}$. In other words, we have proven that

$$\mathbb{P}(D_z\mathcal{L}_{s,\gamma}^{(\alpha)} \geq c_\gamma) \geq \mathbb{P}(E_z).$$

To estimate the latter probability, fix a maximal collection $\{B_i\}_{i\in I}$ of disjoint $\varepsilon_\gamma/4$-balls in A_γ. Then, any choice of one point from each ball is clearly ε_γ-dense in A_γ. Let $c_u > 0$ be such that $\sinh(x) \leq c_u x$ for all $x \in [0,\varepsilon]$ and recall that $\sinh(x) \geq x$ for all $x \geq 0$. Then, the number of such balls does not exceed

$$k_\gamma = \frac{\mathcal{H}^d(B_h^d(o, 2c_\gamma^{1/\alpha}))}{\mathcal{H}^d(B_h^d(o, \varepsilon_\gamma/4))} \leq \frac{c_u^{d-1}2^d c_\gamma^{d/\alpha}}{(\varepsilon_\gamma/4)^d} = \frac{c_u^{d-1}4^d 2^d c^{d/\alpha}}{\varepsilon^d} =: k,$$

which is independent from γ. This shows that

$$\begin{aligned}\mathbb{P}(E_z) &\geq \mathbb{P}(\eta(B_i) = 1 \text{ for all } i,\, \eta(\operatorname{int}(B_h^d(z, c_\gamma^{1/\alpha}))) = 0)\\ &\geq \left(\gamma\mathcal{H}^d(B_h^d(o,\varepsilon_\gamma/4))e^{-\gamma\mathcal{H}^d(B_h^d(o,\varepsilon_\gamma/4))}\right)^k e^{-\gamma\mathcal{H}^d(B_h^d(o,c_\gamma^{1/\alpha}))}\\ &\geq \left(\kappa_d\varepsilon^d e^{-c_u^{d-1}\kappa_d\varepsilon^d/4^d}/4^d\right)^k e^{-c_u^{d-1}\kappa_d c^{d/\alpha}} =: C > 0,\end{aligned}$$

which completes the proof. □

Finally, we state three additional lemmas, which are essential for deriving the variance bounds and the central limit theorem in the upcoming subsection. Lemma 4.19 and Lemma 4.20 are derived analogously to Lemma 4.1 and Lemma 4.2 in [100] with the help of Lemma 4.18.

Lemma 4.18. *Let $c_0 > 0$ be fixed. Then, for all $c_1, c_2 > 0$ there exist constants $\tilde{c}_1, \tilde{c}_2 > 0$, only depending on c_1, c_2, d and c_0, such that for $\gamma \geq c_0$ and $t \geq 0$,*

$$\int_{\mathbb{H}^d \setminus B_h^d(o,t)} \gamma e^{-\gamma c_1 V_h(\frac{d_h(x,o)}{c_2})} \, \mathcal{H}^d(\mathrm{d}x) \leq \tilde{c}_1 e^{-\gamma \tilde{c}_2 V_h(\frac{t}{c_2})}.$$

Proof. Let $c \geq c_2 \sinh^{-1}(1)$ be such that for $\gamma \geq c_0$,

$$\gamma c_1 V_h\Big(\frac{r}{c_2}\Big) - (d-1)r \geq C\gamma V_h\Big(\frac{r}{c_2}\Big)$$

for all $r \geq c$ and some constant $C > 0$. Then, for $t \geq c$, together with (2.12) we have

$$\begin{aligned}
\int_{\mathbb{H}^d \setminus B_h^d(o,t)} \gamma e^{-\gamma c_1 V_h(\frac{d_h(x,o)}{c_2})} \, \mathcal{H}^d(\mathrm{d}x) &= \omega_d \int_t^\infty \gamma e^{-\gamma c_1 V_h(\frac{r}{c_2})} \sinh^{d-1}(r) \, \mathrm{d}r \\
&\leq \omega_d \int_t^\infty \gamma e^{-\gamma c_1 V_h(\frac{r}{c_2}) + (d-1)r} \sinh^{d-1}\Big(\frac{r}{c_2}\Big) \, \mathrm{d}r \\
&\leq \omega_d \int_t^\infty \gamma e^{-\gamma C V_h(\frac{r}{c_2})} \sinh^{d-1}\Big(\frac{r}{c_2}\Big) \, \mathrm{d}r \\
&= \frac{c_2}{C} e^{-\gamma C V_h(\frac{t}{c_2})}. \qquad (4.13)
\end{aligned}$$

For $t \in [0, c]$ we divide the integral into two parts. For the first part we use that there exists a constant $\overline{c} > 0$ such that $\sinh(x) \leq \overline{c} \sinh(\frac{x}{c_2})$ for all $x \leq c$. Then it holds

$$\begin{aligned}
\omega_d \int_t^c \gamma e^{-\gamma c_1 V_h(\frac{r}{c_2})} \sinh^{d-1}(r) \, \mathrm{d}r &\leq \omega_d \overline{c}^{d-1} \int_t^c \gamma e^{-\gamma c_1 V_h(\frac{r}{c_2})} \sinh^{d-1}\Big(\frac{r}{c_2}\Big) \, \mathrm{d}r \\
&\leq \frac{c_2}{c_1} \omega_d \overline{c}^{d-1} \int_t^\infty \frac{c_1}{c_2} \gamma e^{-c_1 \gamma V_h(\frac{r}{c_2})} \sinh^{d-1}\Big(\frac{r}{c_2}\Big) \, \mathrm{d}r \\
&= \frac{c_2}{c_1} \overline{c}^{d-1} e^{-\gamma c_1 V_h(\frac{t}{c_2})}.
\end{aligned}$$

Similarly to (4.13) we derive for the second part of the integral

$$\omega_d \int_c^\infty \gamma e^{-\gamma c_1 V_h(\frac{r}{c_2})} \sinh^{d-1}(r) \, \mathrm{d}r \leq \frac{c_2}{C} e^{-\gamma C V_h(\frac{c}{c_2})} \leq \frac{c_2}{C} e^{-\gamma C V_h(\frac{t}{c_2})}.$$

Altogether, this completes the proof. □

Lemma 4.19. *For $c_0 \geq 0$ and $p \in \mathbb{N}$ there exist constants $C_{1,p}, C_{2,p} > 0$, depending on p, α, d and c_0, such that for all $s > 0$, $z, z_1, z_2 \in B_h^d(o,s)$ and $\gamma \geq c_0$,*

$$\gamma^{\alpha p/d} \mathbb{E}[|D_z \mathcal{L}_{s,\gamma}^{(\alpha)}|^p] \leq C_{1,p} \quad \text{and} \quad \gamma^{\alpha p/d} \mathbb{E}[|D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)}|^p] \leq C_{2,p}.$$

Proof. For $\xi \in \mathbf{N}$ and $z \in B_h^d(o,s)$ it holds

$$|D_z \mathcal{L}_{s,\gamma}^{(\alpha)}(\xi)| \leq \ell(z, \xi + \delta_z)^\alpha + \sum_{x \in \xi} \mathbb{1}\{\ell(x,\xi) \geq d_h(z,x)\} \ell(x,\xi)^\alpha. \qquad (4.14)$$

Together with Jensen's inequality and the Cauchy-Schwarz inequality we have

$$\gamma^{\alpha p/d} \mathbb{E}[|D_z \mathcal{L}_{s,\gamma}^{(\alpha)}|^p]$$

$$
\begin{aligned}
&\le 2^{p-1}\gamma^{\alpha p/d}\mathbb{E}[\ell(z,\eta+\delta_z)^{\alpha p}] + 2^{p-1}\gamma^{\alpha p/d}\mathbb{E}\Big[\Big(\sum_{x\in\eta_s}\mathbb{1}\{\ell(x,\eta)\ge d_h(z,x)\}\ell(x,\eta)^{\alpha}\Big)^p\Big]\\
&\le 2^{p-1}\gamma^{\alpha p/d}\mathbb{E}[\ell(z,\eta+\delta_z)^{\alpha p}]\\
&\quad + 2^{p-1}\gamma^{\alpha p/d}\mathbb{E}\Big[\Big(\sum_{x\in\eta_s}\mathbb{1}\{\ell(x,\eta)\ge d_h(z,x)\}\Big)^{2p}\Big]^{1/2}\mathbb{E}\Big[\max_{\substack{x\in\eta_s:\\ \ell(x,\eta)\ge d_h(z,x)}}\ell(x,\eta)^{2\alpha p}\Big]^{1/2}\\
&=: 2^{p-1}\gamma^{\alpha p/d}\mathbb{E}[\ell(z,\eta+\delta_z)^{\alpha p}] + 2^{p-1}\gamma^{\alpha p/d}M_1(z)^{1/2}M_2(z)^{1/2}. \qquad (4.15)
\end{aligned}
$$

By Lemma 4.12 b) the first summand is uniformly bounded in z and γ.

For $z \in B_h^d(o,s)$ the factor $M_1(z)$ can be written as a linear combination of terms of the form

$$
\mathbb{E}\Big[\sum_{(x_1,\dots,x_k)\in\eta_{s,\neq}^k}\mathbb{1}\{\ell(x_i,\eta)\ge d_h(x_i,z), i=1,\dots,k\}\Big]
$$

for $k \in \{1,\dots,2p\}$. Using the monotonicity relation $\ell(x_i,\eta+\sum_{j=1}^k\delta_{x_j}) \le \ell(x_i,\eta+\delta_{x_i})$ for $i\in\{1,\dots,k\}$, the multivariate Mecke equation (2.15), Hölder's inequality and (4.6), we have

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{(x_1,\dots,x_k)\in\eta_{s,\neq}^k}\mathbb{1}\{\ell(x_i,\eta)\ge d_h(x_i,z), i=1,\dots,k\}\Big]\\
&= \gamma^k\int_{B_h^d(o,s)^k}\mathbb{P}\Big(\ell\Big(x_i,\eta_s+\sum_{j=1}^k\delta_{x_i}\Big)\ge d_h(x_i,z), i=1,\dots,k\Big)\,(\mathcal{H}^d)^k(\mathrm{d}(x_1,\dots,x_k))\\
&\le \gamma^k\int_{B_h^d(o,s)^k}\prod_{i=1}^k\mathbb{P}(\ell(x_i,\eta+\delta_{x_i})\ge d_h(x_i,z))^{1/k}\,(\mathcal{H}^d)^k(\mathrm{d}(x_1,\dots,x_k))\\
&\le \gamma^k\Big(\int_{B_h^d(o,s)}e^{-\gamma\mathcal{H}^d(o,\frac{d_h(x,z)}{2})/k}\,\mathcal{H}^d(\mathrm{d}x)\Big)^k \le \Big(\int_{\mathbb{H}^d}\gamma e^{-\gamma V_h(\frac{d_h(x,o)}{2})/k}\,\mathcal{H}^d(\mathrm{d}x)\Big)^k\\
&\le \tilde{c}_1^k
\end{aligned}
$$

by Lemma 4.18, i.e. M_1 is uniformly bounded in z and for all $\gamma\ge c_0$.

For the factor M_2 we have with the Mecke equation (2.15) and Fubini's theorem,

$$
\begin{aligned}
&\gamma^{2\alpha p/d}\mathbb{E}\Big[\max_{\substack{x\in\eta_s:\\ \ell(x,\eta)\ge d_h(z,x)}}\ell(x,\eta)^{2\alpha p}\Big] = \gamma^{2\alpha p/d}\int_0^\infty\mathbb{P}\Big(\max_{\substack{x\in\eta_s:\\ \ell(x,\eta)\ge d_h(z,x)}}\ell(x,\eta)^{2\alpha p}\ge u\Big)\,\mathrm{d}u\\
&= \gamma^{2\alpha p/d}\int_0^\infty\mathbb{P}(\exists x\in\eta_s\colon \ell(x,\eta)\ge\max\{u^{1/(2\alpha p)},d_h(z,x)\})\,\mathrm{d}u\\
&\le \gamma^{2\alpha p/d}\int_0^\infty\mathbb{E}\Big[\sum_{x\in\eta_s}\mathbb{1}\{\ell(x,\eta)\ge\max\{u^{1/(2\alpha p)},d_h(z,x)\}\}\Big]\,\mathrm{d}u\\
&= \gamma^{2\alpha p/d}\mathbb{E}\Big[\sum_{x\in\eta_s}\int_0^\infty\mathbb{1}\{\ell(x,\eta)\ge\max\{u^{1/(2\alpha p)},d_h(z,x)\}\}\,\mathrm{d}u\Big]\\
&= \gamma^{2\alpha p/d}\mathbb{E}\Big[\sum_{x\in\eta_s}\ell(x,\eta)^{2\alpha p}\mathbb{1}\{\ell(x,\eta)\ge d_h(z,x)\}\Big]
\end{aligned}
$$

$$= \int_{B_h^d(o,s)} \gamma^{2\alpha p/d+1} \mathbb{E}[\ell(x, \eta + \delta_x)^{2\alpha p} \mathbb{1}\{\ell(x, \eta + \delta_x) \geq d_h(z, x)\}] \, \mathcal{H}^d(\mathrm{d}x).$$

Now it follows from Lemma 4.12 b) and Lemma 4.18 that

$$\begin{aligned}\gamma^{2\alpha p/d} \mathbb{E}\Big[\max_{\substack{x \in \eta_s : \\ \ell(x,\eta) \geq d_h(z,x)}} \ell(x, \eta)^{2\alpha p} \Big] &\leq c(\alpha) \int_{B_h^d(o,s)} \gamma e^{-\gamma V_h(d_h(z,x)/2)/2} \, \mathcal{H}^d(\mathrm{d}x) \\ &\leq c(\alpha) \int_{\mathbb{H}^d} \gamma e^{-\gamma V_h(d_h(o,x)/2)/2} \, \mathcal{H}^d(\mathrm{d}x) \leq c(\alpha) \tilde{c}_1,\end{aligned}$$

where $\tilde{c}_1$ does not depend on α, γ and z. Consequently, $\gamma^{2\alpha p/d} M_2(z)$ is uniformly bounded in $\gamma \geq c_0$ and $z \in \mathbb{H}^d$. Finally, the uniform bounds for M_1 and $\gamma^{2\alpha p/d} M_2$ provide the existence of a constant $C_{1,p} > 0$ for which $\gamma^{\alpha p/d} \mathbb{E}[|D_z \mathcal{L}_{s,\gamma}^{(\alpha)}|^p] \leq C_{1,p}$ for all $z \in \mathbb{H}^d$ and $\gamma \geq c_0$.

For the second order difference operator we have

$$\begin{aligned}&\gamma^{\alpha p/d} \mathbb{E}[|D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)}|^p] \\ &\leq 2^{p-1} \gamma^{\alpha p/d} \mathbb{E}[|D_{z_1} \mathcal{L}_{s,\gamma}^{(\alpha)}(\eta_s)|^p] + 2^{p-1} \gamma^{\alpha p/d} \mathbb{E}[|D_{z_1} \mathcal{L}_{s,\gamma}^{(\alpha)}(\eta_s + \delta_{z_2})|^p] \end{aligned} \tag{4.16}$$

for $z_1, z_2 \in B_h^d(o, s)$. The first summand is uniformly bounded since we have already shown that the left-most term in (4.15) is uniformly bounded. For the second expression we can use similar arguments as above. Analogously to (4.14) we have

$$\begin{aligned}|D_{z_1} \mathcal{L}_{s,\gamma}^{(\alpha)}(\eta_s + \delta_{z_2})| &\leq \ell(z_1, \eta + \delta_{z_2} + \delta_{z_1})^\alpha \\ &\quad + \sum_{x \in \eta_s + \delta_{z_2}} \mathbb{1}\{\ell(x, \eta + \delta_{z_2}) \geq d_h(z_1, x)\} \max_{\substack{x \in \eta_s + \delta_{z_2} : \\ \ell(x,\eta+\delta_{z_2}) \geq d_h(z_1,x)}} \ell(x, \eta + \delta_{z_2})^\alpha.\end{aligned}$$

Due to monotonicity it holds

$$\ell(z_1, \eta + \delta_{z_2} + \delta_{z_1})^\alpha \leq \ell(z_1, \eta + \delta_{z_1})^\alpha,$$

$$\sum_{x \in \eta_s + \delta_{z_2}} \mathbb{1}\{\ell(x, \eta + \delta_{z_2}) \geq d_h(z_1, x)\} \leq 1 + \sum_{x \in \eta_s} \mathbb{1}\{\ell(x, \eta) \geq d_h(z_1, x)\}$$

and

$$\max_{\substack{x \in \eta_s + \delta_{z_2} : \\ \ell(x,\eta+\delta_{z_2}) \geq d_h(z_1,x)}} \ell(x, \eta + \delta_{z_2})^\alpha \leq \ell(z_2, \eta + \delta_{z_2})^\alpha + \max_{\substack{x \in \eta_s : \\ \ell(x,\eta) \geq d_h(z_1,x)}} \ell(x, \eta)^\alpha,$$

which provides that the second summand in (4.16) is also uniformly bounded for $z_1, z_2 \in \mathbb{H}^d$ and $\gamma \geq c_0$. □

Lemma 4.20. *For $c_0 \geq 0$ there are constants $c_1, c_2 > 0$, depending on α, d and c_0, such that for all $s > 0$, $z_1, z_2 \in B_h^d(o, s)$ and $\gamma \geq c_0$,*

$$\mathbb{P}(D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)} \neq 0) \leq c_1 e^{-\gamma c_2 V_h(d_h(z_1,z_2)/4)}.$$

Proof. We have

$$D^2_{z_1,z_2}\mathcal{L}^{(\alpha)}_{s,\gamma} = \sum_{x\in\eta_s} D^2_{z_1,z_2}\ell(x,\eta)^\alpha + D_{z_1}\ell(z_2,\eta+\delta_{z_2})^\alpha + D_{z_2}\ell(z_1,\eta+\delta_{z_1})^\alpha.$$

This second order difference operator can only be non-zero if at least one of the three summands above is non-zero. Hence,

$$\begin{aligned}\mathbb{P}(D^2_{z_1,z_2}\mathcal{L}^{(\alpha)}_{s,\gamma}\neq 0) \le \mathbb{P}(\exists x\in\eta_s\colon D^2_{z_1,z_2}\ell(x,\eta)^\alpha \neq 0) + \mathbb{P}(D_{z_1}\ell(z_2,\eta+\delta_{z_2})^\alpha\neq 0)\\ + \mathbb{P}(D_{z_2}\ell(z_1,\eta+\delta_{z_1})^\alpha\neq 0).\end{aligned}$$

Since $D_{z_2}\ell(z_1,\eta+\delta_{z_1})^\alpha \neq 0$ requires $\ell(z_1,\eta+\delta_{z_1})\ge d_h(z_1,z_2)$ and $d_h(z_1,z_2)\le d_h(o,z_1)$, we get

$$\begin{aligned}\mathbb{P}(D_{z_2}\ell(z_1,\eta+\delta_{z_1})^\alpha\neq 0) &\le \mathbb{P}(\ell(z_1,\eta+\delta_{z_1})\ge d_h(z_1,z_2))\mathbb{1}\{d_h(z_1,z_2)\le d_h(o,z_1)\}\\ &= \mathbb{P}(\eta_s(B^d_h(z_1,d_h(z_1,z_2))\cap B^d_h(o,d_h(z_1,o)))=0)\\ &\quad\times \mathbb{1}\{d_h(z_1,z_2)\le d_h(o,z_1)\}\\ &\le e^{-\gamma V_h(d_h(z_1,z_2)/2)},\end{aligned}$$

where we used (4.6) in the last step. Analogously, one can show that

$$\mathbb{P}(D_{z_1}\ell(z_2,\eta+\delta_{z_2})^\alpha\neq 0)\le e^{-\gamma V_h(d_h(z_1,z_2)/2)}.$$

For the first summand we use the Mecke equation (2.15) and the fact that $D^2_{z_1,z_2}\ell(x,\eta)^\alpha\neq 0$ requires $\ell(x,\eta)\ge\max\{d_h(z_1,x),d_h(z_2,x)\}$ and $\max\{d_h(z_1,x),d_h(z_2,x)\}\le d_h(o,x)$ for $x\in B^d_h(0,s)$. Then, we get with Lemma 4.18 and (4.6),

$$\begin{aligned}&\mathbb{P}(\exists x\in\eta_s\colon D^2_{z_1,z_2}\ell(x,\eta)^\alpha\neq 0)\\ &\le \mathbb{E}\Big[\sum_{x\in\eta_s}\mathbb{1}\{D^2_{z_1,z_2}\ell(x,\eta)^\alpha\neq 0\}\Big]\\ &= \int_{B^d_h(o,s)}\gamma\mathbb{P}(D^2_{z_1,z_2}\ell(x,\eta+\delta_x)^\alpha\neq 0)\,\mathcal{H}^d(\mathrm{d}x)\\ &\le \int_{B^d_h(o,s)}\gamma\mathbb{1}\{\max\{d_h(z_1,x),d_h(z_2,x)\}\le d_h(o,x)\}\\ &\qquad\qquad\times\mathbb{P}(\ell(x,\eta+\delta_x)\ge\max\{d_h(z_1,x),d_h(z_2,x)\})\,\mathcal{H}^d(\mathrm{d}x)\\ &\le \int_{B^d_h(o,s)}\gamma e^{-\gamma V_h(\max\{d_h(z_1,x),d_h(z_2,x)\}/2)}\,\mathcal{H}^d(\mathrm{d}x)\\ &\le \int_{\mathbb{H}^d\setminus B^d_h(z_1,\frac{d_h(z_1,z_2)}{2})}\gamma e^{-\gamma V_h(d_h(z_1,x)/2)}\,\mathcal{H}^d(\mathrm{d}x)\\ &\quad+\int_{\mathbb{H}^d\setminus B^d_h(z_2,\frac{d_h(z_1,z_2)}{2})}\gamma e^{-\gamma V_h(d_h(z_2,x)/2)}\,\mathcal{H}^d(\mathrm{d}x)\\ &\le 2\tilde{c}_1 e^{-\tilde{c}_2\gamma V_h(d_h(z_1,z_2)/4)},\end{aligned}$$

which provides the result. □

Proof of Proposition 4.15. The upper bound follows directly from the Poincaré inequality (2.20) and Lemma 4.19, which provide together

$$\begin{aligned}\mathrm{Var}[\mathcal{L}_{s,\gamma}^{(\alpha)}] &\le \gamma \int_{B_h^d(o,s)} \mathbb{E}\Big[\Big(D_z \mathcal{L}_{s,\gamma}^{(\alpha)}\Big)^2\Big]\,\mathcal{H}^d(\mathrm{d}z) \le \gamma \int_{B_h^d(o,s)} \gamma^{-2\alpha/d} C_{1,2}\,\mathcal{H}^d(\mathrm{d}z)\\ &= C_{1,2}\gamma^{1-2\alpha/d} V_h(s).\end{aligned}$$

For the lower bound we use the reverse Poincaré inequality, i.e. we aim to apply Theorem 3.1 for $n = 1$. This means that the proof is complete once we find constants $m, M > 0$ such that

$$\gamma \int_{B_h^d(o,s)} \mathbb{E}\left[\Big(D_z \mathcal{L}_{s,\gamma}^{(\alpha)}\Big)^2\right] \mathcal{H}^d(\mathrm{d}z) \ge m\gamma^{1-2\alpha/d} V_h(s) \tag{4.17}$$

and

$$\gamma^2 \int_{B_h^d(o,s)} \int_{B_h^d(o,s)} \mathbb{E}\left[(D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)})^2\right] \mathcal{H}^d(\mathrm{d}z_1)\,\mathcal{H}^d(\mathrm{d}z_2) \le M\gamma^{1-2\alpha/d} V_h(s). \tag{4.18}$$

For the lower bound in (4.17), we use Lemma 4.17 and obtain

$$\begin{aligned}\gamma \int_{B_h^d(o,s)} \mathbb{E}\left[(D_z \mathcal{L}_{s,\gamma}^{(\alpha)})^2\right]\, \mathcal{H}^d(dz) &\ge \gamma \int_{B_h^d(o,s)\setminus B_h^d(o,3)} c_\gamma^2 \mathbb{P}(D_z \mathcal{L}_{s,\gamma}^{(\alpha)} \ge c_\gamma)\, \mathcal{H}^d(dz)\\ &\ge \gamma c_\gamma^2 C \cdot \big(V_h(s) - V_h(3)\big)\\ &\ge \gamma^{1-2\alpha/d} \frac{c^2 C}{2} \cdot V_h(s)\end{aligned}$$

for sufficiently large $s > 0$, which shows (4.17).

To prove (4.18) we combine the Cauchy-Schwarz inequality with Lemma 4.19 to derive

$$\begin{aligned}\mathbb{E}[|D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)}|^2] &\le \mathbb{E}[|D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)}|^4]^{1/2} \mathbb{P}(D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)} \neq 0)^{1/2}\\ &\le \gamma^{-2\alpha/d} C_{2,4}^{1/2} \mathbb{P}(D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)} \neq 0)^{1/2}.\end{aligned}$$

Together with Lemma 4.20 this implies

$$\begin{aligned}&\gamma^2 \int_{B_h^d(o,s)} \int_{B_h^d(o,s)} \mathbb{E}\left[\Big(D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)}\Big)^2\right]\, \mathcal{H}^d(\mathrm{d}z_1)\, \mathcal{H}^d(\mathrm{d}z_2)\\ &\le \gamma^{-2\alpha/d} C_{2,4}^{1/2} \gamma^2 \int_{B_h^d(o,s)} \int_{B_h^d(o,s)} \mathbb{P}(D^2_{z_1,z_2} \mathcal{L}_{s,\gamma}^{(\alpha)} \neq 0)^{1/2}\, \mathcal{H}^d(\mathrm{d}z_1)\, \mathcal{H}^d(\mathrm{d}z_2)\\ &\le \gamma^{1-2\alpha/d} c_1^{1/2} C_{2,4}^{1/2} \int_{B_h^d(o,s)} \int_{B_h^d(o,s)} \gamma e^{-\gamma c_2 V_h(d_h(z_1,z_2)/4)/2}\, \mathcal{H}^d(\mathrm{d}z_1)\, \mathcal{H}^d(\mathrm{d}z_2)\\ &\le \gamma^{1-2\alpha/d} c_1^{1/2} C_{2,4}^{1/2} \int_{B_h^d(o,s)} \left[\int_{\mathbb{H}^d} \gamma e^{-\gamma c_2 V_h(d_h(z_1,z_2)/4)/2}\, \mathcal{H}^d(\mathrm{d}z_2)\right] \mathcal{H}^d(\mathrm{d}z_1)\\ &\le \gamma^{1-2\alpha/d} c_1^{1/2} C_{2,4}^{1/2} \left[\int_{\mathbb{H}^d} \gamma e^{-\gamma c_2 V_h(d_h(o,w)/4)/2}\, \mathcal{H}^d(\mathrm{d}w)\right] \cdot V_h(s).\end{aligned}$$

By Lemma 4.18 the integral inside the brackets converges and is uniformly bounded for all $\gamma \ge c_0$. This completes the proof of (4.18) and with it, the proof of the proposition. □

4.3.4 Central limit theorems for total edge length functionals

Finally, we turn to the central limit theorem. Note that we mainly use Theorem 2.19 to derive rates in Wasserstein and Kolmogorov distance. Since the bound for the Kolmogorov distance is also a bound for the Wasserstein distance, we only use the result for the Kolmogorov distance to derive an upper bound in both distances. Nevertheless, using the result for the Wasserstein distance in Theorem 2.19 leads to the same rate of convergence.

Theorem 4.21 (Central limit theorem for edge-length functionals)**.** *Let $\alpha > 0$ and $\gamma \geq c_0$ for some $c_0 > 0$. Denote by N a standard Gaussian random variable. Then, there exist constants $C > 0$ and $s_0 > 0$ only depending on d, α and c_0 such that*

$$d_\diamondsuit\left(\frac{\mathcal{L}_{s,\gamma}^{(\alpha)} - \mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}{\sqrt{\operatorname{Var}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}}, N\right) \leq \frac{C}{\sqrt{\gamma V_h(s)}}$$

for $s \geq s_0$ and $\diamondsuit \in \{W, K\}$.

Proof. We use Lemma 4.19, which shows that the moments of the first and second order difference operator of $\gamma^{\alpha/d}\mathcal{L}_{s,\gamma}^{(\alpha)}$ are uniformly bounded by some constant $\bar{c} \geq 1$ for all $\gamma \geq c_0$. Then, Theorem 2.19 implies the inequality

$$\begin{aligned} d_\diamondsuit\left(\frac{\mathcal{L}_{s,\gamma}^{(\alpha)} - \mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}{\sqrt{\operatorname{Var}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}}, N\right) &= d_\diamondsuit\left(\frac{\gamma^{\alpha/d}\mathcal{L}_{s,\gamma}^{(\alpha)} - \mathbb{E}[\gamma^{\alpha/d}\mathcal{L}_{s,\gamma}^{(\alpha)}]}{\sqrt{\operatorname{Var}[\gamma^{\alpha/d}\mathcal{L}_{s,\gamma}^{(\alpha)}]}}, N\right) \\ &\leq \frac{\bar{c} I_1^{1/2}}{\operatorname{Var}[\gamma^{\alpha/d}\mathcal{L}_{s,\gamma}^{(\alpha)}]} + \frac{2\bar{c} I_1}{\operatorname{Var}[\gamma^{\alpha/d}\mathcal{L}_{s,\gamma}^{(\alpha)}]^{3/2}} + \frac{\bar{c} I_1^{5/4} + 2\bar{c} I_1^{3/2}}{\operatorname{Var}[\gamma^{\alpha/d}\mathcal{L}_{s,\gamma}^{(\alpha)}]^2} \\ &\quad + \frac{5\bar{c}}{\operatorname{Var}[\gamma^{\alpha/d}\mathcal{L}_{s,\gamma}^{(\alpha)}]} I_2 + \frac{\sqrt{6}\bar{c} + \sqrt{3}\bar{c}}{\operatorname{Var}[\gamma^{\alpha/d}\mathcal{L}_{s,\gamma}^{(\alpha)}]} I_3 \end{aligned}$$

for $\diamondsuit = W$ and $\diamondsuit = K$, where N is a standard Gaussian random variable and the terms I_1, I_2 and I_3 are defined by

$$\begin{aligned} I_1 &= \gamma \int_{B_h^d(o,s)} \mathbb{P}(D_z \mathcal{L}_{s,\gamma}^{(\alpha)} \neq 0)^{1/10} \, \mathcal{H}^d(\mathrm{d}z), \\ I_2 &= \left(\gamma \int_{B_h^d(o,s)} \left(\gamma \int_{B_h^d(o,s)} \mathbb{P}(D_{x_1,x_2}^2 \mathcal{L}_{s,\gamma}^{(\alpha)} \neq 0)^{1/20} \, \mathcal{H}^d(\mathrm{d}x_2)\right)^2 \mathcal{H}^d(\mathrm{d}x_1)\right)^{1/2} \end{aligned}$$

and

$$I_3 = \left(\gamma \int_{B_h^d(o,s)} \gamma \int_{B_h^d(o,s)} \mathbb{P}(D_{x_1,x_2}^2 \mathcal{L}_{s,\gamma}^{(\alpha)} \neq 0)^{1/10} \, \mathcal{H}^d(\mathrm{d}x_1) \, \mathcal{H}^d(\mathrm{d}x_2)\right)^{1/2}.$$

For I_1 we use the trivial bound $\mathbb{P}(D_z \mathcal{L}_{s,\gamma}^{(\alpha)} \neq 0) \leq 1$ which gives $I_1 \leq \gamma V_h(s)$. To bound I_2 we note that Lemma 4.20 and Lemma 4.18 yield

$$\gamma \int_{B_h^d(o,s)} \mathbb{P}(D_{z_1,z_2}^2 \mathcal{L}_{s,\gamma}^{(\alpha)} \neq 0)^{1/20} \mathrm{d}z_1 \leq c_1 \int_{\mathbb{H}^d} \gamma e^{-\gamma c_2 V_h(d_h(z_1,o)/4)/20} \mathrm{d}z_1 \leq C_2$$

for some constant $C_2 > 0$, which does not depend on γ, and therefore

$$I_2 \leq C_2 \gamma^{1/2} V_h(s)^{1/2}.$$

Similarly, we have $I_3 \leq C_3 \gamma^{1/2} V_h(s)^{1/2}$ for a suitable constant $C_3 > 0$. Together with the lower variance bound provided by Proposition 4.15, which states that $\mathrm{Var}[\gamma^{\alpha/d} \mathcal{L}_{s,\gamma}^{(\alpha)}] \geq c_\ell \gamma V_h(s)$ for s sufficiently large, we obtain for $s \geq s_0$,

$$\begin{aligned} d_\diamondsuit \left(\frac{\mathcal{L}_{s,\gamma}^{(\alpha)} - \mathbb{E}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}{\sqrt{\mathrm{Var}[\mathcal{L}_{s,\gamma}^{(\alpha)}]}}, N \right) &\leq \frac{\bar{c}\gamma^{1/2} V_h(s)^{1/2}}{c_\ell \gamma V_h(s)} + \frac{2\bar{c}\gamma V_h(s)}{c_\ell^{3/2} \gamma^{3/2} V_h(s)^{3/2}} \\ &\quad + \frac{\bar{c}\gamma^{5/4} V_h(s)^{5/4} + 2\bar{c}\gamma^{3/2} V_h(s)^{3/2}}{c_\ell^2 \gamma^2 V_h(s)^2} \\ &\quad + \frac{5\bar{c}}{c_\ell \gamma V_h(s)} C_2 \gamma^{1/2} V_h(s)^{1/2} + \frac{\sqrt{6}\bar{c} + \sqrt{3}\bar{c}}{c_\ell \gamma V_h(s)} C_3 \gamma^{1/2} V_h(s)^{1/2} \\ &\leq \frac{C}{\sqrt{\gamma V_h(s)}} \end{aligned}$$

for some constant $C > 0$, which completes the proof of the theorem. □

Remark 4.22. One can also study the edge-length functionals for $\alpha < 0$. For the Euclidean case quantitative central limit theorems for $\alpha \in (-d/2, 0)$ were derived in [110]. Note that for $\alpha < 0$ short edges become crucial and that the restriction $\alpha > -\frac{d}{2}$ comes from the fact that second moments do otherwise not exist. We believe that similar results can be established for the hyperbolic case as well. Anyway, since this requires different proofs than for $\alpha > 0$, we refrained from considering this situation.

Chapter 5

Random polytopes

In this chapter the vector of two different L^p surface areas of a random polytope is considered and lower variance bounds for linear combinations as well as a result on the two-dimensional normal approximation are derived. If not stated otherwise, this chapter is based on [101, Section 4 and Appendix A.2] by M. Schulte and V. Trapp.

5.1 The model

For $s \geq 1$ let η_s be a homogeneous Poisson process on $B^d(\mathbf{0}, 1)$ with intensity s, i.e. a Poisson process on $\mathbb{R}^d$ with intensity measure $\lambda = s\lambda_d|_{B^d(\mathbf{0},1)}$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$. Let $\mathrm{Conv}(\cdot)$ denote the convex hull. We consider the random polytope Q generated by $\eta_s + \delta_{\mathbf{0}}$, i.e. Q is the convex hull $\mathrm{Conv}(\eta_s + \delta_{\mathbf{0}})$. A simulation of such a random polytope can be seen in Figure 5.1. Note that we add the origin as an extra point to the Poisson process mainly for technical reasons. However, since we are only interested in asymptotic statements for $s \to \infty$, this does not make a difference since $\mathbf{0}$ is with high probability contained in the convex hull of the points of η_s as $s \to \infty$. The following lemma, which is similar to [69, Lemma 3.9], even shows that the probability that a ball of fixed size is not contained in the convex hull of η_s decreases exponentially in s as $s \to \infty$. Throughout this chapter, we fix $\rho_0 \in (0, \frac{1}{4})$ and define $B_{-\rho_0} = B^d(\mathbf{0}, 1) \backslash B^d(\mathbf{0}, 1 - \rho_0)$.

Lemma 5.1. *There are constants $c_0, C_0 > 0$, depending on ρ_0 and d, such that*

$$\mathbb{P}(B^d(\mathbf{0}, 1 - \rho_0) \nsubseteq \mathrm{Conv}(\eta_s)) \leq C_0 e^{-c_0 s}$$

for all $s \geq 1$.

Proof. The proof is similar to the one of [69, Proof of Lemma 3.6]. First note that there exists $m \in \mathbb{N}$ such that one can choose disjoint sets $A_1, \dots, A_m \subseteq B_{-\rho_0}$ with $\lambda_d(A_i) > 0$ for $i \in \{1, \dots, m\}$ and such that for all $\nu \in \mathbf{N}$, with $\nu(A_i) \geq 1$ for $i \in \{1, \dots, m\}$,

$$B^d(\mathbf{0}, 1 - \rho_0) \subseteq \mathrm{Conv}(\nu).$$

This implies

$$\begin{aligned} \mathbb{P}(B^d(\mathbf{0}, 1 - \rho_0) \nsubseteq \mathrm{Conv}(\eta_s)) &\leq \mathbb{P}(\exists i \in \{1, \dots, m\} \colon \eta_s(A_i) = 0) \\ &= 1 - \mathbb{P}(\eta_s(A_i) \geq 1 \text{ for } i \in \{1, \dots, m\}) \end{aligned}$$

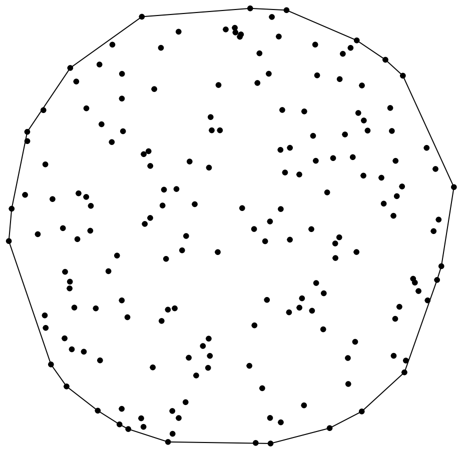

Figure 5.1: Simulation of a random polytope in a circle

$$= 1 - \prod_{i=1}^{m} (1 - \exp[-s\lambda_d(A_i)])$$
$$\leq C_0 \exp[-c_0 s]$$

for suitable constants $c_0, C_0 > 0$, which might depend on ρ_0, by Bernoulli's inequality. □

The study of the convex hull of random points started with the works [93] and [94]. In [90] central limit theorems for the volume and number of k-faces as well as variance bounds were shown. Variance asymptotics and central limit theorems for all intrinsic volumes of the convex hull in a ball were derived in [34]. In [69] the rates of convergence for the central limit theorems were further improved.

In this chapter we are interested in the L^p surface area of a random polytope for $p \in [0, 1]$, which can be interpreted as a generalisation of the volume and the surface area. For $p \in [0, 1]$ the L^p surface area of a random polytope Q is given by

$$A_p = A_p(Q) = \sum_{F \text{ facet of } Q} d(\mathbf{0}, F)^{1-p} \lambda_{d-1}(F) \tag{5.1}$$

(see for instance [54, Section 1]). The L^p surface area measure for a convex body was introduced in [76], where the L^p Minkowski problem was described. The Minkowski problem asks for conditions under which a Borel measure on the sphere is the L^p surface area of a convex body. The discrete L^p Minkowski problem is obtained in the special case where this convex body is a polytope. This situation can, for example, be found in [57] and the references therein. In [54] the expected L^p surface area of random polytopes was considered as a special case of T-functionals of random polytopes.

