Fractional Powers of Linear Operators in Locally Convex Vector Spaces

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Jan Meichsner

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Gutachter:	Prof. Dr. Marko Lindner
	Dr. habil. Christian Seifert
	Prof. Tom ter Elst
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DOI:	https://doi.org/10.15480/882.3674
ORCID:	https://orcid.org/0000-0002-9900-7864

Summary

The work at hand deals with fractional powers of non-negative operators, closed linear but typically discontinuous operators fulfilling a resolvent growth estimate, in quasi-complete Hausdorff locally convex spaces. In particular, the special case of a Banach space is incorporated. Basic properties of such operators in locally convex spaces, among them many properties known from the Banach space case but also things characteristic for more general locally convex spaces such as the stability of the class under formation of inductive and projective limits, are studied. The proofs for the properties which also could be formulated in Banach spaces are similar to the proofs there and do not cause greater problems.

Afterwards the ε-product, a concept to describe certain classes of vector-valued functions, will be introduced and used to formulate what is meant by a functional calculus in this particular setting. When using a concrete representation of the objects in the studied algebras, the formulae arising from this approach take their expected form. Two well known functional calculi, namely the Hille-Phillips as well as the Stieltjes functional calculus, will be introduced and extended.

The Stieltjes calculus then will be used to introduce and study fractional powers of non-negative operators. A couple of expected properties known from the Banach space situation, such as power laws and the possibility of using the Balakrishnan formula, will be investigated and confirmed.

As an application of the so far introduced theory, existence and uniqueness of solutions of the Caffarelli–Silvestre Problem in locally convex spaces with the above mentioned properties will be studied. The uniqueness and existence result will be proved under the slightly more restrictive assumption that the involved operator is actually sectorial which is in general a strict sub–class of the non–negative operators in locally convex spaces. Still the achieved result generalises the situation from the Banach space setting to the framework of locally convex spaces.

Contents

List of Figures

1. Introduction

Fractional powers of closed linear operators in Banach spaces are a classical topic in operator theory. To be more specific, it plays a significant role with numerous applications in the classical theory of ODEs, stochastics, interpolation theory, maximal regularity, and Cauchy-type initial value problems. The main question of the entire theory may be boiled down to an essence by asking whether, for a given closed linear operator A in some space X and a complex number $\alpha \in \mathbb{C}$, we can define an operator A^α which we will call the fractional power of the operator A in such a way that the fractional power inherits properties of the 'base' operator A (one may think of continuity or closedness) and such that the fractional powers behave as we would expect from the study of the possibly simplest instance of the here pictured scenario which is considering the Banach space $\mathbb{C}$. So for example one would expect the power laws $A^\alpha A^\beta = A^{\alpha+\beta}$ as well as $(A^\alpha)^\beta = A^{\alpha\beta}$ assuming the latter makes sense.

Historically, and surely subject to different interpretations, one can say that the research on this topic actually dates back to the 17^{th} century and the early days of calculus when mathematicians found laws comparable to power laws hidden in the newly created theory. It is an interesting question by itself to study possible interpolations between n-folded applications of differentiation and integration apparently first raised by L'Hôpital in a letter to Leibniz in 1695 and subsequently studied by Euler, Lagrange, Laplace, and Fourier. All the so far mentioned mathematicians contributed directly or indirectly to the above raised question, though, they did not have a particular application in mind which changed when Abel used the calculus indirectly to solve a generalised Tautochrone problem. Abel's solution stimulated further deep research in the field carried out by Liouville and Riemann. The historical roots of the subject also explain the today's term 'fractional' as from the interpolation point of view one naturally first considers operators with rational exponents. At the present time the term is somewhat misleading since even the consideration of complex exponents found their applications. For many more interesting details on the history, the reader be adviced to have a look at [55].

At the beginning of the 20^{th} century functional analysis and in particular operator theory developed and provided the language and the tools for a more abstract study of fractional powers but, to the best of the authors knowledge, it took almost 50 more years till new studies on fractional powers took place. This was not due to a lack of possibilities. The spectral theory of normal operators provided already a first instance of what is now commonly called a functional calculus, and one could have easily explored properties of fractional powers for A being a normal operator on a Hilbert space whose spectrum lies inside a fixed sector of the complex plane not intersecting the negative reals and with A^α defined by means of the spectral theorem. It seems that it was the missing application for a general theory outside the realm of calculus which rendered the topic somehow uninteresting. This gap was closed in 1949 by Bochner who studied in [9] the notion of subordination (a term introduced by the same author in [10]) in stochastics which provided a first alternative to the description of fractional powers via spectral theory. The underlying theory was actually already abstracted some time before from the stochastic context to general Banach spaces by Hille in [29] and subsequently extended in [4, 53, 54] to what now is called the Hille-Phillips calculus. In terms of operator theory one obtains fractional powers of an operator A when $-A$ is the generator of a bounded C_0-semigroup by plugging in the operator A in the function $\lambda \mapsto \mathrm{e}^{-z\lambda^\alpha}$ and considering the negative generator of the so obtained new C_0-semigroup which is A^α. For a thorough treatment of subordination and further details on the history including sources, the reader my consult the book [57, Chap. 13] while more information on the Hille-Phillips calculus are available in [27].

A. V. Balakrishnan extended the construction of fractional powers in 1960 in [5] to the wider class of non-negative operators, a term coined by Komatsu in [36] who studied the topic intensively as well in a series of papers beginning with [34]. He also introduced interpolation space methods in the study of fractional powers in [35] and connected them to the real interpolation technique introduced by Lions and Peetre in [41]. The complex interpolation technique introduced by Calderón ([14]) seems to enter the stage the first time in [60]. The general coincidence of domains of fractional powers for base operators with the property to have so-called bounded imaginary powers and complex interpolation spaces was shown in [68].

In the context of Banach spaces a non-negative operator A has the negative reals contained in its resolvent set and fulfills an additional resolvent estimate on them. In Banach spaces these operators coincide with sectorial operators as introduced in [25, 33]. An operator is sectorial if an entire sector symmetric around the negative real axis is contained in the resolvent set and if an estimate on the resolvent operators is available on all proper subsectors. Sectorial operators in this sense are automatically non-negative, and the converse implication in Banach spaces follows from the Neumann series. They are, however, by no means the most general class of operators for which one can define fractional powers. Indeed, all classes typically used as base for fractional powers share the property that the negative reals are contained in the operators resolvent sets but weaker resolvent estimates than those used for sectorial operators suffice to define fractional powers. For an example see [18] where the authors studied operators with polynomial growing resolvents as well as so-called regularised operators. However, from a functional calculus point of view the resolvent estimates defining sectorial operators feel, at least for the author, to be the most natural conditions providing a good compromise between applicability, generality, and richness of results.

Coming back to Balakrishnan's work, in reflexive spaces such as Hilbert spaces sectorial operators are automatically densely defined. Studying the Balakrishnan construction carefully it is revealed that using his construction actually means to consider a certain part of the operator A which is always densely defined. Without this assumption a reasonable spectral mapping theorem cannot be proved and one even fails to obtain $A^1 = A$. Thus, until 1988 a sensible way to define fractional powers for non-densely defined operators was missing. This gap got closed by Marco, Martinez and Sanz in [42] who provided an at first glance more complicated definition which turned out to be a proper generalisation of the Balakrishnan definition including the spectral mapping theorem and the above mentioned equation $A^1 = A$ even for A not being densely defined.

From a today's point of view Balakrishnans construction is part of the modern calculus of sectorial operators as it was introduced in the groundbreaking paper [45]. A very detailed study of the entire topic is available in [25]. Indeed, it is a very natural approach to define A^α by trying to plug in the operator A in the function $\lambda \mapsto \lambda^\alpha$. Other calculi besides the sectorial calculus can be used to define fractional powers as well. To mention two examples, one could directly make use of the halfplane calculus, the natural choice for generators of bounded C_0-semigroups which are not analytic ([6]), or defines the powers as generators of subordinated semigroups defined by means of the Hille-Phillips calculus, i.e., make use of Bochner's subordination mentioned before and thus defines them using the calculus indirectly.

We already mentioned that in Banach spaces there is no need to distinguish between sectorial and non-negative operators. This is not anymore true in more general locally convex spaces. For this reason the development of a new calculus, refraining from using the property of sectoriality, became necessary in order to extend the theory to non-negative operators in locally convex spaces. A first version of such a calculus was already contructed in 1977 by Hirsch ([30]) and extended and used in [15] to generalise the theory to Fréchet spaces and beyond the realm of sectorial operators. Further simplifications and generalisations as well as a quite complete overview of many applications of fractional powers are contained in the book [44]. However, because of the interest of many people in the Banach space situation only, the book merely sketches what to do in locally convex spaces and leaves it to the reader to fill the details. Moreover, the very systematic and appealing view on general functional calculi, as applied in [25], did not enter in [44] (for the obvious reason that the former work was carried out 5 years after [44] was written).

In the beginning we mentioned manifold applications of fractional powers in different fields of mathematics. A rather recent one is the description of a generalised Dirichlet-to-Neumann operator. This application was first considered for the special case of the Laplacian by Caffarelli and Silvestre in 2007 in the celebrated work [13]. In this work the viewpoint was converse to what was just said. The authors used the Dirichlet-to-Neumann operator to describe the fractional power. Later the situation got generalised to the abstract realm by various others introducing completely new (in fact functional calculus) techniques, see [3, 23, 48, 47, 49, 63]. As it finally turned out, the Dirichlet-to-Neumann operator is very much equivalent to the fractional power.

Based on this short summary and overview of the research field, the thesis at hand is structured as follows. In Chapter 2 we will collect basic notions of locally convex vector spaces, introduce the main class of operators we are going to study (namely non-negative operators), and prove some properties for them. The following Chapter 3 will be devoted to the study of the functional calculi in locally convex spaces. We will introduce the notion of ε-products and use it to establish the Hille-Phillips as well as the Stieltjes calculus. The latter will be used in Chapter 4 to define fractional powers and study their properties. Afterwards, the equivalence of the 'real' powers with the indirect description as generators of holomorphic semigroups stemming from subordination will be shown. The final Chapter 5 then deals with the Caffarelli–Silvestre Extension Problem and its solution.

2. Basics

In this first chapter we shall define the basic structures which will be considered in the dissertation in its full generality, and we will prove the first elementary properties. The structure of the chapter is motivated by [44] where (among many other books) the corresponding properties for sectorial operators in Banach spaces are proven and where aspects of the situation in locally convex spaces are discussed. From now on X shall denote a locally convex vector space (LCS) over the field $\mathbb{C}$ unless otherwise specified. A system of continuous seminorms $\|\cdot\|_p$ generating the topology of X will be denoted by $\mathcal{P}_X$. Every such system will, w.l.o.g., be assumed to be *directed* which means that for every $\|\cdot\|_p, \|\cdot\|_q \in \mathcal{P}_X$ we can find $\|\cdot\|_r \in \mathcal{P}_X$ and $C > 0$ such that $\max\{\|\cdot\|_p, \|\cdot\|_q\} \leq C\|\cdot\|_r$. All LCS in the thesis shall always assumed to be Hausdorff. In terms of seminorms the Hausdorff property may be characterised by

$$(\forall \|\cdot\|_p \in \mathcal{P}_X : \|x\|_p = 0) \qquad \Rightarrow \; x = 0.$$

By $\mathcal{U}$ we shall denote the set of all 0-neighborhoods.

A set $A \subseteq X$ is said to be *convex* if for all $x, y \in A$, $t \in [0,1]$ we also have $tx+(1-t)y \in A$ and it is *balanced* if $x \in A$ implies $rx \in A$ for all $r \in \mathbb{C}$ with $|r| \leq 1$. A balanced, convex set is called *absolutely convex*. Equivalently, one can characterise absolute convexity by the fact that $x, y \in A$, $\alpha, \beta \in \mathbb{C}$, $|\alpha| + |\beta| \leq 1$ implies $\alpha x + \beta y \in A$. Every LCS X admits a basis of 0-neighbourhoods which are absolutely convex.