Note that although A_p can also be computed for polytopes, which do not contain $\mathbf{0}$, the definition in (5.1) is only useful for polytopes, which contain $\mathbf{0}$, which is the reason why we add $\mathbf{0}$ to the collection of random points. Anyway, if we denote by $\widetilde{Q}$ the random polytope that is generated by η_s, i.e. $\widetilde{Q} = \mathrm{Conv}(\eta_s)$, and define $A_p(\widetilde{Q})$ by the right-hand side of (5.1), Lemma 5.1 provides

$$\mathbb{P}(A_p(Q) \neq A_p(\widetilde{Q})) \leq \mathbb{P}(B^d(\mathbf{0}, 1 - \rho_0) \not\subseteq \mathrm{Conv}(\eta_s)) \leq C_0 e^{-c_0 s} \tag{5.2}$$

for $s \geq 1$ with suitable constants $C_0, c_0 > 0$. For square integrable random variables X, Y the triangle inequality implies

$$\begin{aligned}\sqrt{\mathrm{Var}[X]} &= \sqrt{\mathbb{E}[(X-\mathbb{E}[X])^2]} \leq \sqrt{\mathbb{E}[(X-Y-\mathbb{E}[X-Y])^2]} + \sqrt{\mathbb{E}[(Y-\mathbb{E}[Y])^2]} \\ &= \sqrt{\mathrm{Var}[X-Y]} + \sqrt{\mathrm{Var}[Y]}.\end{aligned}$$

Moreover, we have $\mathrm{Var}[X] \leq \mathbb{E}[X^2]$ for any square integrable random variable X. Then, since $d(\mathbf{0}, F)^{1-p} \leq 1$ for all $p \in [0,1]$ and facets $F \subseteq B^d(\mathbf{0}, 1)$, it holds $|A_p(Q) - A_p(\widetilde{Q})| \leq d\kappa_d$, which provides together with the previous estimates the bound

$$\begin{aligned}\left|\sqrt{\mathrm{Var}[A_p(Q)]} - \sqrt{\mathrm{Var}[A_p(\widetilde{Q})]}\right|^2 &\leq \mathrm{Var}[A_p(Q) - A_p(\widetilde{Q})] \leq \mathbb{E}[(A_p(Q) - A_p(\widetilde{Q}))^2] \\ &\leq (d\kappa_d)^2 C_0 e^{-c_0 s} \end{aligned} \tag{5.3}$$

and similarly

$$|\mathbb{E}[A_p(Q)] - \mathbb{E}[A_p(\widetilde{Q})]| \leq d\kappa_d C_0 e^{-c_0 s}. \tag{5.4}$$

Altogether, as already indicated in the beginning, (5.2), (5.3) and (5.4) show that for the asymptotic statements that we are interested in it does not matter, whether we analyse Q or $\widetilde{Q}$.

5.2 Representation via scores

Before we provide the main results, we embed our functionals in the framework of stabilising functionals in this section. To this end we consider the space $\mathbb{X} = B^d(\mathbf{0}, 1)$ with the metric

$$d_{\max}(x, y) = \max\left\{\|x-y\|, \sqrt{|d(x, \partial B^d(\mathbf{0},1)) - d(y, \partial B^d(\mathbf{0},1))|}\right\}$$

for $x, y \in B^d(\mathbf{0}, 1)$. To prove condition (3.2) for $n = 1$, we start with writing the difference of the surface area of the ball $B^d(\mathbf{0}, 1)$ and the L^p surface area of the random polytope $\widetilde{Q}$ as a sum of scores. The notation and the following arguments are mostly analogous to [69, Section 3.4], where similar representations for intrinsic volumes were derived.

We start with using the scores from [69, Section 3.4] for intrinsic volumes to define the scores for the surface area, denoted by $\xi_{1,s}$. To this end, we aim to use the notation introduced in the context of Lemma 2.4. For $\nu \in \mathbf{N}$ let $\mathcal{F}$ denote the set of all facets of the random polytope generated by ν, $V(F)$ the set of all vertices of $F \in \mathcal{F}$ and $\mathrm{Cone}(F) = \{ry \colon y \in F, r > 0\}$. Then, for $x \in \nu$ we define

$$\xi_{1,s}(x, \nu) = \frac{2s\kappa_{d,d-1}}{d} \sum_{F \in \mathcal{F} \colon x \in V(F)} \int_{\mathrm{Cone}(F) \cap (B^d(\mathbf{0},1) \setminus \mathrm{Conv}(\nu))} \|t\|^{-1} \vartheta_{d-1}^{B^d(\mathbf{0},1), \mathrm{Conv}(\nu)}(t) \, \mathrm{d}t,$$

where $\kappa_{d,d-1}$ is defined as in Lemma 2.4 and $\vartheta_{d-1}^{B^d(\mathbf{0},1), \mathrm{Conv}(\nu)}$ is the projection avoidance function defined in Subsection 2.2.1. Then, if $|V(F)| = d$ for all $F \in \mathcal{F}$ and $0 \in \mathrm{int}(\mathrm{Conv}(\nu))$, Lemma 2.4 implies

$$\sum_{x \in \nu} \xi_{1,s}(x, \nu) = 2s\kappa_{d,d-1} \sum_{F \in \mathcal{F}} \int_{\mathrm{Cone}(F) \cap (B^d(\mathbf{0},1) \setminus \mathrm{Conv}(\nu))} \|t\|^{-1} \vartheta_{d-1}^{B^d(\mathbf{0},1), \mathrm{Conv}(\nu)}(t) \, \mathrm{d}t$$

$$
\begin{aligned}
&= 2s\kappa_{d,d-1} \int_{B^d(\mathbf{0},1)\setminus \mathrm{Conv}(\nu)} \|t\|^{-1} \vartheta_{d-1}^{B^d(\mathbf{0},1),\mathrm{Conv}(\nu)}(t) \,\mathrm{d}t \\
&= 2s(V_{d-1}(B^d(\mathbf{0},1)) - V_{d-1}(\mathrm{Conv}(\nu))) \\
&= s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - \lambda_{d-1}(\partial \mathrm{Conv}(\nu))), \qquad (5.5)
\end{aligned}
$$

where we use in the last step that $\lambda_{d-1}(\partial K) = 2V_{d-1}(K)$ for any d-dimensional set $K \in \mathcal{K}^d$.

With the help of this score, we can introduce the scores ξ_s for the L^p surface area. For $x \in \nu$ we define

$$
\begin{aligned}
\xi_s(x,\nu) &= \xi_{1,s}(x,\nu) + \frac{s}{d} \sum_{F\in\mathcal{F}:\, x\in V(F)} (1 - d(\mathbf{0},F)^{1-p})\lambda_{d-1}(F) \\
&=: \xi_{1,s}(x,\nu) + \xi_{2,s}(x,\nu). \qquad (5.6)
\end{aligned}
$$

Note that for $\nu = \eta_s$, $|V(F)| = d$ almost surely for all $F \in \mathcal{F}$. Then, if $0 \in \mathrm{int}(\widetilde{Q})$, we have almost surely with (5.5),

$$
\begin{aligned}
&\sum_{x\in\eta_s} \xi_s(x,\eta_s) = \sum_{x\in\eta_s} \xi_{1,s}(x,\eta_s) + \frac{s}{d} \sum_{x\in\eta_s} \sum_{\substack{F\in\mathcal{F}:\\ x\in V(F)}} (1 - d(\mathbf{0},F)^{1-p})\lambda_{d-1}(F) \\
&= s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - \lambda_{d-1}(\partial\widetilde{Q}) + \lambda_{d-1}(\partial\widetilde{Q}) - A_p(\widetilde{Q})) \\
&= s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_p(\widetilde{Q})).
\end{aligned}
$$

In the following Lemmas 5.3, 5.4 and 5.5 and in the proof of Theorem 5.6 we consider slightly modified scores, which are defined by

$$
\tilde{\xi}_s(x,\eta_s) = \mathbb{1}\{x \in B_{-\rho_0}\} \xi_s(x, \eta_s|_{B_{-\rho_0}} + \delta_{\mathbf{0}})
$$

for $x \in \eta_s$, $s \geq 1$, and

$$
\widetilde{A}_p = \sum_{x\in\eta_s} \tilde{\xi}_s(x,\eta_s).
$$

This means, we restrict ourselves to points in $B_{-\rho_0}$. Note that these scores have asymptotically the same behaviour as the scores ξ_s. Combining Lemma 5.1 with the fact that

$$
|s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_p(\widetilde{Q})) - \widetilde{A}_p| \leq s d \kappa_d,
$$

we derive similarly to the proof of (5.2), (5.3) and (5.4) that there exist constants $\bar{C}_p, \bar{c}_p > 0$ such that

$$
\begin{aligned}
\max\big\{&\mathbb{P}(s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_p(\widetilde{Q})) \neq \widetilde{A}_p), |\mathbb{E}[s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_p(\widetilde{Q}))] - \mathbb{E}[\widetilde{A}_p]|, \\
&|\mathrm{Var}[s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_p(\widetilde{Q}))] - \mathrm{Var}[\widetilde{A}_p]|\big\} \leq \bar{C}_p \exp[-\bar{c}_p s]
\end{aligned}
$$

for $s \geq 1$. Together with (5.2), (5.3) and (5.4) we obtain

$$
\begin{aligned}
\max\big\{&\mathbb{P}(s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_p) \neq \widetilde{A}_p), |\mathbb{E}[s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_p)] - \mathbb{E}[\widetilde{A}_p]|, \\
&|\mathrm{Var}[s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_p)] - \mathrm{Var}[\widetilde{A}_p]|\big\} \leq \hat{C}_p \exp[-\hat{c}_p s] \qquad (5.7)
\end{aligned}
$$

for $s \geq 1$ with constants $\hat{C}_p, \hat{c}_p > 0$.

In the remaining part of this section we show that $\widetilde{A}_p$ are stabilising functionals, i.e. we show that the scores $(\tilde{\xi}_s)_{s\geq 1}$ fulfil (2.23), (2.24) and (2.25). For the first property we need a radius of stabilisation. To this end we use a similar construction as in [69, Section 3.4]. We only slightly modify the definition of the map R from [69, Section 3.4] since the proof from [69, Section 3.4] showing that R is a radius of stabilisation is only valid for $d = 2$. As in [69, Section 3.4] we define $u_x = \frac{x}{\|x\|}$ for $x \in B^d(\mathbf{0}, 1)\backslash\{\mathbf{0}\}$, $u_{\mathbf{0}} = e_1$ and

$$H_x = \{y \in \mathbb{R}^d \colon \langle u_x, y\rangle = \|x\|\}.$$

Moreover, for $r \geq 0$ and $x \in B^d(\mathbf{0}, 1)$ let

$$A_{x,r} = \begin{cases} \operatorname{Conv}((H_x \cap B^d(x,r)) \cup \{x + r^2 u_x\}) & \text{if } r \leq \sqrt{d(x, \partial B^d(\mathbf{0},1))}, \\ B^d(\mathbf{0},1)\backslash \operatorname{Conv}((B^d(\mathbf{0},1)\backslash B^d(x,\sqrt{2}r)) \cup \{x\}) & \text{if } r > \sqrt{d(x, \partial B^d(\mathbf{0},1))}. \end{cases}$$

Denote by $B_{d_{\max}}(x, r)$ the ball of radius r with respect to the $d_{\max}$-distance. The properties of $A_{x,r}$ that we require are summarised in the following lemma. For a proof, the reader is referred to the proof of Lemma 3.7 in [69].

Lemma 5.2. *a) There exists a constant $c_{\max} \geq 1$ satisfying $A_{x,r} \subseteq B_{d_{\max}}(x, c_{\max} r)$ for all $r > 0$ and $x \in B_{-\rho_0}$.*

b) There exist constants $c_1, c_2 > 0$ such that for $r \in [0, 1)$ and $x \in B_{-\rho_0}$,

$$c_1 r^{d+1} \leq \lambda_d(A_{x,r}) \leq c_2 r^{d+1}.$$

We consider in the following the case $r > \sqrt{d(x, \partial B^d(\mathbf{0},1))}$ and want to construct disjoint cuboids $C_{1,r}(x), \ldots, C_{m,r}(x)$ for $x \in B^d(\mathbf{0}, 1)$ and some $m \in \mathbb{N}$ with $C_{i,r}(x) \subset A_{x,r}$ for $i = 1, \ldots, m$, $\lambda_d(C_{i,r}(x)) \geq c r^{d+1}$ for some constant $c > 0$, $i \in \{1, \ldots, m\}$ and $B^d(\mathbf{0}, 1)\backslash A_{x,r} \subset \mathcal{C}(x, y_1, \ldots, y_m)$ for all $y_i \in C_{i,r}(x)$, $i \in \{1, \ldots, m\}$ satisfying $x \notin \operatorname{Conv}(y_1, \ldots, y_m, \mathbf{0})$, where

$$\mathcal{C}(x, y_1, \ldots, y_m) = \Big\{x + \sum_{i=1}^{m} t_i(y_i - x) \colon t_i \geq 0 \text{ for } i = 1, \ldots, m\Big\}.$$

We start with the case $x = e_1$ and consider $r \in (0, 1)$. Let $h_r, \ell_r > 0$ be defined such that $(1 - h_r)e_1 + \ell_r e_2 \in \partial B^d(\mathbf{0}, 1) \cap B^d(x, \sqrt{2}r)$ (see Figure 5.2). Then, the equations

$$h_r^2 + \ell_r^2 = 2r^2 \quad \text{and} \quad (1 - h_r)^2 + \ell_r^2 = 1$$

imply $h_r = r^2$ and $\ell_r = \sqrt{2r^2 - r^4}$. Moreover, let $\bar{h}_r, \bar{\ell}_r > 0$ be defined such that $(1 - \frac{h_r}{2} + \bar{h}_r)e_1 + \frac{\ell_r}{2} e_2 \in \partial B^d(\mathbf{0}, 1)$ and $(1 - \frac{h_r}{2})e_1 + (\frac{\ell_r}{2} + \bar{\ell}_r)e_2 \in \partial B^d(\mathbf{0}, 1)$ (see again Figure 5.2). Then, the equation $(1 - \frac{h_r}{2} + \bar{h}_r)^2 + \frac{\ell_r^2}{4} = 1$ provides with $r^4 < r^2$ due to $r < 1$,

$$\begin{aligned} \bar{h}_r &= \sqrt{1 - \frac{2r^2 - r^4}{4}} - \Big(1 - \frac{r^2}{2}\Big) = \sqrt{\Big(1 - \frac{r^2}{2}\Big)^2 + \frac{r^2}{2}} - \Big(1 - \frac{r^2}{2}\Big) \\ &= \frac{1}{2}\int_0^{r^2/2} \Big(\Big(1 - \frac{r^2}{2}\Big)^2 + z\Big)^{-1/2} \,\mathrm{d}z \geq \frac{r^2}{4}\Big(\Big(1 - \frac{r^2}{2}\Big)^2 + \frac{r^2}{2}\Big)^{-1/2} \end{aligned}$$

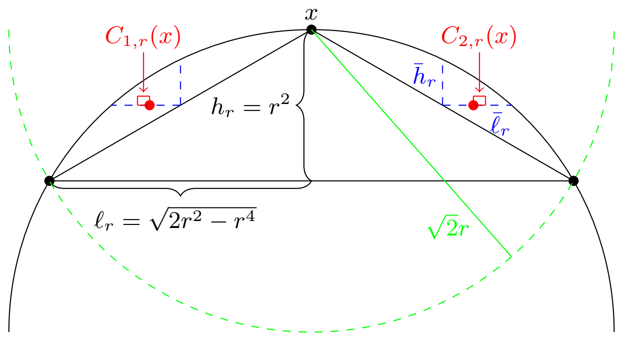

Figure 5.2: Construction for $x = e_1$

$$\geq \frac{r^2}{4}\Big(1 - \frac{r^2}{4}\Big)^{-1/2} \geq \frac{r^2}{4} = \frac{h_r}{4}. \tag{5.8}$$

For $\bar{\ell}_r$, the equation $(1 - \frac{h_r}{2})^2 + (\frac{\ell_r}{2} + \bar{\ell}_r)^2 = 1$ provides together with the estimate $\ell_r = \sqrt{2r^2 - r^4} \leq \sqrt{2}r$,

$$\begin{aligned}\bar{\ell}_r &= \sqrt{r^2 - \frac{r^4}{4}} - \frac{1}{2}\sqrt{2r^2 - r^4} = r\left(\sqrt{1 - \frac{r^2}{4}} - \sqrt{\frac{1}{2} - \frac{r^2}{4}}\right) \\ &= \frac{r}{2}\left(\frac{1}{\sqrt{1 - \frac{r^2}{4}} + \sqrt{\frac{1}{2} - \frac{r^2}{4}}}\right) \geq \frac{r}{2 + \sqrt{2}} \\ &= \frac{1}{2(\sqrt{2}+1)}\sqrt{2}r \geq \frac{1}{2(\sqrt{2}+1)}\ell_r. \end{aligned} \tag{5.9}$$

Let

$$H_r = \Big\{(x_1, \dots, x_d) \in \mathbb{R}^d \colon x_1 = 1 - \frac{h_r}{2}, \|(0, x_2, \dots, x_d)\| \in \Big(\frac{1}{2}\ell_r, \frac{9}{16}\ell_r\Big)\Big\}.$$

Then, by (5.9) and rotational invariance, $\emptyset \neq H_r \subseteq A_{x,r}$.

By (5.8) and (5.9), for fixed $\hat{r} > 0$ we can choose a constant $m \in \mathbb{N}$ and disjoint cuboids $C_{1,\hat{r}}(x), \dots, C_{m,\hat{r}}(x) \subseteq A_{x,\hat{r}}$ of height $\frac{h_{\hat{r}}}{16}$ whose bases are regular cubes in $H_{\hat{r}}$ of side length $\varepsilon\ell_{\hat{r}}$ for some $\varepsilon > 0$. In particular, the variables ε, m and the positions of the cubes can be chosen in such a way that

$$B^d(\mathbf{0}, 1)\backslash A_{x,\hat{r}} \subseteq \mathcal{C}(x, y_1, \dots, y_m) \tag{5.10}$$

for any choices of $y_i \in C_{i,\hat{r}}(x)$ for $i \in \{1, \dots, m\}$. Note that (5.10) is fulfilled if and only if

$$B^d\Big(\Big(1 - \frac{h_{\hat{r}}}{2}\Big)e_1, \frac{\ell_{\hat{r}}}{2}\Big) \cap \Big\{(x_1, \dots, x_d) \in \mathbb{R}^d \colon x_1 = 1 - \frac{h_{\hat{r}}}{2}\Big\} \subseteq \mathrm{Conv}(y_1, \dots, y_m) \tag{5.11}$$

for any choices of $y_i \in C_{i,\hat{r}}(x) \cap H_{\hat{r}}$ for $i \in \{1, \dots, m\}$. In the following we show that the variables ε and m can be chosen independently from r since we can transfer the existing configuration for $\hat{r}$ to each $r \in (0,1)$ such that

$$B^d\Big(\Big(1 - \frac{h_r}{2}\Big)e_1, \frac{\ell_r}{2}\Big) \cap \Big\{(x_1, \dots, x_d) \in \mathbb{R}^d \colon x_1 = 1 - \frac{h_r}{2}\Big\} \subseteq \mathrm{Conv}(\tilde{y}_1, \dots, \tilde{y}_m) \tag{5.12}$$

for any choices of $\tilde{y}_i \in C_{i,r}(x) \cap H_r$ for $i \in \{1, \ldots, m\}$. To this end we consider the transformation $T_r \colon \mathbb{R}^d \to \mathbb{R}^d$ with

$$T_r(\mathbf{z}) = \Big(1 - h_r/2 + \frac{h_r}{h_{\hat{r}}}\Big(z_1 - 1 + \frac{h_{\hat{r}}}{2}\Big), \frac{\ell_r}{\ell_{\hat{r}}} z_2, \ldots, \frac{\ell_r}{\ell_{\hat{r}}} z_d\Big)$$

for $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$. For $i \in \{1, \ldots, m\}$ we define $C_{i,r}(x) = T_r(C_{i,\hat{r}}(x))$. Then, $C_{i,r}(x)$ is a cuboid, whose base is a regular cube in H_r of side length $\varepsilon \ell_r$ and height $\frac{h_r}{16}$. Since $r^4 < r^2$ for $r \in (0,1)$ and $\ell_r^2 \leq 2r^2$ it follows for $x \in C_{i,r}(x)$,

$$\begin{aligned} \|x\|^2 &\leq \frac{9^2}{16^2} \ell_r^2 + \Big(1 - \frac{7}{16} h_r\Big)^2 \leq \frac{2 \cdot 81}{16^2} r^2 + \Big(1 - \frac{7}{16} r^2\Big)^2 \\ &\leq 1 - \Big(\frac{14}{16} - \frac{2 \cdot 81 + 49}{16^2}\Big) r^2 < 1. \end{aligned}$$

Thus, $C_{i,r}(x) \subseteq B^d(\mathbf{0}, 1)$ and hence, $C_{i,r}(x) \subseteq A_{x,r}$. With (5.11) we get

$$\begin{aligned} &d\Big(\partial \mathrm{Conv}(\tilde{y}_1, \ldots, \tilde{y}_m), \Big(1 - \frac{h_r}{2}\Big) e_1\Big) \\ &= \frac{\ell_r}{\ell_{\hat{r}}} d\Big(\partial \mathrm{Conv}(T_r^{-1}(\tilde{y}_1), \ldots, T_r^{-1}(\tilde{y}_m)), \Big(1 - \frac{h_{\hat{r}}}{2}\Big) e_1\Big) \geq \frac{\ell_r}{2} \end{aligned}$$

for any choices of $\tilde{y}_i \in C_{i,r}(x) \cap H_r$ for $i \in \{1, \ldots, m\}$, which shows (5.12). Thus, m and ε can be chosen independently from r and $\lambda_d(C_{i,r}(x)) = \frac{\varepsilon^{d-1}}{16} h_r \ell_r^{d-1} \geq \frac{\varepsilon^{d-1}}{16} r^{d+1}$ for all $r \in (0,1)$.

Since $A_{x,r} \subseteq A_{z,r}$ for any z with $z = a e_1$ for some $a \in [0,1]$, we can define $C_{i,r}(z) = C_{i,r}(x)$ and all desired properties remain valid for $r > \sqrt{d(x, \partial B^d(\mathbf{0}, 1))}$. Moreover, by rotation invariance we can choose the analogous rotated cubes for any $x \in B^d(\mathbf{0}, 1)$. This means that we can find constants ε, m and cubes $C_{1,r}(x), \ldots, C_{m,r}(x)$ for any $x \in B^d(\mathbf{0}, 1)$, which fulfil that $B^d(\mathbf{0}, 1) \backslash A_{x,r} \subset \mathcal{C}(x, y_1, \ldots, y_m)$ for all $y_i \in C_{i,r}(x)$, $i \in \{1, \ldots, m\}$ satisfying $x \notin \mathrm{Conv}(y_1, \ldots, y_m, \mathbf{0})$ and $\lambda_d(C_{i,r}(x)) \geq c r^{d+1}$ for some constant $c > 0$, $i \in \{1, \ldots, m\}$ and $r > \sqrt{d(x, \partial B^d(\mathbf{0}, 1))}$.

The hyperplane H_x can be divided in 2^{d-1} open orthants, which we denote by $G_{x,i}$ for $i \in \{1, \ldots, 2^{d-1}\}$ such that $\cap_{i=1}^{2^{d-1}} \overline{G}_{x,i} = \{x\}$ and define $H_{x,i} = G_{x,i} + \mathrm{Span}(x)$ for $i \in \{1, \ldots, 2^{d-1}\}$. Then we define for $i \in \{1, \ldots, t\}$,

$$D_{i,r}(x) = \begin{cases} A_{x,r} \cap H_{x,i} \cap B_{-\rho_0} & \text{if } r \leq \sqrt{d(x, \partial B^d(\mathbf{0}, 1))}, \\ C_{i,r}(x) \cap B_{-\rho_0} & \text{if } r > \sqrt{d(x, \partial B^d(\mathbf{0}, 1))}, \end{cases}$$

where $t = 2^{d-1}$ for $r \leq \sqrt{d(x, \partial B^d(\mathbf{0}, 1))}$ and $t = m$ if $r > \sqrt{d(x, \partial B^d(\mathbf{0}, 1))}$. Now fix $c_R > 1$ and define the map $R \colon B^d(\mathbf{0}, 1) \times \mathbf{N} \to \mathbb{R}$ by

$$R(x, \nu + \delta_x) = \begin{cases} \min\{2, c_{\max} c_R \inf\{r \in [0,1) \colon \nu(D_{i,r}(x)) \geq 1, i \in \{1, \ldots, t\}\}\} & \text{if } x \in B_{-\rho_0}, \\ 0 & \text{if } x \notin B_{-\rho_0} \end{cases} \tag{5.13}$$

for $x \in B^d(\mathbf{0}, 1)$ and $\nu \in \mathbf{N}$.

Lemma 5.3. *R is a radius of stabilisation for $\tilde{\xi}_s$ and there exist constants $C, c > 0$ such that*

$$\mathbb{P}(R(x, \eta_s + \delta_x) \geq r) \leq C \exp[-csr^{d+1}] \tag{5.14}$$

for $r \geq 0$, $x \in B^d(\mathbf{0}, 1)$ and $s \geq 1$.

The following proof uses similar ideas as the proofs of Lemma 3.10 and Lemma 3.11 in [69].

Proof. To show that R is a radius of stabilisation, it is sufficient to show that $\tilde{\xi}_s(x, \eta_s + \delta_x)$ is completely determined by the points in $B_{d_{\max}}(x, R(x, \eta_s + \delta_x))$ for all $x \in B_{-\rho_0}$.

We consider at first the case where the infimum in the definition of R is equal to some $r^* \in [0, 1)$ and abbreviate $d_x = \sqrt{d(x, \partial B^d(\mathbf{0}, 1))}$. Then, if $r^* = d_x$ and $\eta_s(D_{i,d_x}) \geq 1$ for all $i \in \{1, \dots, t\}$ or $r^* < d_x$, there exists 2^{d-1} points $y_1, \dots, y_{2^{d-1}}$ in η_s with $y_i \in A_{x,r} \cap H_{x,i} \cap B_{-\rho_0}$, $i \in \{1, \dots, 2^{d-1}\}$ for $r = d_x < c_R r^*$ or for $r \in (r^*, \min\{c_R r^*, d_x\}]$, respectively. This provides $x \in \operatorname{Conv}(y_1, \dots, y_m, \mathbf{0})$ and hence, $\tilde{\xi}_s(x, \eta_s + \delta_x) = 0$. Note that this guarantees that $\tilde{\xi}_s(x, \eta_s + \delta_x)$ can be computed by just considering $\eta_s|_{A_{x,c_R r^*}}$. On the other hand if $r^* = d_x$ and $\eta_s(D_{i,d_x}) = 0$ for some $i \in \{1, \dots, t\}$ or $r^* > d_x$, there exists $r \in [r^*, c_R r^*)$ with $r > d_x$ such that $\eta_s(D_{i,r}) \geq 1$ for all $i \in \{1, \dots, t\}$. This means that there exist m points $y_1, \dots, y_m$ in η_s with $y_i \in C_{i,r}(x) \cap B_{-\rho_0}$ for $i \in \{1, \dots, m\}$. By the definition of $C_{i,r}(x)$ it holds that either $x \in \operatorname{Conv}(y_1, \dots, y_m, \mathbf{0})$, which provides $\tilde{\xi}_s(x, \eta_s + \delta_x) = 0$ or $B^d(\mathbf{0}, 1) \backslash A_{x,r} \subseteq \mathcal{C}(x, y_1, \dots, y_m)$. Then, it is guaranteed that all facets of the random polytope that have x as a vertex are contained in $A_{x,r}$. Thus, since $r < c_R r^*$, $\tilde{\xi}_s(x, \eta_s + \delta_x)$ can be determined by just considering $\eta_s|_{A_{x,c_R r^*}}$. Finally, in both cases we know by Lemma 5.2 that $A_{x,c_R r^*} \subseteq B_{d_{\max}}(x, c_{\max} c_R r^*)$ for all $r^* \in (0, 1)$. Together with $B^d(\mathbf{0}, 1) \subseteq B_{d_{\max}}(x, 2)$ for all $x \in B^d(\mathbf{0}, 1)$, this provides that R is a radius of stabilisation.

For the proof of (5.14) it is sufficient to consider only $x \in B_{-\rho_0}$ and $r \in [0, r_0]$ for some $r_0 \in (0, 1)$ small enough such that $A_{x,r/(c_{\max} c_R)} \cap B_{-\rho_0} = A_{x,r/(c_{\max} c_R)}$ for all $r \in [0, r_0]$. Then, by Lemma 5.2, $\lambda_d(A_{x,r/(c_{\max} c_R)} \cap H_{x_i}) \geq \frac{c_1}{2^{d-1}(c_{\max} c_R)^{d+1}} r^{d+1}$. Together with $\lambda_d(C_{i,r/(c_{\max} c_R)}(x)) \geq \frac{\varepsilon^{d-1}}{16(c_{\max} c_R)^{d+1}} r^{d+1}$ for $i \in \{1, \dots, m\}$, this provides $\lambda_d(D_{i,r/(c_{\max} c_R)}(x)) \geq \hat{c} r^{d+1}$ for $i \in \{1, \dots, t\}$ and some constant $\hat{c} > 0$. Thus,

$$\begin{aligned}
\mathbb{P}(R_s(\eta_s + \delta_x) \geq r) &\leq \mathbb{P}(\exists i \in \{1, \dots, t\} \colon \eta_s(D_{i,r/(c_{\max} c_R)}(x)) = 0) \\
&= 1 - \mathbb{P}(\eta_s(D_{i,r/(c_{\max} c_R)}(x)) \geq 1 \text{ for } i \in \{1, \dots, t\}) \\
&= 1 - \prod_{i=1}^{t} (1 - \exp[-s\lambda_d(D_{i,r/(c_{\max} c_R)}(x))]) \\
&\leq 1 - (1 - \exp[-s\hat{c} r^{d+1}])^t \\
&\leq t \exp[-s\hat{c} r^{d+1}],
\end{aligned}$$

where the last step follows from Bernoulli's inequality. This completes the proof. □

The previous lemma shows that the scores $\tilde{\xi}_s$ are exponentially stabilising with $\alpha_{stab} = d + 1$. The following two lemmas prove that they also decay exponentially fast with the distance to $\partial B^d(\mathbf{0}, 1)$ with $\alpha_K = d + 1$ and fulfil a q-th moment condition for $q \geq 1$.

Lemma 5.4. *There are constants $C, c > 0$ such that for $s \geq 1$, $x \in B^d(\mathbf{0}, 1)$ and $A \subseteq B^d(\mathbf{0}, 1)$ with $|A| \leq 9$,*

$$\mathbb{P}\Big(\tilde{\xi}_s\Big(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a\Big) \neq 0\Big) \leq C \exp[-csd_{\max}(x, \partial B^d(\mathbf{0}, 1))^{d+1}].$$

The proof is similar to [69, proof of Lemma 3.12].

Proof. Let $x \in B_{-\rho_0}$. Then, since $c_{\max} c_R \geq 1$ and

$$R\Big(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a\Big) \leq R(x, \eta_s + \delta_x),$$

it holds $\tilde{\xi}_s\Big(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a\Big) = 0$ if

$$R(x, \eta_s + \delta_x) \leq \sqrt{d(x, \partial B^d(\mathbf{0}, 1))} = d_{\max}(x, \partial B^d(\mathbf{0}, 1)).$$

Hence, with Lemma 5.3,

$$\begin{aligned}\mathbb{P}\Big(\tilde{\xi}_s\Big(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a\Big) \neq 0\Big) &\leq \mathbb{P}(R(x, \eta_s + \delta_x) > d_{\max}(x, \partial B^d(\mathbf{0}, 1))) \\ &\leq C \exp[-csd_{\max}(x, \partial B^d(\mathbf{0}, 1))^{d+1}],\end{aligned}$$

which provides the statement. □

Lemma 5.5. *The scores $\tilde{\xi}_s$ fulfil*

$$\sup_{s \geq 1} \sup_{x \in B^d(\mathbf{0}, 1)} \mathbb{E}\Big[\Big|\tilde{\xi}_s\Big(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a\Big)\Big|^q\Big] \leq C_q$$

for $A \subset B^d(\mathbf{0}, 1)$ with $|A| \leq 9$, $q \geq 1$ and some constant $C_q > 0$.

Proof. In order to show a q-th moment condition for $q \geq 1$, we consider the scores $\tilde{\xi}_{1,s}$ and $\tilde{\xi}_{2,s}$, where $\tilde{\xi}_{i,s}(x, \nu) = \mathbb{1}\{x \in B_{-\rho_0}\} \xi_{i,s}(x, \nu|_{B_{-\rho_0}} + \delta_{\mathbf{0}})$ for $x \in B^d(\mathbf{0}, 1)$, $\nu \in \mathbf{N}$ and $\xi_{i,s}$ is defined as in (5.6) for $i \in \{1, 2\}$. Since $\tilde{\xi}_{1,s}$ is twice the score $\tilde{\xi}_{d-1}$ introduced in [69, p. 960] and [69, Lemma 3.13] provides a q'th moment assumption for $\tilde{\xi}_{d-1}(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a)$ we know that $\tilde{\xi}_{1,s}(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a)$ fulfils the required q'th moment condition. (Note that in [69, Lemma 3.13] only sets $A \subset B^d(\mathbf{0}, 1)$ with $|A| \leq 7$ are considered but since the proof does not rely on the specific number of points it also holds for all $A \subset B^d(\mathbf{0}, 1)$ with $|A| \leq 9$ as demanded in our lemma.) This means that it only remains to show the q'th moment condition for $\tilde{\xi}_{2,s}(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a)$. To this end note at first that due to $R(x, \eta_s + \delta_x + \sum_{a \in A} \delta_a) \leq R(x, \eta_s + \delta_x)$,

$$\bigcup_{F \in \mathcal{F}_A \colon x \in V(F)} F \subseteq B^d_{\max}\left(x, R(x, \eta_s + \delta_x)\right) \subseteq B^d\left(x, R(x, \eta_s + \delta_x)\right),$$

where $\mathcal{F}_A$ is the set of vertices of the random polytope generated by $\eta_s + \delta_x + \sum_{a \in A} \delta_a$. Hence, due to monotonicity of the surface area of convex sets we have

$$\sum_{F \in \mathcal{F}_A \colon x \in V(F)} \lambda_{d-1}(F) \leq d\kappa_d R(x, \eta_s + \delta_x)^{d-1}. \tag{5.15}$$

Let now $x \in B_{-\rho_0}$ be a vertex of the random polytope and denote the hyperplane through $\partial B^d\big(x, R(x,\eta_s+\delta_x)\big) \cap \partial B^d(\mathbf{0},1)$ by $\widetilde{H}$. By the definition of R in (5.13), we know that if $R(x,\eta_s+\delta_x) \leq 1$, $[\mathbf{0},x]$ intersects $\widetilde{H}$, where $[\mathbf{0},x]$ denotes the line connecting $\mathbf{0}$ and x. Moreover, the distance of the origin to a facet that contains x is at least as large as the distance from the origin to the hyperplane $\widetilde{H}$. Hence, for a facet F that contains x we have

$$d(\mathbf{0},F) \geq d(\mathbf{0},\widetilde{H}) \geq \sqrt{1-R(x,\eta_s+\delta_x)^2} \geq 1-R(x,\eta_s+\delta_x)^2 \tag{5.16}$$

since the radius of the $(d-1)$-dimensional ball $\widetilde{H} \cap B^d(\mathbf{0},1)$ can be bounded from above by $R(x,\eta_s+\delta_x)$. The bound in (5.16) is obviously also true for $R(x,\eta_s+\delta_x) > 1$.

Since $d(\mathbf{0},F) \leq 1$, it holds that $d(\mathbf{0},F)^{1-p} \geq d(\mathbf{0},F)$ for $p \in [0,1]$ and thus with (5.15) and (5.16) we have for $x \in B_{-\rho_0}$ if $B^d(\mathbf{0},1-\rho_0) \subseteq \mathrm{Conv}(\eta_s)$,

$$\begin{aligned}
\Big|\tilde{\xi}_{2,s}\Big(x,\eta_s+\delta_x+\sum_{a\in A}\delta_a\Big)\Big| &= \Big|\frac{s}{d}\sum_{F\in\mathcal{F}_A\colon x\in V(F)}(1-d(\mathbf{0},F)^{1-p})\lambda_{d-1}(F)\Big| \\
&\leq \frac{s}{d}\sum_{F\in\mathcal{F}_A\colon x\in V(F)}|(1-d(\mathbf{0},F))|\lambda_{d-1}(F) \\
&\leq \frac{s}{d}R(x,\eta_s+\delta_x)^2\sum_{F\in\mathcal{F}_A\colon x\in V(F)}\lambda_{d-1}(F) \\
&\leq \kappa_d sR(x,\eta_s+\delta_x)^{d+1}.
\end{aligned}$$

Moreover, Lemma 5.3 provides $\mathbb{E}[(sR(x,\eta_s+\delta_x)^{d+1})^q] \leq \tilde{C}_q$ for some constant $\tilde{C}_q > 0$. Combining this with $|\tilde{\xi}_{2,s}(x,\eta_s+\delta_x+\sum_{a\in A}\delta_a)| \leq sd\kappa_d$ and Lemma 5.1 implies

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\tilde{\xi}_{2,s}\Big(x,\eta_s+\delta_x+\sum_{a\in A}\delta_a\Big)\Big|^q\Big] \\
&= \mathbb{E}\Big[\Big|\tilde{\xi}_{2,s}\Big(x,\eta_s+\delta_x+\sum_{a\in A}\delta_a\Big)\Big|^q \mathbb{1}\{B^d(\mathbf{0},1-\rho_0) \subseteq \mathrm{Conv}(\eta_s)\}\Big] \\
&\quad + \mathbb{E}\Big[\Big|\tilde{\xi}_{2,s}\Big(x,\eta_s+\delta_x+\sum_{a\in A}\delta_a\Big)\Big|^q \mathbb{1}\{B^d(\mathbf{0},1-\rho_0) \nsubseteq \mathrm{Conv}(\eta_s)\}\Big] \\
&\leq \kappa_d^q\tilde{C}_q + (sd\kappa_d)^q C_0\exp[-c_0 s] \leq 2\kappa_d^q\tilde{C}_q
\end{aligned}$$

for s large enough, which completes the proof. □

5.3 Lower variance bounds for linear combinations of L^p surface areas

In this section, the reverse Poincaré inequality is applied to a vector of two L^p surface areas to study the behaviour of its asymptotic covariance matrix.

Theorem 5.6. *The asymptotic covariance matrix of the vector $s^{(d+3)/(2(d+1))}(A_{p_1},A_{p_2})$ for $p_1,p_2 \in [0,1]$ with $p_1 \neq p_2$ is positive definite, i.e. for any $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2\backslash\{\mathbf{0}\}$ there exists a constant $c > 0$ such that for s sufficiently large*

$$\mathrm{Var}[\alpha_1 A_{p_1} + \alpha_2 A_{p_2}] \geq cs^{-(d+3)/(d+1)}.$$

Lower and upper variance bounds of the same order as in Theorem 5.6 were already derived for the volume in [90]. For binomial input, analogous variance bounds for intrinsic volumes were shown in [11]. The case of an underlying Poisson process and, in particular, variance asymptotics for intrinsic volumes were discussed in [34]. We expect that variance asymptotics for the L^p surface area and especially the positivity of the asymptotic variance can be derived using the same method as in [34]. However, the proof in [34] cannot be directly transferred to linear combinations of two L^p surface areas because for linear combinations with scalars of different signs the monotonicity argument in [34, p. 100] does not work.

Let $S(y^{(1)}, \dots, y^{(m)})$ denote the simplex with vertices $y^{(1)}, \dots, y^{(m)}$ for $m \in \{1, \dots, d+1\}$. For the proof of Theorem 5.6, the representation via scores from the previous section will help showing an upper bound of the form (3.4). To show a lower bound as in (3.5) for $n = 1$, we need to know how the L^p surface area of a polytope changes if we add a simplex on one of its facets. Let this d-dimensional simplex be given by $S(z^{(1)}, \dots, z^{(d+1)})$ for points $z^{(1)}, \dots, z^{(d+1)} \in B^d(\mathbf{0}, 1)$, where $z^{(d+1)}$ denotes the point that is added and $S(z^{(1)}, \dots, z^{(d)})$ is the original facet of the polytope. The facets of the simplex are given by $F_i = S(z^{(1)}, \dots, z^{(i-1)}, z^{(i+1)}, \dots, z^{(d+1)})$ and the distance of a facet to the origin is denoted by $\rho_i = d(F_i, \mathbf{0})$ for $i \in \{1, \dots, d+1\}$. We are interested in

$$\Delta_p = \sum_{i=1}^{d} \rho_i^{1-p} \lambda_{d-1}(F_i) - \rho_{d+1}^{1-p} \lambda_{d-1}(F_{d+1}), \tag{5.17}$$

which is the change of the L^p surface area after adding the simplex.

In the following we also use the notation $\bar{h} = d(z^{(d+1)}, F_{d+1})$ for the height of the added simplex, $T_i = S(z^{(1)}, \dots, z^{(i-1)}, z^{(i+1)}, \dots, z^{(d)})$ for the $(d-2)$-dimensional faces of the base of the simplex and $h_i = d(\bar{z}_{d+1}, T_i)$ for $i \in \{1, \dots, d\}$, where $\bar{z}_{d+1}$ is the projection of $z^{(d+1)}$ to F_{d+1}. The behaviour of Δ_p is described in the following geometric lemma.