Also recall that a set $B \subseteq X$ is called *bounded* if for all $U \in \mathcal{U}$ there is $c \in \mathbb{C}$ such that $B \subseteq cU$. Define the *ball of radius $r > 0$ around $x \in X$ w.r.t. the seminorm $\|\cdot\|_p$* by $B_p(x,r) := \{x \in X \mid \|x\|_p < r\}$. Then it is already sufficient for a set to be bounded if

$$\forall \|\cdot\|_p \in \mathcal{P}_X \; \exists r > 0 : \; B \subseteq B_p(0,r).$$

Furthermore, one can rephrase boundedness in terms of seminorms and gets that a set $B \subseteq X$ is bounded if and only if

$$\forall \|\cdot\|_p \in \mathcal{P}_X : \sup_{x \in B} \|x\|_p < \infty.$$

An LCS is said to be *quasi-complete* if every bounded Cauchy net is convergent in X. From now on we also want to assume that X is quasi-complete. Other additional assumptions about X will be introduced at the points where they are needed. For two LCS X and Y we shall denote by $\mathcal{C}(X,Y)$ the set of all closed linear operators A defined on some subspace $\mathcal{D}(A) \subseteq X$ to Y. In case $X = Y$ we briefly write $\mathcal{C}(X)$. By $\mathcal{L}(X,Y)$ we denote the set of all continuous linear operators from X to Y which we assume to be defined on the whole of X. One should have in mind that every continuous linear operator is *locally bounded*, i.e., maps bounded sets into bounded sets but the converse may be wrong unless X has the property to be *bornological*, see [56, Chap. II, Prop. 8.3]. As in the case for closed operators, we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,X)$. For an overview and more information on LCS the reader may consult [50, 56].

A family $(A_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathcal{L}(X)$ is said to be *equicontinuous* if

$$\forall \left\|\cdot\right\|_p \in \mathcal{P}_X \, \exists C > 0, \left\|\cdot\right\|_q \in \mathcal{P}_X \, \forall x \in X : \sup_{\alpha\in\mathcal{A}} \left\|A_\alpha x\right\|_p \leq C \left\|x\right\|_q .$$

Similar to the case of a Banach space we shall define the *resolvent set* of a linear operator A on X to be $\rho(A) := \{\lambda \in \mathbb{C} \mid (\lambda - A)^{-1} \in \mathcal{L}(X)\}$, and we will call $\sigma(A) := \mathbb{C} \setminus \rho(A)$ the *spectrum* of the operator A. We are now going to introduce the basic class of operators which we will study.

Definition 2.0.1. Let $A \in \mathcal{C}(X)$. The operator A is called *non-negative* if $(-\infty, 0) \subseteq \rho(A)$ and if the family

$$\big(\lambda(\lambda + A)^{-1}\big)_{\lambda>0}$$

is equicontinuous which means that

$$\forall \left\|\cdot\right\|_p \in \mathcal{P}_X \, \exists M > 0, \; \left\|\cdot\right\|_q \in \mathcal{P}_X \, \forall x \in X : \sup_{\lambda>0} \left\|\lambda(\lambda + A)^{-1}x\right\|_p \leq M \left\|x\right\|_q .$$

We denote by $\mathcal{M}(X)$ the set of all non-negative operators on an LCS X.

The here introduced notion of non-negativity is taken from [44] and generalises the common concepts, as defined for instance in [5], for Banach spaces. Occasionally authors also include further assumptions such as dense domain, injectivity, or dense range. This shall not be included in our definition and will explicitly be stated when needed. Let us have a look at examples and non-examples. Before, let us agree that for $z \in \mathbb{C}\setminus(-\infty, 0]$ the symbol $\arg(z) \in (-\pi, \pi)$ denotes the unique number such that $z = |z| \, \mathrm{e}^{i\arg(z)}$ and denote for $\omega \in [0, \pi)$ the *closed sector of angle* ω by $S_\omega := \{z \in \mathbb{C} \setminus (-\infty, 0] \mid |\arg(z)| \leq \omega\} \cup \{0\}$. Furthermore, the reader may recall that an *operator semigroup* is a family $(\mathrm{e}^{-zA})_{z\in S_\omega}$

in $\mathcal{L}(X)$, $\omega \in [0, \frac{\pi}{2}]$, such that $\mathrm{e}^{-0A} = 1_X$, 1_X being the identity, and $\forall z, w \in S_\omega$: $\mathrm{e}^{-zA}\mathrm{e}^{-wA} = \mathrm{e}^{-(z+w)A}$. From this property it follows that the single operators of the family commute with each other. The semigroup is called *strongly continuous* or a C_0*-semigroup* if $\forall x \in X : \lim_{z\to 0,\, z\in S_\omega} \mathrm{e}^{-zA}x = x$. Having these definitions in mind, let us have a look on known examples.

Example 2.0.2.

(a) Let Ω be a Hausdorff locally compact space with Radon measure μ. For $p \in [1, \infty]$ set $X := L^p_{loc}(\Omega)$, and choose $f \in L^0(\Omega)$ to define $A : \mathcal{D}(A) \to X$ via $Ag := fg$ with domain $\mathcal{D}(A) := \{g \in X \mid fg \in X\}$. Similar to the case of $L^p_{loc}(\Omega)$ being replaced with $L^p(\Omega)$, one shows that A is a closed operator, and $A \in \mathcal{L}(X)$ if and only if $f \in L^\infty_{loc}(\Omega)$. If $f \notin L^\infty_{loc}(\Omega)$, then A is densely defined if and only if $p \in [1, \infty)$. Moreover, it is always true that $\sigma(A) = \mathcal{R}_{ess}(f)$. Finally, $A \in \mathcal{M}(X)$ if and only if for every compact set $K \subseteq \Omega$ there is a number $\omega_K \in [0, \pi)$ with the property that $\mathcal{R}_{ess}(f \cdot \mathbb{1}_K) \subseteq S_{\omega_K}$.

(b) Let A be a normal operator in a Hilbert space X. Then A is non-negative if and only if there is $\omega \in [0, \pi)$ with the property that $\sigma(A) \subseteq S_\omega$. This is proven in [44, Thm. 1.3.5].

(c) Let X be an LCS and $-A$ the generator of an equicontinuous C_0-semigroup. Then A is non-negative. This follows from an extension of the Hille–Yosida Theorem to LCS, see [69, Chap. IX, Sect. 7].

(d) Let X be a Banach space, and let $-A$ be the generator of a C_0-semigroup with growth bound $\omega_0 \in \mathbb{R}$. Let $\omega > \omega_0$. Then $A + \omega$ is non-negative. This follows from the classical Hille–Yosida Theorem for Banach spaces ([20, Sect, II, Thm. 3.8]).

(e) An operator which will often cross our path is the (negative) Laplacian $-\Delta$ defined on various spaces. Many of its realisations (sometimes involving a small shift) are actually generators of equicontinuous semigroups and, hence, are non-negative. So for example, the negative Laplacian defined on $H^1_0(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open (Dirichlet-Laplacian) or its Neumann equivalent are such realisations. One may also consider other L^p-spaces (which will lead to geometric constraints on Ω, see [67, Thm. 3.8]) or spaces of continuous functions. As for distributional realisations, one can say that $-\Delta + \varepsilon$, $\varepsilon > 0$, is non-negative on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions but just $-\Delta$ is not though, see [44, Rem. 5.6.2].

Equicontinuity of the family of resolvent operators implies strong continuity and even better differentiability of the family as the following proposition shows.

Proposition 2.0.3. *Let $A \in \mathcal{M}(X)$. Then*

$$\forall x \in X : \big((0,\infty) \ni \lambda \mapsto (\lambda + A)^{-1} x\big) \in C^\infty\big((0,\infty); X\big)$$

Proof. By the resolvent identity it follows that for every $\|\cdot\|_p \in \mathcal{P}_X$ and $\lambda, \mu > 0$ there is $C > 0$ and $\|\cdot\|_q \in \mathcal{P}_X$ such that

$$\left\|(\lambda + A)^{-1} x - (\mu + A)^{-1} x\right\|_p = |\lambda - \mu| \left\|(\mu + A)^{-1} (\lambda + A)^{-1} x\right\|_p \leq C \frac{|\lambda - \mu|}{\lambda\, \mu} \|x\|_q .$$

Hence, $\lambda \mapsto (\lambda + A)^{-1} x$ is even locally Lipschitz continuous. The continuity in turn implies, again by the resolvent identity, differentiability since for $\lambda > 0$, $h \in \mathbb{R}$ such that $\lambda + h > 0$ there is $C > 0$ with

$$\begin{aligned} &\left\|(\lambda + h + A)^{-1} x - (\lambda + A)^{-1} x + h\, (\lambda + A)^{-2} x\right\|_p \\ \leq &\frac{C\,|h|}{\lambda} \left\|(\lambda + h + A)^{-1} x - (\lambda + A)^{-1} x\right\|_q . \end{aligned}$$

Continuity of the derivative is a consequence of the geometric sum formula which states that for $a, b \in \mathbb{C}$, $n \in \mathbb{N}$ it holds that

$$(a^n - b^n) = (a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

Applied to the first derivative this gives, again for some $C > 0$, for instance

$$\begin{aligned} &\left\|(\lambda + A)^{-2} x - (\mu + A)^{-2} x\right\|_p \\ = &\left\|\big((\lambda + A)^{-1} + (\mu + A)^{-1}\big)\big((\lambda + A)^{-1} - (\mu + A)^{-1}\big) x\right\|_p \\ \leq &C\, \frac{\lambda + \mu}{\lambda\, \mu} \left\|(\lambda + A)^{-1} x - (\mu + A)^{-1} x\right\|_q \end{aligned}$$

and similar for higher derivates. Arguing inductively, one finds

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} (\lambda + A)^{-1} x = (-1)^n n!\, (\lambda + A)^{-(n+1)} x$$

and the claim is proven. ∎

Let X be a Banach space and $A \in \mathcal{C}(X)$ such that $\rho(A) \neq \varnothing$. In this situation $\rho(A)$ is open in $\mathbb{C}$ and the mapping $\rho(A) \ni \lambda \to (\lambda - A)^{-1} \in \mathcal{L}(X)$ is holomorphic even w.r.t. the operator topology. This is not necessarily true in LCS as the following example shows.

Example 2.0.4. *We consider the Fréchet space $X := \{f \in C^\infty\big([0,1]\big) \mid \forall n \in \mathbb{N}_0 : f^{(n)}(0) = 0\}$ with its usual topology induced by seminorms*

$$\|f\|_n := \max_{0 \le k \le n} \sup_{x \in [0,1]} \left|f^{(k)}(x)\right| \quad (n \in \mathbb{N}_0).$$

Consider the operator A given by

$$(Af)(x) := \int_0^x f(t)\mathrm{d}t.$$

This is a continuous operator with continuous inverse given by the derivative. Both operators have resolvent set $\mathbb{C}$ and the resolvent of A^{-1} is given by ($\lambda \in \mathbb{C}$, $g \in X$)

$$\Big(\big(\lambda - A^{-1}\big)^{-1} g\Big)(x) = -\mathrm{e}^{\lambda x} \int_0^x \mathrm{e}^{-\lambda t} g(t)\mathrm{d}t.$$

Now let $\lambda > 0$ and note that

$$(\lambda - A)^{-1} = -\frac{1}{\lambda}\left(\frac{1}{\lambda} - A^{-1}\right)^{-1} A^{-1}.$$

Choose $x \in (0,1]$ and $g \in X$ defined by $g(t) := \mathrm{e}^{-\frac{1}{t}}$, $t > 0$, $g(0) := 0$. For this particular g we have $g' \ge 0$ on $[0,1]$ and for every $[a,b] \subset (0,1)$ there is $c > 0$ such that $g'(t) \ge c$ for $t \in [a,b]$. So

$$\begin{aligned}
\lambda\Big(\big(\lambda - A^{-1}\big)^{-1} g\Big)(x) &= -\lambda \mathrm{e}^{\lambda x} \int_0^x \mathrm{e}^{-\lambda t} g(t)\mathrm{d}t = g(x) - \mathrm{e}^{\lambda x} \int_0^x \mathrm{e}^{-\lambda t} g'(t)\mathrm{d}t \\
&\le g(x) - \min_{t \in \left[\frac{x}{2}, x\right]} g'(t) \mathrm{e}^{\lambda x} \int_{\frac{x}{2}}^x \mathrm{e}^{-\lambda t}\mathrm{d}t \\
&= g(x) + \min_{t \in \left[\frac{x}{2}, x\right]} g'(t) \frac{1 - \mathrm{e}^{\lambda \frac{x}{2}}}{\lambda} \to -\infty
\end{aligned}$$

as $\lambda \to \infty$. Hence, if we choose $f := Ag$ with g as above we conclude that

$$\lim_{\lambda \to 0+} (\lambda - A)^{-1} f = \lim_{\lambda \to 0+} \frac{-1}{\lambda} \left(\frac{1}{\lambda} - A^{-1} \right)^{-1} g$$

does not even exists pointwise for $x \in (0, 1]$.

Remark 2.0.5. Let $A \in \mathcal{M}(X)$. Note that because of $A(\lambda + A)^{-1} = 1 - \lambda(\lambda + A)^{-1}$ also the family $\left(A(\lambda + A)^{-1}\right)_{\lambda > 0}$ is equicontinuous.

Proposition 2.0.6. *Let $A \in \mathcal{C}(X)$ and such that $(-\infty, 0) \subseteq \rho(A)$. Then $A \in \mathcal{M}(X)$ and $0 \in \rho(A)$ if and only if the family $\left((\lambda + \varepsilon)(\lambda + A)^{-1}\right)_{\lambda > 0}$ is equicontinuous for some, and hence all, $\varepsilon > 0$.*

Proof. Assume $A \in \mathcal{M}(X)$ and $0 \in \rho(A)$. We only have to show that $\left((\lambda + A)^{-1}\right)_{\lambda > 0}$ is equicontinuous in order to establish the first part of the equivalence. This follows from Remark 2.0.5 and the continuity of A^{-1} since for a given seminorm $\|\cdot\|_p \in \mathcal{P}_X$ we have

$$\begin{aligned} &\exists C, D > 0,\ \|\cdot\|_q, \|\cdot\|_r \in \mathcal{P}_X\ \forall x \in X,\ \lambda > 0 : \\ &\left\|(\lambda + A)^{-1} x\right\|_p = \left\|AA^{-1}(\lambda + A)^{-1} x\right\|_p = \left\|A(\lambda + A)^{-1} A^{-1} x\right\|_p \leq C \left\|A^{-1} x\right\|_q \leq CD \|x\|_r . \end{aligned}$$

So $\left((\lambda + A)^{-1}\right)_{\lambda > 0}$ is equicontinuous.

Conversely, let $\left((\lambda + \varepsilon)(\lambda + A)^{-1}\right)_{\lambda > 0}$ be equicontinuous. From

$$(\lambda + A)^{-1} = \frac{1}{\lambda + \varepsilon} (\lambda + \varepsilon)(\lambda + A)^{-1},$$

it follows that $\left((\lambda + A)^{-1}\right)_{\lambda > 0}$ is equicontinuous. Consider its closure in $\mathcal{L}(X)$ with respect to the topology of pointwise convergence. By [56, Chap. III, Prop. 4.3] this closure is again equicontinuous and since X is quasi-complete it is even complete by [56, Chap. III, Prop. 4.4]. By the resolvent identity and equicontinuity the net $\left((\lambda + A)^{-1}\right)_{\lambda > 0}$ is a Cauchy net as $\lambda \to 0+$ and hence it is convergent. Using the closedness of A, one finds

$$\forall x \in X : \lim_{\lambda \to 0+} (\lambda + A)^{-1} x = A^{-1} x.$$

So $0 \in \rho(A)$. What is left, namely $A \in \mathcal{M}(X)$, follows directly from the equicontinuity of $\left((\lambda + \varepsilon)(\lambda + A)^{-1}\right)_{\lambda > 0}$ and $\left((\lambda + A)^{-1}\right)_{\lambda > 0}$ since

$$\lambda(\lambda + A)^{-1} = (\lambda + \varepsilon)(\lambda + A)^{-1} - \varepsilon(\lambda + A)^{-1}. \qquad \square$$

Remark 2.0.7. A system $\mathcal{B} \subseteq 2^X$ of bounded sets is called *bornology* if it covers X and if it is stable under inclusion and finite unions. Every such bornology gives rise to a locally convex topology on $\mathcal{L}(X)$. Namely, for $A \in \mathcal{L}(X)$ and $B \in \mathcal{B}$ we can define a seminorm by

$$\|A\|_{B,p} := \sup_{x \in B} \|Ax\|_p$$

where $\|\cdot\|_p$ is a continuous seminorm on X. The supremum is always finite since by continuity of A the set $A(B) \subseteq X$ is bounded. In the following $\mathcal{L}_{\mathcal{B}}(X)$ will denote $\mathcal{L}(X)$ equipped with the topology induced by $\mathcal{B}$. The topology of pointwise convergence is the coarsest possible choice. The corresponding bornology is the bornology of finite subsets of X. In the above proof every finer topology induced by a bornology could have been used.

We denote by β the *strong topology* on X', i.e., the topology of uniform convergence on the bounded sets of X. The resulting LCS shall be denoted by X'_β and is called the *strong dual* of X.

Lemma 2.0.8. *Let $A \in \mathcal{M}(X)$.*

1. *If A is injective, one has $A^{-1} \in \mathcal{M}(X)$.*
2. *Let A be densely defined and $A' : \mathcal{D}(A') \to X'_\beta$. Then $A' \in \mathcal{M}(X'_\beta)$.*

Proof.

1. The first part follows from the identity

$$\lambda \left(\lambda + A^{-1}\right)^{-1} = A \left(\frac{1}{\lambda} + A\right)^{-1}$$

which one has to combine with Remark 2.0.5.

2. Since A is densely defined, one can define the operator A'. Let now $B \subset X$ bounded be given. By equicontinuity it holds that

$$\tilde{B} := \bigcup_{\lambda \in (0,\infty)} \lambda \left(\lambda + A\right)^{-1}(B)$$

is bounded. This can be seen as follows. Let $V \in \mathcal{U}$ be given. By equicontinuity we can choose $U \in \mathcal{U}$ such that

$$\bigcup_{\lambda \in (0,\infty)} \lambda \left(\lambda + A\right)^{-1}(U) \subseteq V,$$

see [56, Chap. III, 4.1]. For this U there is $c \in \mathbb{C}$ with $B \subseteq cU$. It follows that $\tilde{B} \subset cV$. Now one can estimate for $f \in X'$

$$\sup_{x \in B} \left| \langle \lambda \left(\lambda + A'\right)^{-1} f, x \rangle \right| = \sup_{x \in B} \left| \langle f, \lambda \left(\lambda + A\right)^{-1} x \rangle \right| \leq \sup_{y \in \tilde{B}} \left| \langle f, y \rangle \right| .$$

This proves the claim. □

We continue with a number of standard approximation results.

Lemma 2.0.9. *Let $A \in \mathcal{M}(X)$ and $x \in X$. Then*

1. $x \in \overline{\mathcal{D}(A)} \Leftrightarrow \forall k \in \mathbb{N} : \lim_{\lambda \to \infty} \lambda^k (\lambda + A)^{-k} x = x \Leftrightarrow \forall k \in \mathbb{N} : \lim_{\lambda \to \infty} A^k (\lambda + A)^{-k} x = 0.$
In particular, $\forall k \in \mathbb{N} : \overline{\mathcal{D}(A)} = \overline{\mathcal{D}(A^k)}$.

2. $x \in \overline{\mathcal{R}(A)} \Leftrightarrow \forall k \in \mathbb{N} : \lim_{\lambda \to 0+} \lambda^k (\lambda + A)^{-k} x = 0 \Leftrightarrow \forall k \in \mathbb{N} : \lim_{\lambda \to 0+} A^k (\lambda + A)^{-k} x = x.$
In particular, $\forall k \in \mathbb{N} : \overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^k)}$.

3. $\forall k, n \in \mathbb{N} : \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)} = \overline{\mathcal{D}(A^k) \cap \mathcal{R}(A^n)}$.

Proof.

1. Let $x \in \overline{\mathcal{D}(A)}$. The geometric sum formula yields

$$x - \lambda^k (\lambda + A)^{-k} x = \sum_{l=0}^{k-1} \lambda^l (\lambda + A)^{-l} \left(x - \lambda (\lambda + A)^{-1} x \right) .$$

Choose $\|\cdot\|_p \in \mathcal{P}_X$ and note that for every $l \in \mathbb{N}$ the family $\left(\lambda^l (\lambda + A)^{-l}\right)_{\lambda > 0}$ is equicontinuous to estimate

$$\left\| x - \lambda^k (\lambda + A)^{-k} x \right\|_p \leq \sum_{l=0}^{k-1} C_l \left\| x - \lambda (\lambda + A)^{-1} x \right\|_{q_l} .$$

Hence, we can concentrate on the expression $x - \lambda(\lambda + A)^{-1} x$. Assume for the beginning even $x \in \mathcal{D}(A)$. Then

$$\left\| x - \lambda (\lambda + A)^{-1} x \right\|_p \leq \frac{C \left\| Ax \right\|_q}{\lambda} \to 0$$

as $\lambda \to \infty$. The result follows for all $x \in \overline{\mathcal{D}(A)}$ by density since we can choose a net

(x_α) in $\mathcal{D}(A)$, $x_\alpha \to x$ and estimate

$$\begin{aligned}
&\|\lambda(\lambda + A)x - x\|_p \\
\leq &\left\|\lambda(\lambda + A)^{-1}(x - x_\alpha)\right\|_p + \left\|\lambda(\lambda + A)^{-1}x_\alpha - x_\alpha\right\|_p + \|x_\alpha - x\|_p \\
\leq &C\,\|x_\alpha - x\|_q + \left\|\lambda(\lambda + A)^{-1}x_\alpha - x_\alpha\right\|_p + \|x_\alpha - x\|_p
\end{aligned}$$

where we used again the equicontinuity of $\left(\lambda(\lambda + A)^{-1}\right)_{\lambda>0}$.

Suppose now

$$\forall k \in \mathbb{N} : \lim_{\lambda\to\infty} \lambda^k(\lambda + A)^{-k}x = x.$$

The second implication follows from the binomial theorem since

$$A^k(\lambda + A)^{-k}x = \sum_{l=0}^{k} \binom{k}{l}(-1)^l\lambda^l(\lambda + A)^{-l}x \to \sum_{l=0}^{k} \binom{k}{l}(-1)^l x = 0.$$

For the last step let

$$\forall k \in \mathbb{N} : \lim_{\lambda\to\infty} A^k(\lambda + A)^{-k}x = 0$$

and argue as before to get

$$\lambda^k(\lambda + A)^{-k}x = \sum_{l=0}^{k} \binom{k}{l}(-1)^l A^l(\lambda + A)^{-l}x \to x,$$

i.e.,

$$\mathcal{D}(A^k) \ni \lambda^k(\lambda + A)^{-k}x \to x.$$

Hence, $x \in \overline{\mathcal{D}(A)}$.