Lemma 5.7. *Let $z^{(1)}, \dots, z^{(d+1)} \in B^d(\mathbf{0}, 1)$. For a simplex $S(z^{(1)}, \dots, z^{(d+1)})$, whose vertices are chosen in such a way that $\arg\min_{i=1,\dots,d+1} \rho_i = d+1$ and $\bar{z}_{d+1}$ belongs to the interior of F_{d+1}, we have*

$$\Big| \Delta_p - \frac{1}{d-1} \sum_{i=1}^{d} \lambda_{d-2}(T_i) \Big(\sqrt{h_i^2 + \bar{h}^2} - h_i \Big) \Big| \le \rho_{d+1}^{-p} (1 - \rho_{d+1}) \sum_{i=1}^{d+1} \lambda_{d-1}(F_i)$$

for $p \in [0, 1]$ and

$$\begin{aligned} &\Big| \Delta_{p_1} - \Delta_{p_2} - \sum_{i=1}^{d} (p_2 - p_1)(\rho_i - \rho_{d+1}) \lambda_{d-1}(F_i) \Big| \\ &\le 2 \rho_{d+1}^{-p_2-1} (1 - \rho_{d+1})^2 \sum_{i=1}^{d} \lambda_{d-1}(F_i) + \rho_{d+1}^{-p_2} (1 - \rho_{d+1}) \sum_{i=1}^{d} \lambda_{d-2}(T_i) \Big(\sqrt{h_i^2 + \bar{h}^2} - h_i \Big) \end{aligned}$$

for $p_1, p_2 \in [0, 1]$ with $p_1 < p_2$.

Proof. For $i \in \{1, \dots, d+1\}$ let $F_{d+1}^{(i)} = S(z^{(1)}, \dots, z^{(i-1)}, z^{(i+1)}, \dots, z^{(d)}, \bar{z}_{d+1})$. Then, we have

$$\lambda_{d-1}(F_{d+1}) = \sum_{i=1}^{d} \lambda_{d-1}(F_{d+1}^{(i)}) = \frac{1}{d-1} \sum_{i=1}^{d} \lambda_{d-2}(T_i) h_i$$

and

$$\sum_{i=1}^{d} \lambda_{d-1}(F_i) = \sum_{i=1}^{d} \frac{1}{d-1} \lambda_{d-2}(T_i) d(z^{(d+1)}, T_i) = \frac{1}{d-1} \sum_{i=1}^{d} \lambda_{d-2}(T_i) \sqrt{h_i^2 + \bar{h}^2}.$$

Hence,

$$\sum_{i=1}^{d} \lambda_{d-1}(F_i) - \lambda_{d-1}(F_{d+1}) = \frac{1}{d-1} \sum_{i=1}^{d} \lambda_{d-2}(T_i) \Big(\sqrt{h_i^2 + \bar{h}^2} - h_i \Big). \tag{5.18}$$

Note that, due to the mean value theorem and the assumption $\arg\min_{i=1,\dots,d+1} \rho_i = d+1$, one has for $i \in \{1, \dots, d+1\}$,

$$0 \le 1 - \rho_i^{1-p} \le (1-p)\rho_i^{-p}(1-\rho_i) \le (1-p)\rho_{d+1}^{-p}(1-\rho_{d+1}).$$

Thus,

$$0 \le \sum_{i=1}^{d+1} \lambda_{d-1}(F_i)(1 - \rho_i^{1-p}) \le \sum_{i=1}^{d+1} \lambda_{d-1}(F_i)(1-p)\rho_{d+1}^{-p}(1-\rho_{d+1}).$$

Therefore, it follows with (5.17) and (5.18) that

$$\begin{aligned}
&\Big| \Delta_p - \frac{1}{d-1} \sum_{i=1}^{d} \lambda_{d-2}(T_i) \Big(\sqrt{h_i^2 + \bar{h}^2} - h_i \Big) \Big| \\
&= \Big| \sum_{i=1}^{d} (\rho_i^{1-p} - 1) \lambda_{d-1}(F_i) - (\rho_{d+1}^{1-p} - 1) \lambda_{d-1}(F_{d+1}) \Big| \\
&\le \sum_{i=1}^{d+1} \lambda_{d-1}(F_i)(1 - \rho_i^{1-p}) \le \sum_{i=1}^{d+1} \lambda_{d-1}(F_i)(1-p)\rho_{d+1}^{-p}(1-\rho_{d+1}) \\
&\le \rho_{d+1}^{-p}(1-\rho_{d+1}) \sum_{i=1}^{d+1} \lambda_{d-1}(F_i).
\end{aligned}$$

For the second inequality we have for $p_1 < p_2$,

$$\begin{aligned}
\Delta_{p_1} - \Delta_{p_2} &= \sum_{i=1}^{d} (\rho_i^{1-p_1} - \rho_i^{1-p_2}) \lambda_{d-1}(F_i) - (\rho_{d+1}^{1-p_1} - \rho_{d+1}^{1-p_2}) \lambda_{d-1}(F_{d+1}) \\
&= \sum_{i=1}^{d} (\rho_i^{1-p_1} - \rho_i^{1-p_2} - (\rho_{d+1}^{1-p_1} - \rho_{d+1}^{1-p_2})) \lambda_{d-1}(F_i) \\
&\quad + (\rho_{d+1}^{1-p_1} - \rho_{d+1}^{1-p_2}) \Big(\sum_{i=1}^{d} \lambda_{d-1}(F_i) - \lambda_{d-1}(F_{d+1}) \Big).
\end{aligned}$$

The mean value theorem leads to

$$|\rho_{d+1}^{1-p_1} - \rho_{d+1}^{1-p_2}| = \rho_{d+1}^{1-p_1} |1 - \rho_{d+1}^{p_1-p_2}| \le \rho_{d+1}^{1-p_1} (p_2 - p_1) \rho_{d+1}^{p_1-p_2-1} (1 - \rho_{d+1})$$

$$= (p_2 - p_1)\rho_{d+1}^{-p_2}(1 - \rho_{d+1}).$$

Together with (5.18) it follows that

$$\Big|\Delta_{p_1} - \Delta_{p_2} - \sum_{i=1}^{d}(\rho_i^{1-p_1} - \rho_i^{1-p_2} - (\rho_{d+1}^{1-p_1} - \rho_{d+1}^{1-p_2}))\lambda_{d-1}(F_i)\Big|$$
$$\leq (p_2 - p_1)\rho_{d+1}^{-p_2}(1 - \rho_{d+1})\frac{1}{d-1}\sum_{i=1}^{d}\lambda_{d-2}(T_i)\Big(\sqrt{h_i^2 + \bar{h}^2} - h_i\Big). \tag{5.19}$$

For $u, v \in [0, 1]$ with $u \geq v$ and $\tau \in [0, 1]$, Taylor approximation of the function $g(x) = x^\tau$ centred at $x = 1$ provides

$$|u^\tau - v^\tau - \tau(u - v)| \leq \tau(1-\tau)\Big(\frac{u^{\tau-2}}{2}(1-u)^2 + \frac{v^{\tau-2}}{2}(1-v)^2\Big) \leq \tau(1-\tau)v^{\tau-2}(1-v)^2.$$

Applying this inequality for $\tau = 1 - p_1$ or $\tau = 1 - p_2$, $u = \rho_i$ and $v = \rho_{d+1}$, we derive together with (5.19) and $\rho_{d+1} \leq 1$,

$$\Big|\Delta_{p_1} - \Delta_{p_2} - \sum_{i=1}^{d}(p_2 - p_1)(\rho_i - \rho_{d+1})\lambda_{d-1}(F_i)\Big|$$
$$\leq \sum_{i=1}^{d}((1-p_1)p_1\rho_{d+1}^{-p_1-1} + (1-p_2)p_2\rho_{d+1}^{-p_2-1})(1-\rho_{d+1})^2\lambda_{d-1}(F_i)$$
$$+ (p_2 - p_1)\rho_{d+1}^{-p_2}(1 - \rho_{d+1})\frac{1}{d-1}\sum_{i=1}^{d}\lambda_{d-2}(T_i)\Big(\sqrt{h_i^2 + \bar{h}^2} - h_i\Big)$$
$$\leq 2\rho_{d+1}^{-p_2-1}(1-\rho_{d+1})^2\sum_{i=1}^{d}\lambda_{d-1}(F_i) + \rho_{d+1}^{-p_2}(1-\rho_{d+1})\sum_{i=1}^{d}\lambda_{d-2}(T_i)\Big(\sqrt{h_i^2 + \bar{h}^2} - h_i\Big),$$

which completes the proof. □

In order to derive Theorem 5.6 from Theorem 3.1, we consider the situation that adding an additional point increases the random polytope by exactly one simplex over an existing facet. Lemma 5.7 allows us to control the corresponding change of the L^p surface area. In order to calculate explicit values for the height and volume of such a simplex, we give in the following lemma the corresponding formulas for regular simplices, which can be found for example in [30, Section 6, p. 367]. Since the proof of the following lemma is only indicated in [30, Section 6, p. 367], we present it here for the sake of completeness.

Lemma 5.8. *Let R_d denote a d-dimensional regular simplex with side length L. Then, it holds*

$$h_d(R_d) = \frac{L}{\sqrt{2}}\sqrt{\frac{d+1}{d}} \quad \textit{and} \quad \lambda_d(R_d) = \frac{L^d}{d!}\sqrt{\frac{d+1}{2^d}}, \tag{5.20}$$

where $h_d(R_d)$ denotes the height of R_d, i.e. the distance of each vertex to the opposite $(d-1)$-dimensional regular simplex.

Proof. We show the general formula by induction. For $d = 1$ we have $h_1(R_1) = \lambda_1(R_1) = L$. Now, assuming (5.20) for fixed d, since the centre of R_d has distance $\frac{h_d(R_d)}{d+1}$ to each of the facets of the simplex due to symmetry, we get with Pythagoras' theorem

$$\begin{aligned} h_{d+1}(R_{d+1})^2 &= L^2 - \frac{d^2}{(d+1)^2} h_d(R_d)^2 = L^2 - \frac{d^2}{(d+1)^2} \frac{L^2}{2} \frac{d+1}{d} = \frac{L^2}{2}\Big(2 - \frac{d}{d+1}\Big) \\ &= \frac{L^2(d+2)}{2(d+1)}, \end{aligned}$$

which shows the formula for the height. Using this formula provides for the volume

$$\begin{aligned} \lambda_{d+1}(R_{d+1}) &= \frac{1}{d+1} \lambda_d(R_d) h_{d+1}(R_{d+1}) = \frac{L^d}{(d+1)d!} \sqrt{\frac{d+1}{2^d}} \times \frac{L}{\sqrt{2}} \sqrt{\frac{d+2}{d+1}} \\ &= \frac{L^{d+1}}{(d+1)!} \sqrt{\frac{d+2}{2^{d+1}}}, \end{aligned}$$

which finishes the proof. □

Finally, we can combine all results from this and the previous section to show Theorem 5.6 using the reverse Poincaré inequality. The main challenge of the following proof is to show that the described situation is sufficiently likely. In order to improve the readability of the proof, the details of some arguments are postponed to the following section.

Proof of Theorem 5.6. Let $a > 0$ be fixed. Throughout the proof we choose $s \geq 1$ depending on a large enough such that several conditions hold. Recall that e_i denotes the standard unit vector in the i-th direction and define $x^{(d+1)} = (1 - as^{-2/(d+1)})e_1$. Let $x^{(1)}, \dots, x^{(d)} \in B^d(\mathbf{0}, 1)$ be points in the hyperplane

$$H = \{y = (y_1, \dots, y_d) \in \mathbb{R}^d \colon y_1 = 1 - (a + a^2)s^{-2/(d+1)}\}$$

of pairwise distance $2\ell = 2\sqrt{a}s^{-1/(d+1)}$ that form a regular $(d-1)$-dimensional simplex S such that all points have the same distance to $x^{(d+1)}$. Then, $x^{(1)}, \dots, x^{(d+1)}$ are the vertices of a d-dimensional simplex with height $h = a^2 s^{-2/(d+1)}$. For a set $A \subset B^d(\mathbf{0}, 1)$ and $x \in B^d(\mathbf{0}, 1) \backslash \mathrm{int}(A)$ let

$$\mathrm{Vis}(x, A) = \{y \in B^d(\mathbf{0}, 1) \colon [y, x] \cap \mathrm{int}(A) = \emptyset\}$$

denote the visibility region at x. Recall that $[y, x]$ denotes the line connecting x and y. Let $\varepsilon_h, \varepsilon_\ell \in (0, 1/4)$, which will be chosen sufficiently small such that some properties are satisfied throughout this proof. Now, we choose d cuboids $C_1^x, \dots, C_d^x \subset \mathrm{Vis}(x^{(d+1)}, \mathrm{Conv}(x^{(1)}, \dots, x^{(d+1)}))$ containing $x^{(1)}, \dots, x^{(d)}$ each with height $\varepsilon_h a^2 s^{-2/(d+1)}$ and such that its $(d-1)$-dimensional base is a cube of side length $\varepsilon_\ell \sqrt{a} s^{-1/(d+1)}$ that is contained in the hyperplane H.

Let S_{d-1} denote a regular $(d-1)$-dimensional simplex of side length 2ℓ. Then, $\varepsilon_h, \varepsilon_\ell \in (0, 1/4)$ can be chosen small enough such that $C_1^x, \dots, C_d^x \subset B^d(\mathbf{0}, 1)$ because by Lemma 5.8 we have for $y \in C_i^x$ with $i \in \{1, \dots, d\}$,

$$\|y\|^2 \leq (1 - (a + a^2 - \varepsilon_h a^2)s^{-2/(d+1)})^2 + \Big(\frac{d-1}{d} h_{d-1}(S_{d-1}) + (d-1)\varepsilon_\ell \sqrt{a} s^{-1/(d+1)}\Big)^2$$

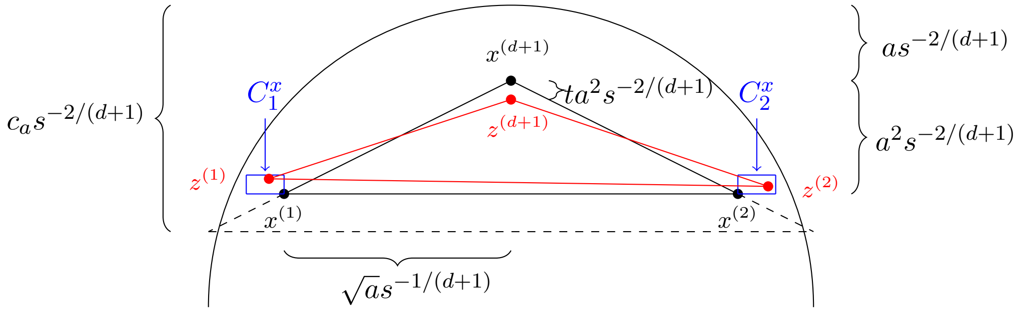

Figure 5.3: Construction in $B^d(\mathbf{0},1)$ for $d=2$ ([101, Figure 2]).

$$
\begin{aligned}
&= (1-(a+a^2-\varepsilon_h a^2)s^{-2/(d+1)})^2 + \left[\sqrt{\frac{2(d-1)a}{d}}s^{-1/(d+1)} + (d-1)\varepsilon_\ell\sqrt{a}s^{-1/(d+1)}\right]^2 \\
&= 1-\left[2\Big(\frac{a}{d}+a^2-\varepsilon_h a^2\Big) - 2(d-1)\sqrt{\frac{2(d-1)}{d}}\varepsilon_\ell a - (d-1)^2\varepsilon_\ell^2 a \right. \\
&\qquad \left. - (a+a^2-\varepsilon_h a^2)^2 s^{-2/(d+1)}\right] s^{-2/(d+1)} < 1
\end{aligned}
$$

for $\varepsilon_h, \varepsilon_\ell \in (0,1/4)$ small enough and s sufficiently large.

In the sequel, we use the same notation as in the context of Lemma 5.7. We consider the simplex $S(z^{(1)},\dots,z^{(d+1)})$, where $z^{(i)} \in C_i^x$ for $i \in \{1,\dots,d\}$ and $z^{(d+1)} = x^{(d+1)} - ta^2 s^{-2/(d+1)} e_1$ for $t \in [0,1/2]$ (see Figure 5.3). Due to the choice of C_i^x we have for s sufficiently large and $t \in [0,1/2]$,

$$
\rho_{d+1} \geq 1-(a+a^2)s^{-2/(d+1)} \tag{5.21}
$$

and

$$
\frac{a^2}{4}s^{-2/(d+1)} \leq \Big(a^2-\varepsilon_h a^2 - \frac{a^2}{2}\Big)s^{-2/(d+1)} \leq \bar{h} \leq a^2 s^{-2/(d+1)}, \tag{5.22}
$$

where we used $\varepsilon_h \in (0,1/4)$.

Moreover, for $i \in \{1,\dots,d\}$ we can control $h_i = d(T_i, \bar{z}_{d+1})$, $\lambda_{d-1}(F_i)$ and $\lambda_{d-2}(T_i)$ with the choice of $\varepsilon_h, \varepsilon_\ell$ uniformly for s sufficiently large. One can show that there exist constants $c_{h,l}, c_{h,u}, c_{T,l}, c_{T,u}, c_{F,l}, c_{F,u} > 0$ such that for s sufficiently large,

$$
c_{h,l}\sqrt{a}s^{-1/(d+1)} \leq h_i \leq c_{h,u}\sqrt{a}s^{-1/(d+1)}, \tag{5.23}
$$

$$
c_{T,l}a^{(d-2)/2}s^{-(d-2)/(d+1)} \leq \lambda_{d-2}(T_i) \leq c_{T,u}a^{(d-2)/2}s^{-(d-2)/(d+1)} \tag{5.24}
$$

and

$$
c_{F,l}a^{(d-1)/2}s^{-(d-1)/(d+1)} \leq \lambda_{d-1}(F_i) \leq c_{F,u}a^{(d-1)/2}s^{-(d-1)/(d+1)}. \tag{5.25}
$$

Here the constants do not depend on a, while the bounds for s such that the inequalities hold may depend on a. The same applies to the inequalities and constants in the sequel if not stated otherwise. For the detailed estimates see Section 5.4.

Due to the fundamental theorem of calculus we have for $x > y > 0$,

$$\sqrt{x^2+y^2} - x = \int_0^{y^2} \frac{1}{2\sqrt{x^2+z}} \,\mathrm{d}z \geq y^2 \frac{1}{2\sqrt{x^2+y^2}} \geq \frac{y^2}{2\sqrt{2}x} \tag{5.26}$$

and

$$\sqrt{x^2+y^2} - x \leq \frac{y^2}{2x}. \tag{5.27}$$

We can assume without loss of generality that $p_1 < p_2$ and that $(\alpha_1, \alpha_2) \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ satisfies $\alpha_1 + \alpha_2 \geq 0$. In the following we distinguish the cases $\alpha_1 \neq -\alpha_2$ and $\alpha_1 = -\alpha_2$. For $\alpha_1 \neq -\alpha_2$, we have by Lemma 5.7, whose assumptions are satisfied by the choice of z_{d+1} and (5.23),

$$\begin{aligned}
&\Big|\alpha_1 \Delta_{p_1} + \alpha_2 \Delta_{p_2} - (\alpha_1+\alpha_2)\frac{1}{d-1}\sum_{i=1}^{d} \lambda_{d-2}(T_i)\Big(\sqrt{h_i^2 + \bar{h}^2} - h_i\Big)\Big| \\
&\leq (|\alpha_1| + |\alpha_2|)\rho_{d+1}^{-p_2}(1-\rho_{d+1})\sum_{i=1}^{d+1}\lambda_{d-1}(F_i).
\end{aligned}$$

Together with (5.21), (5.22), (5.23), (5.24), (5.25) and (5.26) we obtain for $\alpha_1 + \alpha_2 > 0$, $t \in [0, 1/2]$ and s sufficiently large,

$$\begin{aligned}
&\alpha_1 \Delta_{p_1} + \alpha_2 \Delta_{p_2} \\
&\geq \frac{\alpha_1+\alpha_2}{d-1}\sum_{i=1}^{d} c_{T,l}a^{(d-2)/2}s^{-(d-2)/(d+1)} \frac{\frac{a^4}{16}s^{-4/(d+1)}}{2\sqrt{2}c_{h,u}a^{1/2}s^{-1/(d+1)}} \\
&\quad - (|\alpha_1| + |\alpha_2|)2^{p_2}(a+a^2)s^{-2/(d+1)}\sum_{i=1}^{d+1} c_{F,u}a^{(d-1)/2}s^{-(d-1)/(d+1)} \\
&\geq \tilde{c}_d a^{(d+5)/2}s^{-1} - \tilde{c}_{d,p_1,p_2}(a^{(d+3)/2} + a^{(d+1)/2})s^{-1}
\end{aligned}$$

for suitable constants $\tilde{c}_d, \tilde{c}_{d,p_1,p_2} > 0$, where we used that $\rho_{d+1} \geq \frac{1}{2}$ for s sufficiently large. Hence, we can fix $a > 0$ large enough such that this estimate provides for $\alpha_1 \neq -\alpha_2$ the existence of a constant $\tilde{c}_1 > 0$ such that

$$|\alpha_1 \Delta_{p_1} + \alpha_2 \Delta_{p_2}| \geq \tilde{c}_1 a^{(d+5)/2}s^{-1} \tag{5.28}$$

for s sufficiently large and $t \in [0, 1/2]$.

For $\alpha_1 = -\alpha_2$ we fix $a \in (0,1)$. To use the second part of Lemma 5.7 we show in Section 5.4 that

$$\rho_i - \rho_{d+1} \geq c_{\rho,l,a}s^{-2/(d+1)} \tag{5.29}$$

for s sufficiently large with a suitable constant $c_{\rho,l,a} > 0$ that depends on a.

Together with Lemma 5.7 and the inequalities (5.21), (5.22), (5.23), (5.24), (5.25), (5.27), (5.29) this provides for a fixed $a \in (0,1)$, $t \in [0,1/2]$ and s sufficiently large,

$$\Delta_{p_1} - \Delta_{p_2} \geq \sum_{i=1}^{d} (p_2 - p_1)c_{\rho,l,a}s^{-2/(d+1)}c_{F,l}a^{(d-1)/2}s^{-(d-1)/(d+1)}$$

$$
\begin{aligned}
&- 2^{p_2+2}(a+a^2)^2 s^{-4/(d+1)} \sum_{i=1}^{d} c_{F,u} a^{(d-1)/2} s^{-(d-1)/(d+1)} \\
&- 2^{p_2}(a+a^2) s^{-2/(d+1)} \sum_{i=1}^{d} c_{T,u} a^{(d-2)/2} s^{-(d-2)/(d+1)} \frac{a^4 s^{-4/(d+1)}}{2c_{h,l} a^{1/2} s^{-1/(d+1)}} \\
&=: C_{a,1} s^{-1} - C_{a,2} s^{-(d+3)/(d+1)} - C_{a,3} s^{-(d+3)/(d+1)},
\end{aligned}
\tag{5.30}
$$

which can be bounded from below by $\frac{1}{2}C_{a,1}s^{-1}$ for s sufficiently large.

Altogether, for $\alpha_1 \neq -\alpha_2$ we fix $a > 0$ sufficiently large such that (5.28) holds and for $\alpha_1 = -\alpha_2$ we fix $a \in (0,1)$ such that (5.30) holds. Then, for

$$
C_\alpha = \begin{cases} \frac{1}{2} C_{a,1}, & \text{for } \alpha_1 = -\alpha_2, \\ \tilde{c}_1 a^{(d+5)/2}, & \text{else,} \end{cases}
$$

it holds that

$$
|\alpha_1 \Delta_{p_1} + \alpha_2 \Delta_{p_2}| \geq C_\alpha s^{-1} \tag{5.31}
$$

for all $t \in [0, 1/2]$ and s sufficiently large.

For the application of Theorem 3.1 we consider the situation that $z^{(1)}, \dots, z^{(d)}$ are points of the Poisson process and the point $z^{(d+1)}$ is added. To ensure that the change of $\alpha_1 \widetilde{A}_{p_1} + \alpha_2 \widetilde{A}_{p_2}$ is given by $s(\alpha_1 \Delta_{p_1} + \alpha_2 \Delta_{p_2})$, we require that no further points of η_s are present which prevent that $z^{(1)}, \dots, z^{(d)}$ form a facet of the random polytope or which could be connected to $z^{(d+1)}$ by edges. Therefore, we consider the set

$$
M_s^x = \{ y = (y_1, \dots, y_d) \in B^d(\mathbf{0}, 1) \colon y_1 \geq 1 - c_a s^{-2/(d+1)} \} \tag{5.32}
$$

for some constant $c_a > 0$, which might depend on a and can be chosen independently from s such that $(B^d(\mathbf{0},1) \backslash M_s^x) \cap \mathrm{Vis}(x^{(d+1)}, \mathrm{Conv}(z^{(1)}, \dots, z^{(d)}, x^{(d+1)})) = \emptyset$ for all $z^{(1)} \in C_1^x, \dots, z^{(d)} \in C_d^x$ (for more details see Section 5.4). In particular, this implies that $(B^d(\mathbf{0},1) \backslash M_s^x) \cap \mathrm{Vis}(z^{(d+1)}, \mathrm{Conv}(z^{(1)}, \dots, z^{(d+1)})) = \emptyset$ for all $t \in [0, 1/2]$. From now on let s be sufficiently large such that $1 - c_a s^{-2/(d+1)} \geq \rho_0$.

Due to rotation invariance, the same configuration of sets can be constructed for any $x \in B^d(\mathbf{0}, 1)$ with $\|x\| = 1 - (a + ta^2)s^{-2/(d+1)}$ for $t \in [0, 1/2]$ by defining $M_s^x, C_1^x, \dots, C_d^x$ for each x as the suitable rotated regions. Define

$$
A = \{ x \in B^d(\mathbf{0}, 1) \colon \|x\| = 1 - (a + ta^2) s^{-2/(d+1)} \text{ and } t \in [0, 1/2] \}.
$$

Combining our previous considerations leads to

$$
\alpha_1 D_x \widetilde{A}_{p_1} + \alpha_2 D_x \widetilde{A}_{p_2} = s(\alpha_1 \Delta_{p_1} + \alpha_2 \Delta_{p_2})
$$

if

$$
\eta_s(C_i^x) = 1 \quad \text{for} \quad i \in \{1, \dots, d\} \quad \text{and} \quad \eta_s\Big(M_s^x \backslash \bigcup_{i=1}^{d} C_i^x \Big) = 0
$$

for s sufficiently large. Together with (5.31) we obtain for s sufficiently large

$$
\mathbb{E}\left[\int |\alpha_1 D_x \widetilde{A}_{p_1} + \alpha_2 D_x \widetilde{A}_{p_2}|^2 \, \lambda(\mathrm{d}x) \right] \geq \mathbb{E}\left[\int_A |\alpha_1 D_x \widetilde{A}_{p_1} + \alpha_2 D_x \widetilde{A}_{p_2}|^2 \, \lambda(\mathrm{d}x) \right]
$$

$$
\begin{aligned}
&\geq s\int_A \mathbb{P}(|\alpha_1 D_x\widetilde{A}_{p_1}+\alpha_2 D_x\widetilde{A}_{p_2}|\geq C_\alpha)C_\alpha^2\,\mathrm{d}x\\
&\geq C_\alpha^2 s\int_A \mathbb{P}\Big(\eta_s\Big(M_s^x\backslash\bigcup_{i=1}^d C_i^x\Big)=0,\eta_s(C_1^x)=1,\dots,\eta_s(C_d^x)=1\Big)\,\mathrm{d}x\\
&= C_\alpha^2 s\int_A \mathbb{P}\Big(\eta_s\Big(M_s^x\backslash\bigcup_{i=1}^d C_i^x\Big)=0\Big)\prod_{i=1}^d \mathbb{P}(\eta_s(C_i^x)=1)\,\mathrm{d}x. \qquad (5.33)
\end{aligned}
$$

Due to the definition of C_i^x we know that $\lambda_d(C_i^x)=\varepsilon_h a^2(\varepsilon_\ell\sqrt{a})^{d-1}s^{-1}$ for $i\in\{1,\dots,d\}$, i.e. the volume of the sets C_i^x is of order s^{-1}.

For $\lambda_d(M_s^x)$ we consider at first the radius r of the $(d-1)$-dimensional ball $B_C=\{(y_1,\dots,y_d)\in B^d(\mathbf{0},1)\colon y_1=1-c_a s^{-2/(d+1)}\}$. This radius fulfils $r^2+(1-c_a s^{-2/(d+1)})^2=1$. Hence,

$$
r^2=2c_a s^{-2/(d+1)}-c_a^2 s^{-4/(d+1)}\leq 2c_a s^{-2/(d+1)}
$$

and therefore

$$
\lambda_d(M_s^x)\leq\kappa_{d-1}r^{d-1}c_a s^{-2/(d+1)}\leq\tilde{c}_a s^{-1}
$$

for $\tilde{c}_a=\kappa_{d-1}c_a(\sqrt{2c_a})^{d-1}$. Thus, $\lambda_d(M_s^x\backslash\bigcup_{i=1}^d C_i^x)$ is at most of order s^{-1}. Therefore, since the Poisson process has intensity s, the order of the whole term in (5.33) can be bounded from below by a multiple of $s\lambda_d(A)$, where

$$
\lambda_d(A)=\kappa_d\bigg((1-as^{-2/(d+1)})^d-\Big(1-\Big(a+\frac{a^2}{2}\Big)s^{-2/(d+1)}\Big)^d\bigg)\geq\tilde{c}s^{-2/(d+1)}
$$

for a suitable constant $\tilde{c}>0$ and s sufficiently large. Altogether we have

$$
\mathbb{E}\left[\int|\alpha_1 D_x\widetilde{A}_{p_1}+\alpha_2 D_x\widetilde{A}_{p_2}|^2\,\mathrm{d}\lambda(x)\right]\geq\tilde{C}s^{(d-1)/(d+1)}
$$

for some constant $\tilde{C}>0$ and s sufficiently large.

Next, we check condition (3.2). Due to Lemma 5.3, Lemma 5.4 and Lemma 5.5 we can use (2.26) and (2.27). With (2.26), the Hölder inequality and Jensen's inequality one can show that

$$
\begin{aligned}
&\mathbb{E}[|D_{x,y}^2\widetilde{A}_{p_i}|^2]=\mathbb{E}\left[|D_{x,y}^2\widetilde{A}_{p_i}|^2\mathbb{1}\{D_{x,y}^2\widetilde{A}_{p_i}\neq 0\}\right]\\
&\leq(\mathbb{E}[|D_{x,y}^2\widetilde{A}_{p_i}|^5])^{2/5}\mathbb{P}(D_{x,y}^2\widetilde{A}_{p_i}\neq 0)^{3/5}\\
&=(\mathbb{E}[|D_x\widetilde{A}_{p_i}(\eta_s+\delta_y)-D_x\widetilde{A}_{p_i}(\eta_s)|^5])^{2/5}\mathbb{P}(D_{x,y}^2\widetilde{A}_{p_i}\neq 0)^{3/5}\\
&\leq\Big(2^4\Big(\mathbb{E}[|D_x\widetilde{A}_{p_i}(\eta_s+\delta_y)|^5]+\mathbb{E}[|D_x\widetilde{A}_{p_i}(\eta_s)|^5]\Big)\Big)^{2/5}\mathbb{P}(D_{x,y}^2\widetilde{A}_{p_i}\neq 0)^{3/5}\\
&\leq 4C^{2/5}\mathbb{P}(D_{x,y}^2\widetilde{A}_{p_i}\neq 0)^{3/5}
\end{aligned}
$$

for $i\in\{1,2\}$ and the constant $C>0$ from (2.26). Therefore, using Jensen's inequality and (2.27), it follows

$$
\mathbb{E}\left[\int_{B^d(\mathbf{0},1)}\int_{B^d(\mathbf{0},1)}\left(D_{x,y}^2\sum_{i=1}^2\alpha_i\widetilde{A}_{p_i}\right)^2\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y)\right]
$$

$$\leq 2\sum_{i=1}^{2}\alpha_i^2\int_{B^d(\mathbf{0},1)}\int_{B^d(\mathbf{0},1)}\mathbb{E}[|D^2_{x,y}\widetilde{A}_{p_i}|^2]\,\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y)$$
$$\leq 2\sum_{i=1}^{2}\alpha_i^2 s\int_{B^d(\mathbf{0},1)} s\int_{B^d(\mathbf{0},1)} 4C^{2/5}\mathbb{P}(D^2_{x,y}\widetilde{A}_{p_i}\neq 0)^{3/5}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq 8\sum_{i=1}^{2}\alpha_i^2 C^{2/5}s\int_{B^d(\mathbf{0},1)} C_{3/5}\exp[-c_{3/5}s d_{\max}(x,\partial B^d(\mathbf{0},1))^{(d+1)}]\,\mathrm{d}x$$
$$\leq c_\alpha^{(1)}s\int_{B^d(\mathbf{0},1)}\exp[-c_{3/5}s(1-\|x\|)^{(d+1)/2}]\,\mathrm{d}x$$
$$\leq c_\alpha^{(2)}s\int_0^1\exp[-c_{3/5}s(1-r)^{(d+1)/2}]\,\mathrm{d}r = c_\alpha^{(2)}s\int_0^1\exp[-c_{3/5}su^{(d+1)/2}]\,\mathrm{d}u$$
$$\leq c_\alpha^{(3)}s\int_0^{(c_{3/5}s)^{2/(d+1)}} e^{-t^{(d+1)/2}}s^{-2/(d+1)}\mathrm{d}t \leq c_\alpha^{(4)}ss^{-2/(d+1)} = c_\alpha^{(4)}s^{(d-1)/(d+1)}$$

for suitable constants $c_\alpha^{(i)}>0$ for $i\in\{1,2,3,4\}$, the constants $c_{3/5},C_{3/5}>0$ from (2.27) and s sufficiently large. This shows together with an application of Theorem 3.1 for $n=1$ that $\mathrm{Var}[\alpha_1\widetilde{A}_{p_1}+\alpha_2\widetilde{A}_{p_2}]\geq cs^{(d-1)/(d+1)}$ for a suitable constant $c>0$. Now (5.7) yields a lower bound of the same order for $\alpha_1 sA_{p_1}+\alpha_2 sA_{p_2}$, which completes the proof. □

In contrast to the examples of spatial random graphs in Euclidean space, the variance is here not of order s. The reason for this is that in the case of random graphs the difference operator can be bounded away from 0 no matter where exactly the corresponding point is added (apart from some small effects near the boundary). In the case of random polytopes, the difference operator can only be not equal to 0 if the added point is not in the convex hull of the points of the underlying Poisson process. Especially, for increasing s this means that the difference operator can only be distinct from 0 (with sufficiently large probability) if the added point is close enough to the boundary. As one can see in the proof, this leads to an additional factor of $s^{-2/(d+1)}$ which results from the shape of the boundary.

Remark 5.9. A natural extension of Theorem 5.6 is to consider linear combinations of more than two L^p surface areas, i.e. to study $\mathrm{Var}[\sum_{i=1}^m\alpha_i A_{p_i}]$ for $(\alpha_1,\dots,\alpha_m)\in\mathbb{R}^m$, distinct $p_1,\dots,p_m\in[0,1]$ and $m\in\mathbb{N}$ with $m>2$. For $\sum_{i=1}^m\alpha_i\neq 0$ we can use the same strategy as in the proof of Theorem 5.6 and apply the first part of Lemma 5.7, which provides a lower variance bound of the desired order. However, for $\sum_{i=1}^m\alpha_i=0$ it is not clear how to generalise the second part of Lemma 5.7 so that we restricted ourselves to the case of two L^p surface areas.

5.4 Details of the proof of Theorem 5.6

In this section we derive the inequalities (5.23), (5.24), (5.25) and (5.29), which we use in the proof of Theorem 5.6, and show that c_a from (5.32) can be chosen independently from s such that $(B^d(\mathbf{0},1)\backslash M_s^x)\cap\mathrm{Vis}(x^{(d+1)},\mathrm{Conv}(z^{(1)},\dots,z^{(d)},x^{(d+1)}))=\emptyset$ for all $z^{(1)}\in C_1^x,\dots,z^{(d)}\in C_d^x$.

Estimates for h_i in (5.23)

For $i \in \{1, \dots, d\}$ we show in the following that we can control $h_i = d(T_i, \bar{z}_{d+1})$ with the choice of $\varepsilon_h, \varepsilon_\ell$ uniformly for s sufficiently large. Define $\widetilde{F}_{d+1} = S(x^{(1)}, \dots, x^{(d)})$ and let $\bar{z}_{d+1}$ and $\bar{x}_{d+1}$ denote the projections of $z^{(d+1)}$ to F_{d+1} and $\widetilde{F}_{d+1}$, respectively. Moreover, let $\widetilde{T}_i = S(x^{(1)}, \dots, x^{(i-1)}, x^{(i+1)}, \dots, x^{(d)})$. Then, for each $y \in T_i$ there exists $\tilde{y} \in \widetilde{T}_i$ such that $\|y - \tilde{y}\| \le (d-1)\varepsilon_\ell \sqrt{a} s^{-1/(d+1)} + \varepsilon_h a^2 s^{-2/(d+1)}$. Hence, with Lemma 5.8,

$$\begin{aligned}
\sqrt{\frac{2}{d(d-1)}}\sqrt{a}s^{-1/(d+1)} &= d(\bar{x}_{d+1}, \widetilde{T}_i) \\
&\le d(\bar{x}_{d+1}, T_i) + (d-1)\varepsilon_\ell\sqrt{a}s^{-1/(d+1)} + \varepsilon_h a^2 s^{-2/(d+1)} \\
&\le \|\bar{x}_{d+1} - \bar{z}_{d+1}\| + h_i + (d-1)\varepsilon_\ell\sqrt{a}s^{-1/(d+1)} + \varepsilon_h a^2 s^{-2/(d+1)}.
\end{aligned}$$

For the distance of the projections we have

$$\|\bar{x}_{d+1} - \bar{z}_{d+1}\| \le \|\bar{x}_{d+1} - z^{(d+1)}\| + \|z^{(d+1)} - \bar{z}_{d+1}\| \le 2a^2 s^{-2/(d+1)}. \tag{5.34}$$

Hence, we derive for h_i,

$$h_i \ge \sqrt{\frac{2}{d(d-1)}}\sqrt{a}s^{-1/(d+1)} - 2a^2 s^{-2/(d+1)} - (d-1)\varepsilon_\ell\sqrt{a}s^{-1/(d+1)} - \varepsilon_h a^2 s^{-2/(d+1)}.$$

Note that $2a^2 s^{-2/(d+1)} \le \frac{1}{2}\sqrt{\frac{2}{d(d-1)}}\sqrt{a}s^{-1/(d+1)}$ for s sufficiently large. Therefore, we can choose $\varepsilon_\ell, \varepsilon_h > 0$ small enough such that for all $t \in [0, 1/2]$ and s sufficiently large,

$$h_i \ge c_{h,l}\sqrt{a}s^{-1/(d+1)} \tag{5.35}$$

with a constant $c_{h,l} > 0$. Using again (5.34) as well as $\varepsilon_h, \varepsilon_\ell \le 1/4$, we have

$$\begin{aligned}
h_i &\le \|\bar{x}_{d+1} - \bar{z}_{d+1}\| + d(\bar{x}_{d+1}, T_i) \\
&\le \|\bar{x}_{d+1} - \bar{z}_{d+1}\| + d(\bar{x}_{d+1}, \widetilde{T}_i) + (d-1)\varepsilon_\ell\sqrt{a}s^{-1/(d+1)} + \varepsilon_h a^2 s^{-2/(d+1)} \\
&\le \frac{9}{4}a^2 s^{-2/(d+1)} + \left(\sqrt{\frac{2}{d(d-1)}} + \frac{d-1}{4}\right)\sqrt{a}s^{-1/(d+1)} \\
&\le c_{h,u}\sqrt{a}s^{-1/(d+1)}
\end{aligned} \tag{5.36}$$

for a suitable constant $c_{h,u} > 0$, $t \in [0, 1/2]$, and s sufficiently large, which provides (5.23).

Estimates for $\lambda_{d-2}(T_i)$ in (5.24)

By definition, $\widetilde{T}_i$ with $i \in \{1, \dots, d\}$ is a regular $(d-2)$-dimensional simplex of side length $2\ell = 2\sqrt{a}s^{-1/(d+1)}$. We know that the $(d-2)$-dimensional volume of a $(d-2)$-dimensional regular simplex of side length $2\sqrt{a}$ in $\mathbb{R}^d$ is continuous with regard to translations of the vertices. Therefore, we can choose a cube around each vertex small enough such that moving each vertex within the corresponding cube changes the $(d-2)$-dimensional volume of the $(d-2)$-dimensional simplex only slightly. Due to homogeneity we can transfer this result to a regular simplex of side length $2\sqrt{a}s^{-1/(d+1)}$ for all $s \ge 1$, where each side of the cubes is scaled by $s^{-1/(d+1)}$. Hence, we can choose $\varepsilon_h, \varepsilon_\ell \in (0, 1/4)$ small enough such that with (5.20) for s sufficiently large,

$$\begin{aligned}\lambda_{d-2}(T_i) \geq \frac{1}{2}\lambda_{d-2}(\widetilde{T}_i) &= \frac{2^{(d-2)/2}\sqrt{d-1}}{2(d-2)!}(\sqrt{a}s^{-1/(d+1)})^{d-2} \\ &=: c_{T,l}a^{(d-2)/2}s^{-(d-2)/(d+1)}\end{aligned}$$

and

$$\lambda_{d-2}(T_i) \leq c_{T,u}a^{(d-2)/2}s^{-(d-2)/(d+1)} \tag{5.37}$$

for a suitable constant $c_{T,u} > 0$, which finishes the proof of (5.24).

Estimates for $\lambda_{d-1}(F_i)$ in (5.25)

Together with (5.36) and (5.37), it holds

$$\begin{aligned}\lambda_{d-1}(F_{d+1}) &= \frac{1}{d-1}\sum_{i=1}^{d}\lambda_{d-2}(T_i)h_i \\ &\leq \frac{1}{d-1}\sum_{i=1}^{d} c_{T,u}a^{(d-2)/2}s^{-(d-2)/(d+1)}c_{h,u}\sqrt{a}s^{-1/(d+1)}\end{aligned}$$

and with (5.22),

$$\begin{aligned}\lambda_{d-1}(F_i) &= \frac{1}{d-1}\lambda_{d-2}(T_i)\sqrt{h_i^2+\bar{h}^2} \\ &\leq \frac{1}{d-1}c_{T,u}a^{(d-2)/2}s^{-(d-2)/(d+1)}\sqrt{c_{h,u}^2 a s^{-2/(d+1)} + a^4 s^{-4/(d+1)}}\end{aligned}$$

for $i \in \{1, \ldots, d\}$. Hence, we have for $j \in \{1, \ldots, d+1\}$ and s sufficiently large,

$$\lambda_{d-1}(F_j) \leq c_{F,u}a^{(d-1)/2}s^{-(d-1)/(d+1)}$$

for a suitable constant $c_{F,u} > 0$. Analogously, we have for s sufficiently large,

$$\lambda_{d-1}(F_j) \geq c_{F,l}a^{(d-1)/2}s^{-(d-1)/(d+1)}$$

for a suitable constant $c_{F,l} > 0$ and $j \in \{1, \ldots, d+1\}$, which provides (5.25).

Lower bound for $\rho_i - \rho_{d+1}$ in (5.29)

In the following we show the estimate for $\rho_i - \rho_{d+1}$ for $i \in \{1, \ldots, d\}$. Let u_i be the projection of $\mathbf{0}$ to F_i for $i \in \{1, \ldots, d+1\}$ and note that $\bar{x}_{d+1}$, which we introduced in the estimate for h_i as the projection of $z^{(d+1)}$ to $\widetilde{F}_{d+1}$, is also the projection of $\mathbf{0}$ on $\widetilde{F}_{d+1}$. Then, for every $i \in \{1, \ldots, d\}$, there exist a constant $\beta_i \geq 0$ and a vector v_i orthogonal to u_{d+1} such that

$$u_i = (1 + \beta_i h)u_{d+1} + v_i$$

and, thus,

$$\rho_i^2 = \|u_i\|^2 = (1+\beta_i h)^2 \|u_{d+1}\|^2 + \|v_i\|^2 = (1+\beta_i h)^2 \rho_{d+1}^2 + \|v_i\|^2.$$

Let $\bar{u}$ be the projection of u_{d+1} to $\widetilde{F}_{d+1}$, while $\bar{z}_0$ is the intersection point of F_{d+1} with the line through $\mathbf{0}$ and $z^{(d+1)}$ (see Figure 5.4). We show that we can choose $\varepsilon_h > 0$ small enough such that u_{d+1} is very close to $\bar{z}_0$ to ensure a minimum distance from u_{d+1} to T_i. It holds

$$\|\bar{x}_{d+1}\|^2 + \|\bar{x}_{d+1} - \bar{u}\|^2 = \|\bar{u}\|^2 \le \|u_{d+1}\|^2 \le \|\bar{z}_0\|^2 \le (\|\bar{x}_{d+1}\| + \varepsilon_h a^2 s^{-2/(d+1)})^2,$$

which implies

$$\|\bar{x}_{d+1} - \bar{u}\|^2 \le 2\|\bar{x}_{d+1}\| \varepsilon_h a^2 s^{-2/(d+1)} + \varepsilon_h^2 a^4 s^{-4/(d+1)}.$$

This provides

$$\|\bar{z}_0 - u_{d+1}\|^2 \le \|\bar{x}_{d+1} - \bar{u}\|^2 + \varepsilon_h^2 a^4 s^{-4/(d+1)} \le 2\varepsilon_h a^2 s^{-2/(d+1)} + 2\varepsilon_h^2 a^4 s^{-4/(d+1)}.$$

Hence, we can choose $\varepsilon_h \in (0, 1/4)$ small enough such that

$$\|\bar{z}_0 - u_{d+1}\| \le \frac{1}{4}\sqrt{\frac{2}{d(d-1)}} a s^{-1/(d+1)} = \frac{\sqrt{a}}{4}\sqrt{\frac{2}{d(d-1)}}\ell \le \frac{1}{4}\sqrt{\frac{2}{d(d-1)}}\ell \qquad (5.38)$$

since $a \in (0,1)$. For $\varepsilon_\ell > 0$ small enough such that for s sufficiently large,

$$d(\bar{z}_0, T_i) \ge d(\bar{x}_{d+1}, \widetilde{T}_i) - 2\varepsilon_h a^2 s^{-2/(d+1)} - (d-1)\varepsilon_\ell \sqrt{a} s^{-1/(d+1)} \ge \frac{1}{2}\sqrt{\frac{2}{d(d-1)}}\ell,$$

(5.38) implies that $d(u_{d+1}, T_i) \ge \frac{1}{4}\sqrt{\frac{2}{d(d-1)}}\ell$ for $i \in \{1, \dots, d\}$ and s sufficiently large. Then, for $\|v_i\| \le \frac{1}{8}\sqrt{\frac{2}{d(d-1)}}\ell$, $d(u_i, T_i)$ is at least $\frac{1}{8}\sqrt{\frac{2}{d(d-1)}}\ell$ since $d(u_{d+1}, T_i) \le \|v_i\| + d(u_i, T_i)$ (see Figure 5.5). Hence, with the intercept theorem, (5.22) and (5.36) we have

$$\begin{aligned}
\rho_i - \rho_{d+1} \ge \beta_i h \|u_{d+1}\| &= \bar{h}\frac{d(u_i, T_i)}{d(z^{(d+1)}, T_i)} \\
&\ge \frac{1}{8}\sqrt{\frac{2}{d(d-1)}}\ell \cdot \frac{\bar{h}}{\sqrt{\bar{h}^2 + h_i^2}} \\
&\ge \frac{1}{8}\sqrt{\frac{2a}{d(d-1)}} s^{-1/(d+1)} \cdot \frac{\frac{1}{4}a^2 s^{-2/(d+1)}}{\sqrt{a^4 s^{-4/(d+1)} + c_{h,u}^2 a s^{-2/(d+1)}}} \\
&\ge c_{\rho,l} a^2 s^{-2/(d+1)}
\end{aligned}$$

for a suitable constant $c_{\rho,l} > 0$. If $\|v_i\| > \frac{1}{8}\sqrt{\frac{2}{d(d-1)}}\ell$, we have

$$\rho_i^2 - \rho_{d+1}^2 \ge \rho_i^2 - (1+\beta_i h)^2 \rho_{d+1}^2 = \|v_i\|^2 > \frac{1}{64}\frac{2}{d(d-1)}\ell^2.$$

Hence,

$$\rho_i - \rho_{d+1} \ge \frac{2}{64(\rho_i + \rho_{d+1})d(d-1)}\ell^2 \ge \frac{1}{64 d(d-1)}\ell^2 = \frac{a}{64 d(d-1)} s^{-2/(d+1)},$$

which completes the proof of (5.29).

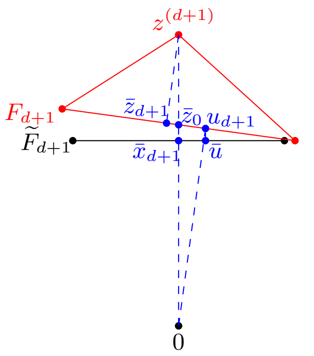

Figure 5.4: Point configuration on F_{d+1} and $\widetilde{F}_{d+1}$ ([101, Figure 3])

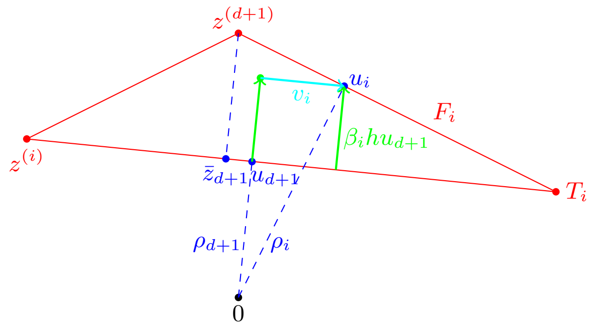

Figure 5.5: Decomposition of the projection of 0 to F_i ([101, Figure 4])

How to choose $c_a > 0$ in (5.32)

In the following we show that the constant $c_a > 0$ from the definition of M_s^x can be chosen independently from s such that $(B^d(\mathbf{0},1)\backslash M_s^x) \cap \mathrm{Vis}(x^{(d+1)}, \mathrm{Conv}(z^{(1)},\dots,z^{(d)},x^{(d+1)})) = \emptyset$ for all $z^{(1)} \in C_1^x, \dots, z^{(d)} \in C_d^x$, i.e. that $c_a > 0$ can be chosen in such a way that any line on the boundary of $\mathrm{Conv}(z^{(1)},\dots,z^{(d)},x^{(d+1)})$ through $x^{(d+1)}$ meets the hyperplane $\{y = (y_1,\dots,y_d) \in \mathbb{R}^d \colon y_1 = 1 - c_a s^{-2/(d+1)}\}$ outside the ball $B^d(\mathbf{0},1)$. With (5.34) and (5.35) it holds that

$$\begin{aligned} d(\bar{x}_{d+1}, T_i) &\geq h_i - \|\bar{x}_{d+1} - \bar{z}_{d+1}\| \geq c_{h,l}\sqrt{a}s^{-1/(d+1)} - 2a^2 s^{-2/(d+1)} \\ &\geq \tilde{c}_{h,l}\sqrt{a}s^{-1/(d+1)} \end{aligned} \tag{5.39}$$

for $\tilde{c}_{h,l} = \frac{c_{h,l}}{2}$, $i \in \{1,\dots,d\}$ and s sufficiently large.

Let $B_C^{d-1} = B^d(\bar{x}_{d+1}, \tilde{c}_{h,l}\sqrt{a}s^{-1/(d+1)}) \cap H$. Then, because of (5.39),

$$\mathrm{Vis}(x^{(d+1)}, \mathrm{Conv}(z^{(1)},\dots,z^{(d)},x^{(d+1)}))$$

is a subset of the visibility region at $x^{(d+1)}$ of the smallest cone K with apex $x^{(d+1)}$ that contains B_C^{d-1}. Hence, if we choose $c_a > 0$ such that $(B^d(\mathbf{0},1)\backslash M_s^x) \cap \mathrm{Vis}(x^{(d+1)}, K) = \emptyset$, then also $(B^d(\mathbf{0},1)\backslash M_s^x) \cap \mathrm{Vis}(x^{(d+1)}, \mathrm{Conv}(z^{(1)},\dots,z^{(d)},x^{(d+1)})) = \emptyset$. Because of symmetry it suffices to ensure that the line through $x^{(d+1)}$ and

$$\hat{y} = (1 - (a+a^2)s^{-2/(d+1)})e_1 + \tilde{c}_{h,l}\sqrt{a}s^{-1/(d+1)}e_2$$

meets H outside of $B^d(\mathbf{0},1)$. A point x_r on the line through $x^{(d+1)}$ and $\hat{y}$ can be described by

$$x_r = (1 - as^{-2/(d+1)})e_1 + r(-a^2 s^{-2/(d+1)}e_1 + \tilde{c}_{h,l}\sqrt{a}s^{-1/(d+1)}e_2) \tag{5.40}$$

for $r \in \mathbb{R}$. To determine a possible constant $c_a > 0$, we need an $r > 1$ such that the point $x_r = (x_{r,1},\dots,x_{r,d})$ fulfils $\|x_r\| > 1$. If $x_{r,1} > 1 - \frac{1}{2}\sum_{i=2}^d x_{r,i}^2 \geq \sqrt{1 - \sum_{i=2}^d x_{r,i}^2}$, it holds that $x_r \notin B^d(\mathbf{0},1)$, i.e. $x_r \notin B^d(\mathbf{0},1)$ if

$$1 - (a + ra^2)s^{-2/(d+1)} > 1 - \frac{r^2}{2}\tilde{c}_{h,l}^2 a s^{-2/(d+1)} \qquad \Longleftrightarrow \qquad \frac{r^2}{2}\tilde{c}_{h,l}^2 - ra - 1 > 0. \tag{5.41}$$

This inequality is fulfilled for $r > 1$ large enough independently of s. Hence, inserting a possible $\hat{r} > 1$, which fulfils (5.41), in (5.40) provides that $c_a > 0$ can be chosen independently from s as $c_a = a + \hat{r}a^2$.

5.5 Central limit theorems for vectors of L^p surface areas

As a consequence of the lower variance bound in Theorem 5.6, one can derive quantitative bounds for the multivariate normal approximation of two L^p surface areas.

Theorem 5.10. *Let (A_{p_1}, A_{p_2}) be the vector of L^p surface areas for $p_1, p_2 \in [0,1]$ with $p_1 \neq p_2$. Denote by $\Sigma(s)$ the covariance matrix of $s^{(d+3)/(2(d+1))}(A_{p_1}, A_{p_2})$. Let $N_{\Sigma(s)}$ be a centred Gaussian random vector with covariance matrix $\Sigma(s)$. Then there exists a constant $c > 0$ such that*

$$d_{\text{convex}}(s^{(d+3)/(2(d+1))}(A_{p_1} - \mathbb{E}[A_{p_1}], A_{p_2} - \mathbb{E}[A_{p_2}]), N_{\Sigma(s)}) \leq cs^{-(d-1)/(2(d+1))}$$

for $s \geq 1$.

Let us compare our results to previous results from the literature. In [50] the multivariate normal approximation of the vector of all intrinsic volumes and all numbers of lower-dimensional faces of the convex hull of Poisson points in a smooth convex body is considered. As in Theorem 5.10, one compares it to a multivariate normal distribution with the same covariance matrix, but as the so-called d_3-distance is studied, no information about the regularity of the asymptotic covariance matrix is required. In the same work positive linear combinations of intrinsic volumes were considered since for coefficients with different signs it could not be ensured that the corresponding asymptotic variance is positive. For the special case of volume and surface area and an underlying ball, this problem is resolved by Theorem 5.6. In contrast to the findings in [50], Theorem 5.10 deals with non-smooth test functions and the obtained bounds are of a better order since a logarithmic factor could be removed. The rates of convergence derived in [69, Section 3] for the univariate normal approximation of intrinsic volumes in Kolmogorov distance are of the order $s^{-(d-1)/(2(d+1))}$ just like the rates in Theorem 5.10.

For the proof of Theorem 5.10 we use the following lemma, which is a direct consequence of [22, Corollary 3.2].

Lemma 5.11. *Let N_I be a standard normally distributed random vector in $\mathbb{R}^d$ and let $K \in \mathcal{K}^d$. Then there exists a constant $c_d > 0$ such that for all $\mu \geq 0$,*

$$\mathbb{P}(N_I \in K \backslash K_{-\mu}) \leq c_d \mu,$$

where $K_{-\mu} = \{x \in K : d(x, \partial K) \geq \mu\}$.

Proof of Theorem 5.10. For $s \geq 1$ we define $\widetilde{Z}_s = (\widetilde{Z}_s^{(1)}, \widetilde{Z}_s^{(2)}) = s^{-(d-1)/(2(d+1))}(\widetilde{A}_{p_1}, \widetilde{A}_{p_2})$. By Lemmas 5.3, 5.4 and 5.5, $\widetilde{A}_{p_i}$ is stabilising for $i \in \{1,2\}$. Hence, we can apply Theorem 2.21 for $\tau = (d-1)/(2(d+1))$, which provides that

$$d_{\text{convex}}(\widetilde{Z}_s - \mathbb{E}[\widetilde{Z}_s], N_{\Sigma(s)}) \leq \tilde{c}s^{-(d-1)/(2(d+1))} \tag{5.42}$$

for $s \geq 1$ with a constant $\tilde{c} > 0$ if we can check that

i) for any constant $c_I > 0$ there exists a constant $\tilde{c}_I > 0$ such that

$$s \int_{B^d(\mathbf{0},1)} \exp[-c_I s d_{\max}(x, \partial B^d(\mathbf{0},1))^{(d+1)}] \, \mathrm{d}x \leq \tilde{c}_I s^{(d-1)/(d+1)}$$

for $s \geq 1$,

ii) $|(\Sigma(s))_{u,v} - \operatorname{Cov}(\widetilde{Z}_s^{(u)}, \widetilde{Z}_s^{(v)})|$ is at most of order $s^{-(d-1)/(2(d+1))}$ for all $u, v \in \{1,2\}$,

iii) $\|\Sigma(s)^{-1}\|_{\mathrm{op}}$ is uniformly bounded for s sufficiently large, where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm.

Analogously to the calculation at the end of the proof of Theorem 5.6 one can show i) since

$$\begin{aligned} s \int_{B^d(\mathbf{0},1)} \exp[-c_I s d_{\max}(x, \partial B^d(\mathbf{0},1))^{(d+1)}] \, \mathrm{d}x &\leq s \int_{B^d(\mathbf{0},1)} \exp[-c_I s (1 - \|x\|))^{(d+1)/2}] \, \mathrm{d}x \\ &\leq c_1 s \int_0^1 \exp[-c_I s (1-r)^{(d+1)/2}] \, \mathrm{d}r \\ &\leq c_2 s \int_0^\infty s^{-2/(d+1)} \exp[-t^{(d+1)/2}] \, \mathrm{d}t \\ &\leq c_3 s^{(d-1)/(d+1)} \end{aligned} \tag{5.43}$$

for suitable constants $c_1, c_2, c_3 > 0$. Next, we want to show ii) for all $u, v \in \{1,2\}$. We start with the case $u = v$. Then, we have with (5.7),

$$\begin{aligned} |\operatorname{Var}[s^{(d+3)/(2(d+1))} A_{p_u}] - \operatorname{Var}[\widetilde{Z}_s^{(u)}]| &= s^{-(d-1)/(d+1)} |\operatorname{Var}[s A_{p_u}] - \operatorname{Var}[\widetilde{A}_{p_u}]| \\ &\leq s^{-(d-1)/(d+1)} \hat{C}_p \exp[-\hat{c}_p s] \end{aligned}$$

with the constants $\hat{C}_p, \hat{c}_p > 0$ from (5.7), which shows ii) for $u = v$. For $u \neq v$ we use that

$$\begin{aligned} &2|\operatorname{Cov}(s A_{p_u}, s A_{p_v}) - \operatorname{Cov}(\widetilde{A}_{p_u}, \widetilde{A}_{p_v})| \\ &= \big|\operatorname{Var}[s A_{p_u} + s A_{p_v}] - \operatorname{Var}[\widetilde{A}_{p_u} + \widetilde{A}_{p_v}] - \operatorname{Var}[s A_{p_u}] + \operatorname{Var}[\widetilde{A}_{p_u}] \\ &\quad - \operatorname{Var}[s A_{p_v}] + \operatorname{Var}[\widetilde{A}_{p_v}]\big| \\ &\leq |\operatorname{Var}[s A_{p_u} + s A_{p_v}] - \operatorname{Var}[\widetilde{A}_{p_u} + \widetilde{A}_{p_v}]| + |\operatorname{Var}[s A_{p_u}] - \operatorname{Var}[\widetilde{A}_{p_u}]| \\ &\quad + |\operatorname{Var}[s A_{p_v}] - \operatorname{Var}[\widetilde{A}_{p_v}]|. \end{aligned}$$

The second and third term can be bounded by (5.7) as before, while for the first summand we can apply an analogous version of (5.7) for the distance of the variances of these sums, which can be derived in the same way as (5.7). This provides the existence of constants $C, c > 0$ such that

$$\begin{aligned} &\big|\operatorname{Cov}(s^{(d+3)/(2(d+1))} A_{p_u}, s^{(d+3)/(2(d+1))} A_{p_v}) - \operatorname{Cov}(\widetilde{Z}_s^{(u)}, \widetilde{Z}_s^{(v)})\big| \\ &= s^{-(d-1)/(d+1)} |\operatorname{Cov}(s A_{p_u}, s A_{p_v}) - \operatorname{Cov}(\widetilde{A}_{p_u}, \widetilde{A}_{p_v})| \\ &\leq s^{-(d-1)/(d+1)} C \exp[-cs], \end{aligned}$$

which implies ii).

It remains to establish iii). To this end we assume that there is a subsequence $(s_n)_{n\in\mathbb{N}}$ such that $\|\Sigma(s_n)^{-1}\|_{\text{op}} \to \infty$ and $s_n \to \infty$ as $n \to \infty$. From the Poincaré inequality (see (2.20)), Lemma 2.20 a) and b) and (5.43) one deduces that for $\varepsilon \in (4, 4+p)$ and $q = (1-\frac{2}{\varepsilon})^{-1}$,

$$\begin{aligned}
\operatorname{Var}[\widetilde{Z}_s^{(u)}] &= s^{-(d-1)/(d+1)}\operatorname{Var}[\widetilde{A}_{p_u}] \le s^{-(d-1)/(d+1)}\mathbb{E}\Big[\int_{B^d(\mathbf{0},1)} (D_x\widetilde{A}_{p_u})^2\,\lambda(\mathrm{d}x)\Big] \\
&= s^{-(d-1)/(d+1)} s\mathbb{E}\Big[\int_{B^d(\mathbf{0},1)} \mathbb{1}\{D_x\widetilde{A}_{p_u} \neq 0\}(D_x\widetilde{A}_{p_u})^2\,\mathrm{d}x\Big] \\
&\le s^{-(d-1)/(d+1)} s\int_{B^d(\mathbf{0},1)} \mathbb{P}(D_x\widetilde{A}_{p_u} \neq 0)^{1/q}\mathbb{E}[(D_x\widetilde{A}_{p_u})^{\varepsilon}]^{2/\varepsilon}\,\mathrm{d}x \\
&\le s^{-(d-1)/(d+1)} C^{2/\varepsilon}\tilde{C}_{1/q} s\int_{B^d(\mathbf{0},1)} \exp[-\tilde{c}_{1/q} s d_{\max}(x, \partial B^d(\mathbf{0},1))^{(d+1)}]\,\mathrm{d}x \\
&\le c
\end{aligned}$$

for a suitable constant $c > 0$. This means that all variances of the components of $\widetilde{Z}_s$ are uniformly bounded for $s \ge 1$. As before, one can show the same for all covariances as well. By ii) the same holds for the entries of $\Sigma(s)$. Thus, there exists a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ and a matrix $\Sigma \in \mathbb{R}^{2\times 2}$ such that $\Sigma(s_{n_k}) \to \Sigma$ as $k \to \infty$. From Theorem 5.6 it follows that Σ is positive definite as $\alpha^T\Sigma\alpha = \lim_{k\to\infty} \alpha^T\Sigma(s_{n_k})\alpha > 0$ for any $\alpha \in \mathbb{R}^2\backslash\{\mathbf{0}\}$. Thus, $\|\Sigma^{-1}\|_{\text{op}}$ is well-defined and $\|\Sigma(s_{n_k})^{-1}\|_{\text{op}} \to \|\Sigma^{-1}\|_{\text{op}}$ as $k \to \infty$. Since this is a contradiction to the assumption, we have shown that $\|\Sigma(s)^{-1}\|_{\text{op}}$ is uniformly bounded for s sufficiently large, which is iii) and completes the proof of (5.42).

Now let

$$Z_s = s^{(d+3)/(2(d+1))}(A_{p_1}, A_{p_2}) = s^{-(d-1)/(2(d+1))}(sA_{p_1}, sA_{p_2})$$

and

$$\widehat{Z}_s = s^{-(d-1)/(2(d+1))}(s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_{p_1}), s(\lambda_{d-1}(\partial B^d(\mathbf{0},1)) - A_{p_2})).$$

Then, $Z_s - \mathbb{E}[Z_s]$ and $-(\widehat{Z}_s - \mathbb{E}[\widehat{Z}_s])$ have the same distribution. Together with the symmetry of the normal distribution and the triangle inequality it holds that

$$\begin{aligned}
&d_{\text{convex}}(Z_s - \mathbb{E}[Z_s], N_{\Sigma(s)}) = d_{\text{convex}}(-(\widehat{Z}_s - \mathbb{E}[\widehat{Z}_s]), N_{\Sigma(s)}) = d_{\text{convex}}(\widehat{Z}_s - \mathbb{E}[\widehat{Z}_s], N_{\Sigma(s)}) \\
&\le d_{\text{convex}}(\widehat{Z}_s - \mathbb{E}[\widehat{Z}_s], \widetilde{Z}_s - \mathbb{E}[\widehat{Z}_s]) + d_{\text{convex}}(\widetilde{Z}_s - \mathbb{E}[\widehat{Z}_s], N_{\Sigma(s)}) \\
&\le \mathbb{P}(\widehat{Z}_s \neq \widetilde{Z}_s) + d_{\text{convex}}(\widetilde{Z}_s - \mathbb{E}[\widetilde{Z}_s], N_{\Sigma(s)} + \mathbb{E}[\widehat{Z}_s] - \mathbb{E}[\widetilde{Z}_s]) \\
&\le \mathbb{P}(\widehat{Z}_s \neq \widetilde{Z}_s) + d_{\text{convex}}(\widetilde{Z}_s - \mathbb{E}[\widetilde{Z}_s], N_{\Sigma(s)}) + d_{\text{convex}}(N_{\Sigma(s)}, N_{\Sigma(s)} + \mathbb{E}[\widehat{Z}_s] - \mathbb{E}[\widetilde{Z}_s]).
\end{aligned}$$

Since the first term on the right-hand side vanishes exponentially fast by (5.7) and the second one was treated in (5.42), it remains to study the third term. We have that

$$\begin{aligned}
&d_{\text{convex}}(N_{\Sigma(s)}, N_{\Sigma(s)} + \mathbb{E}[\widehat{Z}_s] - \mathbb{E}[\widetilde{Z}_s]) \\
&= d_{\text{convex}}(N_I, N_I + \Sigma(s)^{-1/2}(\mathbb{E}[\widehat{Z}_s] - \mathbb{E}[\widetilde{Z}_s]))
\end{aligned}$$

$$\leq \sup_{K \subseteq \mathbb{R}^2 \text{ convex}} \mathbb{P}(N_I \in K \setminus K_{-\mu}), \tag{5.44}$$

where N_I is distributed according to a two-dimensional standard normal distribution and

$$K_{-\mu} = \{x \in K : d(x, \partial K) \geq \mu\} \text{ for } \mu = \|\Sigma(s)^{-1/2}(\mathbb{E}[\widehat{Z}_s] - \mathbb{E}[\widetilde{Z}_s])\|.$$

From Lemma 5.11 one obtains that (5.44) is bounded by a constant times

$$\mu \leq \|\Sigma(s)^{-1}\|_{\text{op}}^{1/2} \|\mathbb{E}[\widehat{Z}_s] - \mathbb{E}[\widetilde{Z}_s]\|.$$

Now iii) from above and (5.7) imply that this expression vanishes exponentially fast for $s \to \infty$, which completes the proof. □

Remark 5.12. Theorem 5.6 and Theorem 5.10 especially provide a lower variance bound and a result on the multivariate normal approximation for the vector of surface area and volume of a random polytope since $A_0 = dV_d$ and $A_1 = S_{d-1}$, where V_d and S_{d-1} denote the volume and surface area, respectively.

Remark 5.13. The results of this chapter prevail if we assume that the Poisson processes have underlying intensity measures $s\mu$ for $s \geq 0$, where μ is a measure with a density $g\colon B^d(\mathbf{0}, 1) \to [0, \infty)$ satisfying $\underline{c} \leq g(x) \leq \overline{c}$ for all $x \in B^d(\mathbf{0}, 1)$ and some constants $\underline{c}, \overline{c} > 0$ (see also Remark 4.7). Moreover, we expect that it is possible to replace the d-dimensional unit ball by a compact convex non-empty subset of $\mathbb{R}^d$ with C^2-boundary and positive Gaussian curvature. Since the boundaries of these sets are locally between two paraboloids as the boundary of the unit ball, we believe that similar arguments as in [69, Section 3.4] allow to prove our results for this larger class of underlying bodies.

However, note that for our arguments a smooth boundary is necessary. Considering random polytopes in other convex sets like polytopes is a different problem since random polytopes show different behaviours close to corners of the underlying set. Results for functionals like the volume or the number of ℓ-dimensional faces of random polytopes in polytopes for $\ell \in \{0, \dots, d-1\}$ can be found for example in [12, 13, 33, 35, 49, 52].

Chapter 6

Excursion sets of Poisson shot noise processes

In this chapter we consider geometric functionals of excursion sets of Poisson shot noise processes and aim to analyse their asymptotic behaviours for growing observation windows. We start with analysing the asymptotic variance of the volume of excursion sets for a non-marked underlying Poisson process in Section 6.2. In Section 6.3 we consider a marked underlying Poisson process and analyse more general geometric functionals of a more restricted set of kernel functions, for which we can guarantee that the corresponding excursion sets are almost surely polyconvex. For these geometric functionals we study the asymptotic expectation, the behaviour of the asymptotic variance and derive qualitative and quantitative central limit theorems. This chapter is mainly based on [101, Section 5] by M. Schulte and V. Trapp and [109] by V. Trapp. Sections 6.1 and 6.3 are mainly taken from [109], while Section 6.2 is based on [101, Section 5].

6.1 The model

Let $(\mathbb{M}, \mathcal{F}_{\mathbb{M}}, \mathbb{Q})$ be a mark space with σ-field $\mathcal{F}_{\mathbb{M}}$ and probability measure $\mathbb{Q}$. For $d \in \mathbb{N}$ we consider a marked Poisson process η on $\hat{\mathbb{R}}^d = \mathbb{R}^d \times \mathbb{M}$ with intensity measure $\lambda = \gamma\lambda_d \otimes \mathbb{Q}$ for some $\gamma > 0$. For a family of measurable kernel functions $(g_m)_{m\in\mathbb{M}}$ with $g_m \colon \mathbb{R}^d \to \mathbb{R}$, a *Poisson shot noise process* $(f_\eta(y))_{y\in\mathbb{R}^d}$ is defined by

$$f_\eta(y) = \sum_{(x,m)\in\eta} g_m(y-x) \tag{6.1}$$

for $y \in \mathbb{R}^d$. To simplify the model, one can also consider Poisson shot noise processes with underlying non-marked Poisson process and fixed kernel function g. In this case f_η is analogously defined as in (6.1) by replacing g_m with g and summing over all points of a Poisson process in $\mathbb{R}^d$ with intensity measure $\gamma\lambda_d$. (Note that this corresponds to the model with $\mathbb{M} = \{1\}$, $\mathbb{Q}(\{1\}) = 1$ and $g_1 = g$.) In Section 6.2 we will consider this simplified version of the Poisson shot noise process, while Section 6.3 deals with the more general case of Poisson shot noise processes with an underlying marked Poisson process.

For a fixed $u > 0$, the *excursion set* of f_η is given by

$$Z_u = \{y \in \mathbb{R}^d \colon f_\eta(y) \geq u\}.$$

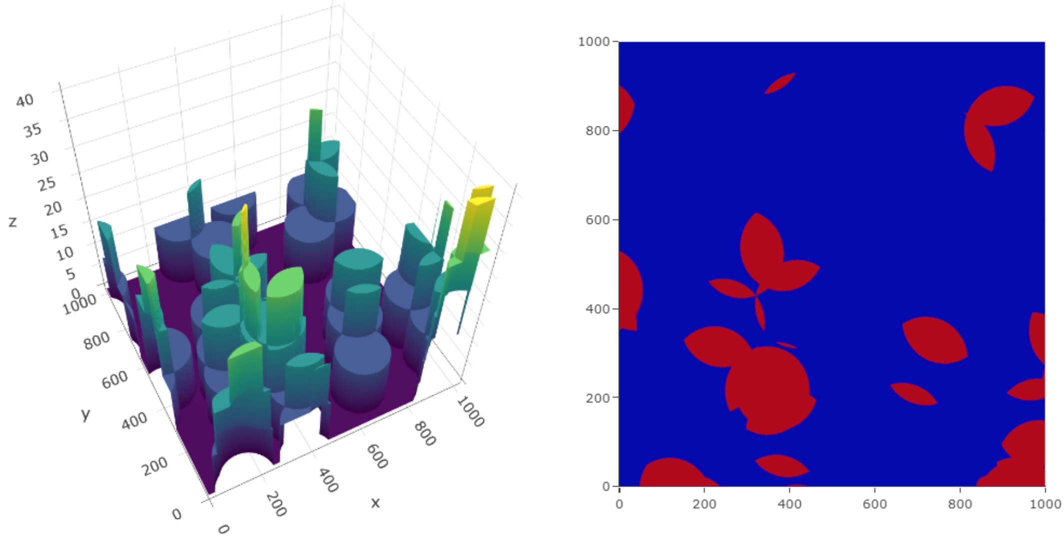

Figure 6.1: Simulation of a Poisson shot noise process (left) and its excursion set (right) for $u = 15$, $d = 2$ and $g_m(x) = \frac{u}{2}$ for $x \in K_m$ and all $m \in \mathbb{M}$.

A simulation of a Poisson shot noise process and the corresponding excursion set can be seen in Figure 6.1.

Random fields and their excursion sets are an important topic of probability theory with a wide range of applications. They are used in various fields for modelling random signals, with excursion sets corresponding to the regions where these signals interfere and exceed some given threshold. In 1944, for example, random fields were used in [95] to model random noise in electronic devices and analyse shot effects, which occur if a lot of disturbances overlap. Beyond this, random fields have a variety of applications in different areas such as wireless communication networks (see e.g. [7, 8]), insurance mathematics (see for instance [29, 62, 108]) and medicine (see e.g. [107, 112]), to name just a few. For a general introduction to random fields we refer the reader to [2, 6]. While the best studied underlying fields in the literature are Gaussian random fields, we focus in this thesis on Poisson shot noise processes and aim to study the behaviour of their excursion sets within compact convex observation windows for fixed $u > 0$ for growing observation windows. This means, we consider a family of windows $(W_s)_{s\geq 1}$ with $W_s = sW$ for $s \geq 1$ and some compact convex set W with $\lambda_d(W) > 0$ and are interested in the asymptotic behaviour of $\varphi(Z_u \cap W_s)$, where φ is a functional that provides geometric information about $Z_u \cap W_s$.

In the literature there already exist a few results for specific geometric functionals of excursion sets of different random fields. For Gaussian random fields, [46] establishes a central limit theorem for the Euler characteristic of excursion sets, while [26, 43] use the Euler characteristic to construct tests for Gaussianity. Central limit theorems for Lipschitz-Killing curvatures of Gaussian excursion sets are proven in [63, 78]. More general random fields are considered for instance in [1, 25, 26, 32, 105]. Among other things, the Euler characteristic, surface area and volume of excursion sets of two-dimensional stationary isotropic random fields are analysed in [26]. In [1, 25] the asymptotic expectation of functionals like Lipschitz-Killing curvatures, the level perimeter or level total curvature integrals are calculated for different fields such as stable fields or two-dimensional smooth stationary fields. An overview on asymptotic results for the volume of stationary random fields is given in [105]. In [32] variance asymptotics and central limit theorems for the volume of excursion sets of quasi-associated random fields were considered, which include a large class of Poisson shot noise processes (see [32, Proposition 1]). Further research on the asymptotic behaviour of the volume or a smoothed version of the volume of excursion sets of Poisson shot noise processes is for example done in [66, 67], which show limit theorems and variance asymptotics under different conditions on the kernel functions. Moreover, the perimeter or a smoothed version of the perimeter of Poisson shot noise processes is considered in [24, 66, 67]. While [24] analyses its expectation, the works [66, 67] show central limit theorems. Additionally, [66] shows asymptotic results for other functionals like the Euler characteristic. In the following section we start with analysing the volume of excursion sets of Poisson shot noise processes and complement the findings from [32, 66, 67] to a wider range of kernel functions.

6.2 Volumes of excursion sets

We consider a specific class of Poisson shot noise processes with underlying non-marked Poisson processes with intensity 1. Recall that this means that we consider a stationary Poisson process η on $\mathbb{R}^d$ with intensity measure λ_d and a fixed integrable kernel function $g\colon \mathbb{R}^d \to \mathbb{R}$. The corresponding Poisson shot noise process $(f_\eta(y))_{y\in\mathbb{R}^d}$ is given by

$$f_\eta(y) = \sum_{x\in\eta} g(y-x) \tag{6.2}$$

for $y \in \mathbb{R}^d$. Note that all arguments, which are used in the following, also work for Poisson shot noise processes with underlying non-marked Poisson processes with intensity $\gamma > 0$ for fixed $\gamma > 0$. Hence, all results of this section continue to hold in this case with constants that may depend on γ.

We are interested in the volume of the excursion set in the observation window $B^d(0,s)$ with $s \geq 1$, i.e. we want to analyse the behaviour of

$$F_s = \lambda_d(\{y \in B^d(0,s)\colon f_\eta(y) \geq u\})$$

as $s \to \infty$.

Since the volume of the excursion set can be written as integral over indicator functions, one obtains with Fubini's theorem and translation invariance of the Poisson shot noise process

$$\begin{aligned}
\operatorname{Var}[F_s] &= \mathbb{E}\left[\left(\int_{B^d(0,s)} \mathbb{1}\{f_\eta(y) \geq u\}\,\mathrm{d}y\right)^2\right] - \mathbb{E}\left[\int_{B^d(0,s)} \mathbb{1}\{f_\eta(y) \geq u\}\,\mathrm{d}y\right]^2 \\
&= \int_{B^d(0,s)}\int_{B^d(0,s)} \mathbb{P}(f_\eta(y_1) \geq u, f_\eta(y_2) \geq u) - \mathbb{P}(f_\eta(y_1) \geq u)\mathbb{P}(f_\eta(y_2) \geq u)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
&= \int_{\mathbb{R}^d} \lambda_d(\{y \in B^d(0,s)\colon y+z \in B^d(0,s)\}) \\
&\qquad \times \left(\mathbb{P}(f_\eta(0) \geq u, f_\eta(z) \geq u) - \mathbb{P}(f_\eta(0) \geq u)\mathbb{P}(f_\eta(z) \geq u)\right)\,\mathrm{d}z.
\end{aligned}$$

Note that $\lambda_d(\{y \in B^d(0,s)\colon y+z \in B^d(0,s)\})/\lambda_d(B^d(0,s)) \leq 1$ for all $z \in \mathbb{R}^d$ and that it converges to one as $s \to \infty$ for all $z \in \mathbb{R}^d$. Thus, the dominated convergence theorem yields

$$\lim_{s\to\infty} \frac{\operatorname{Var}[F_s]}{\lambda_d(B^d(0,s))} = \int_{\mathbb{R}^d} \mathbb{P}(f_\eta(0) \geq u, f_\eta(z) \geq u) - \mathbb{P}(f_\eta(0) \geq u)\mathbb{P}(f_\eta(z) \geq u)\, \mathrm{d}z$$

if the integral on the right-hand side is well defined. However, since the difference in the integral can be either positive or negative, this representation does not imply the positivity of the asymptotic variance. This is the reason why we use Theorem 3.1 in this section to derive a lower variance bound for F_s under different assumptions on the kernel function g. Among other things, we use the following assumption.