As a byproduct we even get $\overline{\mathcal{D}(A^k)} = \overline{\mathcal{D}(A)}$.

2. The proof of the second part is completely analogous to the first part and will be omitted. Note however, that similar as before $\overline{\mathcal{R}(A^k)} = \overline{\mathcal{R}(A)}$ holds.

3. From the first and the second part it follows that

$$\mathcal{D}(A^k) \cap \mathcal{R}(A^n) \subseteq \overline{\mathcal{D}(A^k)} \cap \overline{\mathcal{R}(A^n)} = \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$$

which shows the first inclusion.

For the other inclusion let $x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$, $\lambda > 0$, and set $x_\lambda := A^n(\lambda +$

$A)^{-n}\left(\frac{1}{\lambda}\right)^k\left(\frac{1}{\lambda}+A\right)^{-k}x$. Then $x_\lambda \in \mathcal{D}(A^k)\cap\mathcal{R}(A^n)\subseteq\overline{\mathcal{D}(A^k)\cap\mathcal{R}(A^n)}$ and

$$\begin{aligned}\|x_\lambda - x\|_p &\le \left\|A^n(\lambda+A)^{-n}\left(\left(\frac{1}{\lambda}\right)^k\left(\frac{1}{\lambda}+A\right)^{-k}x-x\right)\right\|_p + \left\|A^n(\lambda+A)^{-n}x-x\right\|_p\\ &\le C\left\|\left(\frac{1}{\lambda}\right)^k\left(\frac{1}{\lambda}+A\right)^{-k}x-x\right\|_q + \left\|A^n(\lambda+A)^{-n}x-x\right\|_p\\ &\to 0 \text{ as } \lambda\to 0+.\end{aligned}$$

□

The presented approximation results lead to a couple of interesting corollaries.

Corollary 2.0.10. *Let $A\in\mathcal{M}(X)$. Then we have that*

1. $\mathcal{N}(A)\cap\overline{\mathcal{R}(A)}=\{0\}$. *Consequently,* $\overline{\mathcal{R}(A)}=X \Rightarrow \mathcal{N}(A)=\{0\}$.
2. *if* $\overline{\mathcal{D}(A)}$ *is equipped with the subspace topology, then* $A\in\mathcal{L}(\overline{\mathcal{D}(A)}) \Rightarrow \mathcal{D}(A)=X$.
3. *if* $\overline{\mathcal{D}(A)}\subsetneq X$*, then for all* $n\in\mathbb{N}$ $\mathcal{R}(A^n)\not\subset\overline{\mathcal{D}(A)}$*, and if* $\overline{\mathcal{R}(A)}\subsetneq X$*, then* $n\in\mathbb{N}$ $\mathcal{D}(A^n)\not\subset\overline{\mathcal{R}(A)}$.
4. $\forall k\in\mathbb{N}:\mathcal{N}(A)=\mathcal{N}(A^k)$.

Proof.

1. Let $x\in\mathcal{N}(A)\cap\overline{\mathcal{R}(A)}$. Then

$$x=\lim_{\lambda\to 0+}A(\lambda+A)^{-1}x=0.$$

2. Let $x\in X$. Then there are constants $M,C>0$ such that

$$\left\|A(\lambda+A)^{-1}x\right\|_p\le C\left\|(\lambda+A)^{-1}x\right\|_q\le C\cdot M\frac{\|x\|_r}{\lambda}\to 0 \quad\text{as}\quad \lambda\to\infty.$$

Hence, $x\in\overline{\mathcal{D}(A)}$ by Lemma 2.0.9. Furthermore, since A is closed, we also have $\overline{\mathcal{D}(A)}=\mathcal{D}(A)$. So by arbitraryness of x we get $\mathcal{D}(A)=X$.

3. For the first claim choose $x\notin\overline{\mathcal{D}(A)}$, $\lambda>0$ and $n\in\mathbb{N}$. Then

$$\mathcal{R}(A^n)\ni A^n(\lambda+A)^{-n}x=x+\sum_{k=1}^{n}\binom{n}{k}(-1)^k\lambda^k(\lambda+A)^{-k}x\notin\overline{\mathcal{D}(A)}.$$

Similarly for the second part choose $x \notin \overline{\mathcal{R}(A)}$ and λ, n as above and derive analogously

$$\mathcal{D}(A^n) \ni \lambda^n(\lambda + A)^{-n}x = x + \sum_{k=1}^{n} \binom{n}{k} (-1)^k A^k (\lambda + A)^{-k} x \notin \overline{\mathcal{R}(A)}.$$

4. The inclusion $\mathcal{N}(A) \subseteq \mathcal{N}(A^k)$ is clear. Conversely, let $x \in \mathcal{N}(A^k)$. For $\lambda > 0$ we conclude
$$0 = (\lambda + A)^{-1}A^k x \to A^{k-1}x \text{ as } \lambda \to 0+.$$
Arguing inductively yields the claim. □

More things can be proven by strengthening the assumptions on the space X. For this let again β denote the strong topology on X' and denote the strong dual by X'_β. We define the *bidual* X'' to be the dual space of X'_β, i.e., $X'' := (X'_\beta)'$. An LCS X is called *semi-reflexive* if $X'' \simeq X$, i.e., X'' is linearly isomorphic to X. Moreover, one may equip X'' with the topology of uniform convergence on the bounded sets of X'_β (by analogy to what has been said we may denote the result by X''_β), call it strong bidual, and say that X is *reflexive* if $X''_\beta \simeq X$, i.e., the spaces are even topologically isomorphic. For semi-reflexive spaces we can improve on the first part of Corollary 2.0.10.

Corollary 2.0.11. *If X is semi-reflexive and $A \in \mathcal{M}(X)$, it follows that $\overline{\mathcal{D}(A)} = X$ and $X = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$. In particular, we obtain $\overline{\mathcal{R}(A)} = X \Leftrightarrow \mathcal{N}(A) = \{0\}$.*

Proof. From semi-reflexivity, it follows that every bounded set is relatively compact w.r.t. the *weak topology* $\sigma(X, X')$ [56, Chap. IV, Thm. 5.5], i.e., bounded nets contain weakly convergent subnets. Let $x \in X$. Consider the bounded net given by $\big(\lambda(\lambda + A)^{-1}x\big)_{\lambda>0}$ in $\mathcal{D}(A)$ directed towards ∞. Choose a weakly convergent subnet $\big(\lambda_\alpha(\lambda_\alpha + A)^{-1}x\big)_{\alpha\in\mathcal{A}}$ and denote its weak limit by y. Then also the net $\big(A(\lambda_\alpha + A)^{-1}x\big)_{\alpha\in\mathcal{A}}$ converges weakly. The net $\big((\lambda + A)^{-1}x\big)_{\lambda>0}$, and therefore also its subnet $\big((\lambda_\alpha + A)^{-1}x\big)_{\alpha\in\mathcal{A}}$, converges to 0 in X. In particular, it converges weakly to 0. The operator A is closed and therefore also weakly closed. We conclude that the subnet $\big(A(\lambda_\alpha + A)^{-1}x\big)_{\alpha\in\mathcal{A}}$ converges weakly to 0 which implies that $\big(\lambda_\alpha(\lambda_\alpha + A)^{-1}x\big)_{\alpha\in\mathcal{A}}$ converges weakly towards x. So x is in the weak closure of $\mathcal{D}(A)$ which coincides with the closure in the given locally convex topology of X since $\mathcal{D}(A)$ is convex, see also ([31, Sect. 8.2, Prop. 4]).

For the second part of the statement we proceed similarly. Use again semi-reflexivity and choose for given $x \in X$ one (of possibly many) weak accumulation points of the net $\big(A(\lambda + A)^{-1}x\big)_{\lambda>0}$ which, in contrast to before, shall be directed towards 0. Call the accumulation point y and set $z := x - y$. Then $x = y + z$ with $y \in \overline{\mathcal{R}(A)}$. Furthermore,

$$A(1+A)^{-1}z = A(1+A)^{-1}x - A(1+A)^{-1}y = A(1+A)^{-1}x - \lim_{\alpha} A(\lambda_\alpha + A)^{-1}A(1+A)^{-1}x = 0$$

by Lemma 2.0.9. So $z \in \mathcal{N}(A(1+A)^{-1}) = \mathcal{N}(A)$. So far we found one possible decomposition of the arbitrary element x in the desired manner. Uniqueness follows now from $\mathcal{N}(A) \cap \overline{\mathcal{R}(A)} = \{0\}$ (Corollary 2.0.10). □

Another possibility to state more is to employ the open mapping theorem. In order to prove it in an LCS X the notation of a web and an inductive limit is required. A *web* W in an LCS X is a mapping $W : \bigcup_{k=1}^{\infty} \mathbb{N}^k \to 2^X$ such that $\mathcal{R}(W)$ is contained in the absolutely convex sets and

$$\bigcup_{l=1}^{\infty} W(l) = X \quad \text{as well as} \quad \forall k \in \mathbb{N},\, n_1, \ldots, n_k \in \mathbb{N} : \bigcup_{l=1}^{\infty} W(n_1, \ldots, n_k, l) = W(n_1, \ldots, n_k).$$

Furthermore, one requires a rather mild completeness property which says

$$\forall (n_k)_{k\in\mathbb{N}} \text{ in } \mathbb{N}\, \exists (\lambda_k)_{k\in\mathbb{N}} \text{ in } (0,\infty)\, \forall (x_k)_{k\in\mathbb{N}},\, x_k \in W(n_1, \ldots, n_k) : \sum_{k=1}^{\infty} \lambda_k x_k \text{ converges.}$$

An LCS X admitting a web is said to be a *webbed space*. In order to understand the last statement a bit better recall that a general topological space Ω is a Baire space if every intersection of countable many open dense subsets is again dense in Ω. Baire's category theorem states that among others completely metrisable spaces are Baire spaces.

One can show that precisely the Fréchet spaces are the LCS which are webbed spaces and Baire spaces at the same time ([31, Sect. 5.4, Thm. 4]).

A second notion is still needed to formulate the open mapping theorem in a very general context. For this let $\mathcal{A}$ be a directed set, $(X_\alpha)_{\alpha\in\mathcal{A}}$ a family of LCS and $(l_{\beta\alpha})_{\alpha\leq\beta\in\mathcal{A}}$ a family of linear continuous maps: $l_{\beta\alpha} : X_\alpha \to X_\beta$ with the property that $l_{\alpha\alpha} = 1_{X_\alpha}$ and $l_{\gamma\beta} \circ l_{\beta\alpha} = l_{\gamma\alpha}$.

Define an equivalence relation $(x_\alpha, \alpha) \sim (x_\beta, \beta) \;:\Leftrightarrow\; \exists \gamma \geq \alpha, \beta : l_{\gamma\alpha}(x_\alpha) = l_{\gamma\beta}(x_\beta)$. Now define $X := \{[(x_\alpha, \alpha)] \mid x_\alpha \in X_\alpha,\, \alpha \in \mathcal{A}\}$. This set becomes a vector space by setting

$$\forall \lambda, \mu \in \mathbb{C},\, [(x_\alpha, \alpha)], [(x_\beta, \beta)] \in X : \lambda[(x_\alpha, \alpha)] + \mu[(x_\beta, \beta)] := [(\lambda l_{\gamma\alpha}(x_\alpha) + \mu l_{\gamma\beta}(x_\beta), \gamma)],$$

for some $\gamma \geq \alpha, \beta$. Linear maps from all X_α to X are given by $L_\alpha : X_\alpha \to X$, $L_\alpha(x_\alpha) := [(x_\alpha, \alpha)]$. A locally convex topology on X may then be induced by defining that a seminorm $\|\cdot\| : X \to \mathbb{R}$ is continuous if and only if the seminorms $\|\cdot\| \circ L_\alpha : X_\alpha \to \mathbb{R}$ are continuous for every $\alpha \in \mathcal{A}$. Note that no non-trivial seminorm on X at all has to be continuous in which case we interpret X as equipped with the indiscrete topology. This topology is locally convex but lacks the Hausdorff property. This also can happen in more general situations. We shall always assume the resulting topology on X to be a Hausdorff topology and refer in this case to $\operatorname{ind}_{\alpha\in\mathcal{A}} X_\alpha := X$ as the *inductive limit* of the so-called *inductive spectrum* $(X_\alpha)_{\alpha\in\mathcal{A}}$.