Assumption 6.1. There exist constants $\underline{c}_g, \overline{c}_g, \delta, \gamma > 0$ and $c_g \geq 1$ such that $\delta + d/2 > \gamma \geq \delta > 3d$ and

$$\underline{c}_g \|x\|^{-\gamma} \leq |g(x)| \leq \overline{c}_g \|x\|^{-\delta}$$

for all $x \in \mathbb{R}^d$ with $\|x\| \geq c_g$.

Then we can derive the following lower variance bound.

Theorem 6.2. *Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function with $g(0) > 0$.*

a) If g fulfils Assumption 6.1, there exists a constant $c > 0$ such that

$$\operatorname{Var}[F_s] \geq cs^d$$

for all $s \geq 1$.

b) Assume that g has compact support S. Then, there exists a constant $c > 0$ such that

$$\operatorname{Var}[F_s] \geq cs^d$$

for all $s \geq 1$.

Replacing g by $g(\cdot - z)$ for any $z \in \mathbb{R}^d$ leads to a translation of the Poisson shot noise field and, thus, by translation invariance, to a Poisson shot noise process with the same distribution. Thus, the assumption $g(0) > 0$ is no loss of generality because any g that can take positive values can be modified accordingly, while the case of a non-positive function g is trivial because then the level set for $u > 0$ becomes empty.

Since statements of the form that the variance is at least of the order of the volume of the observation window as in Theorem 6.2 were already proven in [32, Proposition 1] and [66, Theorem 4.1], let us compare the assumptions of Theorem 6.2 a) with those made before. In [32, Proposition 1], it is required that g is a bounded and uniformly continuous function on $\mathbb{R}^d$ with $|g(x)| \leq c\|x\|^\alpha$ for some constant $c > 0$ and $\alpha > 3d$ (as in our Assumption 6.1). A crucial difference is that we allow g to take positive and negative values, while it has to be non-negative in [32], where this assumption might be essential since it ensures that the Poisson shot noise process is positively associated. A lower bound on the decay of $|g|$ as in Assumption 6.1 is not present in [32], but we use it only to ensure the boundedness of the density of $f_\eta(0)$, which is assumed in [32], and to guarantee that

$g(x)$ for $\|x\|$ sufficiently large is either positive or negative. The result in [32] deals with marks in the sense that in (6.2) each summand is multiplied by an i.i.d. copy of a non-negative random variable. It might be possible to generalise our results in this direction as well. The assumptions in [66, Theorem 4.1] seem to be more restrictive than in our case. There it is supposed that g depends only on the norm of its argument and that $|g(x)|$ has an upper bound as in Assumption 1 but with $\delta = 11d$. Instead of a lower bound on $|g(x)|$, a rather technical assumption (see (4.3) in [66]) is made, which even requires differentiability of g. We are not aware of any results dealing with the situation of part b) of Theorem 6.2. The compact support implies that $f_\eta(0)$ does not possess a density since

$$\mathbb{P}(f_\eta(0) = 0) \geq \mathbb{P}(\eta(-S) = 0) = e^{-\lambda_d(S)} > 0.$$

We prepare the proof of Theorem 6.2 with the following lemma.

Lemma 6.3. *Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous, bounded function with $g(0) > 0$ that fulfils Assumption 6.1. Then, $f_\eta(y)$ has a bounded density for all $y \in \mathbb{R}^d$.*

Proof. We use the fact that $f_\eta(y)$ has a bounded density if its characteristic function φ is integrable. By [31, Chapter 1, Lemma 3.7] the characteristic function of $f_\eta(y)$ is given by

$$\varphi(t) = \exp\left[-\int_{\mathbb{R}^d} 1 - e^{\mathbf{i}tg(y-x)}\,\mathrm{d}x\right],$$

where $\mathbf{i}$ is the imaginary unit. Thus, $f_\eta(y)$ has a bounded density if

$$\int_{\mathbb{R}} |\varphi(t)|\,\mathrm{d}t = \int_{\mathbb{R}} \left|\exp\left[-\int_{\mathbb{R}^d} 1 - e^{\mathbf{i}tg(y-x)}\,\mathrm{d}x\right]\right|\,\mathrm{d}t < \infty.$$

Choose $c > 0$ small enough such that $1 - \cos(\hat{x}) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\hat{x}^{2k}}{(2k)!} \geq \frac{\hat{x}^2}{4}$ for $\hat{x} \in [-c, c]$. Then it holds

$$\begin{aligned}
\int_{\mathbb{R}^d} 1 - \cos(tg(y-x))\,\mathrm{d}x &\geq \int_{\{z\in\mathbb{R}^d\colon t^2 g(y-z)^2 \leq c^2, \|y-z\|\geq c_g\}} \frac{(tg(y-x))^2}{4}\,\mathrm{d}x \\
&\geq \int_{\{z\in\mathbb{R}^d\colon t^2 \overline{c}_g^2 \|y-z\|^{-2\delta} \leq c^2, \|y-z\|\geq c_g\}} \frac{t^2 \underline{c}_g^2 \|y-x\|^{-2\gamma}}{4}\,\mathrm{d}x \\
&\geq \frac{d\kappa_d t^2 \underline{c}_g^2}{4} \int_{\max\{(t\overline{c}_g/c)^{1/\delta}, c_g\}}^{\infty} r^{-2\gamma} r^{d-1}\,\mathrm{d}r \\
&= \frac{d\kappa_d t^2 \underline{c}_g^2}{4(2\gamma - d)} \cdot \max\left\{(t\overline{c}_g/c)^{1/\delta}, c_g\right\}^{(d-2\gamma)}
\end{aligned}$$

and, therefore,

$$\begin{aligned}
\int_{\mathbb{R}} |\varphi(t)|\,\mathrm{d}t &= \int_{\mathbb{R}} \left|\exp\left[-\int_{\mathbb{R}^d} 1 - e^{\mathbf{i}tg(y-x)}\,\mathrm{d}x\right]\right|\,\mathrm{d}t \\
&= 2\int_{\mathbb{R}_+} \exp\left[-\int_{\mathbb{R}^d} 1 - \cos(tg(y-x))\,\mathrm{d}x\right]\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\int_{\mathbb{R}_+} \exp\left[-\frac{d\kappa_d t^2 \underline{c}_g^2}{4(2\gamma-d)} \cdot \max\left\{(t\overline{c}_g/c)^{1/\delta}, c_g\right\}^{(d-2\gamma)}\right] \,\mathrm{d}t \\
&= 2\int_0^{c_g^\delta c/\overline{c}_g} \exp[-c_{1,\gamma,\delta,d}t^2] \,\mathrm{d}t + 2\int_{c_g^\delta c/\overline{c}_g}^{\infty} \exp[-c_{2,\gamma,\delta,d}t^{(2(\delta-\gamma)+d)/\delta}] \,\mathrm{d}t \\
&< \infty
\end{aligned}
$$

with suitable constants $c_{1,\gamma,\delta,d}, c_{2,\gamma,\delta,d} > 0$ since $\delta - \gamma + d/2 > 0$. This shows that $f_\eta(y)$ has a bounded density. □

Proof of Theorem 6.2. Let $z \in \mathbb{R}^d$ be fixed. In the first part of the proof we derive lower bounds for $|D_z F_s|$ for particular point configurations. For the proof of a) we distinguish the cases

$$g(x) < 0 \quad \text{for all} \quad x \in \mathbb{R}^d \quad \text{with} \quad \|x\| \geq c_g$$

and

$$g(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}^d \quad \text{with} \quad \|x\| \geq c_g,$$

which is sufficient since Assumption 6.1 and the continuity of g imply that $g(x)$ has the same sign for all $x \in \mathbb{R}^d$ with $\|x\| \geq c_g$. We start with the first case. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$. Then, we can find $k_1 \in \mathbb{N}_0$ such that

$$g(x) + k_1 g(x + 2c_g e_1) \leq u - 2 \tag{6.3}$$

for all $x \in B^d(0, c_g)$. Moreover, by the intermediate value theorem we can choose $k_2 \in \mathbb{N}_0$ and $x_0 \in \mathbb{R}^d \setminus B^d(0, 2c_g)$ such that

$$k_1 g(5c_g e_1) + k_2 g(3c_g e_1 - x_0) = u - \frac{g(3c_g e_1)}{2}. \tag{6.4}$$

By the continuity of g we can choose $\varepsilon > 0$ such that $B^d(x_0, \varepsilon) \subset \mathbb{R}^d \setminus B^d(0, 2c_g)$ and such that for all $x \in B^d(z, c_g)$, $\hat{x}_1, \dots, \hat{x}_{k_1} \in B^d(z - 2c_g e_1, \varepsilon)$ and $\tilde{x}_1, \dots, \tilde{x}_{k_2} \in B^d(z + x_0, \varepsilon)$ it holds that $g(x - z) + \sum_{i=1}^{k_1} g(x - \hat{x}_i) \leq u - 1$ due to (6.3), and thus,

$$g(x - z) + \sum_{i=1}^{k_1} g(x - \hat{x}_i) + \sum_{i=1}^{k_2} g(x - \tilde{x}_i) \leq u - 1 \tag{6.5}$$

since $\|x - \tilde{x}_i\| \geq c_g$ for $i \in \{1, \dots, k_2\}$. Furthermore, by (6.4) we can choose $\varepsilon > 0$ so small that for all $y \in B^d(z + 3c_g e_1, \varepsilon)$, $\hat{x}_1, \dots, \hat{x}_{k_1} \in B^d(z - 2c_g e_1, \varepsilon)$ and $\tilde{x}_1, \dots, \tilde{x}_{k_2} \in B^d(z + x_0, \varepsilon)$,

$$g(y - z) \leq \frac{7}{8} g(3c_g e_1) \tag{6.6}$$

and

$$\sum_{i=1}^{k_1} g(y - \hat{x}_i) + \sum_{i=1}^{k_2} g(y - \tilde{x}_i) \in \left(u - \frac{g(3c_g e_1)}{4}, u - \frac{3g(3c_g e_1)}{4}\right). \tag{6.7}$$

We abbreviate $D_1 = B^d(z - 2c_g e_1, \varepsilon)$ and $D_2 = B^d(z + x_0, \varepsilon)$ and let A_1 be the event that

$$\eta(D_1) = k_1, \quad \eta(D_2) = k_2 \quad \text{and} \quad \sum_{x \in \eta|_{\mathbb{R}^d \setminus (D_1 \cup D_2)}} |g(y - x)| \leq \frac{\min\{|g(3c_g e_1)|, 1\}}{8} \tag{6.8}$$

for all $y \in B^d(z + 3c_g e_1, \varepsilon) \cup B^d(z, c_g)$. Assuming that the event A_1 is satisfied, adding z to the underlying point configuration does not increase the excursion set since the Poisson shot noise process can only increase on $B^d(z, c_g)$, where it does not exceed u after adding z because of (6.5) and (6.8). On the other hand, the ball $B^d(z + 3c_g e_1, \varepsilon)$ belongs to the excursion set before adding z but not thereafter due to (6.6), (6.7) and (6.8). Thus, we have shown that

$$\mathbb{1}_{A_1} |D_z F_s| \geq \mathbb{1}_{A_1} \kappa_d \varepsilon^d \tag{6.9}$$

if $B^d(z + 3c_g e_1, \varepsilon) \subseteq B^d(0, s)$.

We continue with the second case where

$$g(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}^d \quad \text{with} \quad \|x\| \geq c_g.$$

If g can become negative, we choose $k_1 \in \mathbb{N}_0$ such that

$$g(x) + k_1 g(x + 2c_g e_1) \geq u + 2$$

for all $x \in B^d(0, c_g)$. In case that g is non-negative, we let $k_1 = 0$. For this part of the proof we assume that $g(3c_g e_1) < 2u$. Note that this is not a restriction because $g(x) \to 0$ as $\|x\| \to \infty$ and hence, we can set c_g large enough such that $g(3c_g e_1) < 2u$ is fulfilled. Then, we can find $k_2 \in \mathbb{N}_0$ and $x_0 \in \mathbb{R}^d \setminus B^d(0, 2c_g)$ such that

$$k_1 g(5c_g e_1) + k_2 g(3c_g e_1 - x_0) = u - \frac{g(3c_g e_1)}{2}.$$

Similarly to the first case we can choose $\varepsilon > 0$ sufficiently small so that $B^d(x_0, \varepsilon) \subset \mathbb{R}^d \setminus B^d(0, 2c_g)$ and such that for all $x \in B^d(z, c_g)$, $\hat{x}_1, \dots, \hat{x}_{k_1} \in B^d(z - 2c_g e_1, \varepsilon)$ and $\tilde{x}_1, \dots, \tilde{x}_{k_2} \in B^d(z + x_0, \varepsilon)$,

$$g(x - z) + \sum_{i=1}^{k_1} g(x - \hat{x}_i) + \sum_{i=1}^{k_2} g(x - \tilde{x}_i) \geq u + 1 \tag{6.10}$$

if g can become negative. Moreover, we choose $\varepsilon > 0$ such that for all $y \in B^d(z + 3c_g e_1, \varepsilon)$, $\hat{x}_1, \dots, \hat{x}_{k_1} \in B^d(z - 2c_g e_1, \varepsilon)$ and $\tilde{x}_1, \dots, \tilde{x}_{k_2} \in B^d(z + x_0, \varepsilon)$,

$$g(y - z) \geq \frac{7}{8} g(3c_g e_1) \tag{6.11}$$

and

$$\sum_{i=1}^{k_1} g(y - \hat{x}_i) + \sum_{i=1}^{k_2} g(y - \tilde{x}_i) \in \left(u - \frac{3g(3c_g e_1)}{4}, u - \frac{g(3c_g e_1)}{4} \right). \tag{6.12}$$

If the event A_1 occurs, after adding z to the underlying point configuration, $B^d(z+3c_g e_1, \varepsilon)$ is included in the excursion set, whereas no point of $B^d(z + 3c_g e_1, \varepsilon)$ was part of the excursion set before adding z by (6.8), (6.11) and (6.12). If g is non-negative, the excursion set cannot decrease after adding z. If g is somewhere negative on $B^d(0, c_g)$, the excursion set cannot decrease either as all points of $B^d(z, c_g)$ belong to the excursion set after adding z by (6.8) and (6.10). Thus, we can conclude that

$$\mathbb{1}_{A_1}|D_z F_s| \geq \mathbb{1}_{A_1}\kappa_d \varepsilon^d$$

if $B^d(z + 3c_g e_1, \varepsilon) \subseteq B^d(0, s)$.

For b) we first assume that $\lambda_d(\{x \in \mathbb{R}^d \colon g(x) \geq u\}) = 0$. Let $k \in \mathbb{N}$ be the largest possible number such that

$$\lambda_d^{k-1}\Big(\Big\{(x_1, \ldots, x_{k-1}) \in (\mathbb{R}^d)^{k-1} \colon \lambda_d\Big(\Big\{y \in \mathbb{R}^d \colon g(y) + \sum_{i=1}^{k-1} g(y - x_i) \geq u\Big\}\Big) > 0\Big\}\Big) = 0 \tag{6.13}$$

with the convention $\lambda_d^0(\cdot) = 0$. Then, there exists $\varepsilon > 0$ such that the set

$$V = \Big\{(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k \colon \lambda_d\Big(\Big\{y \in \mathbb{R}^d \colon g(y) + \sum_{i=1}^{k} g(y - x_i) \geq u\Big\}\Big) > \kappa_d \varepsilon^d\Big\}$$

satisfies $\lambda_d^k(V) > 0$. Let

$$\widetilde{V} = \Big\{(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k \colon \lambda_d\Big(\Big\{y \in \mathbb{R}^d \colon \sum_{i=1}^{k} g(y - x_i) \geq u\Big\}\Big) > 0\Big\}.$$

With (6.13) it holds

$$\begin{aligned}
&\lambda_d^k(\widetilde{V}) \\
&= \int_{\mathbb{R}^d} \lambda_d^{k-1}\Big(\Big\{(x_1, \ldots, x_{k-1}) \in (\mathbb{R}^d)^{k-1} \colon \lambda_d\Big(\Big\{y \in \mathbb{R}^d \colon \sum_{i=1}^{k} g(y - x_i) \geq u\Big\}\Big) > 0\Big\}\Big) \, \mathrm{d}x_k \\
&= \int_{\mathbb{R}^d} \lambda_d^{k-1}\Big(\Big\{(x_1, \ldots, x_{k-1}) \in (\mathbb{R}^d)^{k-1} \colon \\
&\qquad \lambda_d\Big(\Big\{y \in \mathbb{R}^d \colon g(y) + \sum_{i=1}^{k-1} g(y - x_i) \geq u\Big\}\Big) > 0\Big\}\Big) \, \mathrm{d}x_k = 0.
\end{aligned}$$

We choose $R_0 > 0$ large enough such that $g(y - x) = 0$ for all $y \in S$ and $x \in B^d(0, R_0)^c$. For $z \in \mathbb{R}^d$ this means that the points of η in $B^d(z, R_0)^c$ do not influence the excursion set on $S + z$. Define $\widehat{V} = ((V + z)\backslash\widetilde{V}) \cap B^d(z, R_0)^k$, where $M + z = \{(x_1 + z, \ldots, x_k + z) \colon (x_1, \ldots, x_k) \in M\}$ for $M \subseteq (\mathbb{R}^d)^k$ and let A_2 be the event that

$$\eta(B^d(z, R_0)) = k \quad \text{and} \quad \eta_{\neq}^k(\widehat{V}) \neq 0.$$

The second condition guarantees that the k points in $B^d(z, R_0)$ are arranged in such a way that the volume of the excursion set in $S + z$ is 0 before adding z and larger than $\kappa_d \varepsilon^d$ after adding z. This implies

$$\mathbb{1}_{A_2}|D_z F_s| \geq \mathbb{1}_{A_2}\kappa_d \varepsilon^d$$

for all $z \in B^d(0,s)$ with $S+z \subseteq B^d(0,s)$. For $\lambda_d(\{x \in \mathbb{R}^d \colon g(x) \geq u\}) > 0$ this is obviously true for a suitable $\varepsilon > 0$ if $A_2 = \{\eta(B^d(z,R_0)) = 0\}$.

To control $\mathbb{P}(|D_z F_s| \geq \kappa_d \varepsilon^d)$ we bound in the following $\mathbb{P}(A_1)$ and $\mathbb{P}(A_2)$. We abbreviate $D_3 = B^d(z,c_g) \cup B^d(z+3c_g e_1, \varepsilon)$. For a) let $\widehat{R}_0 > 0$ be such that $D_3 \subseteq B^d(z,\widehat{R}_0)$ and define $B_R = B^d(z,R)$ for some $R \geq \widehat{R}_0 + c_g$. Then, for $c > 0$ the Markov inequality and the Mecke equation lead to

$$\begin{aligned}
\mathbb{P}\Bigg(\exists y \in D_3\colon \sum_{x \in \eta|_{B_R^c}} |g(y-x)| > c\Bigg) &\leq \mathbb{P}\Bigg(\sum_{x \in \eta|_{B_R^c}} \max_{y \in D_3} |g(y-x)| > c\Bigg) \\
&\leq \frac{1}{c}\mathbb{E}\Bigg[\sum_{x \in \eta|_{B_R^c}} \max_{y \in D_3} |g(y-x)|\Bigg] \\
&\leq \frac{1}{c}\int_{\mathbb{R}^d \setminus B_R} \max_{y \in B^d(z,\widehat{R}_0)} |g(y-x)| \,\mathrm{d}x \\
&\leq \frac{1}{c}\int_{\mathbb{R}^d \setminus B_R} \overline{c}_g (\|x\| - \widehat{R}_0)^{-\delta} \,\mathrm{d}x \\
&= \frac{d\kappa_d}{c}\int_{R-\widehat{R}_0}^{\infty} \overline{c}_g r^{-\delta} (r+\widehat{R}_0)^{d-1} \,\mathrm{d}r.
\end{aligned}$$

Choosing $R \geq \widehat{R}_0 + c_g$ large enough such that the probability above is at most $\frac{1}{2}$ for $c = \frac{\min\{|g(3c_g e_1)|,1\}}{8}$ and $D_1 \cup D_2 \subseteq B_R$ provides for a) with (6.9),

$$\begin{aligned}
\mathbb{P}(|D_z F_s| \geq \kappa_d \varepsilon^d) &\geq \mathbb{P}(A_1) \\
&\geq \mathbb{P}(\eta(D_1) = k_1, \eta(D_2) = k_2, \eta(B_R \setminus (D_1 \cup D_2)) = 0) \\
&\quad \times \mathbb{P}\Big(\sum_{x \in \eta|_{B_R^c}} |g(y-x)| \leq \frac{\min\{|g(3c_g e_1)|,1\}}{8} \text{ for all } y \in D_3\Big) \\
&\geq \frac{1}{2}\mathbb{P}(\eta(D_1) = k_1, \eta(D_2) = k_2, \eta(B_R \setminus (D_1 \cup D_2)) = 0) =: p_1 > 0.
\end{aligned}$$

For b) we get for $\lambda_d(\{x \in \mathbb{R}^d \colon g(x) \geq u\}) = 0$ and $\widehat{V} = ((V+z) \setminus \widetilde{V}) \cap B^d(z,R_0)^k$ with the multivariate Mecke formula

$$\begin{aligned}
\mathbb{P}(|D_z F_s| \geq \kappa_d \varepsilon^d) &\geq \mathbb{P}(A_2) \\
&= \mathbb{P}(\eta^k_{\neq}(\widehat{V}) \neq 0, \eta(B^d(z,R_0)) = k) \\
&= \frac{1}{k!}\mathbb{E}\Bigg[\sum_{(x_1,\dots,x_k) \in \eta^k_{\neq}|_{\widehat{V}}} \mathbb{1}\{\eta(B^d(z,R_0)) = k\}\Bigg] \\
&= \frac{1}{k!}\int_{\widehat{V}} \mathbb{P}(\eta(B^d(z,R_0) \setminus \cup_{i=1}^k \{x_i\}) = 0) \,\mathrm{d}(x_1,\dots,x_k) \\
&=: p_2.
\end{aligned}$$

Clearly,

$$\mathbb{P}(\eta(B^d(z,R_0) \setminus \cup_{i=1}^k \{x_i\}) = 0) = \mathbb{P}(\eta(B^d(z,R_0)) = 0) > 0.$$

From (6.13) it follows that $\lambda_d^k(((V+z)\backslash\widetilde{V})\backslash B^d(z,R_0)^k) = 0$ since as soon as one of $x_1,\dots,x_k$ does not belong to $B^d(z,R_0)$, we are in the situation of (6.13). Together with $\lambda_d^k(\widetilde{V})=0$ and $\lambda_d^k(V)>0$ we see that

$$\lambda_d^k(\widehat{V}) = \lambda_d^k(((V+z)\backslash\widetilde{V})\cap B^d(z,R_0)^k) = \lambda_d^k(V) > 0.$$

This implies $p_2 > 0$. The same holds for $\lambda_d(\{x\in\mathbb{R}^d\colon g(x)\geq u\})>0$, where $A_2 = \{\eta(B^d(z,R_0))=0\}$.

Altogether, for $W_s=\{z\in\mathbb{R}^d\colon B^d(z+3c_g e_1,\varepsilon)\subseteq B^d(0,s)\}$ and $p=p_1$ in case of a) or $W_s=\{z\in\mathbb{R}^d\colon S+z\subset B^d(0,s)\}$ and $p=p_2$ in case of b) we conclude that

$$\begin{aligned}\mathbb{E}\left[\int (D_zF_s)^2\,\mathrm{d}z\right] &\geq \kappa_d^2\varepsilon^{2d}\int_{\mathbb{R}^d}\mathbb{P}(|D_zF_s|\geq\kappa_d\varepsilon^d)\,\mathrm{d}z\\ &\geq \kappa_d^2\varepsilon^{2d}\int_{W_s} p\,\mathrm{d}z \geq \kappa_d^2\varepsilon^{2d}p\lambda_d(W_s)\geq c_{d,\varepsilon}s^d\end{aligned}$$

for some constant $c_{d,\varepsilon}>0$ and s large enough.

In the following we consider the second-order difference operator to check (3.2). For $z_1,z_2\in\mathbb{R}^d$ with $z_1\neq z_2$ we have

$$D^2_{z_1,z_2}F_s = \int_{B^d(0,s)} D^2_{z_1,z_2}\mathbb{1}\{f_\eta(y)\geq u\}\,\mathrm{d}y$$

so that

$$|D^2_{z_1,z_2}F_s| \leq 2\lambda_d(B_s(z_1,z_2)) \tag{6.14}$$

with $B_s(z_1,z_2) = \{y\in B^d(0,s)\colon D^2_{z_1,z_2}\mathbb{1}\{f_\eta(y)\geq u\}\neq 0\}$, where we used the bound $|D^2_{z_1,z_2}\mathbb{1}\{f_\eta(y)\geq u\}|\leq 2$. The inequality (6.14) leads to

$$I := \mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(D^2_{z_1,z_2}F_s)^2\,\mathrm{d}z_1\,\mathrm{d}z_2\right] \leq 4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{E}\left[\lambda_d(B_s(z_1,z_2))^2\right]\,\mathrm{d}z_1\,\mathrm{d}z_2.$$

First we study the situation of a). Let $y\in B^d(0,s)$ and assume that $|g(y-z_2)|\leq|g(y-z_1)|$. It holds

$$\begin{aligned}D^2_{z_1,z_2}\mathbb{1}\{f_\eta(y)\geq u\} &= (\mathbb{1}\{f_\eta(y)+g(y-z_1)+g(y-z_2)\geq u\} - \mathbb{1}\{f_\eta(y)+g(y-z_1)\geq u\})\\ &\quad - (\mathbb{1}\{f_\eta(y)+g(y-z_2)\geq u\} - \mathbb{1}\{f_\eta(y)\geq u\}) =: d_1 - d_2.\end{aligned}$$

Note that $d_1 = 0$ if

$$|f_\eta(y)+g(y-z_1)-u|\geq|g(y-z_2)|$$

and $d_2=0$ if

$$|f_\eta(y)-u|\geq|g(y-z_2)|.$$

Hence, we obtain that

$$D^2_{z_1,z_2}\mathbb{1}\{f_\eta(y)\geq u\}=0$$

if

$$f_\eta(y)+g(y-z_1)\notin[u-|g(y-z_2)|,u+|g(y-z_2)|]$$

and

$$f_\eta(y) \notin [u - |g(y - z_2)|, u + |g(y - z_2)|].$$

Together with the fact that the density of $f_\eta(y)$ is bounded by a constant $C_1 > 0$, which was shown in Lemma 6.3, we derive

$$\begin{aligned}\mathbb{P}(y \in B_s(z_1, z_2)) &\leq \mathbb{P}(f_\eta(y) + g(y - z_1) \in [u - |g(y - z_2)|, u + |g(y - z_2)|]) \\ &\quad + \mathbb{P}(f_\eta(y) \in [u - |g(y - z_2)|, u + |g(y - z_2)|]) \\ &\leq 4C_1|g(y - z_2)|.\end{aligned}$$

Using the same arguments for $|g(y - z_2)| \geq |g(y - z_1)|$, we deduce

$$\mathbb{P}(y \in B_s(z_1, z_2)) \leq 4C_1 \min\{|g(y - z_1)|, |g(y - z_2)|\}$$

so that with Hölder's inequality and the inequality $\min\{a, b\} \leq \sqrt{a}\sqrt{b}$ for $a, b \geq 0$,

$$\begin{aligned}&\mathbb{E}\left[\lambda_d(B_s(z_1, z_2))^2\right] \\ &= \int_{B^d(0,s)} \int_{B^d(0,s)} \mathbb{P}(y_1 \in B_s(z_1, z_2), y_2 \in B_s(z_1, z_2))\, \mathrm{d}y_1\, \mathrm{d}y_2 \\ &\leq \int_{B^d(0,s)} \int_{B^d(0,s)} \mathbb{P}(y_1 \in B_s(z_1, z_2))^{2/3} \mathbb{P}(y_2 \in B_s(z_1, z_2))^{1/3}\, \mathrm{d}y_1\, \mathrm{d}y_2 \\ &\leq 4C_1 \int_{B^d(0,s)} \int_{B^d(0,s)} |g(y_1 - z_1)|^{1/3} |g(y_1 - z_2)|^{1/3} |g(y_2 - z_1)|^{1/3}\, \mathrm{d}y_1\, \mathrm{d}y_2.\end{aligned}$$

From Assumption 6.1 and the continuity of g it follows that g is bounded by a constant $C_2 > 0$. Using the decay of $|g|$ and $\delta > 3d$ in Assumption 6.1, we have for $y \in B^d(0, s)$ that

$$\begin{aligned}\int_{\mathbb{R}^d} |g(y - z)|^{1/3}\, \mathrm{d}z &= \int_{\mathbb{R}^d \setminus B^d(y, c_g)} |g(y - z)|^{1/3}\, \mathrm{d}z + \int_{B^d(y, c_g)} |g(y - z)|^{1/3}\, \mathrm{d}z \\ &\leq \int_{\mathbb{R}^d \setminus B^d(y, c_g)} \bar{c}_g^{1/3} \|y - z\|^{-\delta/3}\, \mathrm{d}z + C_2^{1/3} \kappa_d c_g^d \\ &= d\kappa_d \bar{c}_g^{1/3} \int_{c_g}^{\infty} r^{d-1} r^{-\delta/3}\, \mathrm{d}r + C_2^{1/3} \kappa_d c_g^d \\ &= d\kappa_d \bar{c}_g^{1/3} \frac{c_g^{d-\delta/3}}{\delta/3 - d} + C_2^{1/3} \kappa_d c_g^d =: C_3.\end{aligned}$$

The same estimate holds for $\int_{B^d(0,s)} |g(y - z)|^{1/3}\, \mathrm{d}y$ for $z \in \mathbb{R}^d$. Hence,

$$\begin{aligned}I &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 16C_1 \int_{B^d(0,s)} \int_{B^d(0,s)} |g(y_1 - z_1)|^{1/3} |g(y_1 - z_2)|^{1/3} \\ &\qquad \times |g(y_2 - z_1)|^{1/3}\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}z_1\, \mathrm{d}z_2 \\ &= 16C_1 \int_{B^d(0,s)} \int_{\mathbb{R}^d} |g(y_1 - z_1)|^{1/3} \int_{B^d(0,s)} |g(y_2 - z_1)|^{1/3} \\ &\qquad \times \int_{\mathbb{R}^d} |g(y_1 - z_2)|^{1/3}\, \mathrm{d}z_2\, \mathrm{d}y_2\, \mathrm{d}z_1\, \mathrm{d}y_1\end{aligned}$$

$$\leq 16C_1 \int_{B^d(0,s)} C_3^3 \, \mathrm{d}y_1 =: \tilde{c}_1 s^d.$$

For b) let $\widetilde{R} > 0$ be such that $S \subseteq B^d(0, \widetilde{R})$ and let $z_1, z_2 \in \mathbb{R}^d$. Then, since

$$B_s(z_1, z_2) \subseteq \{y \in B^d(0,s) \colon \|y - z_1\| \leq \widetilde{R}, \|y - z_2\| \leq \widetilde{R}\},$$

it follows

$$\mathbb{E}\left[\lambda_d(B_s(z_1, z_2))^2\right] \leq \lambda_d(\{y \in B^d(0,s) \colon \|y - z_1\| \leq \widetilde{R}, \|y - z_2\| \leq \widetilde{R}\})^2.$$

The triangle inequality implies $\lambda_d(\{y \in B^d(0,s) \colon \|y - z_1\| \leq \widetilde{R}, \|y - z_2\| \leq \widetilde{R}\}) = 0$ for $\|z_1 - z_2\| > 2\widetilde{R}$ or $\|z_2\| > s + \widetilde{R}$ and therefore

$$\begin{aligned} I &\leq 4 \int_{B^d(0,s+\widetilde{R})} \int_{B^d(z_2,2\widetilde{R})} \lambda_d(\{y \in B^d(0,s) \colon \|y - z_1\| \leq \widetilde{R}, \|y - z_2\| \leq \widetilde{R}\})^2 \, \mathrm{d}z_1 \, \mathrm{d}z_2 \\ &\leq 4 \int_{B^d(0,s+\widetilde{R})} \int_{B^d(z_2,2\widetilde{R})} (\kappa_d \widetilde{R}^d)^2 \, \mathrm{d}z_1 \, \mathrm{d}z_2 \leq 4(\kappa_d \widetilde{R}^d)^2 \kappa_d^2 (2\widetilde{R})^d (s + \widetilde{R})^d \\ &\leq C s^d \end{aligned}$$

for a suitable constant $C > 0$. Combining the derived lower and upper bounds with Theorem 3.1 for $n = 1$ completes the proof of both cases. □

Remark 6.4. An alternative approach is to construct the Poisson shot noise process only with respect to points of the Poisson process within the observation window, i.e. to consider

$$f_{\eta|_{B^d(0,s)}}(y) = \sum_{x \in \eta|_{B^d(0,s)}} g(y - x)$$

for $y \in \mathbb{R}^d$ and the functional

$$\widetilde{F}_s = \lambda_d(\{y \in B^d(0,s) \colon f_{\eta|_{B^d(0,s)}}(y) \geq u\}),$$

which is the volume of the excursion set on $B^d(0,s)$. Then, the integrals of the second moments of the first-order difference operator can be bounded from below as in the proof of Theorem 6.2. The arguments from this proof can also be used to control the second-order difference operator in the case where g has compact support. Under Assumption 6.1, $f_{\eta|_{B^d(0,s)}}(y)$ does not posses a density as it has an atom in 0 so that the arguments of the proof of Theorem 6.2 for the second-order difference operator do not carry over. However, if there exists a constant $c > 0$ such that

$$\mathbb{P}(f_{\eta|_{B^d(0,s)}}(y) \in [u - a, u + a]) \leq ca$$

for all $y \in B^d(0,s)$ and $a > 0$, our proof works for the alternative setting as well.

6.3 Geometric functionals of polyconvex excursion sets

In this section we want to study general geometric functionals of excursion sets of Poisson shot noise processes, whose underlying Poisson process is a marked Poisson process on $\mathbb{R}^d \times \mathbb{M}$ with intensity measure $\gamma\lambda_d \times \mathbb{Q}$ as defined in (6.1). Let $(W_s)_{s\geq 1}$ be a family of observation windows, where $W_s = sW$ for $s \geq 1$ and $W \in \mathcal{K}^d$ with $\lambda_d(W) > 0$. Remember that we denote by f_η the Poisson shot noise process, while

$$Z_u \cap W_s = \{y \in W_s \colon f_\eta(y) \geq u\}$$

for $s \geq 1$.

By a geometric functional φ we mean a measurable, translation invariant, additive and locally bounded functional as defined in Definition 2.5. Unlike functionals such as the volume or the Euler characteristic, general geometric functionals are only well defined for polyconvex sets. This requires the introduction of conditions on the family of kernel functions for which the corresponding excursion sets in the observation window are almost surely polyconvex, i.e. for which $Z_u \cap W_s$ can almost surely be written as a union of finitely many compact convex sets. For this reason we denote by $\widetilde{L}^1(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ the space of integrable functions on $\mathbb{R}^d$ with compact convex support K satisfying $\lambda_d(K) > 0$, whose restriction to K is continuous. In the following this space will be equipped with the Borel σ-algebra $\mathcal{L}_1$, which is induced by the L^1-norm. Throughout this section we use the following assumption for our family of kernel functions $(g_m)_{m\in\mathbb{M}}$.

Assumption 6.5. It holds that

i) g_m is a non-negative, measurable function for all $m \in \mathbb{M}$.

ii) g_m has a compact convex support K_m with $\lambda_d(K_m) > 0$ for all $m \in \mathbb{M}$.

iii) $g_m|_{K_m}$ is concave and continuous for all $m \in \mathbb{M}$.

iv) $m \mapsto g_m$ is an $\mathcal{F}_\mathbb{M}$-$\mathcal{L}_1$-measurable map.

Note that in iii) the continuity at inner points of K_m also follows from the concavity on K_m so that the continuity condition describes the behaviour of the function at the boundary of K_m. Since g_m is non-negative by i), this basically guarantees that g_m is upper semi-continuous, i.e. that $\limsup_{x\to x_0} g_m(x) \leq g_m(x_0)$ for all $x_0 \in \mathbb{R}^d$ and $m \in \mathbb{M}$. Under Assumption 6.5 we can also show the following measurability property for $(g_m)_{m\in\mathbb{M}}$.

Lemma 6.6. *Let $(g_m)_{m\in\mathbb{M}}$ be a family of kernel functions satisfying Assumption 6.5. Then, the map $m \mapsto K_m$ is $\mathcal{F}_\mathbb{M}$-$\mathcal{B}(\mathcal{F}^d)$-measurable.*

Proof. Recall that the Fell topology can be generated by $\{\{F \in \mathcal{F}^d \colon F \cap U \neq \emptyset\} \colon U \in \mathcal{U}^d\}$ (see Subsection 2.2.1). We start with showing that the map $h : \widetilde{L}^1(\mathbb{R}^d) \to \mathcal{F}^d$ defined by $h(f) = \operatorname{supp}(f)$ for $f \in \widetilde{L}^1(\mathbb{R}^d)$ is $\mathcal{L}_1$-$\mathcal{B}(\mathcal{F}^d)$-measurable. To this end, for $U \in \mathcal{U}^d$ we need to prove that

$$h^{-1}(\{F \in \mathcal{F}^d \colon F \cap U \neq \emptyset\}) = \{f \in \widetilde{L}^1(\mathbb{R}^d) \colon \operatorname{supp}(f) \cap U \neq \emptyset\}$$

is an open set. Note at first that the facts that $\operatorname{supp}(f)$ is a closed convex set with $\lambda_d(\operatorname{supp}(f)) > 0$ and U is open provides that $\operatorname{supp}(f) \cap U \neq \emptyset$ implies $\int_U |f(x)| \, \mathrm{d}x > 0$. Thus, if $\operatorname{supp}(f) \cap U \neq \emptyset$, we have for all $g \in \widetilde{L}^1(\mathbb{R}^d)$ with $\|f - g\| \leq \frac{1}{2} \int_U |f(x)| \, \mathrm{d}x$,

$$\begin{aligned}\int_U |g(x)| \, \mathrm{d}x &\geq \int_U |f(x)| \, \mathrm{d}x - \int_U |f(x) - g(x)| \, \mathrm{d}x \\ &\geq \frac{1}{2} \int_U |f(x)| \, \mathrm{d}x > 0,\end{aligned}$$

i.e. $\operatorname{supp}(g) \cap U \neq \emptyset$. Hence, $h^{-1}(\{F \in \mathcal{F}^d : F \cap U \neq \emptyset\})$ is open. Together with Assumption 6.5 iv) and since $g_m \in \widetilde{L}^1(\mathbb{R}^d)$ for all $m \in \mathbb{M}$, it follows that $m \mapsto K_m$ is $\mathcal{F}_{\mathbb{M}}$-$\mathcal{B}(\mathcal{F}^d)$-measurable. □

Throughout this section we also need some additional assumptions on the intrinsic volumes of the supports. For the whole section we assume that

$$\int_{\mathbb{M}} V_i(K_m) \, \mathbb{Q}(\mathrm{d}m) < \infty \tag{6.15}$$

for $i \in \{1, \dots, d\}$. In the course of this section we also assume higher order moment conditions of the form

$$\int_{\mathbb{M}} V_i(K_m)^k \, \mathbb{Q}(\mathrm{d}m) < \infty \tag{6.16}$$

of up to order $k = 4$ for $i \in \{1, \dots, d\}$ to derive variance asymptotics and central limit theorems.

We often represent a point $\hat{x} \in \eta$ as $\hat{x} = (x, m) \in \mathbb{R}^d \times \mathbb{M}$ and abbreviate

$$K(\hat{x}) = K_m + x = \{y \in \mathbb{R}^d : y - x \in K_m\}.$$

For $K \in \mathcal{K}^d$ define

$$S_K = \{\hat{x} = (x, m) \in \mathbb{R}^d \times \mathbb{M} : K(\hat{x}) \cap K \neq \emptyset\}.$$

Since

$$V_d(\{x \in \mathbb{R}^d : K(\hat{x}) \cap W_s \neq \emptyset\}) = V_d(\{x \in \mathbb{R}^d : (K_m + x) \cap W_s \neq \emptyset\}) = V_d(-K_m + W_s)$$

and by Steiner's formula (see also (2.1))

$$V_d(-K_m + W_s) \leq V_d(-K_m + B^d(0, R(W_s))) \leq C(s) \sum_{k=0}^{d} V_k(K_m)$$

for some constant $C(s) > 0$, which might depend on the circumradius $R(W_s)$ of W_s, the moment assumption in (6.15) provides that

$$\mathbb{E}[\eta(S_{W_s})] \leq \gamma \int_{\mathbb{M}} V_d(-K_m + W_s) \, \mathbb{Q}(\mathrm{d}m) < \infty$$

and thus, $\eta(S_{W_s}) < \infty$ almost surely for any $s \geq 1$. Let now $n = \eta(S_{W_s})$ and denote the points of the Poisson process in S_{W_s} by $\hat{x}_1 = (x_1, m_1), \ldots, \hat{x}_n = (x_n, m_n)$. For $j \in \{1, \ldots, n\}$ we often use the short notation

$$\hat{K}_j = K(\hat{x}_j) = K_{m_j} + x_j.$$

For $\emptyset \neq I \subseteq \{1, \ldots, n\}$ and $K \in \mathcal{K}^d$ we define

$$X_{I,K} = \Big\{ y \in K \colon \sum_{j \in I} g_{m_j}(y - x_j) \geq u, y \in \hat{K}_j \text{ for all } j \in I \Big\}. \tag{6.17}$$

We use these sets to show that $Z_u \cap W_s$ is almost surely polyconvex.

Proposition 6.7. *Let $u > 0$ be fixed and $n = \eta(S_{W_s})$. Then, X_{I,W_s} is a compact convex set for all $\emptyset \neq I \subseteq \{1, \ldots, n\}$ and*

$$Z_u \cap W_s = \bigcup_{\emptyset \neq I \subseteq \{1,\ldots,n\}} X_{I,W_s}$$

for all $s \geq 1$ almost surely.

Proof. In the following we show that the set X_{I,W_s} for $\emptyset \neq I \subseteq \{1, \ldots, n\}$ is a compact convex set and that we can write $Z_u \cap W_s$ as a union of these sets for any $s \geq 1$. For $y_1, y_2 \in X_{I,W_s}$ and $\alpha \in (0, 1)$ we have $\alpha y_1 + (1 - \alpha) y_2 \in \hat{K}_j$ since $\hat{K}_j$ is convex for all $j \in I$. Due to the concavity of g_{m_j} on K_{m_j} it holds

$$\sum_{j \in I} g_{m_j}(\alpha y_1 + (1 - \alpha) y_2 - x_j) \geq \sum_{j \in I} \alpha g_{m_j}(y_1 - x_j) + (1 - \alpha) g_{m_j}(y_2 - x_j) \geq u.$$

Hence, X_{I,W_s} is convex for $\emptyset \neq I \subseteq \{1, \ldots, n\}$. Since $g_{m_j} \in \widetilde{L}^1(\mathbb{R}^d)$, the restriction of g_{m_j} to K_{m_j} is continuous for all $j \in \{1, \ldots, n\}$. Together with the fact that W_s is a compact convex set and $X_{I,W_s} \subseteq W_s$ for all $s \geq 1$, this ensures that X_{I,W_s} is compact for $\emptyset \neq I \subseteq \{1, \ldots, n\}$.

Moreover, $\bigcup_{\emptyset \neq I \subseteq \{1,\ldots,n\}} X_{I,W_s} \subseteq Z_u \cap W_s$ since $X_{I,W_s} \subseteq Z_u \cap W_s$ for all $\emptyset \neq I \subseteq \{1, \ldots, n\}$ by the non-negativity of g_m for all $m \in \mathbb{M}$. As there is also $\emptyset \neq I \subseteq \{1, \ldots, n\}$ with $y \in X_{I,W_s}$ for all $y \in Z_u \cap W_s$, it follows

$$Z_u \cap W_s = \bigcup_{\emptyset \neq I \subseteq \{1,\ldots,n\}} X_{I,W_s},$$

which completes the proof. □

Because of Proposition 6.7 the geometric functional φ of $Z_u \cap W_s$ is defined via the inclusion exclusion principle (2.8). We have for $n = \eta(S_{W_s})$,

$$\varphi(Z_u \cap W_s) = \sum_{J \subseteq \mathcal{P}(\{1,\ldots,n\}) \setminus \emptyset \colon |J| \geq 1} (-1)^{|J|+1} \varphi(\cap_{I \in J} X_{I,W_s}),$$

where $\mathcal{P}(\{1, \ldots, n\})$ denotes the power set of $\{1, \ldots, n\}$.

An important example of random sets are Boolean models, which correspond to considering the union of the supports $Z = \bigcup_{\hat{x}\in\eta} K(\hat{x})$. The behaviour of $\varphi(Z \cap W_s)$ for the Boolean model Z was studied, for example, in [55, 71, 98]. It can be shown that the expectation grows with order $V_d(W_s)$ (see e.g. [98, Chapter 9.1]). Variance asymptotics of order $V_d(W_s)$ and corresponding quantitative central limit theorems for the Boolean model were derived in [55, Sections 3 and 9] or [71, Chapter 22.2]. One way to generalise the Boolean model is to consider the union of cylinders of Poisson cylinder processes, which have long-range dependencies. Variance asymptotics and central limit theorems for the case of Poisson cylinder processes were shown in [20, Theorem 3.5 and Theorem 3.10].

The model introduced in this section also generalises the Boolean model. In fact, the Boolean model arises as a special case of our model if $g_m(x) \geq u$ for all $x \in K_m$ and $m \in \mathbb{M}$. Since we do not consider the union of the supports but the union of the sets X_{I,W_s} for $\emptyset \neq I \subseteq \{1, \dots, \eta(S_{W_s})\}$, in general our random sets and the Boolean model differ mainly in two aspects in which Boolean models and unions of the cylinders of Poisson cylinder processes do not deviate. First, X_{I,W_s} depends on $|I|$ points while $K(\hat{x})$ only depends on the point $\hat{x}$. This also implies that two sets X_{I,W_s}, X_{J,W_s} can depend on the same point for $I \neq J$ if $I \cap J \neq \emptyset$, while $K(\hat{x})$ and $K(\hat{y})$ cannot depend on the same point if $\hat{x} \neq \hat{y}$. The second main difference arises from the possible number of convex sets needed to represent the excursion set as a union of these convex sets. Let $N(A)$ denote the smallest number needed to write a polyconvex set A as a union of $N(A)$ compact convex sets. Then, for the Boolean model Z, $N(Z \cap C^d) \leq \eta(S_{C^d})$ for $C^d = [0,1]^d$. For our model we might need up to $2^{\eta(S_{C^d})} - 1$ sets (cf. Proposition 6.7). This means that $\mathbb{E}[2^{N(Z_u \cap C^d)}]$ is not necessarily finite (see also Example 6.12) and therefore, Z_u is in contrast to Z in general not a standard random set as defined in [98, Definition 9.2.1].

In this section we want to derive the previously mentioned asymptotic results, which are already known for the Boolean model and Poisson cylinder processes, also for excursion sets of Poisson shot noise processes with polyconvex excursion sets in the observation windows. Similarly to the case of the Boolean model treated in [55] or the case of Poisson cylinder processes in [20], the proofs of this section are based on the Fock space representation and the Malliavin-Stein method. In particular, the proof for the variance asymptotics in Theorem 6.13 follows the strategy of the proof of [55, Theorem 3.1] and the proof for the central limit theorems in Theorem 6.22 uses similar arguments as the proof of [20, Theorem 3.5]. The main difference and difficulty compared to the proofs in [20, 55] is that $\mathbb{E}[2^{N(Z_u \cap C^d)}]$ is in the case of Poisson shot noise processes not necessarily finite as mentioned above. This makes the case of Poisson shot noise processes substantially more intricate and requires the usage of the following dynamic decomposition.

6.3.1 A dynamic decomposition

For a set $K \in \mathcal{K}^d$ we use a dynamic decomposition, which depends on the number of points whose corresponding supports hit but do not cover K. For this, we define $\eta([Q]^*)$ for $Q \in \mathcal{K}^d$ as the number of points $\hat{x} \in \eta$ for which $K(\hat{x}) \cap Q \notin \{\emptyset, Q\}$.

For $n \in \mathbb{N}_0$ and $z \in \mathbb{Z}^d$ let $Q_{n,z} = 2^{-n}(z + [0,1]^d)$ be the cube of side length 2^{-n} and left lower corner $2^{-n}z$. We start covering K with $\bigcup_{z\in\mathbb{Z}^d :\, Q_{0,z}\cap K\neq\emptyset} Q_{0,z}$. Then, for $n = 0, 1, 2, \dots$, if $\eta([Q_{n,z}]^*) > L$ for a fixed constant $L \in \mathbb{N}$, we iteratively decompose $Q_{n,z}$ in 2^d cubes $Q_{n+1,p_1}, \dots, Q_{n+1,p_{2^d}}$ with $\{p_1, \dots, p_{2^d}\} = \{2z + \sum_{j\in J} e_j \text{ for } J \subseteq \{1, \dots, d\}\}$, where e_j denotes the unit vector in direction j. Thus, at the end we cover K with a dynamic grid of cubes, i.e. we consider

$$\bigcup_{(n,z)\in I_{K,L}} Q_{n,z}$$

with

$$I_{K,L} = \{(n,z) \in \mathbb{N}_0 \times \mathbb{Z}^d \colon Q_{n,z} \cap K \neq \emptyset, \eta([Q_{n-1,\lfloor z/2 \rfloor}]^*) > L, \eta([Q_{n,z}]^*) \leq L\},$$

with $\eta([Q_{-1,\lfloor z/2 \rfloor}]^*) = L + 1$ and $\lfloor z/2 \rfloor = (\lfloor z_1/2 \rfloor, \dots, \lfloor z_d/2 \rfloor)$ for $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$, where $\lfloor \cdot \rfloor$ stands for the floor function. For $(m,p) \in I_{K,L}$ let

$$N(m,p) = \{(n,z) \in I_{K,L} \colon Q_{m,p} \cap Q_{n,z} \neq \emptyset, n \leq m\}.$$

Then, the number of adjacent cubes $|N(m,p)|$ which are not smaller than the current cube itself is bounded by 3^d for geometric reasons.

We use the following lemma to control the probability that $(n,z) \in I_{K,L}$ for $n \in \mathbb{N}$ and $z \in \mathbb{Z}^d$.

Lemma 6.8. *For every $L \in \mathbb{N}$ there exists a constant $C > 0$, only depending on the first moments in* (6.15), *d, L and γ, such that for all $n \in \mathbb{N}$ and $z \in \mathbb{Z}^d$,*

$$\mathbb{P}(\eta([Q_{n-1,\lfloor z/2 \rfloor}]^*) > L, \eta([Q_{n,z}]^*) \leq L) \leq C 2^{-n(L+1)}.$$

Proof. For $n \in \mathbb{N}$ and $z \in \mathbb{Z}^d$ define

$$P(n,z) = \{\hat{x} \in \mathbb{R}^d \times \mathbb{M} \colon Q_{n-1,\lfloor z/2 \rfloor} \cap K(\hat{x}) \notin \{\emptyset, Q_{n-1,\lfloor z/2 \rfloor}\}\}.$$

Let $c_{n,z}$ denote the centre of $Q_{n-1,\lfloor z/2 \rfloor}$. Then, $Q_{n-1,\lfloor z/2 \rfloor} \cap K(\hat{x}) \notin \{\emptyset, Q_{n-1,\lfloor z/2 \rfloor}\}$ implies that $d(\partial K(\hat{x}), c_{n,z}) \leq \frac{\sqrt{d}}{2^n}$, because $\partial K(\hat{x})$ has to intersect $Q_{n-1,\lfloor z/2 \rfloor}$. Therefore, since $K(\hat{x}) = K_m + x$ for $\hat{x} = (x,m)$,

$$\begin{aligned}
\lambda(P(n,z)) &\leq \gamma \int_{\mathbb{M}} \lambda_d\Big(\Big\{x \in \mathbb{R}^d \colon d(\partial K(\hat{x}), c_{n,z}) \leq \frac{\sqrt{d}}{2^n}\Big\}\Big)\, \mathbb{Q}(\mathrm{d}m) \\
&= \gamma \int_{\mathbb{M}} \lambda_d\Big(\Big\{x \in \mathbb{R}^d \colon d(\partial K_m, c_{n,z} - x) \leq \frac{\sqrt{d}}{2^n}\Big\}\Big)\, \mathbb{Q}(\mathrm{d}m) \\
&= \gamma \int_{\mathbb{M}} \lambda_d\Big(\Big\{x \in \mathbb{R}^d \colon d(\partial K_m, x) \leq \frac{\sqrt{d}}{2^n}\Big\}\Big)\, \mathbb{Q}(\mathrm{d}m),
\end{aligned}$$

where the last step uses the translation invariance of the volume. For fixed $m \in \mathbb{M}$ it holds by [55, Equation (3.19)] and the convexity of K_m,

$$\begin{aligned}
\lambda_d\Big(\Big\{x \in K_m \colon d(\partial K_m, x) \leq \frac{\sqrt{d}}{2^n}\Big\}\Big) &\leq \lambda_d\Big(\Big(K_m + B^d\Big(\mathbf{0}, \frac{\sqrt{d}}{2^n}\Big)\Big) \Big\backslash K_m\Big) \\
&= \lambda_d\Big(\Big\{x \in \mathbb{R}^d \backslash K_m \colon d(\partial K_m, x) \leq \frac{\sqrt{d}}{2^n}\Big\}\Big).
\end{aligned}$$

Together with Steiner's formula (2.1) we get

$$\lambda(P(n,z)) \leq 2\gamma \int_{\mathbb{M}} \lambda_d\Big(\Big(K_m + B^d\Big(\mathbf{0}, \frac{\sqrt{d}}{2^n}\Big)\Big)\Big\backslash K_m\Big)\, \mathbb{Q}(\mathrm{d}m)$$

$$= 2\gamma \sum_{k=0}^{d-1} \Big(\frac{\sqrt{d}}{2^n}\Big)^{d-k} \kappa_{d-k} \int_{\mathbb{M}} V_k(K_m)\, \mathbb{Q}(\mathrm{d}m) \leq c 2^{-n} \tag{6.18}$$

for a suitable constant $c > 0$, which might depend on the first moments in (6.15), d and γ. This provides

$$\begin{aligned}\mathbb{P}(\eta([Q_{n-1,\lfloor z/2 \rfloor}]^*) > L, \eta([Q_{n,z}]^*) \leq L) &\leq \mathbb{P}(\eta(P(n,z)) > L)\\ &= \sum_{s \geq L+1} \mathbb{P}(\eta(P(n,z)) = s)\\ &= \sum_{s \geq L+1} \frac{\lambda(P(n,z))^s}{s!} e^{-\lambda(P(n,z))}\\ &\leq \lambda(P(n,z))^{L+1} \sum_{s \geq L+1} \frac{c^{s-L-1}}{s!},\end{aligned}$$

which shows together with (6.18) the lemma. □

In the following lemma we use our decomposition to bound moments of geometric functionals of excursion sets. For this we define $Z_u(\xi) = \{y \in \mathbb{R}^d : f_\xi(y) \geq u\}$ for a set ξ of points in $\mathbb{R}^d \times \mathbb{M}$, where $f_\xi(y) = \sum_{(x,m) \in \xi} g_m(y - x)$.

Lemma 6.9. *Let φ be a geometric functional and $A = \{\hat{x}_1, \ldots, \hat{x}_t\}$ be a set of t points in $\mathbb{R}^d \times \mathbb{M}$ for some $t \in \mathbb{N}_0$. Denote $\eta_A = \eta + \sum_{j=1}^t \delta_{\hat{x}_j}$. Then it holds for all $K \in \mathcal{K}^d$ and $L \in \mathbb{N}$,*

$$|\varphi(Z_u(\eta_A) \cap K)| \leq C 2^{2^{L+T}} |I_{K,L}|$$

for $T = |\{j \in \{1, \ldots, t\} : \hat{K}_j \cap K \neq \{\emptyset, K\}\}|$ and some constant $C > 0$, which may depend on d. For $K_1, K_2 \in \mathcal{K}^d$, $k_1, k_2 \in \mathbb{N}_0$ and $L \in \mathbb{N}$ with $L \geq 2d \max\{k_1, k_2\}$ we have

$$\mathbb{E}[|I_{K_1,L}|^{k_1} |I_{K_2,L}|^{k_2}] \leq C_{k_1,k_2} V_d\Big(K_1^{\sqrt{d}}\Big)^{k_1} V_d\Big(K_2^{\sqrt{d}}\Big)^{k_2}$$

for some constant $C_{k_1,k_2} > 0$, which may depend on k_1, k_2, d, γ, L and the first moments in (6.15) and where $K_i^{\sqrt{d}} = (K_i)^{\sqrt{d}}$ for $i \in \{1, 2\}$.

Proof. Denote by $\hat{x}_{t+1}, \ldots, \hat{x}_{t+\eta(S_K)}$ the points of the Poisson process in S_K. Let $\emptyset \neq I \subseteq \{1, \ldots, t + \eta(S_K)\}$ and $X_{I,K}$ be defined as in (6.17). For a set $Q \in \mathcal{K}^d$ with $Q \subseteq K$ and $j \in \{1, \ldots, t + \eta(S_K)\}$ we have $X_{I,K} \cap Q \subseteq X_{I \cup \{j\},K} \cap Q$ if $Q \subseteq \hat{K}_j$ and $X_{I \cup \{j\},K} \cap Q = \emptyset$ if $Q \cap \hat{K}_j = \emptyset$. Thus, for $J_1 = \{j \in \{1, \ldots, t\} : \hat{K}_j \cap Q \neq \{Q, \emptyset\}\}$, $J_2 = \{j \in \{t+1, \ldots, t + \eta(S_K)\} : \hat{K}_j \cap Q \neq \{Q, \emptyset\}\}$ and $M = \{j \in \{1, \ldots, t + \eta(S_K)\} : \hat{K}_j \cap Q = Q\}$ we have

$$Z_u(\eta_A) \cap Q = \bigcup_{\substack{I \subseteq J_1 \cup J_2 :\\ I \cup M \neq \emptyset}} X_{I \cup M,K} \cap Q.$$

Under the condition that $\eta([Q]^*) \leq L$ this is the union of at most $2^{L+|J_1|}$ sets. Note that for $Q \subseteq K$, it holds $|J_1| \leq T$. Hence, if $Q \subseteq K$ is contained in some translate of the unit cube we have with (2.7) and the inclusion exclusion principle (2.8),

$$|\varphi(Z_u(\eta_A) \cap Q)| \leq \sum_{\emptyset \neq J \subseteq \mathcal{P}(J_1 \cup J_2)} |\varphi(\cap_{I \in J} X_{I \cup M, K} \cap Q)| \leq \sum_{\emptyset \neq J \subseteq \mathcal{P}(J_1 \cup J_2)} M_\varphi$$
$$\leq 2^{2^{L+T}} M_\varphi.$$

Thus, we have for $K \in \mathcal{K}^d$,

$$\begin{aligned}
|\varphi(Z_u(\eta_A) \cap K)| &= \Big|\varphi\Big(Z_u(\eta_A) \cap K \cap \bigcup_{(n,z) \in I_{K,L}} Q_{n,z}\Big)\Big| \\
&\leq \sum_{\emptyset \neq J \subseteq I_{K,L}} \Big|\varphi\Big(Z_u(\eta_A) \cap K \cap \bigcap_{(n,z) \in J} Q_{n,z}\Big)\Big| \\
&\leq \sum_{(m,p) \in I_{K,L}} \sum_{\substack{J \subseteq N(m,p): \\ (m,p) \in J}} \Big|\varphi\Big(Z_u(\eta_A) \cap K \cap \bigcap_{(n,z) \in J} Q_{n,z}\Big)\Big| \\
&\leq 2^{3^d} 2^{2^{L+T}} M_\varphi |I_{K,L}|,
\end{aligned}$$

where the last inequality holds because of $|N(m,p)| \leq 3^d$ for $(m,p) \in I_{K,L}$. This provides the first part of the lemma.

Now, let $K_1, K_2 \in \mathcal{K}^d$. For the second part of the lemma note at first that the statement is clear for $k_1 = k_2 = 0$. For the other cases we define $\tilde{I}_r = \{(n,z) \in \mathbb{N}_0 \times \mathbb{Z}^d : Q_{n,z} \cap K_r \neq \emptyset\}$ for $r = 1, 2$ and

$$\ell_r = |\{z \in \mathbb{Z}^d : Q_{0,z} \cap K_r \neq \emptyset\}| \leq V_d(K_r + B^d(\mathbf{0}, \sqrt{d})) = V_d\Big(K_r^{\sqrt{d}}\Big). \tag{6.19}$$

Then, for fixed $n \in \mathbb{N}$, there are up to $2^{nd}\ell_r$ vectors $z \in \mathbb{Z}^d$ such that $(n,z) \in \tilde{I}_r$. Together with Lemma 6.8 we have for $r \in \{1,2\}$ and $k_r \geq 1$,

$$\begin{aligned}
\mathbb{E}[|I_{K_r,L}|^{2k_r}] &= \sum_{\substack{(n_i,z_i) \in \tilde{I}_r, \\ i \in \{1,\dots,2k_r\}}} \mathbb{E}[\mathbb{1}\{(n_i,z_i) \in I_{K_r,L} \text{ for all } i \in \{1,\dots,2k_r\}] \\
&\leq 2k_r \sum_{\substack{(n_i,z_i) \in \tilde{I}_r, \\ n_1 \geq n_i, \\ i \in \{1,\dots,2k_r\}}} \mathbb{E}[\mathbb{1}\{(n_1,z_1) \in I_{K_r,L}\}] \\
&= 2k_r \sum_{\substack{(n_i,z_i) \in \tilde{I}_r, \\ n_1 \geq n_i, \\ i \in \{1,\dots,2k_r\}}} \mathbb{P}(\eta([Q_{n_1-1,\lfloor z_1/2 \rfloor}]^*) > L, \eta([Q_{n_1,z_1}]^*) \leq L) \\
&\leq 2k_r \left[\ell_r^{2k_r} + \sum_{n_1=1}^{\infty} \sum_{\substack{n_i \in \{0,\dots,n_1\}, \\ i \in \{2,\dots,2k_r\}}} \sum_{\substack{z_i \in \mathbb{Z}^d : (n_i,z_i) \in \tilde{I}_r, \\ i \in \{1,\dots,2k_r\}}} C 2^{-n_1(L+1)}\right] \\
&\leq 2k_r \ell_r^{2k_r} \left(1 + C \sum_{n_1=1}^{\infty} (n_1+1)^{2k_r-1} 2^{2k_r n_1 d} 2^{-n_1(L+1)}\right) =: C_{k_r} \ell_r^{2k_r} < \infty
\end{aligned}$$

for $L \geq 2dk_r$ and some constant $C_{k_r} > 0$. Then, the Cauchy inequality implies for $L \geq 2d \max\{k_1, k_2\}$,

$$\mathbb{E}[|I_{K_1,L}|^{k_1} |I_{K_2,L}|^{k_2}] \leq \mathbb{E}[|I_{K_1,L}|^{2k_1}]^{1/2} \mathbb{E}[|I_{K_2,L}|^{2k_2}]^{1/2} \leq \sqrt{C_{k_1} C_{k_2}} \ell_1^{k_1} \ell_2^{k_2},$$

which provides together with (6.19) the lemma for $k_1, k_2 \geq 1$. The cases $k_1 = 0, k_2 \geq 1$ and $k_1 \geq 1, k_2 = 0$ follow analogously. □

6.3.2 Expectation

We start with analysing the asymptotic behaviour of the expectation.

Theorem 6.10. *Let φ be a geometric functional and let $u > 0$ be fixed. If $(g_m)_{m \in \mathbb{M}}$ fulfils Assumption 6.5 and* (6.15) *is valid, we have*

$$\lim_{s \to \infty} \frac{\mathbb{E}[\varphi(Z_u \cap W_s)]}{V_d(W_s)} = \mathbb{E}[\varphi(Z_u \cap C_0^d)],$$

where $C_0^d = [0,1)^d$ denotes the half-open d-dimensional unit cube and $\varphi(Z_u \cap C_0^d)$ is defined as in (2.9).

Remark 6.11. For $\varphi = V_d$ one can calculate the expectation directly. Since the Poisson process is stationary, $f_\eta(y)$ has the same distribution as $f_\eta(0)$. Therefore, $\mathbb{P}(y \in Z_u) = \mathbb{P}(0 \in Z_u)$ for all $y \in W_s$ and we have

$$\mathbb{E}[V_d(Z_u \cap W_s)] = \mathbb{E}\left[\int_{W_s} \mathbb{1}\{y \in Z_u\} \, \mathrm{d}y\right] = \mathbb{P}(0 \in Z_u) V_d(W_s),$$

which shows the result in Theorem 6.10 not only in the limit but for all $s \geq 1$ as $\mathbb{P}(0 \in Z_u)$ does not depend on the observation window W_s.

A similar result as in Theorem 6.10 was shown in [98, Theorem 9.2.1] for the limit of $\mathbb{E}[\varphi(Z \cap W_s)]$, where Z is a so called standard random set. Remember that for a polyconvex set $A \in \mathcal{R}^d$, $N(A)$ denotes the smallest possible $M \in \mathbb{N}$ such that A can be written as a union of M compact convex sets, i.e. $A = \bigcup_{i=1}^M K_i$ for some compact convex sets K_i, $i \in \{1, \ldots, M\}$. A standard random set Z defined as in [98, Definition 9.2.1] fulfils the integrability condition $\mathbb{E}[2^{N(Z \cap C^d)}] < \infty$. Our excursion sets do not necessarily fulfil this integrability condition, which is essential for the proof of the asymptotic expectation in [98, Theorem 9.2.1], and are therefore not necessarily standard random sets. In fact, if $\eta(S_{C^d}) = n$, the construction in Proposition 6.7 may yield up to $2^n - 1$ convex sets and if $N(Z_u \cap C^d)$ becomes with high probability too large, $\mathbb{E}[2^{N(Z_u \cap C^d)}]$ is no longer finite. Since Proposition 6.7 only shows an upper bound for $N(Z_u \cap C^d)$, the following example shows that $\mathbb{E}[2^{N(Z_u \cap C^d)}]$ is indeed not necessarily finite.

Example 6.12. Let $\mathbb{Q}$ be the probability measure on $\mathbb{M} = \mathbb{N}$ with $\mathbb{Q}(\{k\}) = (1-p)p^{k-1}$ for $k \in \mathbb{N}$ and some fixed $p \in (2^{-1/4}, 1)$, i.e. the marks are geometrically distributed with parameter p. We consider the case $\gamma = 1$ and $d = 2$, where for $m \in \mathbb{N}$ and $x \in \mathbb{R}^2$, $g_m \colon \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ is defined by

$$g_m(x) = \begin{cases} \frac{u}{2} & \text{for } x \in K_m, \\ 0 & \text{else,} \end{cases}$$

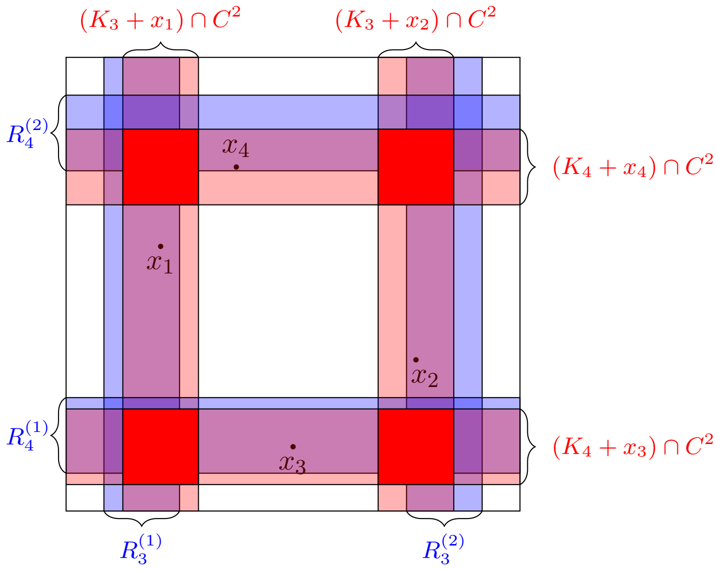

Figure 6.2: Configuration in C^2 in Example 6.12 for $i=2$, if there is exactly one point in each of the sets $R_3^{(1)}$, $R_3^{(2)}$, $R_4^{(1)}$ and $R_4^{(2)}$, denoted by x_1, x_2, x_3 and x_4, where we omit the representation of the marks in the figure for simplicity. If the corresponding support of no other point hits C^2, the red region is the corresponding excursion set $Z_u \cap C^2$, which fulfils $N(Z_u \cap C^d) = i^2 = 4$.

where $K_{2j-1} = [-\frac{1}{4(2j-1)}, \frac{1}{4(2j-1)}] \times [-1,1]$ and $K_{2j} = [-1,1] \times [-\frac{1}{4(2j-1)}, \frac{1}{4(2j-1)}]$ for $j \in \mathbb{N}$. One can show that $\mathbb{E}[2^{N(Z_u \cap C^2)}] = \infty$ by constructing configurations with $2i$ points and $N(Z_u \cap C^2) = i^2$ for all $i \in \mathbb{N}$, which arise with a sufficiently large probability. The idea of the following construction is that we divide the $2i$ points in two groups of size i such that the corresponding shifted supports of two points from different groups overlap in a cube, which does not intersect the shifted support of any other point and hence provides that $N(Z_u \cap C^2) = i^2$.

To this end, for $i \in \mathbb{N}$ and $k \in \{1,\dots,i\}$ let $R_{2i-1}^{(k)} = [\frac{8k-7}{4(2i-1)}, \frac{8k-5}{4(2i-1)}] \times [0,1] \times \{2i-1\}$ and $R_{2i}^{(k)} = [0,1] \times [\frac{8k-7}{4(2i-1)}, \frac{8k-5}{4(2i-1)}] \times \{2i\}$. Then, if for some fixed $i \in \mathbb{N}$,

$$\eta(R_{2i-1}^{(k)}) = \eta(R_{2i}^{(k)}) = 1 \quad \text{for} \quad k \in \{1,\dots,i\} \quad \text{and} \quad \eta\Big(S_{C^2} \Big\backslash \bigcup_{k=1}^{i} (R_{2i-1}^{(k)} \cup R_{2i}^{(k)})\Big) = 0,$$

it holds $N(Z_u \cap C^2) = i^2$. For an illustration in the case $i=2$ see Figure 6.2. Since

$$\lambda(R_{2i-1}^{(k)}) = \lambda_2 \otimes \mathbb{Q}(R_{2i-1}^{(k)}) = \frac{2}{4(2i-1)}\mathbb{Q}(\{2i-1\}) = \frac{1-p}{2(2i-1)}p^{2i-2} \leq \frac{(1-p)}{2}, \quad (6.20)$$

$$\lambda(R_{2i-1}^{(k)}) \geq \lambda(R_{2i}^{(k)}) = \frac{1-p}{2(2i-1)}p^{2i-1} \quad (6.21)$$

and

$$\lambda\Big(S_{C^2} \Big\backslash \bigcup_{k=1}^{i} (R_{2i-1}^{(k)} \cup R_{2i}^{(k)})\Big) \leq \lambda(S_{C^2}) \leq \lambda_2([-2,2]^2) = 16, \quad (6.22)$$

we have

$$\mathbb{E}[2^{N(Z_u\cap C^2)}] \geq \sum_{i=1}^{\infty} 2^{i^2}\mathbb{P}(N(Z_u\cap C^2)=i^2, \eta(S_{C^2})=2i)$$

$$\geq \sum_{i=1}^{\infty} 2^{i^2}\mathbb{P}\Big(\eta(R_{2i-1}^{(k)})=\eta(R_{2i}^{(k)})=1 \text{ for } k\in\{1,\dots,i\}, \eta\Big(S_{C^2}\setminus\bigcup_{k=1}^{i}(R_{2i-1}^{(k)}\cup R_{2i}^{(k)})\Big)=0\Big)$$

$$\geq \sum_{i=1}^{\infty} 2^{i^2}\prod_{k=1}^{i}\lambda(R_{2i-1}^{(k)})\lambda(R_{2i}^{(k)})e^{-\lambda(R_{2i-1}^{(k)})-\lambda(R_{2i}^{(k)})}e^{-\lambda(S_{C^2}\setminus\cup_{k=1}^{i}(R_{2i-1}^{(k)}\cup R_{2i}^{(k)}))}$$

$$\geq c\sum_{i=1}^{\infty} 2^{i^2}\prod_{k=1}^{i}\lambda(R_{2i}^{(k)})^2$$

for a suitable constant $c>0$, where c depends on the bounds in (6.20) and (6.22). Hence, with (6.21),

$$\mathbb{E}[2^{N(Z_u\cap C^2)}] \geq c\sum_{i=1}^{\infty} 2^{i^2}\frac{(1-p)^{2i}}{2^{2i}(2i-1)^{2i}}p^{2i(2i-1)} \geq c\sum_{i=1}^{\infty}(2p^4)^{i^2}\Big(\frac{(1-p)^2}{p^2\cdot 4(2i-1)^2}\Big)^i=\infty$$

because $2p^4>1$, which shows that Z_u is not a standard random set for $p\in(2^{-1/4},1)$.

Proof of Theorem 6.10. By Lemma 6.9 we have for $\mathcal{K}^d\ni K\subseteq C^d$,

$$\mathbb{E}[|\varphi(Z_u\cap K)|]\leq cV_d(K^{\sqrt{d}})\leq cV_d(C^d+B^d(\mathbf{0},\sqrt{d}))<\infty \tag{6.23}$$

for some constant $c>0$, i.e. $\varphi(Z_u\cap K)$ is integrable. By the translation invariance and the additivity of φ, $\varphi(Z_u\cap A)$ is also integrable for $A\in\mathcal{R}^d$. Hence, we can define a function $\phi\colon \mathcal{R}^d\to\mathbb{R}$ by

$$\phi(A)=\mathbb{E}[\varphi(Z_u\cap A)].$$

This function is clearly additive, translation invariant by the stationarity of Z_u and locally bounded due to (6.23). Hence, we can apply Lemma 2.6, which shows Theorem 6.10. □

6.3.3 Variance

Under a second moment condition we now consider the asymptotic behaviour of the variance.

Theorem 6.13. *Assume that $(g_m)_{m\in\mathbb{M}}$ fulfils Assumption 6.5 and* (6.16) *is valid for $k=2$. Let φ be a geometric functional and let $u>0$ be fixed. Then, the limit*

$$\lim_{s\to\infty}\frac{\mathrm{Var}[\varphi(Z_u\cap W_s)]}{V_d(W_s)}=\sigma_0$$

exists, is finite and given by

$$\sigma_0=\sum_{n=1}^{\infty}\frac{\gamma}{n!}\int_{\mathbb{M}}\int_{(\mathbb{R}^d\times\mathbb{M})^{n-1}}(\mathbb{E}[D^n_{(\mathbf{0},m),\hat{x}_2,\dots,\hat{x}_n}\varphi(Z_u\cap K)])^2\,\lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n))\,\mathbb{Q}(\mathrm{d}m),$$

where $K=K_m\cap\bigcap_{j=2}^n\hat{K}_j$ and $\mathbf{0}=(0,\dots,0)\in\mathbb{R}^d$.

Similar results as in Theorem 6.13 were already shown for geometric functionals of the Boolean model in [55, Theorem 3.1] and for Poisson cylinder processes in [20, Theorem 3.10].

In order to calculate the asymptotic variance using the Fock space representation, we need to control the expectation of the n-th difference operator. For the proof of the central limit theorem we additionally require bounds for moments and products of difference operators, which we establish here as well.

Lemma 6.14. *Let $\hat{x}_j \in \mathbb{R}^d \times \mathbb{M}$ for $j \in \{1, \dots, n\}$ and $\hat{K} = \bigcap_{j=1}^n \hat{K}_j$. Then it holds*

$$D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s) = D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s \cap \hat{K}) \tag{6.24}$$

and for $\hat{y}_\ell,\ \hat{z}_s \in \mathbb{R}^d \times \mathbb{M}$ with $\ell \in \{1, \dots, n_1\}$, $s \in \{1, \dots, n_2\}$, $n_1, n_2 \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{N}_0$,

$$\begin{aligned}
&\mathbb{E}[|D^{n_1}_{\hat{y}_1,\dots,\hat{y}_{n_1}} \varphi(Z_u \cap W_s)|^{k_1} |D^{n_2}_{\hat{z}_1,\dots,\hat{z}_{n_2}} \varphi(Z_u \cap W_s)|^{k_2}] \\
&\leq c_{k_1,k_2} 2^{k_1 n_1 + k_2 n_2} V_d\Big(\hat{A}_1^{\sqrt{d}} \cap W_s^{\sqrt{d}}\Big)^{k_1} V_d\Big(\hat{A}_2^{\sqrt{d}} \cap W_s^{\sqrt{d}}\Big)^{k_2}
\end{aligned} \tag{6.25}$$

for some constant $c_{k_1,k_2} > 0$, which depends on k_1 and k_2 and where $\hat{A}_1 = \bigcap_{\ell=1}^{n_1} K(\hat{y}_\ell)$ and $\hat{A}_2 = \bigcap_{s=1}^{n_2} K(\hat{z}_s)$.

Proof. For $j \in \{1, \dots, n\}$ and $\hat{x}_j = (x_j, m_j) \in \mathbb{R}^d \times \mathbb{M}$ we have $g_{m_j} \geq 0$ and $g_{m_j}(y - x_j) = 0$ for $y \notin \hat{K}_j$. With the additivity of geometric functionals we get for $K \in \mathcal{K}^d$,

$$\begin{aligned}
D_{\hat{x}_j} \varphi(Z_u \cap K) &= \varphi(\{y \in K \colon f_\eta(y) \geq u - g_{m_j}(y - x_j)\}) - \varphi(\{y \in K \colon f_\eta(y) \geq u\}) \\
&= \varphi(\{y \in K \colon f_\eta(y) \geq u\} \cup \{y \in K \cap \hat{K}_j \colon f_\eta(y) \geq u - g_{m_j}(y - x_j)\}) \\
&\quad - \varphi(\{y \in K \colon f_\eta(y) \geq u\}) \\
&= \varphi(\{y \in K \cap \hat{K}_j \colon f_\eta(y) \geq u - g_{m_j}(y - x_j)\}) \\
&\quad - \varphi(\{y \in K \cap \hat{K}_j \colon f_\eta(y) \geq u\}) \\
&= D_{\hat{x}_j} \varphi(Z_u \cap K \cap \hat{K}_j).
\end{aligned}$$

Due to (2.17) this implies for the difference operator of order n,

$$\begin{aligned}
D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s) &= D^{n-1}_{\hat{x}_1,\dots,\hat{x}_{n-1}} (D_{\hat{x}_n} \varphi(Z_u \cap W_s)) \\
&= D^{n-1}_{\hat{x}_1,\dots,\hat{x}_{n-1}} (D_{\hat{x}_n} \varphi(Z_u \cap W_s \cap \hat{K}_n)) \\
&= D^n_{\hat{x}_n,\hat{x}_1,\dots,\hat{x}_{n-1}} \varphi(Z_u \cap W_s \cap \hat{K}_n) \\
&= D^{n-1}_{\hat{x}_n,\hat{x}_1,\dots,\hat{x}_{n-2}} (D_{\hat{x}_{n-1}} \varphi(Z_u \cap W_s \cap \hat{K}_n)) \\
&= D^{n-1}_{\hat{x}_n,\hat{x}_1,\dots,\hat{x}_{n-2}} (D_{\hat{x}_{n-1}} \varphi(Z_u \cap W_s \cap \hat{K}_{n-1} \cap \hat{K}_n)) \\
&= D^n_{\hat{x}_{n-1},\hat{x}_n,\hat{x}_1,\dots,\hat{x}_{n-2}} \varphi(Z_u \cap W_s \cap \hat{K}_{n-1} \cap \hat{K}_n) \\
&= \dots = D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s \cap \hat{K}),
\end{aligned}$$

which shows (6.24).