Example 2.0.12.

(a) Let $n \in \mathbb{N}$ and consider for $k \in \mathbb{N}$ the spaces $X_k := C_0^\infty\big(B(0,k)\big)$ of smooth functions on $B(0,k)$ for which all derivates allow for continuous continuation on the boundary $\partial B(0,k)$ and which all vanish there. Here $B(x,r) \subseteq \mathbb{R}^n$ denotes the open ball with centre $x \in \mathbb{R}^n$ and radius $r > 0$. These spaces become Fréchet spaces when being equipped with seminorms of the form

$$\|f\|_l := \max_{|\alpha|\le l} \sup_{x\in B(0,k)} |(D^\alpha f)(x)|, \quad (l \in \mathbb{N}_0).$$

For $m,k \in \mathbb{N}, m \ge k$, define $l_{mk} : X_k \to X_m$ to be the embedding of X_k in X_m. The so constructed inductive spectrum is an example of a so-called countable, strict, regular (see [8] for definitions and details) embedding spectrum. The inductive limit is given by

$$\mathcal{D}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \operatorname{supp}(f) \text{ is compact}\}.$$

(b) Let again $n \in \mathbb{N}$ and consider for $k \in \mathbb{N}$ the space X_k defined by

$$X_k := \{f \in C^\infty(\mathbb{R}^n) \mid \forall l \in \mathbb{N}_0,\ \lambda \in \big(0, \frac{1}{k}\big) : \|f\|_{l,\lambda} := \max_{|\alpha|\le l} \sup_{x\in\mathbb{R}^n} \big|(D^\alpha f)(x)\mathrm{e}^{\lambda\|x\|}\big| < \infty\}.$$

These spaces become also Fréchet spaces when using the seminorms $\|\cdot\|_l$, $l \in \mathbb{N}_0$. The inductive limit X will later be used and also characterised in terms of Fourier transforms of the functions f.

(c) Let $\omega \in (0,\pi)$, $k_0 \in \mathbb{N}$ be such that $\omega + \frac{1}{k_0} < \pi$ and define for $k \in \mathbb{N}$, $k \ge k_0$ the numbers $\omega_k := \omega + \frac{1}{k}$. Consider the spaces

$$X_k := \{f \in \mathcal{H}^\infty(\mathring{S}_{\omega_k}) \mid \forall \alpha \in (-\omega_k, \omega_k) : \int_0^\infty \big|f(s\mathrm{e}^{i\alpha})\big| \frac{\mathrm{d}s}{s} < \infty\}$$

where $\mathcal{H}^\infty(\mathring{S}_{\omega_k})$ is the set of holomorphic, bounded functions defined on the open sector $\mathring{S}_{\omega_k}$ of opening angle ω_k. As before one can turn the single X_k into Fréchet spaces. The inductive limit plays an important role in the functional calculus of sectorial operators.

Remark 2.0.13. Similarly as in Example 2.0.12 (a) one can construct for an open set $\Omega \subseteq \mathbb{R}^n$ the space $\mathcal{D}(\Omega)$ as an inductive limit of a countable, strict, regular embedding spectrum.

Using all the tools we introduced, we formulate the following corollary.

Corollary 2.0.14. *Let X be the inductive limit of a sequence of Fréchet spaces (a so-called LF-space) and $A \in \mathcal{M}(X)$. Then A admits a continuous inverse if and only if it is surjective.*

Proof. If A admits a continuous inverse defined on all of X, it is clearly surjective. So let A be surjective. By [31, Sect. 5.2, Coro. 4] every LF-space carries a web, i.e., it is a webbed space. Since it is at the same time the inductive limit of a family of Baire spaces the open mapping theorem holds by [31, Sect. 5.5, Thm. 2]. By surjectivity we conclude from Corollary 2.0.10 that A is also injective. The continuous invertibility follows now since A is bijective and the open mapping theorem directly implies that every linear bijection admits a continuous inverse. □

Let us come back to the possibility to construct new spaces as 'limits' of given spaces. While the category of Banach spaces is not closed under the formation of inductive and, their dual concept, projective limits, the class of LCS is (except for the already mentioned problem that inductive limits in general may not be Hausdorff). Operators on the building blocks fulfilling certain compatibility assumptions allow for a lift to the limit. Naturally one may ask in which situations the lift of a family of non-negative operators results in a non-negative operator on the limit space.

Proposition 2.0.15. *Let $(X_\alpha)_{\alpha\in\mathcal{A}}$ be an inductive spectrum, and denote its inductive limit by X. Furthermore, assume for every $\alpha \in \mathcal{A}$ an operator $A_\alpha \in \mathcal{M}(X_\alpha)$ be given such that the following compatibility assumptions hold:*

$$\forall \beta \geq \alpha \in \mathcal{A} : l_{\beta\alpha}\left(\mathcal{D}(A_\alpha)\right) \subseteq \mathcal{D}(A_\beta) \quad \textit{and} \quad l_{\beta\alpha}A_\alpha = A_\beta l_{\beta\alpha}.$$

Then

$$\mathcal{D}(A) := \{x \in X \mid \exists \alpha \in \mathcal{A}, x_\alpha \in \mathcal{D}(A_\alpha) : \, x = [(x_\alpha, \alpha)]\}$$
$$Ax = A[(x_\alpha, \alpha)] := [(A_\alpha x_\alpha, \alpha)]$$

is a well-defined linear operator on X which is again non-negative. If all A_α are densely defined, so is A in X.

Proof. Let us first show that A is a well-defined operator. For this let $x_\alpha \in \mathcal{D}(A_\alpha)$ and $x_\beta \in \mathcal{D}(A_\beta)$ be such that $(x_\alpha, \alpha) \sim (x_\beta, \beta)$. Then there is $\gamma \geq \alpha, \beta$ with $l_{\gamma\alpha} x_\alpha = l_{\gamma\beta} x_\beta$. By our compatibility assumption we have $l_{\gamma\alpha} x_\alpha$, $l_{\gamma\beta} x_\beta \in \mathcal{D}(A_\gamma)$, and

$$l_{\gamma\alpha} A_\alpha x_\alpha = A_\gamma l_{\gamma\alpha} x_\alpha = A_\gamma l_{\gamma\beta} x_\beta = l_{\gamma\beta} A_\beta x_\beta.$$

We conclude that A is well-defined and even $(A_\alpha x_\alpha, \alpha) \sim (A_\beta x_\beta, \beta)$.

By the definition of the vector space structure on X linearity follows.

It remains to show $A \in \mathcal{M}(X)$. For the resolvent set of A we shall show the more general fact

$$\bigcap_{\alpha \in \mathcal{A}} \rho(A_\alpha) \subseteq \rho(A).$$

For this let $\lambda \in \bigcap_{\alpha \in \mathcal{A}} \rho(A_\alpha)$ and consider the equation

$$(\lambda - A)[(x_\alpha, \alpha)] = [(y_\beta, \beta)]. \tag{2.0.1}$$

Analogously as above one can see that

$$B_\lambda[(x_\alpha, \alpha)] := [((\lambda - A_\alpha)^{-1} x_\alpha, \alpha)]$$

defines a linear operator on the whole of X. Now a direct calculation verifies that the unique solution to (2.0.1) is given by $[(x_\alpha, \alpha)] = B_\lambda[(y_\beta, \beta)]$, i.e., $\lambda - A$ is bijective.

In order to see its continuity note that a general linear mapping B from any inductive limit X into any LCS Y is continuous if and only if for $\alpha \in \mathcal{A}$ the mappings $B \circ L_\alpha$ are continuous. Apply this to our situation by choosing $Y := X$ and note that

$$B_\lambda \circ L_\alpha = L_\alpha \circ (\lambda - A_\alpha)^{-1}$$

which is continuous as composition of continuous maps. In particular, since all A_α are non-negative, we have $(-\infty, 0) \subseteq \rho(A)$.

It remains to show the equicontinuity of the family $\big(\lambda(\lambda + A)^{-1}\big)_{\lambda>0}$. As a preparation note that a basis of $\mathcal{U}_X$ (0-neighborhoods in X) is given by all absolutely convex sets $U \subseteq X$ such that for all $\alpha \in \mathcal{A}$ it holds that $L_\alpha^{-1}(U) \in \mathcal{U}_{X_\alpha}$, see [8]. So let $U \in \mathcal{U}_X$ be given and assume it to be absolutely convex which can be done w.l.o.g. Define $U_\alpha := L_\alpha^{-1}(U) \in \mathcal{U}_{X_\alpha}$. Use now $A_\alpha \in \mathcal{M}(X_\alpha)$ and choose for every U_α a neighborhood V_α such that

$$\bigcup_{\lambda>0} \lambda(\lambda + A_\alpha)^{-1}\,(V_\alpha) \subseteq U_\alpha.$$

Set $V := \operatorname{acx}\big(\bigcup_{\alpha\in\mathcal{A}} L_\alpha\,(V_\alpha)\big)$. Here $\operatorname{acx}(U)$ denotes the *absolutely convex hull* of U, i.e., the smallest absolutely convex set which contains U. Let $\beta \in \mathcal{A}$ be arbitrary. Then

$$V_\beta \subseteq L_\beta^{-1}\big(L_\beta(V_\beta)\big) \subseteq L_\beta^{-1}\,(V)\,.$$

This shows that $V \in \mathcal{U}_X$. Let finally $\lambda > 0$. Then

$$\lambda(\lambda + A)^{-1}(V) = \operatorname{acx} \bigcup_{\alpha\in\mathcal{A}} L_\alpha \underbrace{\lambda(\lambda + A_\alpha)^{-1}\,(V_\alpha)}_{\subseteq U_\alpha} \subseteq U$$

shows the claimed equicontinuity.

For the last part let $x = L_\alpha x_\alpha$ be given. Since A_α is densely defined one can choose a net $(x_{\alpha\kappa})$ in $\mathcal{D}(A_\alpha)$ such that $\lim_\kappa x_{\alpha\kappa} = x_\alpha$. By continuity of L_α it follows that $x_\kappa := L_\alpha x_{\alpha\kappa} \to L_\alpha x_\alpha = x$ which finishes the proof. □

Remark 2.0.16. In the proof we actually showed a generalisation of the mentioned continuity criterion. Namely, let $(X_\alpha)_{\alpha\in\mathcal{A}}$ be an inductive spectrum with limit X, Y another LCS and $(A_\kappa)_{\kappa\in K}$ a family of linear continuous mappings from X to Y. Then this family is equicontinuous if and only if all the families $(A_\kappa L_\alpha)_{\kappa\in K}$ are equicontinuous.

Example 2.0.17.