Equations (6.24), (2.16) and Lemma 6.9 provide for $L \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$|D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s)| = |D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s \cap \hat{K})|$$

$$= \Big| \sum_{J\subseteq\{1,\dots,n\}} (-1)^{n-|J|}\varphi\Big(Z_u\Big(\eta+\sum_{j\in J}\delta_{\hat{x}_j}\Big)\cap W_s\cap \hat{K}\Big)\Big|$$
$$\leq \sum_{k=0}^{n}\binom{n}{k} C2^{2^L}|I_{\hat{K}\cap W_s,L}| = C2^n 2^{2^L}|I_{\hat{K}\cap W_s,L}|.$$

Note that T from Lemma 6.9 vanishes since $\hat{K}_j\cap\hat{K}\cap W_s=\hat{K}\cap W_s$ for $j\in\{1,\dots,n\}$. This provides

$$|D^{n_1}_{\hat{y}_1,\dots,\hat{y}_{n_1}}\varphi(Z_u\cap W_s)|^{k_1}|D^{n_2}_{\hat{z}_1,\dots,\hat{z}_{n_2}}\varphi(Z_u\cap W_s)|^{k_2}$$
$$\leq \big(C2^{2^L}\big)^{k_1+k_2}2^{k_1n_1+k_2n_2}|I_{\hat{A}_1\cap W_s,L}|^{k_1}|I_{\hat{A}_2\cap W_s,L}|^{k_2}.$$

Now, Lemma 6.9 completes the proof of (6.25) with a choice of $L=2d\max\{k_1,k_2\}$ since $V_d\Big((\hat{A}_i\cap W_s)^{\sqrt{d}}\Big)\leq V_d\Big(\hat{A}_i^{\sqrt{d}}\cap W_s^{\sqrt{d}}\Big)$ for $i\in\{1,2\}$. □

Remark 6.15. Note that for $k\in\mathbb{N}$, Lemma 6.14 especially provides

$$\mathbb{E}[|D^n_{\hat{x}_1,\dots,\hat{x}_n}\varphi(Z_u\cap\hat{K}\cap W_s)|^k]\leq c_{k,0}2^{kn}V_d(\hat{K}^{\sqrt{d}}\cap W_s^{\sqrt{d}})^k$$

for $\hat{K}=\bigcap_{j=1}^n\hat{K}_j$ for all $s\geq 1$, which implies that

$$\mathbb{E}[|D^n_{\hat{x}_1,\dots,\hat{x}_n}\varphi(Z_u\cap\hat{K})|^k]\leq c_{k,0}2^{kn}V_d(\hat{K}^{\sqrt{d}})^k$$

since s can be chosen large enough such that $\hat{K}\subseteq W_s$.

Moreover, we use the following lemma, which is similar to [55, Lemma 3.5].

Lemma 6.16. *There exists a constant $C_3>0$, depending on d, γ and the first moments in* (6.15)*, such that*

$$\int_{(\mathbb{R}^d\times\mathbb{M})^n}\sum_{k=0}^{d}V_k(K\cap\hat{K}_1\cap\ldots\cap\hat{K}_n)\ \lambda^n(\mathrm{d}(\hat{x}_1,\dots,\hat{x}_n))\leq C_3^n\sum_{k=0}^{d}V_k(K) \tag{6.26}$$

for all $K\in\mathcal{K}^d$ and $n\in\mathbb{N}$.

Proof. The proof is analogous to the proof of [55, Lemma 3.5]. With the help of (2.4), we compute

$$\int_{(\mathbb{R}^d\times\mathbb{M})^n}\sum_{k=0}^{d}V_k(K\cap\hat{K}_1\cap\ldots\cap\hat{K}_n)\ \lambda^n(\mathrm{d}(\hat{x}_1,\dots,\hat{x}_n))$$
$$=\gamma\sum_{k=0}^{d}\int_{(\mathbb{R}^d\times\mathbb{M})^{n-1}}\int_{\mathbb{M}}\int_{\mathbb{R}^d}V_k(K\cap(K_{m_1}+x_1)\cap\hat{K}_2\cap\ldots\cap\hat{K}_n)\,\mathrm{d}x_1\ \mathbb{Q}(\mathrm{d}m_1)$$
$$\times\ \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n))$$
$$\leq c\gamma\sum_{k=0}^{d}\sum_{j=k}^{d}\int_{\mathbb{M}}V_j(K_{m_1})\ \mathbb{Q}(\mathrm{d}m_1)\int_{(\mathbb{R}^d\times\mathbb{M})^{n-1}}\sum_{r=0}^{d}V_r(K\cap\hat{K}_2\cap\ldots\cap\hat{K}_n)$$
$$\times\ \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n)).$$

Applying this scheme iteratively completes the proof. □

Proof of Theorem 6.13. Since the second moment of $|\varphi(Z_u \cap W_s)|$ exists by Lemma 6.9, the Fock space representation (2.18) and Equation (6.24) provide

$$\begin{aligned}\operatorname{Var}[\varphi(Z_u \cap W_s)] &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d \times \mathbb{M})^n} (\mathbb{E}[D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s)])^2 \, \lambda^n(\mathrm{d}(\hat{x}_1,\dots,\hat{x}_n)) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d \times \mathbb{M})^n} (\mathbb{E}[D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s \cap \hat{K})])^2 \, \lambda^n(\mathrm{d}(\hat{x}_1,\dots,\hat{x}_n)),\end{aligned}$$

where $\hat{K} = \bigcap_{j=1}^n \hat{K}_j$. Let

$$f_{n,s}(m_1) = \frac{\gamma}{V_d(W_s)} \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d \times \mathbb{M})^{n-1}} (\mathbb{E}[D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s \cap \hat{K})])^2 \, \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n)) \, \mathrm{d}x_1$$

and for $K = K_{m_1} \cap \bigcap_{j=2}^n \hat{K}_j$ and $\mathbf{0} = (0,\dots,0) \in \mathbb{R}^d$,

$$\begin{aligned}f_n(m_1) &= \gamma \int_{(\mathbb{R}^d \times \mathbb{M})^{n-1}} (\mathbb{E}[D^n_{(\mathbf{0},m_1),\hat{x}_2,\dots,\hat{x}_n} \varphi(Z_u \cap K)])^2 \, \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n)) \\ &= \frac{\gamma}{V_d(W_s)} \int_{\mathbb{R}^d} \mathbb{1}\{x_1 + z \in W_s\} \\ &\qquad \times \int_{(\mathbb{R}^d \times \mathbb{M})^{n-1}} (\mathbb{E}[D^n_{(\mathbf{0},m_1),\hat{x}_2,\dots,\hat{x}_n} \varphi(Z_u \cap K)])^2 \, \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n)) \, \mathrm{d}x_1 \\ &= \frac{\gamma}{V_d(W_s)} \int_{\mathbb{R}^d} \mathbb{1}\{x_1 + z \in W_s\} \\ &\qquad \times \int_{(\mathbb{R}^d \times \mathbb{M})^{n-1}} (\mathbb{E}[D^n_{(x_1,m_1),\hat{x}_2,\dots,\hat{x}_n} \varphi(Z_u \cap \hat{K})])^2 \, \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n)) \, \mathrm{d}x_1\end{aligned} \tag{6.27}$$

for any $z \in K_{m_1}$ by translation invariance. We want to show that

$$\frac{\operatorname{Var}[\varphi(Z_u \cap W_s)]}{V_d(W_s)} = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{M}} f_{n,s}(m_1) \, \mathbb{Q}(\mathrm{d}m_1) \to \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{M}} f_n(m_1) \, \mathbb{Q}(\mathrm{d}m_1) = \sigma_0$$

for $s \to \infty$. To this end, we first apply Fubini's theorem and then show the convergence of the integrals using the dominated convergence theorem.

To apply the dominated convergence theorem we start with bounding $|f_{n,s}(m_1)|$. With Remark 6.15 we have together with the monotonicity of intrinsic volumes of convex sets and (2.2),

$$\begin{aligned}|f_{n,s}(m_1)| &\le \frac{\gamma}{V_d(W_s)} \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d \times \mathbb{M})^{n-1}} V_d(\hat{K}^{\sqrt{d}} \cap W_s^{\sqrt{d}})^2 c_{1,0}^2 4^n \, \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n)) \, \mathrm{d}x_1 \\ &\le \frac{C_1 c_{1,0}^2 4^n \gamma}{V_d(W_s)} \int_{\mathbb{R}^d} V_d(\hat{K}_1^{\sqrt{d}} \cap W_s^{\sqrt{d}}) \int_{(\mathbb{R}^d \times \mathbb{M})^{n-1}} \sum_{j=0}^{d} V_j(\hat{K}) \, \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n)) \, \mathrm{d}x_1.\end{aligned}$$

Equation (6.26) and the translation invariance of intrinsic volumes provide

$$\int_{(\mathbb{R}^d \times \mathbb{M})^{n-1}} \sum_{j=0}^{d} V_j\Big(\hat{K}_1 \cap \bigcap_{j=2}^{n} \hat{K}_j\Big) \, \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n)) \le C_3^{n-1} \sum_{k=0}^{d} V_k(K_{m_1}).$$

With (2.2), (2.3) and

$$V_d(W_s^{\sqrt{d}}) = V_d(sW + B^d(\mathbf{0}, \sqrt{d})) \le s^d V_d(W + B^d(\mathbf{0}, \sqrt{d})) = s^d V_d(W^{\sqrt{d}})$$

for $s \ge 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} V_d(\hat{K}_1^{\sqrt{d}} \cap W_s^{\sqrt{d}})\,\mathrm{d}x_1 &= \int_{\mathbb{R}^d} V_d((K_{m_1}^{\sqrt{d}} + x_1) \cap W_s^{\sqrt{d}})\,\mathrm{d}x_1 = V_d(K_{m_1}^{\sqrt{d}})V_d(W_s^{\sqrt{d}})\\ &\le C_1 \sum_{k=0}^{d} V_k(K_{m_1}) s^d V_d(W^{\sqrt{d}}).\end{aligned}$$

Altogether, this leads with the positivity of the intrinsic volumes for convex sets and $V_d(sW) = s^d V_d(W)$ to

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n!}|f_{n,s}(m_1)| &\le \sum_{n=1}^{\infty} \frac{\gamma c_{1,0}^2 4^n C_3^{n-1}}{n!} C_1^2 \frac{V_d(W^{\sqrt{d}})}{V_d(W)} \Big(\sum_{k=0}^{d} V_k(K_{m_1})\Big)^2\\ &\le e^{4C_3}\gamma c_{1,0}^2 \frac{C_1^2 V_d(W^{\sqrt{d}})}{C_3 V_d(W)} \Big(\sum_{k=0}^{d} V_k(K_{m_1})\Big)^2, \qquad (6.28)\end{aligned}$$

which is independent of s and by the second moment assumption in (6.16) integrable.

In the next step we bound $|f_{n,s}(m_1) - f_n(m_1)|$. By (6.27) it holds for any $z \in K_{m_1}$,

$$\begin{aligned}&|f_{n,s}(m_1) - f_n(m_1)|\\ &= \Big|\frac{\gamma}{V_d(W_s)} \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d\times\mathbb{M})^{n-1}} \Big[(\mathbb{E}[D^n_{\hat{x}_1,\dots,\hat{x}_n}\varphi(Z_u \cap W_s \cap \hat{K})])^2\\ &\qquad - \mathbb{1}\{x_1 + z \in W_s\}(\mathbb{E}[D^n_{\hat{x}_1,\dots,\hat{x}_n}\varphi(Z_u \cap \hat{K})])^2\Big]\ \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n))\,\mathrm{d}x_1\Big|.\end{aligned}$$

The term in the integral becomes 0 if $\hat{K}_1 \subseteq W_s$ or $\hat{K}_1 \cap W_s = \emptyset$. In the first case it vanishes because for $\hat{K}_1 \subseteq W_s$ it holds $\varphi(Z_u \cap \hat{K}) = \varphi(Z_u \cap W_s \cap \hat{K})$ and $x_1 + z \in W_s$. In the second case it vanishes since for $\hat{K}_1 \cap W_s = \emptyset$ we have $\varphi(Z_u \cap W_s \cap \hat{K}) = 0$ and $x_1 + z \notin W_s$. Remark 6.15 and the triangle inequality imply

$$\begin{aligned}&|f_{n,s}(m_1) - f_n(m_1)|\\ &= \Big|\frac{\gamma}{V_d(W_s)} \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d\times\mathbb{M})^{n-1}} \mathbb{1}\{\hat{K}_1 \cap \partial W_s \neq \emptyset\}\Big[(\mathbb{E}[D^n_{\hat{x}_1,\dots,\hat{x}_n}\varphi(Z_u \cap W_s \cap \hat{K})])^2\\ &\qquad - \mathbb{1}\{x_1 + z \in W_s\}(\mathbb{E}[D^n_{\hat{x}_1,\dots,\hat{x}_n}\varphi(Z_u \cap \hat{K})])^2\Big]\ \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n))\,\mathrm{d}x_1\Big|\\ &\le \frac{\gamma}{V_d(W_s)} \int_{\mathbb{R}^d} \mathbb{1}\{\hat{K}_1 \cap \partial W_s \neq \emptyset\}\\ &\qquad \times \int_{(\mathbb{R}^d\times\mathbb{M})^{n-1}} 2V_d(\hat{K}^{\sqrt{d}})^2 c_{1,0}^2 4^n\ \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n))\,\mathrm{d}x_1.\end{aligned}$$

Together with the monotonicity of the intrinsic volumes, (2.2) and (6.26), we have for the second part of the integral

$$\int_{(\mathbb{R}^d\times\mathbb{M})^{n-1}} 2V_d(\hat{K}^{\sqrt{d}})^2 c_{1,0}^2 4^n\ \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n))$$

$$\leq 2c_{1,0}^2 4^n V_d(K_{m_1}^{\sqrt{d}}) \int_{(\mathbb{R}^d\times\mathbb{M})^{n-1}} V_d(\hat{K}^{\sqrt{d}})\ \lambda^{n-1}(\mathrm{d}(\hat{x}_2,\dots,\hat{x}_n))$$
$$\leq 2C_3^{n-1} c_{1,0}^2 4^n V_d(K_{m_1}^{\sqrt{d}}) C_1 \sum_{j=0}^{d} V_j(K_{m_1}).$$

Equation (2.5) provides together with the the homogeneity of the intrinsic volumes

$$\begin{aligned}|f_{n,s}(m_1) - f_n(m_1)| &\leq 2\gamma C_2 C_3^{n-1} c_{1,0}^2 4^n C_1 V_d(K_{m_1}^{\sqrt{d}}) \Big(\sum_{j=0}^{d} V_j(K_{m_1})\Big)^2 \sum_{k=0}^{d-1} \frac{V_k(W_s)}{V_d(W_s)}\\ &\leq 2\gamma C_2 C_3^{n-1} c_{1,0}^2 4^n C_1 V_d(K_{m_1}^{\sqrt{d}}) \Big(\sum_{j=0}^{d} V_j(K_{m_1})\Big)^2 \sum_{k=0}^{d-1} \frac{V_k(W)}{s^{d-k} V_d(W)}\end{aligned}$$

for all $s \geq 1$. Hence,

$$\begin{aligned}&\Big|\sum_{n=1}^{\infty} \frac{1}{n!} f_{n,s}(m_1) - \sum_{n=1}^{\infty} \frac{1}{n!} f_n(m_1)\Big|\\ &\leq e^{4C_3} \gamma c_{1,0}^2 \frac{2C_1C_2}{C_3} V_d(K_{m_1}^{\sqrt{d}}) \Big(\sum_{j=0}^{d} V_j(K_{m_1})\Big)^2 \sum_{k=0}^{d-1} \frac{V_k(W)}{s^{d-k} V_d(W)},\end{aligned}$$

which goes to 0 for $s \to \infty$ and provides the theorem together with (6.28) by the dominated convergence theorem. □

Theorem 6.13 does not answer the question under which conditions $\sigma_0 > 0$. In [55, Section 4] this question was discussed for the special case of the Boolean model and for example in [32, 66, 67] different conditions on the kernel functions were introduced to show the positivity of the asymptotic variance and derive central limit theorems for functionals like the volume of Poisson shot noise processes. As done in the previous section for the volume, we use Theorem 3.1 to show in the following for different families of kernel functions that the asymptotic variance is positive for specific geometric functionals. In contrast to all previous examples we do not only use Theorem 3.1 for $n = 1$ but the choice of n depends on the properties of $(g_m)_{m\in\mathbb{M}}$.

Proposition 6.17. *Let φ be a geometric functional, $u > 0$ and let σ_0 be defined as in Theorem 6.13. Assume* (6.16) *for $k = 2$ and Assumption 6.5 for $(g_m)_{m\in\mathbb{M}}$.*

a) If $\mathbb{Q}(\{m \in \mathbb{M}\colon \varphi(\{x \in \mathbb{R}^d\colon g_m(x) \geq u\}) \neq 0\}) > 0$, it holds $\sigma_0 > 0$.

b) Assume that there exists $\alpha = (\alpha_0,\dots,\alpha_d) \in \mathbb{R}^{d+1}\setminus\{\mathbf{0}\}$ satisfying

$$\varphi = \sum_{i=0}^{d} \alpha_i V_i. \tag{6.29}$$

Then, if $\mathbb{Q}(\{m \in \mathbb{M}\colon \max_{x\in\mathbb{R}^d} g_m(x) \leq u, g_m \text{ strictly concave on } K_m\}) > 0$, it holds $\sigma_0 > 0$.

Remark 6.18. Proposition 6.17 mainly distinguishes the cases $\max_{x\in\mathbb{R}^d} g_m(x) \geq u$ and $\max_{x\in\mathbb{R}^d} g_m(x) \leq u$ for all $m \in M$ and some $M \subseteq \mathbb{M}$ satisfying $\mathbb{Q}(M) > 0$. In the first case, our proof works for general geometric functionals if $\varphi(\{x \in \mathbb{R}^d : g_m(x) \geq u\}) \neq 0$ for enough $m \in M$. In the second case we can show the positivity of the variance for geometric functionals that can be represented as a linear combination of intrinsic volumes if g_m is strictly concave for enough $m \in M$. Note that by Hadwiger's theorem (see for example [98, Theorem 14.4.6]), this class of geometric functionals includes all geometric functionals that are continuous and invariant under rigid motions.

For the proof of the positivity of the variance in *b*), we use three geometric lemmas for which we introduce the following notation. Let $m_1, \dots, m_n \in \mathbb{M}$ and $x_n \in \mathbb{R}^d$ for some $n \in \mathbb{N}$ with $n \geq 2$. For $\mathbf{v} = (v_1, \dots, v_{n-1}) \in (\mathbb{R}^d)^{n-1}$ let $\mathbf{K}(\mathbf{v}) = \bigcap_{i=1}^{n-1}(K_{m_i} + v_i) \cap (K_{m_n} + x_n)$ and let $f_{\mathbf{v}} : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ with $f_{\mathbf{v}}(y) = \sum_{i=1}^{n-1} g_{m_i}(y - v_i) + g_{m_n}(y - x_n)$. Moreover, we define

$$\widetilde{Z}_u(\mathbf{v}) = \Big\{ y \in \mathbf{K}(\mathbf{v}) : f_{\mathbf{v}}(y) \geq u \Big\}.$$

Lemma 6.19. *Let $n \in \mathbb{N}$ with $n \geq 2$, $x_n \in \mathbb{R}^d$ and $m_1, \dots, m_n \in \mathbb{M}$ be such that g_{m_i} is strictly concave on K_{m_i} and $\max_{x\in\mathbb{R}^d} g_{m_i}(x) \leq \frac{u}{n-1}$ for $i \in \{1, \dots, n\}$. Let $(\mathbf{x}_k)_{k\in\mathbb{N}}$ be a sequence in $(\mathbb{R}^d)^{n-1}$ with $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$ for some $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$. Then, for $t \in \{1, \dots, d\}$,*

$$\limsup_{k\to\infty} V_t(\widetilde{Z}_u(\mathbf{x}_k)) \leq V_t(\widetilde{Z}_u(\mathbf{x})).$$

Proof. Let $\mathbf{x} = (x_1, \dots, x_{n-1}) \in (\mathbb{R}^d)^{n-1}$ and $\mathbf{x}_k = (x_{k,1}, \dots, x_{k,n-1}) \in (\mathbb{R}^d)^{n-1}$ for $k \in \mathbb{N}$ with $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$. We start with showing that for each compact set $C \subseteq \mathbb{R}^d$ there exists $K_0 \in \mathbb{N}$ such that

$$\widetilde{Z}_u(\mathbf{x}) \cap C = \emptyset \implies \widetilde{Z}_u(\mathbf{x}_k) \cap C = \emptyset \quad \forall k \geq K_0. \tag{6.30}$$

To this end let $C \subseteq \mathbb{R}^d$ be a compact set with $\widetilde{Z}_u(\mathbf{x}) \cap C = \emptyset$ and define $C_1 = \{y \in C : f_{\mathbf{x}}(y) = u\}$. For each $y \in C_1$ there exists $i \in \{1, \dots, n\}$ with $y \notin K_{m_i} + x_i$ because $y \notin \widetilde{Z}_u(\mathbf{x})$ and hence also $B^d(y, 2\delta(y)) \cap (K_{m_i} + x_i) = \emptyset$ for some $\delta(y) > 0$. Since $K_{m_i} + x_{k,i} \to K_{m_i} + x_i$ in the Hausdorff metric as $k \to \infty$, there exists $K_1(y) \in \mathbb{N}$ with $B^d(y, \delta(y)) \cap (K_{m_i} + x_{k,i}) = \emptyset$ and hence $B^d(y, \delta(y)) \cap \widetilde{Z}_u(\mathbf{x}_k) = \emptyset$ for $k \geq K_1(y)$. Note that by the strict concavity of g_{m_i} on K_{m_i} for $i \in \{1, \dots, n\}$, the maximum of g_{m_i} is only attained at one point for $i \in \{1, \dots, n\}$. Together with $\max_{x\in\mathbb{R}^d} g_{m_i}(x) \leq \frac{u}{n-1}$ it follows $|C_1| \leq 1$ (for $n > 2$) or $|C_1| \leq 2$ (for $n = 2$) for fixed $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$. This is due to the fact that $f_{\mathbf{x}}(y) = u$ outside of $\mathbf{K}(\mathbf{x})$ is only possible if for $\mathbf{x} = (x_1, \dots, x_{n-1})$, $g_{m_i}(y - x_i) = \max_{x\in\mathbb{R}^d} g_{m_i}(x)$ for $n-1$ indices $i \in \{1, \dots, n\}$ and $g_{m_j}(y - x) = 0$ for the remaining index j.

Now, because of $|C_1| \leq 2$ there exists $K_1 \in \mathbb{N}$ such that $\bigcup_{y\in C_1} B^d(y, \delta(y)) \cap \widetilde{Z}_u(\mathbf{x}_k) = \emptyset$ for $k \geq K_1$. Let $C_2 = C \backslash \operatorname{int}\big(\bigcup_{y\in C_1} B^d(y, \delta(y))\big)$. Then, $f_{\mathbf{x}}(y) < u$ for all $y \in C_2$. Assume that there exists a sequence $(k_j)_{j\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$ there exists $y_j \in C_2$ with $f_{\mathbf{x}_{k_j}}(y_j) \geq u$. Since C_2 is compact, there exists a subsequence $(y_{j_\ell})_{\ell\in\mathbb{N}}$ which converges to some $y \in C_2$ as $\ell \to \infty$. As g_{m_i} is upper semi-continuous by Assumption 6.5 for $i \in \{1, \dots, n\}$, we have

$$\limsup_{\ell\to\infty} f_{\mathbf{x}_{k_{j_\ell}}}(y_{j_\ell}) \leq f_{\mathbf{x}}(y) < u,$$

which is a contradiction to the assumption. Hence, $C_2 \cap \widetilde{Z}_u(\mathbf{x}_k) = \emptyset$ for almost every $k \in \mathbb{N}$. Together with the result for C_1 there exists $K_0 \in \mathbb{N}$ with $C \cap \widetilde{Z}_u(\mathbf{x}_k) = \emptyset$ for all $k \geq K_0$.

Additionally, there exists a compact set $Z \subseteq \mathbb{R}^d$ such that $\widetilde{Z}_u(\mathbf{x}_k) \subseteq Z$ for all $k \in \mathbb{N}$ since $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$. Then, if $\widetilde{Z}_u(\mathbf{x}) = \emptyset$, applying (6.30) for $C = Z$ provides $\widetilde{Z}_u(\mathbf{x}_k) = \emptyset$ for k large enough. Thus, $V_t(\widetilde{Z}_u(\mathbf{x}_k)) = V_t(\widetilde{Z}_u(\mathbf{x})) = 0$ for k large enough in the case of $\widetilde{Z}_u(\mathbf{x}) = \emptyset$. If $\widetilde{Z}_u(\mathbf{x}) \neq \emptyset$, for any $\varepsilon > 0$ we can use (6.30) for $C = Z \backslash \operatorname{int}(\widetilde{Z}_u(\mathbf{x}) + B^d(\mathbf{0}, \varepsilon))$, which yields the existence of $K_0(\varepsilon) \in \mathbb{N}$ with $\widetilde{Z}_u(\mathbf{x}_k) \subseteq \widetilde{Z}_u(\mathbf{x}) + B^d(\mathbf{0}, \varepsilon)$ for $k \geq K_0(\varepsilon)$. Hence, $V_t(\widetilde{Z}_u(\mathbf{x}_k)) \leq V_t(\widetilde{Z}_u(\mathbf{x}) + B^d(\mathbf{0}, \varepsilon))$ for any $\varepsilon > 0$ if k is large enough. Since $K \mapsto V_t(K)$ is continuous with respect to the Hausdorff metric for all non-empty compact convex sets (see also Subsection 2.2.1), this provides $\limsup_{k\to\infty} V_t(\widetilde{Z}_u(\mathbf{x}_k)) \leq V_t(\widetilde{Z}_u(\mathbf{x}))$. □

Lemma 6.20. *Let $n \in \mathbb{N}$ with $n \geq 2$, $x_n \in \mathbb{R}^d$ and $m_1, \ldots, m_n \in \mathbb{M}$. Define $h\colon (\mathbb{R}^d)^{n-1} \to \mathbb{R}_{\geq 0}$ by $h(\mathbf{x}) = \max_{y \in \mathbb{R}^d} f_{\mathbf{x}}(y)$ for $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$. Then, h is continuous for all $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ with $r(\mathbf{K}(\mathbf{x})) > 0$, where $r(\cdot)$ denotes the inradius.*

Proof. For $\widetilde{\mathbf{x}} \in (\mathbb{R}^d)^{n-1}$ with $r(\mathbf{K}(\widetilde{\mathbf{x}})) > 0$ let $y_{\max}(\widetilde{\mathbf{x}})$ be a point where the maximum of $f_{\widetilde{\mathbf{x}}}$ is attained. For $k \in \mathbb{N}$ let $\mathbf{x}_k \in (\mathbb{R}^d)^{n-1}$ with $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$ and let $(y_{\max}(\mathbf{x}_{k_j}))_{j\in\mathbb{N}}$ be a sequence which converges to some $\tilde{y}$. Then, since g_{m_i} is upper semi-continuous for $i \in \{1, \ldots, n\}$, we have

$$\limsup_{j\to\infty} f_{\mathbf{x}_{k_j}}(y_{\max}(\mathbf{x}_{k_j})) \leq f_{\mathbf{x}}(\tilde{y}) \leq f_{\mathbf{x}}(y_{\max}(\mathbf{x})) = h(\mathbf{x}).$$

Since $(y_{\max}(\mathbf{x}_k))_{k\in\mathbb{N}}$ is bounded, every subsequence of $(y_{\max}(\mathbf{x}_k))_{k\in\mathbb{N}}$ has a convergent subsequence. Thus,

$$\limsup_{k\to\infty} h(\mathbf{x}_k) = \limsup_{k\to\infty} f_{\mathbf{x}_k}(y_{\max}(\mathbf{x}_k)) \leq h(\mathbf{x}).$$

For the reverse direction we use that for any $\tau > 0$ there exists some $y_\tau \in \mathbb{R}^d$ with $y_\tau \notin \partial(K_{m_i} + x_i)$ for $i \in \{1, \ldots, n\}$ and such that $f_{\mathbf{x}}(y_\tau) \geq f_{\mathbf{x}}(y_{\max}(\mathbf{x})) - \tau = h(\mathbf{x}) - \tau$ as $r(\mathbf{K}(\mathbf{x})) > 0$ and since $g_{m_i}|_{K_{m_i}}$ is continuous and K_{m_i} is convex for all $i \in \{1, \ldots, n\}$. Due to the continuity of g_{m_i} in $y_\tau - x_i$ for $i \in \{1, \ldots, n\}$, we get

$$\liminf_{k\to\infty} h(\mathbf{x}_k) \geq \liminf_{k\to\infty} f_{\mathbf{x}_k}(y_\tau) = f_{\mathbf{x}}(y_\tau) \geq h(\mathbf{x}) - \tau$$

for any $\tau > 0$ and hence $\liminf_{k\to\infty} h(\mathbf{x}_k) \geq h(\mathbf{x})$. Altogether, $\lim\limits_{k\to\infty} h(\mathbf{x}_k) = h(\mathbf{x})$ for any $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ with $r(\mathbf{K}(\mathbf{x})) > 0$. □

Lemma 6.21. *Let $n \in \mathbb{N}$ with $n \geq 2$, $x_n \in \mathbb{R}^d$ and $m_1, \ldots, m_n \in \mathbb{M}$ be such that g_{m_i} is strictly concave on K_{m_i} and $\frac{u}{n} < \max_{x\in\mathbb{R}^d} g_{m_i}(x) \leq \frac{u}{n-1}$ for $i \in \{1, \ldots, n\}$. Then, for $t \in \{1, \ldots, d\}$,*

$$\lambda_d^{n-1}(\{\mathbf{x} \in (\mathbb{R}^d)^{n-1} \colon V_t(\widetilde{Z}_u(\mathbf{x})) \in (0, R)\}) > 0$$

for all $R > 0$.

Proof. As in Lemma 6.20 let $h\colon (\mathbb{R}^d)^{n-1} \to \mathbb{R}_{\geq 0}$ be defined by $h(\mathbf{x}) = \max_{y\in\mathbb{R}^d} f_{\mathbf{x}}(y)$ for $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ and let $y_{\max}(\mathbf{x})$ be a point where the maximum of $f_{\mathbf{x}}$ is attained. Note at first that $r(\widetilde{Z}_u(\mathbf{x})) > 0$ for all $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ with $h(\mathbf{x}) > u$ and $r(\mathbf{K}(\mathbf{x})) > 0$ since $f_{\mathbf{x}}|_{\mathbf{K}(\mathbf{x})}$ is continuous by the continuity of $g_{m_i}|_{K_{m_i}}$ for $i \in \{1, \dots, n\}$ and because of $y_{\max}(\mathbf{x}) \in \mathbf{K}(\mathbf{x})$ for the case when $h(\mathbf{x}) > u$ and $r(\mathbf{K}(\mathbf{x})) > 0$ as $\max_{x\in\mathbb{R}^d} g_{m_i}(x) \leq \frac{u}{n-1}$ for $i \in \{1, \dots, n\}$.

Moreover, the space of all $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ satisfying $r(\mathbf{K}(\mathbf{x})) > 0$ is path-connected for the following reason. Let c_i denote the centre of a ball of maximal radius inscribed in K_{m_i} for $i \in \{1, \dots, n\}$ and let $\mathbf{v} = (v_1, \dots, v_{n-1}) \in (\mathbb{R}^d)^{n-1}$ be such that $v_i + c_i = x_n + c_n$ for $i \in \{1, \dots, n-1\}$. Note that by the choice of c_i this especially implies $r(\mathbf{K}(\mathbf{v})) > 0$. Now, for each $\mathbf{x} = (x_1, \dots, x_{n-1}) \in (\mathbb{R}^d)^{n-1}$ with $r(\mathbf{K}(\mathbf{x})) > 0$ there exists a path from $\mathbf{x}$ to $\mathbf{v}$ such that $r(\mathbf{K}(\mathbf{w})) > 0$ for all $\mathbf{w} = (w_1, \dots, w_{n-1}) \in (\mathbb{R}^d)^{n-1}$ on the path by the convexity of $K_{m_1}, \dots, K_{m_n}$. For example, such a path could be constructed in the following way. In a first step, $(x_1, \dots, x_{n-1})$ is shifted simultaneously by some $w_1 \in \mathbb{R}^d$ to $\mathbf{w}_1 = (x_1+w_1, \dots, x_{n-1}+w_1) \in (\mathbb{R}^d)^{n-1}$ such that $r(\mathbf{K}(\mathbf{w})) > 0$ for all $\mathbf{w} \in (\mathbb{R}^d)^{n-1}$ on the path and $x_n + c_n \in \operatorname{int}(\mathbf{K}(\mathbf{w}_1))$ at the end. Now, iteratively for each $i \in \{1, \dots, n-1\}$ we can take the direct line to v_i as path. By the convexity of $K_{m_1}, \dots, K_{m_n}$ it is guaranteed that $x_n + c_n \in \operatorname{int}(\mathbf{K}(\mathbf{w}))$ and hence $r(\mathbf{K}(\mathbf{w})) > 0$ for all $\mathbf{w} \in (\mathbb{R}^d)^{n-1}$ on the path throughout the whole process, which shows that the space of all $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ satisfying $r(\mathbf{K}(\mathbf{x})) > 0$ is path-connected.

In the following we consider two cases. If there exists $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ with $h(\mathbf{x}) \leq u$ and $r(\mathbf{K}(\mathbf{x})) > 0$, we use that there exists $\bar{\mathbf{x}} \in (\mathbb{R}^d)^{n-1}$ satisfying $h(\bar{\mathbf{x}}) > u$ and $r(\mathbf{K}(\bar{\mathbf{x}})) > 0$ because of $\max_{x\in\mathbb{R}^d} g_{m_i}(x) > \frac{u}{n}$ for $i \in \{1, \dots, n\}$. Then, by the continuity of h from Lemma 6.20 and the fact that the space of all $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ satisfying $r(\mathbf{K}(\mathbf{x})) > 0$ is path-connected, there exist $\bar{\mathbf{z}} \in \mathbb{R}^d$ and a sequence $(\bar{\mathbf{z}}_k)_{k\in\mathbb{N}}$ with $\bar{\mathbf{z}}_k \to \bar{\mathbf{z}}$ as $k \to \infty$, $h(\bar{\mathbf{z}}_k) > u$, $h(\bar{\mathbf{z}}) = u$, $r(\mathbf{K}(\bar{\mathbf{z}}_k)) > 0$ for all $k \in \mathbb{N}$ and $r(\mathbf{K}(\bar{\mathbf{z}})) > 0$. Additionally, since g_{m_i} is strictly concave on K_{m_i} for $i \in \{1, \dots, n\}$, $f_{\mathbf{v}}$ is strictly concave on $\mathbf{K}(\mathbf{v})$ for any $\mathbf{v} \in (\mathbb{R}^d)^{n-1}$. Thus, there is at most one point in $\widetilde{Z}_u(\bar{\mathbf{z}})$, hence $V_t(\widetilde{Z}_u(\bar{\mathbf{z}})) = 0$. As discussed in the first paragraph, note that the condition $h(\bar{\mathbf{z}}_k) > u$ for the sequence $(\bar{\mathbf{z}}_k)_{k\in\mathbb{N}}$ guarantees that $r(\widetilde{Z}_u(\bar{\mathbf{z}}_k)) > 0$ since $r(\mathbf{K}(\bar{\mathbf{z}}_k)) > 0$ and hence, $V_t(\widetilde{Z}_u(\bar{\mathbf{z}}_k)) > 0$. By Lemma 6.19 it holds $0 \leq \limsup_{k\to\infty} V_t(\widetilde{Z}_u(\bar{\mathbf{z}}_k)) \leq V_t(\widetilde{Z}_u(\bar{\mathbf{z}})) = 0$. Hence, for any $R > 0$ there exists $k_R \in \mathbb{N}$ such that $0 < V_t(\widetilde{Z}_u(\bar{\mathbf{z}}_{k_R})) \leq \frac{R}{2}$. By Lemma 6.19 and Lemma 6.20, there exists $\delta > 0$ such that $h(\mathbf{x}) > u$, $r(\mathbf{K}(\mathbf{x})) > 0$ and $V_t(\widetilde{Z}_u(\mathbf{x})) < R$ for all $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ with $\|\mathbf{x} - \bar{\mathbf{z}}_{k_R}\| < \delta$ and thus $V_t(\widetilde{Z}_u(\mathbf{x})) \in (0, R)$ for all $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ with $\|\mathbf{x} - \bar{\mathbf{z}}_{k_R}\| < \delta$. This provides

$$\lambda_d^{n-1}(\{\mathbf{x} \in (\mathbb{R}^d)^{n-1} \colon V_t(\widetilde{Z}_u(\mathbf{x})) \in (0, R)\}) \geq \lambda_d^{n-1}(\{\mathbf{x} \in (\mathbb{R}^d)^{n-1} \colon \|\mathbf{x} - \bar{\mathbf{z}}_{k_R}\| < \delta\}) > 0$$

for any $R > 0$.

If $h(\mathbf{x}) > u$ for all $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ with $r(\mathbf{K}(\mathbf{x})) > 0$, it holds $V_t(\widetilde{Z}_u(\mathbf{x})) > 0$ for any $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ with $r(\mathbf{K}(\mathbf{x})) > 0$ by the arguments from the first paragraph. Since $\hat{K}_n$ is a compact convex set, for any $x \in \mathbb{R}^d \backslash \hat{K}_n$ there exists $y_n \in \hat{K}_n$ with $\|y_n - x\| = \max_{y\in\hat{K}_n} \|y - x\|$. Hence, the hyperplane H through y_n, which is orthogonal to $y_n - x$, divides $\mathbb{R}^d$ in two half spaces H^+ and H^-, where H^+ denotes the half space that fulfils $\hat{K}_n \subseteq H^+$, and we have $H \cap \hat{K}_n = \{y_n\}$. By choosing $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_{n-1}) \in (\mathbb{R}^d)^{n-1}$ such that $y_n \in K_{m_i} + \bar{z}_i$ and $K_{m_i} + \bar{z}_i \subseteq H^-$ for $i \in \{1, \dots, n-1\}$, we obtain $\mathbf{K}(\bar{\mathbf{z}}) = \{y_n\}$. Clearly, there exists a sequence $(\bar{\mathbf{z}}_k)_{k\in\mathbb{N}}$ with $\bar{\mathbf{z}}_k \to \bar{\mathbf{z}}$ as $k \to \infty$ and $r(\mathbf{K}(\bar{\mathbf{z}}_k)) > 0$ for all $k \in \mathbb{N}$. Similarly to the first case, by Lemma 6.19, it holds $0 \leq \limsup_{k\to\infty} V_t(\widetilde{Z}_u(\bar{\mathbf{z}}_k)) \leq V_t(\widetilde{Z}_u(\bar{\mathbf{z}})) = 0$, i.e. there exist $k_R \in \mathbb{N}$ and $\delta > 0$ such that $V_t(\widetilde{Z}_u(\mathbf{x})) \in (0, R)$ for all $\mathbf{x} \in (\mathbb{R}^d)^{n-1}$ with $\|\mathbf{x} - \bar{\mathbf{z}}_{k_R}\| < \delta$. With

$$\lambda_d^{n-1}(\{\mathbf{x} \in (\mathbb{R}^d)^{n-1} \colon V_t(\widetilde{Z}_u(\mathbf{x})) \in (0, R)\}) \geq \lambda_d^{n-1}(\{\mathbf{x} \in (\mathbb{R}^d)^{n-1} \colon \|\mathbf{x} - \bar{\mathbf{z}}_{k_R}\| < \delta\}) > 0$$

the proof of Lemma 6.21 is complete. □

Proof of Proposition 6.17. We use Theorem 3.1 for the difference operator of order n for a suitable $n \in \mathbb{N}$. To verify the assumption in (3.2) we derive at first an upper bound for the integral of the expected squared difference operator of order $n+1$. To this end we use (6.25) and (2.3) to show that

$$\begin{aligned}
&\mathbb{E}\Big[\int \cdots \int |D^{n+1}_{\hat{x}_1,\dots,\hat{x}_{n+1}} \varphi(Z_u \cap W_s)|^2 \, \lambda^{n+1}(\mathrm{d}(\hat{x}_1, \dots, \hat{x}_{n+1}))\Big] \\
&\leq \int \cdots \int c_{2,0} 4^{n+1} V_d\Big(\bigcap_{i=1}^{n+1} \hat{K}_i^{\sqrt{d}} \cap W_s^{\sqrt{d}}\Big)^2 \, \lambda^{n+1}(\mathrm{d}(\hat{x}_1, \dots, \hat{x}_{n+1})) \\
&\leq \gamma^{n+1} c_{2,0} 4^{n+1} \int_{\mathbb{M}^{n+1}} V_d\Big(K_{m_1}^{\sqrt{d}}\Big) \\
&\qquad\qquad \times \int_{(\mathbb{R}^d)^{n+1}} V_d\Big(\bigcap_{i=1}^{n+1} \hat{K}_i^{\sqrt{d}} \cap W_s^{\sqrt{d}}\Big) \, \mathrm{d}(x_1, \dots, x_{n+1}) \, \mathbb{Q}^{n+1}(\mathrm{d}(m_1, \dots, m_{n+1})) \\
&= \gamma^{n+1} c_{2,0} 4^{n+1} \lambda_d\Big(W_s^{\sqrt{d}}\Big) \int_{\mathbb{M}^{n+1}} V_d\Big(K_{m_1}^{\sqrt{d}}\Big)^2 \prod_{i=2}^{n+1} V_d\Big(K_{m_i}^{\sqrt{d}}\Big) \, \mathbb{Q}^{n+1}(\mathrm{d}(m_1, \dots, m_{n+1})) \\
&\leq c_1 \lambda_d(W_s)
\end{aligned}$$

for a suitable constant $c_1 > 0$. Note that due to the moment assumptions of up to order two, the integrals are finite. For the proof of the positivity of the variance it remains to show that

$$\mathbb{E}\Big[\int \cdots \int |D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s)|^2 \, \lambda^n(\mathrm{d}(\hat{x}_1, \dots, \hat{x}_n))\Big] \geq c_2 \lambda_d(W_s)$$

for some $c_2 > 0$ and a suitable $n \in \mathbb{N}$. Then, assumption (3.2) is fulfilled and $\sigma_0 > 0$ by Theorem 3.1.