(a) *The negative Laplacian $-\Delta$, nor any shift $-\Delta + \varepsilon$ ($\varepsilon > 0$) of it, is not non-negative on $\mathcal{D}(\mathbb{R}^n)$. The problem can already be located when considering the building blocks $C_0^\infty\big(B(0,k)\big)$. Suppose the equation $(-\Delta + \lambda)f = g$ had for every $\lambda > 0$ and every $g \in C_0^\infty\big(B(0,k)\big)$ a unique solution $f \in C_0^\infty\big(B(0,k)\big)$. Taking Fourier transform on both sides, this would imply that on the one hand*

$$(\mathcal{F}f)(z) = \frac{1}{(z^1)^2 + \cdots + (z^n)^2 + \lambda}(\mathcal{F}g)(z), \quad \big(z = (z^1, \ldots, z^n) \in \mathbb{C}^n\big)$$

while on the other hand $\mathcal{F}f$ has to be entire (analytic on the whole of $\mathbb{C}^n$) by the *assumption on the compact support of f. From the above formula however, one can see that $\mathcal{F}f$ will in general just be analytic on the tubular domain $\mathbb{C}^n_{\|\mathrm{Im}\|<\sqrt{\lambda}} := \{z = (z^1, \ldots, z^n) \in \mathbb{C}^n \mid \|\mathrm{Im}\, z\|^2 := \sum_{k=1}^n (\mathrm{Im}\, z^k)^2 < \lambda\}$.*

(b) The problem which arises when using the testfunction space $\mathcal{D}(\mathbb{R}^n)$ can be overcome by using instead the spaces

$$X_k = \{f \in C^\infty(\mathbb{R}^n) \mid \forall l \in \mathbb{N}_0,\ \lambda \in \big(0, \frac{1}{k}\big) : \ \|f\|_{l,\lambda} := \max_{|\alpha| \leq l} \sup_{x \in \mathbb{R}^n} \big|(D^\alpha f)(x) \mathrm{e}^{\lambda \|x\|}\big| < \infty\}$$

introduced in Example 2.0.12 (b). Analogously to the classical Paley–Wiener Theorem for functions (see [69, Chap. VI, Sect. 4]) one can show that for $f \in X_k$ it holds that $\mathcal{F}f \in \mathcal{H}(\mathbb{C}^n_{\|\mathrm{Im}\|<\frac{1}{k}})$ and for all $N \in \mathbb{N}_0$, $\lambda \in \big(0, \frac{1}{k}\big)$ there is $C_{N,\lambda} > 0$ such that

$$|(\mathcal{F}f)(z)| \leq C_{N,\lambda} (1 + \left|z^1\right| + \cdots + |z^n|)^{-N} (\lambda - \|\mathrm{Im}\, z\|)^{-n}$$

holds for all $z \in \mathbb{C}^n_{\|\mathrm{Im}\|<\lambda}$ while conversely every function $g \in \mathcal{H}(\mathbb{C}^n_{\|\mathrm{Im}\|<\frac{1}{k}})$ with the above growth property is a Fourier transform of some $f \in X_k$. Choosing now $\varepsilon > 0$ one can see by arguing as above that X_k is invariant under $(-\Delta + \varepsilon + \lambda)^{-1}$ for $k > \frac{1}{\varepsilon}$ and all $\lambda > 0$. By Proposition 2.0.15 the operator $-\Delta + \varepsilon$ is non-negative on the inductive limit for every $\varepsilon > 0$.

As mentioned earlier the algebraic (and to a certain extent also topological) dual concept of an inductive limit is a *projective limit*. For this let again be given a directed set $\mathcal{A}$, a family $(X_\alpha)_{\alpha \in \mathcal{A}}$ of LCS, and continuous linear maps $(\pi_{\beta\alpha})_{\alpha \leq \beta \in \mathcal{A}}$ such that $\pi_{\beta\alpha} : X_\beta \to X_\alpha$, $\pi_{\alpha\alpha} = 1_{|X_\alpha}$ and $\pi_{\beta\alpha} \circ \pi_{\gamma\beta} = \pi_{\gamma\alpha}$. One defines $X := \{(x_\alpha)_{\alpha \in \mathcal{A}} \mid x_\alpha \in X_\alpha \wedge \forall \beta \geq \alpha : \pi_{\beta\alpha} x_\beta = x_\alpha\}$. The space X is a subspace of the cartesian product $\Pi_{\alpha \in \mathcal{A}} X_\alpha$ and shall be equipped with the subspace topology, i.e., the topology of pointwise convergence. More explicitly, a system of continuous seminorms defining the topology is given by

$$\|(x_\alpha)\|_p := \sum_{i=1}^n \|x_{\alpha_i}\|_{p_i}$$

where $\|\cdot\|_{p_i}$ are continuous seminorms in the single spaces X_{α_i}. So the continuous seminorms on X are finite sums of continuous seminorms from the single X_α. Linear maps from X to all X_α are given by $P_\alpha : X \to X_\alpha$, $P_\alpha\big((x_\alpha)\big) := x_\alpha$. The space $\mathrm{proj}_{\alpha \in \mathcal{A}} X_\alpha := X$ is called *projective limit* of the *projective spectrum* $(X_\alpha)_{\alpha \in \mathcal{A}}$.

Example 2.0.18.

(a) *Let $1 \le p \le \infty$ and define for $m, k \in \mathbb{N}$, $m \ge k$ the space $X_k := L^p\big(B(0,k)\big)$ and $\pi_{mk} : X_m \to X_k$ by $\pi_{mk} f := f|_{X_k}$. This defines a projective spectrum and the limit is $X = L^p_{loc}(\mathbb{R}^n)$ with its usual topology.*

(b) *For $n, k \in \mathbb{N}$ consider $X_k := C^k(\mathbb{R}^n)$ and let π_{mk} be the inclusion map from X_m to X_k. The projective limit of this spectrum is $X = C^\infty(\mathbb{R}^n)$.*

(c) *Take $\mathcal{A} := \mathbb{N}_0^2$ and order as usual via $(m_1, k_1) \ge (m_2, k_2) :\Leftrightarrow m_1 \ge m_2 \wedge k_1 \ge k_2$. Consider for $(m,k) \in \mathbb{N}_0^2$ the space*

$$X_{(m,k)} := \{f \in C^k(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}_0^n, |\alpha| \le k : \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^m \, |(D^\alpha f)(x)| < \infty\}.$$

This defines again a projective spectrum with limit $X = \mathscr{S}(\mathbb{R}^n)$.

An analogous statement to Proposition 2.0.15 could be proven for projective limits.

Proposition 2.0.19. *Let $(X_\alpha)_{\alpha \in \mathcal{A}}$ be a projective spectrum and denote its projective limit by X. Furthermore, assume a family $(A_\alpha)_{\alpha \in \mathcal{A}}$ of non-negative operators given on every X_α such that the following compatibility assumptions hold:*

$$\forall \beta \ge \alpha \in \mathcal{A} : \pi_{\beta\alpha}\left(\mathcal{D}(A_\beta)\right) \subseteq \mathcal{D}(A_\alpha) \quad \text{and} \quad \pi_{\beta\alpha} A_\beta = A_\alpha \pi_{\beta\alpha}.$$

Then

$$\mathcal{D}(A) := \{x \in X \mid \forall \alpha \in \mathcal{A} :\, x_\alpha \in \mathcal{D}(A_\alpha)\}, \quad Ax = A(x_\alpha) := (A_\alpha x_\alpha)$$

is a well-defined linear operator on X which is again non-negative. If all A_α are densely defined, so is A in X.

Proof. To begin with we need to show that again $Ax \in X$. For this let $x \in \mathcal{D}(A)$, $\beta \ge \alpha$ and $x_\beta \in \mathcal{D}(A_\beta)$. Then $\pi_{\beta\alpha} x_\beta = x_\alpha \in \mathcal{D}(A_\alpha)$ and

$$\pi_{\beta\alpha} A_\beta x_\beta = A_\alpha \pi_{\beta\alpha} x_\beta = A_\alpha x_\alpha$$

which shows $Ax \in X$. Furthermore, for $\lambda \in \bigcap_{\alpha \in \mathcal{A}} \rho(A_\alpha)$, $x \in \mathcal{D}(A)$ and $\beta \ge \alpha$ we conclude from

$$\pi_{\beta\alpha}(\lambda - A_\beta) x_\beta = (\lambda - A_\alpha) \pi_{\beta\alpha} x_\beta$$

that for all $y \in X$

$$(\lambda - A_\alpha)^{-1} \pi_{\beta\alpha} y_\beta = \pi_{\beta\alpha} (\lambda - A_\beta)^{-1} y_\beta.$$

So as expected $\bigcap_{\alpha\in\mathcal{A}}\rho(A_\alpha)\subseteq\rho(A)$ and

$$(\lambda-A)^{-1}y=\big((\lambda-A_\alpha)^{-1}y_\alpha\big).$$

The non-negativity of A follows from

$$\left\|\lambda(\lambda+A)^{-1}x\right\|_p=\sum_{i=1}^{k}\underbrace{\left\|\lambda(\lambda+A_{\alpha_i})^{-1}x_{\alpha_i}\right\|_{p_i}}_{\le M_i\left\|x_{\alpha_i}\right\|_{q_i}}\le M\left\|x\right\|_q$$

where $M=\max\{M_1,\dots,M_k\}$.

Finally, let $x=(x_\alpha)\in X$ be given. Choose for every x_α a net $(x_{\alpha\kappa_\alpha})_{\kappa_\alpha\in\mathcal{K}_\alpha}$ in $\mathcal{D}(A_\alpha)$ being convergent towards x_α and set $\mathcal{K}:=\Pi_{\alpha\in\mathcal{A}}\mathcal{K}_\alpha$ as well as $x_\kappa:=(x_{\alpha\kappa_\alpha})$. Direct this set in the usual way by defining $\kappa\ge\sigma:\Leftrightarrow\forall\alpha\in\mathcal{A}:\ \kappa_\alpha\ge\sigma_\alpha$ and note that this implies $\lim_\kappa x_\kappa=x$. $\square$

Example 2.0.20. *Let us come back to the space $\mathscr{S}(\mathbb{R}^n)$. In Example 2.0.18 (c) we already saw that this space can be considered as a projective limit. In order to apply Proposition 2.0.19 we will need another system though. For $k\in\mathbb{N}_0$ and $\alpha\in\mathbb{N}_0^n$, $|\alpha|\le k$, consider the closed operators Q^α and D^α where $(Q^\alpha f)(x):=x^\alpha f(x)$ with maximal domain in $L^2(\mathbb{R}^n)$. Set*

$$X_k:=\bigcap_{\substack{\alpha\in\mathbb{N}_0^n,\\|\alpha|\le k}}\mathcal{D}(D^\alpha)\cap\mathcal{D}(Q^\alpha)=\{f\in H^k(\mathbb{R}^n)\mid\mathcal{F}f\in H^k(\mathbb{R}^n)\}$$

These spaces also form a projective spectrum by using the natural embeddings between them and its limit is also $\mathscr{S}(\mathbb{R}^n)$. Furthermore, applying the same Fourier type arguments one can see that $-\Delta+\varepsilon$ is non-negative on every X_k. Hence, by Proposition 2.0.19 $-\Delta+\varepsilon$ is non-negative on $\mathscr{S}(\mathbb{R}^n)$.

Note that by dualisation every inductive spectrum $(X_\alpha,\,l_{\beta\alpha})$ gives rise to a projective spectrum $(X'_\alpha,\pi_{\beta\alpha})$ with $\pi_{\beta\alpha}:=l'_{\beta\alpha}$ where $l'_{\beta\alpha}:\ X'_\beta\to X'_\alpha$ denotes the dual mapping. This relation also holds in the other direction. It is always true that

$$\Big(\operatorname*{ind}_{\alpha\in\mathcal{A}}X_\alpha\Big)'=\operatorname*{proj}_{\alpha\in\mathcal{A}}X'_\alpha\quad\text{(just algebraically, , see [22, §26, Satz 1.2]).}$$

One can show that the pairing is given by

$$\big\langle(x'_\beta),[(x_\alpha,\alpha)]\big\rangle=\langle x'_\alpha,x_\alpha\rangle$$

which is well defined. Note also that both constructions are purely algebraic and do not require any topology on the building blocks unless one wishes to construct a topology on the limit space. If further for a projective spectrum it holds that $P_\alpha(X)$ is dense in X_α (one says that the projective spectrum is reduced and this can w.l.o.g. always be assumed, [31, Sect. 2.6, Prop. 2]), then one also has

$$\left(\operatorname*{proj}_{\alpha\in\mathcal{A}} X_\alpha\right)' = \operatorname*{ind}_{\alpha\in\mathcal{A}} X'_\alpha \quad \text{(just algebraically)}.$$

More interesting, however, is the question whether the stated algebraic isomorphisms are also topological isomorphisms. Here one has the following result:

Lemma 2.0.21. *Let $(X_\alpha)_{\alpha\in\mathcal{A}}$ be a regular inductive spectrum. Then*

$$\left(\operatorname*{ind}_{\alpha\in\mathcal{A}} X_\alpha\right)'_\beta = \operatorname*{proj}_{\alpha\in\mathcal{A}} (X_\alpha)'_\beta$$

in the sense of an isomorphism between LCS.

For a proof see [22, §26, Satz 2.1].

This statement may be combined with Lemma 2.0.8 to obtain the following relation between Propositions 2.0.15 and 2.0.19.

Theorem 2.0.22. *Let $(X_\alpha)_{\alpha\in\mathcal{A}}$ be a regular inductive spectrum and denote its inductive limit by X. Furthermore, assume a family $(A_\alpha)_{\alpha\in\mathcal{A}}$ of non-negative densely defined operators given on every X_α as in Proposition 2.0.15. Define A as in this proposition. Then $X'_\beta = \operatorname{proj}_{\alpha\in\mathcal{A}} (X_\alpha)'_\beta$, $(A' : X'_\beta \to X'_\beta) \in \mathcal{M}(X'_\beta)$ and A' is given by*

$$\mathcal{D}(A') = \{x' = (x'_\alpha) \in X'_\beta \mid \forall \alpha \in \mathcal{A} : x'_\alpha \in \mathcal{D}(A'_\alpha)\}, \quad A'(x'_\alpha) = (A'_\alpha x'_\alpha).$$

Moreover,

$$\forall \gamma \geq \alpha \in \mathcal{A} : l'_{\gamma\alpha}\left(\mathcal{D}(A'_\gamma)\right) \subseteq \mathcal{D}(A'_\alpha) \quad \textit{and} \quad l'_{\gamma\alpha}A'_\gamma = A'_\alpha l'_{\gamma\alpha}.$$

That means A' is the operator which one can construct from the operators $A'_\alpha \in \mathcal{M}\big((X_\alpha)'_\beta\big)$ as it happened in Proposition 2.0.19.

Proof. The first part of the statement ist just Lemma 2.0.21.

The part about the non-negativity of A' on X'_β follows from Lemma 2.0.8.

For the compatibility properties let $\gamma \geq \alpha$ and $x'_\alpha = l'_{\gamma\alpha}x'_\gamma \in l'_{\gamma\alpha}\big(\mathcal{D}(A'_\gamma)\big)$. We need to show that $x'_\alpha \in \mathcal{D}(A'_\alpha)$. Let $x_\alpha \in \mathcal{D}(A_\alpha)$ and calculate

$$\langle x'_\alpha, A_\alpha x_\alpha\rangle = \langle x'_\gamma, l_{\gamma\alpha}A_\alpha x_\alpha\rangle = \langle x'_\gamma, A_\gamma l_{\gamma\alpha} x_\alpha\rangle = \langle l'_{\gamma\alpha}A'_\gamma x'_\gamma, x_\alpha\rangle.$$

We conclude that $x'_\alpha \in \mathcal{D}(A'_\alpha)$ and $A'_\alpha x'_\alpha = A'_\alpha l'_{\gamma\alpha} x'_\gamma = l'_{\gamma\alpha} A'_\gamma x'_\gamma$.

It remains to show the special form of A' which follows from the form of A. Let $x' \in \mathcal{D}(A')$ and $x \in \mathcal{D}(A)$. Choose $\alpha \in \mathcal{A}$ such that $x_\alpha \in \mathcal{D}(A_\alpha)$ and $x = [(x_\alpha, \alpha)]$. Then

$$\langle A'x', x\rangle = \langle A'(x'_\gamma), [(x_\alpha, \alpha)]\rangle = \langle \big(A'(x'_\gamma)\big)_\alpha, x_\alpha\rangle = \langle x'_\alpha, A_\alpha x_\alpha\rangle.$$

Hence, $x'_\alpha \in \mathcal{D}(A'_\alpha)$ and $\big(A'(x'_\gamma)\big)_\alpha = A'_\alpha x'_\alpha$ for this particular α. Now let $\gamma \in \mathcal{A}$ be arbitrary and choose $\delta \geq \alpha, \gamma$. It holds that $x_\delta := l_{\delta\alpha} x_\alpha \in \mathcal{D}(A_\delta)$ and repeating the already presented argument gives $x'_\delta \in \mathcal{D}(A'_\delta)$. But now $x'_\gamma = l'_{\delta\gamma} x'_\delta \in \mathcal{D}(A'_\gamma)$ by the already established compatibility. This shows

$$\mathcal{D}(A') \subseteq \{x' = (x'_\alpha) \in X'_\beta \mid \forall \alpha \in \mathcal{A}:\, x'_\alpha \in \mathcal{D}(A'_\alpha)\}.$$

Conversely, let $x' = (x'_\gamma)$ be such that $x'_\gamma \in \mathcal{D}(A'_\gamma)$ for all $\gamma \in \mathcal{A}$. From

$$\forall \alpha \in \mathcal{A}:\, \langle (A'_\gamma x'_\gamma), [(x_\alpha, \alpha)]\rangle = \langle A'_\alpha x'_\alpha, x_\alpha\rangle$$

we conclude that

$$X \ni x = [(x_\alpha, \alpha)] \mapsto \langle (A'_\gamma x'_\gamma), [(x_\alpha, \alpha)]\rangle \in X',$$

in particular when being restricted to $\mathcal{D}(A)$ where it is equal to $\mathcal{D}(A) \ni x \mapsto \langle x', Ax\rangle$. Hence $x' \in \mathcal{D}(A')$ and the proof is finished. $\square$

Example 2.0.23. *In Example 2.0.17 (b) we learnt that $-\Delta + \varepsilon$ is non-negative on the inductive limit X of the spaces*

$$X_k := \{f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}_0^n:\, \sup_{x \in \mathbb{R}^n} |(D^\alpha f)(x) \mathrm{e}^{\frac{\|x\|}{k}}| < \infty\}$$

for any $\varepsilon > 0$. The considered spectrum is regular by [8, Thm. 3] since the mappings $l_{mk} : X_m \to X_k$ are embeddings. By Proposition 2.0.22, $-\Delta' + \varepsilon$ is non-negative on the strong dual X'_β and can be obtained from the non-negative operators $-\Delta' + \varepsilon$ restricted to the building blocks $(X_k)'_\beta$.

We finish this chapter by a standard proposition. For a closed subspace $Y \subseteq X$ and $A \in \mathcal{C}(X)$ we denote by A_Y the *part of A in Y* which is the operator A with domain

$$\mathcal{D}(A_Y) := \{x \in Y \cap \mathcal{D}(A) \mid Ax \in Y\}.$$

If $A \in \mathcal{M}(X)$, one can show that A_Y is non-negative if Y is invariant under all resolvent operators $(\lambda + A)^{-1}$, $\lambda > 0$. For reduction purposes it is of interest to study the part of $A \in \mathcal{M}(X)$ in $D := \overline{\mathcal{D}(A)}$, $R := \overline{\mathcal{R}(A)}$ and $G := \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

Lemma 2.0.24. *Let $A \in \mathcal{M}(X)$. The operators A_D, A_R and A_G are all non-negative in the respective spaces. Furthermore,*

1. *A_D is densely defined,*
2. *A_R has dense range,*
3. *A_G is densely defined with dense range.*

Proof. Let $\lambda > 0$. The spaces $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are invariant under $(\lambda + A)^{-1}$. By continuity of the resolvent this invariance carries over to the closures which means that D and R are invariant. The same holds for G as being the intersection of D and R. From the non–negativity of A in X we can conclude now that all parts of A are non–negative in the spaces D, R, and G. The statements about the density of the domain and range follow now as before by using Lemma 2.0.9 together with the approximations $\lambda(\lambda + A)^{-1}x \to x$ as $\lambda \to \infty$ if $x \in D$ (density of the domain) and $A(\lambda + A)^{-1}x \to x$ as $\lambda \to 0+$ if $x \in R$ (density of the range). $\square$

Example 2.0.25.

(a) *We would like to demonstrate Lemma 2.0.24 by a standard application. Assume X to be a semi-reflexive LCS and $A \in \mathcal{M}(X)$. By Corollary 2.0.11 we know that A is densely defined. Hence $A = A_D$ and $A_R = A_G$. Additionally, $X = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$. So every $x \in \mathcal{D}(A)$ may be decomposed in $x = y+z$ with $y \in \mathcal{N}(A)$ and $z \in \overline{\mathcal{R}(A)} \cap \mathcal{D}(A)$. Hence $Ax = A_D x = A_G z$ and so in case of semi-reflexive spaces it is always enough to study A_G instead of A.*

(b) *Consider $-\Delta$ on $X := C_b^\infty(\mathbb{R}^n)$ of which $Y := C_0^\infty(\mathbb{R}^n)$ is a proper subspace. The subspace Y is invariant under the resolvents $((-\Delta+\lambda)^{-1})_{\lambda>0}$ which can be seen when considering the semigroup generated by $-(-\Delta)$ which is given through*

$$(\mathrm{e}^{\Delta t} f)(x) := \begin{cases} f & t = 0, \\ k_t * f & t > 0. \end{cases}$$

where $k\colon (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$,

$$k(t, x) := k_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x\|^2}{(4t)}} \quad (t > 0, x \in \mathbb{R}^n)$$

is the Gauß-Weierstraß kernel. Restricted to Y, the operator $-\Delta$ is additonally injective.

For later use, we shall also introduce the subclass of sectorial operators. Considering Definition 2.0.1 of non-negative operators, one can see that the more common sectoriality can be generalised in the same fashion to LCS.

Definition 2.0.26. Let $A \in \mathcal{C}(X)$. The operator A is called *sectorial* if there is $\phi \in (0, \pi)$ such that $\mathbb{C} \setminus S_\phi \subseteq \rho(A)$ and if for every $\sigma \in (\phi, \pi)$ the family

$$\left(\lambda(\lambda - A)^{-1}\right)_{\lambda \in \mathbb{C} \setminus S_\sigma}$$

is equicontinuous. Let ω be the infimum of all angles ϕ such that A is sectorial. Then ω is called *angle of sectoriality* of the sectorial operator A. We denote by $\mathcal{S}_\omega(X)$ the set of all sectorial operators with angle of sectoriality ω and set $\mathcal{S}(X) := \bigcup_{\omega \in [0,\pi)} \mathcal{S}_\omega(X)$.

One has $\mathcal{S}(X) \subseteq \mathcal{M}(X)$ while equality holds for example in Banach spaces, see [44, Prop. 1.2.1]. In general LCS one has $\mathcal{S}(X) \subsetneq \mathcal{M}(X)$ though. See [44, Ex. 1.4.3 and Ex. 1.4.4] for two examples in Fréchet spaces. All the so far produced results for non-negative operators also hold for the class of sectorial operators with almost identical proofs. The following two chapters will only deal with non-negative operators. However, the reader will encounter sectorial operators again in the last chapter where sectoriality will be essential for the proof of uniqueness of the Caffarelli-Silvestre Problem.