For the proof of a) we choose $n = 1$. Let $U_m = \{x \in \mathbb{R}^d \colon g_m(x) \geq u\}$ for $m \in \mathbb{M}$ and $M = \{m \in \mathbb{M} \colon \varphi(U_m) \neq 0\}$. By assumption, $\mathbb{Q}(M) > 0$. Then, if $\eta(S_{K_m+x}) = 0$ for $m \in M$ and $x \in W_s$ such that $K_m + x \subseteq W_s$ we have with (6.24),

$$D_{\hat{x}} \varphi(Z_u \cap W_s) = D_{\hat{x}} \varphi(Z_u \cap (K_m + x)) = \varphi(U_m) - \varphi(\emptyset) = \varphi(U_m)$$

for $\hat{x} = (x, m)$. This provides

$$\mathbb{P}(D_{\hat{x}} \varphi(Z_u \cap W_s) = \varphi(U_m)) \geq \mathbb{P}(\eta(S_{K_m+x}) = 0) = e^{-\lambda(S_{K_m+x})} = e^{-\lambda(S_{K_m})}. \tag{6.31}$$

Let $r_0 > 0$ be such that $\mathbb{Q}(\{m \in M \colon R(K_m) \leq r_0\}) > 0$, where $R(K_m)$ denotes the circumradius of K_m, and $\widehat{W}_s = \{x \in W_s \colon d(x, \partial W_s) \geq r_0\}$. Then, $\lambda_d(\widehat{W}_s) \geq \frac{1}{2}\lambda_d(W_s)$ for s large enough, which implies together with (6.31) for s large enough,

$$\mathbb{E}\Big[\int |D_{\hat{x}} \varphi(Z_u \cap W_s)|^2 \, \lambda(\mathrm{d}\hat{x})\Big] \geq \int \mathbb{P}(D_{\hat{x}} \varphi(Z_u \cap W_s) = \varphi(U_m)) \varphi(U_m)^2 \, \lambda(\mathrm{d}\hat{x})$$

$$\geq \gamma\lambda_d(\widehat{W}_s)\int_M \mathbb{1}\{R(K_m)\leq r_0\}e^{-\lambda(S_{K_m})}\varphi(U_m)^2\,\mathbb{Q}(\mathrm{d}m)$$
$$\geq c_3\lambda_d(W_s)$$

for a suitable constant $c_3 > 0$, which completes the proof of a).

For the proof of b) we use a similar proof strategy. This means, we start with point configurations that occur with a positive probability and for which the excursion set on a specific compact convex set is the empty set. Then, we add just enough points such that it is not empty anymore and such that the corresponding difference operator is not equal to zero. For the start let

$$\widetilde{\mathbb{M}} = \Big\{m \in \mathbb{M}\colon \max_{x\in\mathbb{R}^d} g_m(x) \leq u, g_m \text{ strictly concave on } K_m\Big\}.$$

Note that $\mathbb{Q}(\widetilde{\mathbb{M}}) > 0$ by assumption. Moreover, let

$$M(v,\varepsilon) = \Big\{m \in \widetilde{\mathbb{M}}\colon \max_{x\in\mathbb{R}^d} g_m(x) \in [v-\varepsilon, v+\varepsilon]\Big\}$$

for $v \in [0,u]$ and $\varepsilon > 0$. Define the map $f\colon \widetilde{\mathbb{M}} \to [0,u]$ by $m \mapsto \max_{x\in\mathbb{R}^d} g_m(x)$ and denote by $\mathbb{Q}_f$ the corresponding push-forward measure. If $\mathbb{Q}_f$ has an atom in a point $v \in [0,u]$ with $\frac{u}{v} \in \mathbb{N}$, it holds $\mathbb{Q}(M(v,0)) > 0$. If $\mathbb{Q}_f$ does not have such a point, we can apply [60, Lemma 1.19], which provides that

$$\mathbb{Q}_f\Big(\Big\{v \in (0,u)\Big\backslash \bigcup_{i\in\mathbb{N}}\Big\{\frac{u}{i}\Big\}\colon \mathbb{Q}(M(v,\varepsilon)) > 0 \text{ for all } \varepsilon > 0\Big\}\Big)$$
$$= \mathbb{Q}_f(\{v \in [0,u]\colon \mathbb{Q}(M(v,\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}) > 0,$$

where we used in the first step that $\mathbb{Q}_f(\{0\}) = 0$ since K_m is non-empty for all $m \in \widetilde{\mathbb{M}}$. Altogether, this guarantees that there either exists $v \in (0,u)$ with $\frac{u}{v} \notin \mathbb{N}$ such that $\mathbb{Q}(M(v,\varepsilon)) > 0$ for all $\varepsilon > 0$ or there exists $v \in (0,u]$ with $\frac{u}{v} \in \mathbb{N}$ and $\mathbb{Q}(M(v,0)) > 0$. This will be used to divide the proof in different cases. For the first two cases we will further define $n \in \mathbb{N}$, $M \subseteq \widetilde{\mathbb{M}}^n$ with $\mathbb{Q}^n(M) > 0$ and $S_{m_1,\dots,m_n} \subset (\mathbb{R}^d)^n$ for $(m_1,\dots,m_n) \in M$ with

$$(x_1,\dots,x_n) \in S_{m_1,\dots,m_n} \Rightarrow (x_1+z,\dots,x_n+z) \in S_{m_1,\dots,m_n} \tag{6.32}$$

for any $z \in \mathbb{R}^d$ and $x_1,\dots,x_n \in \mathbb{R}^d$,

$$\lambda_d^{n-1}(\{(x_1,\dots,x_{n-1}) \in (\mathbb{R}^d)^{n-1}\colon (x_1,\dots,x_{n-1},\mathbf{0}) \in S_{m_1,\dots,m_n}\}) > 0 \tag{6.33}$$

in case that $n \geq 2$ and

$$(x_1,\dots,x_n) \notin S_{m_1,\dots,m_n} \text{ if } \bigcap_{i=1}^n \hat{K}_i = \emptyset. \tag{6.34}$$

Let $t_{\min} = \min\{i \in \{0,\dots,d\}\colon \alpha_i \neq 0\}$. We consider the following three cases.

Case 1: There is some $v \in (0,u)$ with $\mathbb{Q}(M(v,\varepsilon)) > 0$ for all $\varepsilon > 0$ and $\frac{u}{v} \notin \mathbb{N}$. Then, we can choose $\varepsilon > 0$ small enough such that there exists an $n \in \mathbb{N}$ with $n \geq 2$ satisfying

$$\frac{u}{v-\varepsilon} < n < \frac{u}{v+\varepsilon} + 1.$$

Now, we define $M = M(v,\varepsilon)^n$ and for $\hat{t} = \max\{t_{\min}, 1\}$,

$$S_{m_1,\dots,m_n} = \{(x_1,\dots,x_n) \in (\mathbb{R}^d)^n \colon V_{\hat{t}}(\widetilde{Z}_u(\mathbf{x})) \in (0,R) \text{ with } \mathbf{x} = (x_1,\dots,x_{n-1})\}$$

for some $R > 0$. Then, (6.32), (6.33) and (6.34) are fulfilled by the translation invariance of intrinsic volumes, Lemma 6.21 and since $V_{\hat{t}}(\emptyset) = 0$.

Case 2: We can find $v \in (0,u]$ with $\frac{u}{v} \in \mathbb{N}$, $\mathbb{Q}(M(v,0)) > 0$ and $t_{\min} \neq 0$ or $v \neq u$. Then, we define $n = \frac{u}{v} + 1$, $M = M(v,0)^n$ and for $\hat{t} = \max\{t_{\min}, 1\}$ and $U_m = \{x \in \mathbb{R}^d \colon g_m(x) \geq v\}$,

$$\begin{aligned} S_{m_1,\dots,m_n} = \Big\{&(x_1,\dots,x_n) \in (\mathbb{R}^d)^n \colon V_{\hat{t}}(\widetilde{Z}_u(\mathbf{x})) \in (0,R) \text{ with } \mathbf{x} = (x_1,\dots,x_{n-1}),\\ &V_t\Big(\bigcap_{j=1}^{u/v}(U_{m_{i_j}} + x_{i_j})\Big) = 0 \text{ for } \{i_1,\dots,i_{u/v}\} \subseteq \{1,\dots,n\},\\ &i_j \neq i_\ell \text{ for } j \neq \ell \text{ and all } t \geq t_{\min}\Big\}, \end{aligned}$$

for some $R > 0$. Note that since U_m consists by the strict concavity of g_m only of one point for all $m \in M(v,0)$, it holds

$$\lambda_d^{u/v}\Big(\Big\{(x_{i_1},\dots,x_{i_{u/v}}) \in (\mathbb{R}^d)^{u/v} \colon V_t\Big(\bigcap_{j=1}^{u/v}(U_{m_{i_j}} + x_{i_j})\Big) \neq 0\Big\}\Big) = 0$$

for $\{i_1,\dots,i_{u/v}\} \subseteq \{1,\dots,n\}$ with $i_j \neq i_\ell$ for $j \neq \ell$ and $t \geq t_{\min}$. Together with the translation invariance of the intrinsic volumes, properties (6.32), (6.33) and (6.34) are fulfilled as in Case 1.

Case 3: It holds $\mathbb{Q}(M(v,0)) > 0$ for $v = u$ and $t_{\min} = 0$. Then, with (6.29), we have for $U_m = \{x \in \mathbb{R}^d \colon g_m(x) \geq v\}$,

$$\varphi(U_m) = \sum_{i=0}^{d} \alpha_i V_i(U_m) = \alpha_0 \neq 0$$

for all $m \in M(v,0)$ since U_m consists only of one point by the strict concavity of g_m for all $m \in M(v,0)$. Hence, $\mathbb{Q}(\{m \in \mathbb{M} \colon \varphi(U_m) \neq 0\}) > 0$. Thus, the assumption of part a) of the proposition is fulfilled and therefore the proof for Case 3 is complete with the proof of a).

It remains to show the proposition for Cases 1 and 2. To this end let $K = \bigcap_{i=1}^{n} \hat{K}_i \cap W_s$. By the choice of n we have for $\eta(S_K) = 0$, $(x_1,\dots,x_n) \in S_{m_1,\dots,m_n}$, $(m_1,\dots,m_n) \in M$ and $I \subseteq \{1,\dots,n\}$,

$$Z_u\Big(\eta + \sum_{i \in I} \delta_{\hat{x}_i}\Big) \cap K = \emptyset$$

if $|I| \leq n-1$ in Case 1 and if $|I| \leq n-2$ in Case 2. If $|I|$ exceeds those thresholds, we have

$$Z_u\Big(\eta + \sum_{i \in I} \delta_{\hat{x}_i}\Big) \cap K = X_{I,K},$$

which is convex by the proof of Proposition 6.7.

Now, we consider $\hat{x}_1, \dots, \hat{x}_n$ with $\hat{K}_i \subseteq W_s$ for $i \in \{1, \dots, n\}$. Note that for $\eta(S_K) = 0$ and fixed $x_n \in \mathbb{R}^d$ we have $Z_u\Big(\eta + \sum_{i=1}^n \delta_{\hat{x}_i}\Big) \cap K = \widetilde{Z}_u(\mathbf{x})$ for $\mathbf{x} = (x_1, \dots, x_{n-1}) \in (\mathbb{R}^d)^{n-1}$.

In the following we use assumption (6.29) to represent the geometric functional as a linear combination of intrinsic volumes. Recall that $t_{\min} = \min\{i \in \{0, \dots, d\} \colon \alpha_i \neq 0\}$. Without loss of generality we assume $\alpha_{t_{\min}} > 0$. Then, for $t_{\min} > 0$ it holds with (2.6),

$$\begin{aligned}
|\varphi(\widetilde{Z}_u(\mathbf{x}))| &= \Big| \sum_{i=0}^d \alpha_i V_i(\widetilde{Z}_u(\mathbf{x}))\Big| \\
&\geq \sum_{i \in \{0,\dots,d\} \colon \alpha_i > 0} \alpha_i V_i(\widetilde{Z}_u(\mathbf{x})) - \sum_{i \in \{0,\dots,d\} \colon \alpha_i < 0} |\alpha_i| V_i(\widetilde{Z}_u(\mathbf{x})) \\
&\geq V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x}))\Big(\alpha_{t_{\min}} - \sum_{i \in \{0,\dots,d\} \colon \alpha_i < 0} |\alpha_i| \frac{V_i(\widetilde{Z}_u(\mathbf{x}))}{V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x}))}\Big) \\
&\geq V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x}))\Big(\alpha_{t_{\min}} - \sum_{i \in \{0,\dots,d\} \colon \alpha_i < 0} |\alpha_i| C(t_{\min}, i) V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x}))^{(i-t_{\min})/t_{\min}}\Big) \\
&\geq \frac{\alpha_{t_{\min}}}{2} V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x}))
\end{aligned}$$

if $V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x}))$ is small enough. Similarly, since $V_0(\widetilde{Z}_u(\mathbf{x})) = 1$ as $\widetilde{Z}_u(\mathbf{x})$ is convex, we have for $t_{\min} = 0$,

$$\begin{aligned}
|\varphi(\widetilde{Z}_u(\mathbf{x}))| &\geq V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x}))\Big(\alpha_{t_{\min}} - \sum_{i \in \{0,\dots,d\} \colon \alpha_i < 0} |\alpha_i| V_i(\widetilde{Z}_u(\mathbf{x}))\Big) \\
&\geq V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x}))\Big(\alpha_{t_{\min}} - \sum_{i \in \{0,\dots,d\} \colon \alpha_i < 0} |\alpha_i| C(1,i) V_1(\widetilde{Z}_u(\mathbf{x}))^i\Big) \\
&\geq \frac{\alpha_{t_{\min}}}{2} V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x}))
\end{aligned}$$

if $V_1(\widetilde{Z}_u(\mathbf{x}))$ is small enough. In Case 2 it additionally holds

$$\varphi\Big(K \cap \bigcap_{j=1}^{n-1} (U_{m_{i_j}} + x_{i_j})\Big) = \varphi\Big(\bigcap_{j=1}^{n-1} (U_{m_{i_j}} + x_{i_j})\Big) = 0$$

for $(x_1, \dots, x_n) \in S_{m_1,\dots,m_n}$, $(m_1, \dots, m_n) \in M$, $i_1, \dots, i_{n-1} \in \{1, \dots, n\}$ with $i_j \neq i_\ell$ for $j \neq \ell$ and $\hat{K}_i \subseteq W_s$ for $i \in \{1, \dots, n\}$. Note that we can choose the parameter $R > 0$ in the definition of $S_{m_1,\dots,m_n}$ sufficiently small by Lemma 6.21 such that all estimations above hold while (6.33) remains valid. Then, if $\eta(S_K) = 0$, it holds

$$|D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap K)| = \Big|\varphi\Big(Z_u\Big(\eta + \sum_{i=1}^n \delta_{\hat{x}_i}\Big) \cap K\Big)\Big| \geq \frac{\alpha_{t_{\min}}}{2} V_{t_{\min}}(\widetilde{Z}_u(\mathbf{x})) =: C_{\hat{x}_1,\dots,\hat{x}_n} > 0$$

for $(x_1,\dots,x_n)\in S_{m_1,\dots,m_n}$ with $\hat{K}_i\subseteq W_s$ for $i\in\{1,\dots,n\}$ and $(m_1,\dots,m_n)\in M$, i.e. we have

$$\mathbb{P}(|D^n_{\hat{x}_1,\dots,\hat{x}_n}\varphi(Z_u\cap K)|\ge C_{\hat{x}_1,\dots,\hat{x}_n})\ge \mathbb{P}(\eta(S_K)=0)\ge e^{-\lambda(S_{K_{m_1}})}>0. \tag{6.35}$$

For $\mathbf{m}=(m_1,\dots,m_n)\in M$ let $h_{\mathbf{m}}\colon\mathbb{R}^d\to\mathbb{R}_{\ge 0}$ be defined as

$$h_{\mathbf{m}}(x_1,\dots,x_n)=C^2_{\hat{x}_1,\dots,\hat{x}_n}\mathbb{1}\{(x_1,\dots,x_n)\in S_{m_1,\dots,m_n}\}e^{-\lambda(S_{K_{m_1}})}.$$

The translation invariance of the intrinsic volumes provides

$$C_{\hat{x}_1,\dots,\hat{x}_n}=C_{(x_1+z,m_1),\dots,(x_n+z,m_n)}$$

for any $z\in\mathbb{R}^d$. Then, by (6.32), $h_{\mathbf{m}}$ is translation invariant and with (6.33) and (6.34) we get

$$\int_{(\mathbb{R}^d)^{n-1}}\mathbb{1}\Big\{K_{m_n}\cap\bigcap_{i=1}^{n-1}(K_{m_i}+y_i)\neq\emptyset\Big\}h_{\mathbf{m}}(y_1,\dots,y_{n-1},\mathbf{0})\,\mathrm{d}(y_1,\dots,y_{n-1})>0. \tag{6.36}$$

Moreover, for $\widehat{W}_s=\{(x_1,\dots,x_n)\in(\mathbb{R}^d)^n\colon \hat{K}_i\subseteq W_s, i\in\{1,\dots,n\}\}$ and $y_1,\dots,y_{n-1}\in\mathbb{R}^d$ satisfying $K_{m_n}\cap\bigcap_{i=1}^{n-1}(K_{m_i}+y_i)\neq\emptyset$, there exists $r_0(m_1,\dots,m_n)>0$ only depending on $m_1,\dots,m_n$ such that

$$\begin{aligned}&\lambda_d(\{y\in\mathbb{R}^d\colon (y_1+y,\dots,y_{n-1}+y,y)\in\widehat{W}_s\})\\&\ge\lambda_d\Big(\Big\{y\in W_s\colon \mathrm{d}(y,\partial W_s)\ge R(K_{m_n})+2\max_{i\in\{1,\dots,n-1\}}R(K_{m_i})\Big\}\Big)\ge\frac{\lambda_d(W_s)}{2}\end{aligned} \tag{6.37}$$

for all $s\ge r_0(m_1,\dots,m_n)$.

Now, let $r_0>0$ be large enough such that $\mathbb{Q}(\{m\in M\colon r_0(m_1,\dots,m_n)\le r_0\})>0$. Then we have with (6.24), (6.35), (6.36), (6.37) and the translation invariance of $h_{\mathbf{m}}$ for $s\ge r_0$,

$$\begin{aligned}&\mathbb{E}\Big[\int\cdots\int|D^n_{\hat{x}_1,\dots,\hat{x}_n}\varphi(Z_u\cap W_s)|^2\,\lambda^n(\mathrm{d}(\hat{x}_1,\dots,\hat{x}_n))\Big]\\
&\ge\int\cdots\int\mathbb{P}\Big(\Big|D^n_{\hat{x}_1,\dots,\hat{x}_n}\varphi\Big(Z_u\cap W_s\cap\bigcap_{i=1}^n\hat{K}_i\Big)\Big|\ge C_{\hat{x}_1,\dots,\hat{x}_n}\Big)C^2_{\hat{x}_1,\dots,\hat{x}_n}\,\lambda^n(\mathrm{d}(\hat{x}_1,\dots,\hat{x}_n))\\
&\ge\int\cdots\int\mathbb{1}\{(m_1,\dots,m_n)\in M,(x_1,\dots,x_n)\in\widehat{W}_s\}h_{\mathbf{m}}(x_1,\dots,x_n)\,\lambda^n(\mathrm{d}(\hat{x}_1,\dots,\hat{x}_n))\\
&\ge\gamma^n\int_M\int_{(\mathbb{R}^d)^n}\mathbb{1}\{(y_1+y_n,\dots,y_{n-1}+y_n,y_n)\in\widehat{W}_s\}\mathbb{1}\Big\{K_{m_n}\cap\bigcap_{i=1}^{n-1}(K_{m_i}+y_i)\neq\emptyset\Big\}\\
&\qquad\qquad\times h_{\mathbf{m}}(y_1,\dots,y_{n-1},\mathbf{0})\,\mathrm{d}(y_1,\dots,y_n)\,\mathbb{Q}^n(\mathrm{d}(m_1,\dots,m_n))\\
&\ge\gamma^n\int_M\int_{(\mathbb{R}^d)^{n-1}}\lambda_d(\{y\in\mathbb{R}^d\colon(y_1+y,\dots,y_{n-1}+y,y)\in\widehat{W}_s\})\\
&\qquad\qquad\times\mathbb{1}\Big\{K_{m_n}\cap\bigcap_{i=1}^{n-1}(K_{m_i}+y_i)\neq\emptyset\Big\}\end{aligned}$$

$$\times h_{\mathbf{m}}(y_1,\dots,y_{n-1},\mathbf{0})\,\mathrm{d}(y_1,\dots,y_{n-1})\,\mathbb{Q}^n(\mathrm{d}(m_1,\dots,m_n))$$
$$\geq c_4\lambda_d(W_s)$$

for a suitable constant $c_4 > 0$, which completes the proof. □

6.3.4 Central limit theorem

Finally we derive a qualitative central limit theorem and corresponding quantitative central limit theorems in Wasserstein and Kolmogorov distance under different moment assumptions if the asymptotic variance is positive.

Theorem 6.22. *Let φ be a geometric functional and let $u > 0$. Denote by N a standard Gaussian random variable, let σ_0 be defined as in Theorem 6.13 and assume $\sigma_0 > 0$. Let $(g_m)_{m\in\mathbb{M}}$ fulfil Assumption 6.5.*

a) If (6.16) *is fulfilled for $k = 2$, it holds*

$$\frac{\varphi(Z_u\cap W_s)-\mathbb{E}[\varphi(Z_u\cap W_s)]}{\sqrt{\operatorname{Var}[\varphi(Z_u\cap W_s)]}}\xrightarrow{d} N \quad \text{as} \quad s\to\infty.$$

b) If (6.16) *is fulfilled for $k = 3$, there exists a constant $C > 0$, depending on γ, d and the first three moments in* (6.16)*, such that*

$$d_W\left(\frac{\varphi(Z_u\cap W_s)-\mathbb{E}[\varphi(Z_u\cap W_s)]}{\sqrt{\operatorname{Var}[\varphi(Z_u\cap W_s)]}},N\right)\leq\frac{C}{\sqrt{V_d(W_s)}}$$

for s large enough.

c) If (6.16) *is fulfilled for $k = 4$, there exists a constant $C > 0$, depending on γ, d and the first four moments in* (6.16)*, such that*

$$d_K\left(\frac{\varphi(Z_u\cap W_s)-\mathbb{E}[\varphi(Z_u\cap W_s)]}{\sqrt{\operatorname{Var}[\varphi(Z_u\cap W_s)]}},N\right)\leq\frac{C}{\sqrt{V_d(W_s)}}$$

for s large enough.

Similar results as in Theorem 6.22 were also shown for geometric functionals of the Boolean model and specific geometric functionals of excursion sets of Poisson shot noise processes. The analogous results for the Boolean model and for Poisson cylinder processes can be found in [55, Theorem 9.3] and in [20, Theorem 3.5]. For Poisson shot noise processes there exist various results for different functionals under several model assumptions. For example, for the volume or smoothed versions of the volume of excursion sets of Poisson shot noise processes central limit theorems were shown in [32, Proposition 3.2.1], [66, Theorem 4.1] and [67, Theorem 4.3] under different conditions on the kernel functions.

For the proof of the qualitative central limit theorem we show that γ_1, γ_2 and $\widetilde{\gamma}_3$ from Theorem 2.17 vanish as $s\to\infty$. For the proof of the quantitative central limit theorems we show that $\gamma_1,\dots,\gamma_6$ from Theorem 2.17 and Theorem 2.18 are of the right order under the additional moment assumptions. To this end we mainly use (6.25).

Proof of Theorem 6.22. We consider $F_s = \frac{\varphi(Z_u \cap W_s) - \mathbb{E}[\varphi(Z_u \cap W_s)]}{\sqrt{\operatorname{Var}[\varphi(Z_u \cap W_s)]}}$. For this standardised random variable it holds that

$$D^n_{\hat{x}_1,\dots,\hat{x}_n} F_s = \frac{D^n_{\hat{x}_1,\dots,\hat{x}_n} \varphi(Z_u \cap W_s)}{\sqrt{\operatorname{Var}[\varphi(Z_u \cap W_s)]}}.$$

To show *a)* we start with bounding γ_1, γ_2 and $\widetilde{\gamma}_3$ from Theorem 2.17. Let $f\big(K_1, \dots, K_t\big) = V_d\big(\bigcap_{i=1}^t K_i^{\sqrt{d}}\big)$ for $K_i \in \mathcal{K}^d$, $i \in \{1, \dots, t\}$ and $t \in \mathbb{N}$. Then, for $\hat{x}_1 = (x_1, m_1), \hat{x}_2 = (x_2, m_2), \hat{x}_3 = (x_3, m_3) \in \mathbb{R}^d \times \mathbb{M}$ we have with (6.25) for suitable constants $c_1, c_2 > 0$,

$$\mathbb{E}[(D_{\hat{x}_1} F_s)^2 (D_{\hat{x}_2} F_s)^2] \leq \frac{c_1}{\operatorname{Var}[\varphi(Z_u \cap W_s)]^2} f(\hat{K}_1, W_s)^2 f(\hat{K}_2, W_s)^2$$

and

$$\mathbb{E}[(D^2_{\hat{x}_1,\hat{x}_3} F_s)^2 (D^2_{\hat{x}_2,\hat{x}_3} F_s)^2] \leq \frac{c_2}{\operatorname{Var}[\varphi(Z_u \cap W_s)]^2} f(\hat{K}_1, \hat{K}_3, W_s)^2 f(\hat{K}_2, \hat{K}_3, W_s)^2.$$

Since with (2.3) it holds for $j \in \{1, 2\}$ that

$$\int_{\mathbb{R}^d} f(\hat{K}_j, W_s) f(\hat{K}_j, \hat{K}_3, W_s) \, \mathrm{d}x_j \leq f(K_{m_j}) \int_{\mathbb{R}^d} f(\hat{K}_j, \hat{K}_3, W_s) \, \mathrm{d}x_j = f(K_{m_j})^2 f(\hat{K}_3, W_s),$$

we get for $s \geq 1$,

$$\begin{aligned}
\gamma_1^2 &\leq \frac{4\sqrt{c_1 c_2}\gamma^3}{\operatorname{Var}[\varphi(Z_u \cap W_s)]^2} \int_{\mathbb{M}^3} f(K_{m_1})^2 f(K_{m_2})^2 \int_{\mathbb{R}^d} f(\hat{K}_3, W_s)^2 \, \mathrm{d}x_3 \, \mathbb{Q}^3(\mathrm{d}(m_1, m_2, m_3)) \\
&\leq \frac{4\sqrt{c_1 c_2}\gamma^3}{\operatorname{Var}[\varphi(Z_u \cap W_s)]^2} \int_{\mathbb{M}^3} f(K_{m_1})^2 f(K_{m_2})^2 f(K_{m_3})^2 f(W_s) \, \mathbb{Q}^3(\mathrm{d}(m_1, m_2, m_3)) \\
&\leq \frac{4\sqrt{c_1 c_2} f(W_s)\gamma^3}{\operatorname{Var}[\varphi(Z_u \cap W_s)]^2} \Big(\int_{\mathbb{M}} f(K_m)^2 \, \mathbb{Q}(\mathrm{d}m) \Big)^3.
\end{aligned}$$

Note that the integral exists because of the second moment assumptions in (6.16). By Theorem 6.13 it holds $\operatorname{Var}[\varphi(Z_u \cap W_s)] \geq \frac{\sigma_0}{2} V_d(W_s) = \frac{\sigma_0 s^d}{2} V_d(W)$ for s large enough. Then, since $f(W_s) \leq s^d f(W)$ for $s \geq 1$ we get for s large enough,

$$\gamma_1^2 \leq \frac{c_3}{V_d(W_s)} \tag{6.38}$$

for a suitable constant $c_3 > 0$ and hence $\gamma_1 \to 0$ as $s \to \infty$. Analogously, we get for γ_2,

$$\gamma_2^2 \leq \frac{c_2 f(W_s)\gamma^3}{\operatorname{Var}[\varphi(Z_u \cap W_s)]^2} \Big(\int_{\mathbb{M}} f(K_m)^2 \, \mathbb{Q}(\mathrm{d}m) \Big)^3 \leq \frac{c_4}{V_d(W_s)} \tag{6.39}$$

for a suitable constant $c_4 > 0$ and s large enough. Thus, $\gamma_2 \to 0$ as $s \to \infty$. To bound $\widetilde{\gamma}_3$ we use that

$$\mathbb{E}[|D_{\hat{x}_1} F_s|^3] \leq \frac{c_5}{\operatorname{Var}[\varphi(Z_u \cap W_s)]^{3/2}} f(\hat{K}_1, W_s)^3 \tag{6.40}$$

from (6.25) for some $c_5 > 0$. Since additionally $\mathbb{E}[|D_{\hat{x}_1}F_s|^{3/2}]^{2/3} \leq \mathbb{E}[|D_{\hat{x}_1}F_s|^3]^{1/3}$ by Jensen's inequality, we have for fixed $t > 0$,

$$\begin{aligned}\widetilde{\gamma}_3 &\leq \gamma \int_{\mathbb{M}}\int_{\mathbb{R}^d} \mathbb{1}\{f(K_{m_1}) \leq t\}\mathbb{E}[|D_{\hat{x}_1}F_s|^3]\, \mathrm{d}x_1\, \mathbb{Q}(\mathrm{d}m_1)\\ &\quad + 2\gamma \int_{\mathbb{M}}\int_{\mathbb{R}^d} \mathbb{1}\{f(K_{m_1}) > t\}\mathbb{E}[|D_{\hat{x}_1}F_s|^3]^{2/3}\, \mathrm{d}x_1\, \mathbb{Q}(\mathrm{d}m_1) =: I_1(s,t) + I_2(s,t).\end{aligned}$$

For $I_1(s,t)$ we have with (6.40),

$$\begin{aligned}I_1(s,t) &\leq \frac{c_5}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^{3/2}} \int_{\mathbb{M}} \mathbb{1}\{f(K_{m_1}) \leq t\} f(K_{m_1})^2 \int_{\mathbb{R}^d} f(\hat{K}_1, W_s)\, \mathrm{d}x_1\, \mathbb{Q}(\mathrm{d}m_1)\\ &\leq \frac{t c_5 f(W_s)}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^{3/2}} \int_{\mathbb{M}} \mathbb{1}\{f(K_{m_1}) \leq t\} f(K_{m_1})^2\, \mathbb{Q}(\mathrm{d}m_1) \leq \frac{c_6 t}{\sqrt{V_d(W_s)}}\end{aligned}$$

for a suitable constant $c_6 > 0$ if s is large enough and hence, $I_1(s,t) \to 0$ as $s \to \infty$ for any $t > 0$. Similarly, we get for $I_2(s,t)$,

$$\begin{aligned}I_2(s,t) &\leq \frac{2c_5^{2/3}}{\mathrm{Var}[\varphi(Z_u \cap W_s)]} \int_{\mathbb{M}} \mathbb{1}\{f(K_{m_1}) > t\} f(K_{m_1}) \int_{\mathbb{R}^d} f(\hat{K}_1, W_s)\, \mathrm{d}x_1\, \mathbb{Q}(\mathrm{d}m_1)\\ &= \frac{2c_5^{2/3} f(W_s)}{\mathrm{Var}[\varphi(Z_u \cap W_s)]} \int_{\mathbb{M}} \mathbb{1}\{f(K_{m_1}) > t\} f(K_{m_1})^2\, \mathbb{Q}(\mathrm{d}m_1)\\ &\leq c_7 \int_{\mathbb{M}} \mathbb{1}\{f(K_{m_1}) > t\} f(K_{m_1})^2\, \mathbb{Q}(\mathrm{d}m_1)\end{aligned}$$

for a suitable constant $c_7 > 0$ if s is large enough. Note that by the second moment assumptions in (6.16), $\int_{\mathbb{M}} \mathbb{1}\{f(K_{m_1}) > t\} f(K_{m_1})^2\, \mathbb{Q}(\mathrm{d}m_1) \to 0$ as $t \to \infty$. This means that for any $\varepsilon > 0$ we can choose $\hat{t} > 0$ such that $I_2(s,\hat{t}) \leq \frac{\varepsilon}{2}$ for s sufficiently large. Then, for s large enough such that also $I_1(s,\hat{t}) \leq \frac{\varepsilon}{2}$, we have $\widetilde{\gamma}_3 < \varepsilon$. Altogether, we have shown that γ_1, γ_2 and $\widetilde{\gamma}_3$ vanish as $s \to \infty$, which provides *a*).

For the proof of the Wasserstein distance in *b*) it is sufficient to additionally bound γ_3 from Theorem 2.17 since we have already shown in (6.38) and (6.39) that γ_1 and γ_2 are of the right order. To this end we use (6.40). Then, with (2.3) we have for s large enough,

$$\begin{aligned}\gamma_3 &\leq \frac{c_5\gamma}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^{3/2}} \int_{\mathbb{M}} f(K_{m_1})^2 \int_{\mathbb{R}^d} f(\hat{K}_1, W_s)\, \mathrm{d}x_1\, \mathbb{Q}(\mathrm{d}m_1)\\ &= \frac{c_5\gamma f(W_s)}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^{3/2}} \int_{\mathbb{M}} f(K_{m_1})^3\, \mathbb{Q}(\mathrm{d}m_1)\\ &\leq \frac{c_8}{\sqrt{V_d(W_s)}}\end{aligned}$$

for a suitable constant $c_8 > 0$. Note that the integral is finite by the moment assumptions from (6.16) of order three, which provides together with (6.38) and (6.39) the result for the Wasserstein distance.

To show *c*) we additionally bound γ_4, γ_5 and γ_6 from Theorem 2.18. From [70, Lemma 4.2] we know that

$$\mathbb{E}[F_s^4] \leq \max\Big\{256\Big(\int (\mathbb{E}[(D_{\hat{x}}F_s)^4]^{1/2})\, \lambda(\mathrm{d}\hat{x})\Big)^2, 4\int \mathbb{E}[(D_{\hat{x}}F_s)^4]\, \lambda(\mathrm{d}\hat{x}) + 2\Big\}. \qquad (6.41)$$

The bound

$$\mathbb{E}[|D_{\hat{x}_1} F_s|^4] \leq \frac{c_9}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} f(\hat{K}_1, W_s)^4 \tag{6.42}$$

from Lemma 6.14 for some constant $c_9 > 0$ provides

$$\begin{aligned}
&\Big(\gamma \int_{\mathbb{M}} \int_{\mathbb{R}^d} \mathbb{E}[(D_{\hat{x}} F_s)^4]^{1/2} \,\mathrm{d}x_1 \,\mathbb{Q}(\mathrm{d}m_1)\Big)^2 \\
&\leq \frac{c_9 \gamma^2}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} \Big(\int_{\mathbb{M}} f(K_{m_1}) \int_{\mathbb{R}^d} f(\hat{K}_1, W_s) \,\mathrm{d}x_1 \,\mathbb{Q}(\mathrm{d}m_1)\Big)^2 \\
&= \frac{c_9 \gamma^2 f(W_s)^2}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} \Big(\int_{\mathbb{M}} f(K_{m_1})^2 \,\mathbb{Q}(\mathrm{d}m_1)\Big)^2
\end{aligned}$$

and

$$\begin{aligned}
&\gamma \int_{\mathbb{M}} \int_{\mathbb{R}^d} \mathbb{E}[(D_{\hat{x}} F_s)^4] \,\mathrm{d}x_1 \,\mathbb{Q}(\mathrm{d}m_1) \\
&\leq \frac{c_9 \gamma}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} \int_{\mathbb{M}} f(K_{m_1})^3 \int_{\mathbb{R}^d} f(\hat{K}_1, W_s) \,\mathrm{d}x_1 \,\mathbb{Q}(\mathrm{d}m_1) \\
&= \frac{c_9 \gamma f(W_s)}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} \int_{\mathbb{M}} f(K_{m_1})^4 \,\mathbb{Q}(\mathrm{d}m_1).
\end{aligned} \tag{6.43}$$

Since the variance and $f(W_s)$ grow with order s^d, this bound leads with the fourth moment assumptions and (6.41) to $\mathbb{E}[F_s^4] \leq c_{10}$ for a suitable constant $c_{10} > 0$ and s large enough. Hence, with (6.25) we have

$$\begin{aligned}
\gamma_4 &\leq \frac{\gamma c_{10}^{1/4}}{2} \int_{\mathbb{M}} \int_{\mathbb{R}^d} (\mathbb{E}[(D_{\hat{x}_1} F_s)^4])^{3/4} \,\mathrm{d}x_1 \,\mathbb{Q}(\mathrm{d}m_1) \\
&\leq \frac{c_{11}}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^{3/2}} \int_{\mathbb{M}} f(K_{m_1})^2 \int_{\mathbb{R}^d} f(\hat{K}_1, W_s) \,\mathrm{d}x_1 \,\mathbb{Q}(\mathrm{d}m_1) \\
&= \frac{c_{11} f(W_s)}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^{3/2}} \int_{\mathbb{M}} f(K_{m_1})^3 \,\mathbb{Q}(\mathrm{d}m_1) \leq \frac{c_{12}}{\sqrt{V_d(W_s)}}
\end{aligned}$$

for suitable constants $c_{11}, c_{12} > 0$ and s large enough. By (6.43) we get for γ_5,

$$\gamma_5^2 \leq \frac{c_{13}}{V_d(W_s)}$$

for s large enough and some constant $c_{13} > 0$. Finally, for γ_6 we need the estimate

$$\begin{aligned}
\mathbb{E}[(D^2_{\hat{x}_1, \hat{x}_2} F_s)^4] &\leq \frac{c_{14}}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} f(\hat{K}_1, \hat{K}_2, W_s)^4 \\
&\leq \frac{c_{14}}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} f(\hat{K}_1, \hat{K}_2, W_s)^2 f(\hat{K}_1, W_s)^2.
\end{aligned}$$

for some $c_{14} > 0$. This provides with (6.42) and $c_{15} = 6\sqrt{c_7 c_{14}} + 3c_{14}$,

$$\gamma_6^2 \leq \frac{c_{15} \gamma^2}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} \int_{\mathbb{M}^2} \int_{(\mathbb{R}^d)^2} f(\hat{K}_1, W_s)^2 f(\hat{K}_1, \hat{K}_2, W_s)^2 \,\mathrm{d}(x_1, x_2) \,\mathbb{Q}^2(\mathrm{d}(m_1, m_2))$$

$$
\begin{aligned}
&\leq \frac{c_{15}\gamma^2}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} \int_{\mathbb{M}^2} f(K_{m_1})^2 f(K_{m_2}) \int_{(\mathbb{R}^d)^2} f(\hat{K}_1, \hat{K}_2, W_s) \, \mathrm{d}(x_1, x_2) \, \mathbb{Q}^2(\mathrm{d}(m_1, m_2)) \\
&\leq \frac{c_{15}\gamma^2 f(W_s)}{\mathrm{Var}[\varphi(Z_u \cap W_s)]^2} \int_{\mathbb{M}} f(K_{m_1})^3 \, \mathbb{Q}(\mathrm{d}m_1) \int_{\mathbb{M}} f(K_{m_2})^2 \, \mathbb{Q}(\mathrm{d}m_2) \\
&\leq \frac{c_{16}}{V_d(W_s)}
\end{aligned}
$$

for a suitable constant $c_{16} > 0$ and s large enough. Altogether, the estimates for $\gamma_1, \ldots, \gamma_6$ complete the proof of the quantitative central limit theorem in Kolmogorov distance. □

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