3. Functional calculus in locally convex spaces

The third chapter shall be devoted to the development of what we shall refer to as functional calculus. Since the goal of this chapter will be a way to define what we mean by the fractional power A^α of a given linear operator A, this calculus should include the function $\lambda \mapsto \lambda^\alpha$. Seeing the vast number of appearances of the term 'functioncal calculus' (notably also in the title of this third chapter), one would assume perfect agreement on what a calculus actually is. Most surprisingly this does not seem to be the case. We shall follow [25] and also [27] for the more abstract point of view and include some minor modifications. As preparation we will denote in this chapter by $\mathscr{A}$ a *locally convex algebra*. We remind the reader that this means that $\mathscr{A}$ is a LCS for which the multiplication $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is continuous and that this property may be characterised by

$$\forall \left\|\cdot\right\|_p \in \mathcal{P}_{\mathscr{A}} \, \exists \left\|\cdot\right\|_q \in \mathcal{P}_{\mathscr{A}} \, \forall a, b \in \mathscr{A} : \left\|ab\right\|_p \leq \left\|a\right\|_q \left\|b\right\|_q .$$

Definition 3.0.1. A *continuous functional calculus* is a triple $(X, \mathscr{A}, \Phi)$ where X is a LCS, $\mathscr{A}$ is a unital locally convex algebra, and $\Phi : \mathscr{A} \to \mathcal{L}_{\mathcal{B}}(X)$ is a continuous algebra homomorphism where $\mathcal{L}_{\mathcal{B}}(X)$ is equipped with some topology induced by the bornology $\mathcal{B}$, cf. Remark 2.0.7.

The here given definition of a calculus differs just slightly from the more general one given in [27] which reduces to the here given definition for calculi having their values in the continuous linear operators.

Example 3.0.2.

(a) Let everything be as in Example 2.0.2 (a), i.e., let Ω be a Hausdorff locally compact space with a Radon measure μ, $X = L^p_{loc}(\Omega)$ for some $p \in [1, \infty)$, and $f \in L^0(\Omega)$. Define $\Lambda := \mathcal{R}_{ess}(f)$, equip this set with the σ-algebra $\mathscr{B}(\Lambda)$, and suppose a measure ν is given on $\mathscr{B}(\Lambda)$ with the property that $|\mu|_f \ll \nu$. Set $\mathscr{A} := L^\infty(\Lambda, \mathscr{B}(\Lambda), \nu)$. This is a Banach algebra. In particular, it is locally convex. Furthermore, define the

mapping $\Phi : \mathscr{A} \to \mathcal{L}(X)$ *by* $\Phi(g)h := (g \circ f)h$ $(h \in X)$. *The absolute continuity of* $|\mu|_f$ *w.r.t.* ν *ensures that* Φ *is well defined. Moreover, the mapping* Φ *is also a homomorphism. Every operator* $\Phi(g)$ *is continuous since for all compacta* $K \subseteq \Omega$

$$\|\Phi(g)h\|_K = \left(\int_K |g(f(x))h(x)|^p \, |\mu| \, (\mathrm{d}x) \right)^{\frac{1}{p}} \leq C \, \|h\|_K \quad (h \in X)$$

where $C := \operatorname{ess\,sup}_{y \in \Lambda} |g(y)|$.

(b) We consider $\mathscr{A} := M_b\big([0,\infty)\big)$ *with convolution as multiplication and total variation norm. This set is a Banach algebra. Assume* X *to be a Banach space and* $-A \in \mathcal{C}(X)$ *being the generator of a* C_0*-semigroup* $(\mathrm{e}^{-At})_{t \geq 0}$. *Define the mapping* $\Phi : \mathscr{A} \to \mathcal{L}(X)$ *by*

$$\Phi(\mu)x := \int_{[0,\infty)} \mathrm{e}^{-At} x \, \mu(\mathrm{d}t)$$

which is a continuous homomorphism. The so defined calculus is called Hille–Phillips calculus and was apparently first developed in [29, Chap. XV].

(c) Let us consider again the inductive limit from Example 2.0.12 (c) and introduce the notation $\mathcal{E}(S_\omega)$ *for it. Choose* $\mathscr{A} := \mathcal{E}(S_\omega)$. *Let again* X *be a Banach space and* $A \in \mathcal{S}_\omega(X)$ *a sectorial operator. Now define the mapping* Φ *via*

$$\Phi(f) := \frac{1}{2\pi i} \int_\gamma f(z)(z - A)^{-1} \mathrm{d}z$$

where, for some $\varepsilon > 0$, *the contour of integration* γ *is chosen to be the boundary of a slightly larger sector* $S_{\omega+\varepsilon}$ *which is still in the domain of* f, *see Figure 3.0.1.*

The so constructed calculus is an extension of the Dunford-Riesz calculus [19] and the here presented version goes back to [45].

The last two examples are prototypical. In both examples, it is already known how to set up a calculus for a special class of functions (like the exponential function or the inverse function) and more functions are derived by means of integral mixtures of the known objects. If this procedure is performed for numbers $\lambda \in \mathbb{C}$ instead of operators A, one typically speaks of kernel definitions of the transforms of distributions. The appearance of the latter term may be explained by the fact that distributions usually refer to some kind of linear form defined on rather small LCS and the connections to measures is the fact that

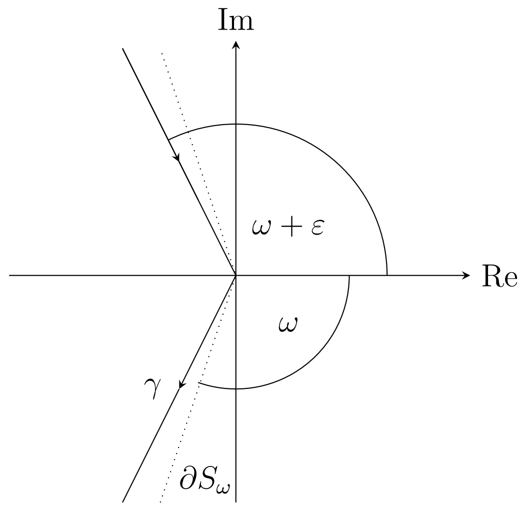

Figure 3.0.1.: Path of integration

those form in many cases subspaces of the space of all distributions. To summarise, one might ask whether an extension of the algebras appearing in the Examples 3.0.2 (b) and (c) is possible. It is indeed doable by using so-called ε-products which will be introduced and studied in the next section.

3.1. ε-products

The ε-product was introduced and studied independently in [24] and [59] with minor differences, see [32]. We will adopt the definition of Schwartz. To anticipate the main result, ε-products can be seen as rather natural constructions generalising injective tensor products. Their main use will be a very elegant definition of vector-valued functions. More details can be found in [58, Lec. 4] as well as [32, 59] where the very good last two sources unfortunately require knowledge of the German and French language. For the next definition, we will denote by κ the *topology of uniform convergence on absolutely convex compact sets in a given LCS X.*

Definition 3.1.1. For two LCS X and Y we define the *ε-product $X\varepsilon Y$* to be the space $\mathcal{L}_{ec}(Y'_\kappa, X)$ of continuous linear mappings from Y'_κ to X where Y'_κ is equipped with the topology κ, and the considered space $\mathcal{L}_{ec}(Y'_\kappa; X)$ is itself topologised by using the topology of uniform convergence on equicontinuous subsets of Y'.

In the following we will need for a subset $U \subseteq Y$ the notion of a so-called *polar* $U^\circ := \{y' \in Y' \mid \forall y \in U : |\langle y', y\rangle| \leq 1\} \subseteq Y'$. The same notation will be used for $V \subseteq Y'$ but w.r.t. the duality pairing (Y, Y') which means $V^\circ := \{y \in Y \mid \forall y' \in V : |\langle y', y\rangle| \leq 1\}$.

For a subset $E \subseteq Y'$ equicontinuity simply means that there is $\|\cdot\|_q \in \mathcal{P}_Y$ and $s > 0$ such that $E \subset B_q(0,s)^\circ$. Hence, every $\|\cdot\|_p \in \mathcal{P}_X$, $\|\cdot\|_q \in \mathcal{P}_Y$ induces a continuous seminorm $\|\cdot\|_r \in \mathcal{P}_{X\varepsilon Y}$ via

$$\forall u \in X\varepsilon Y : \ \|u\|_r := \sup\{\|\langle y', u\rangle\|_p \mid y' \in B_q(0,1)^\circ\}$$

where we wrote $\langle y', u\rangle$ for the application of the linear operator $u \in X\varepsilon Y$ on the element $y' \in Y'$ for reasons which will become clear in what follows.

Remark 3.1.2. It is not obvious but the ε-product is symmetric in the sense that $X\varepsilon Y$ is isomorphic to $Y\varepsilon X$ by transposition, see [32, Satz 10.3]. We shall freely change between these two interpretations and even write $X\varepsilon Y = Y\varepsilon X$ in the sense of this isomorphism. At first glance this identification will seem to be confusing but it is actually very convenient and intuitive.

Example 3.1.3.

(a) *Let X and Y be two Banach spaces. A subset $B \subseteq Y'$ is equicontinuous if and only if it is norm-bounded in Y'. A norm-bounded subset B of Y' is also compact in the w^*-topology, i.e., the topology of pointwise convergence in Y'. This is even true for the topology of uniform convergence on precompact sets, see [31, Chap. 8.5, Thm 1]. In the special here considered example, precompact sets are relatively compact and hence there is no difference between uniform convergence on precompact sets and uniform convergence on compact sets. Putting all together, B is also compact in Y'_κ. Choosing now any $u \in X\varepsilon Y$, we conclude that $u(B)$ is compact in X which shows $X\varepsilon Y \subset \mathcal{K}(Y', X)$, the set of compact operators from Y' equipped with norm topology to X, cf. [32, p. 234]. One can actually do even better. It holds that $\mathcal{K}(Y, X) = Y'\varepsilon X$ ([32, Satz 10.4]). One inclusion follows from the already noted relation $Y'\varepsilon X = X\varepsilon Y' \subseteq \mathcal{K}(Y'', X)$ by restriction while, on the other hand, every $u \in \mathcal{K}(Y, X)$ gives rise to $u' \in \mathcal{K}(X', Y')$ for which one can show $u' \in Y'\varepsilon X$ by making use of the compactness of u. But also the first observation can be improved to $X\varepsilon Y = \{u \in \mathcal{K}(Y', X) \mid u \in \mathcal{L}(Y'_\sigma, X_\sigma)\}$ ([32, p. 258]). Here Y'_σ and X_σ is short for Y' equipped with the topology $\sigma(Y', Y)$ and X equipped with $\sigma(X, X')$, respectively.*

(b) *Let Y be a quasi-complete LCS, and let Ω be any locally compact space. Consider the LCS $C(\Omega; Y)$ of continuous functions defined on Ω with values in Y equipped with the topology of uniform convergence on compact subsets in Ω and set $X := C(\Omega)$ to be its 'scalar' version. Every $f \in C(\Omega; Y)$ gives rise to an element, again denoted by f, in $X\varepsilon Y$ by defining $\langle y', f\rangle(\cdot) := \langle y', f(\cdot)\rangle \in X$ where we used again* *the suggestive notation $\langle y', f\rangle$ for the application of f on y'. From this point of view one has $C(\Omega; Y) \subseteq X\varepsilon Y$ and in fact this mapping is even injective, continuous with continuous inverse defined on its range, i.e., it is an embedding. In the concrete example one can moreover prove that the embedding is onto and therefore it is an isomorphism ([32, Satz 10.2]).*

Remark 3.1.4.

1. The reason for choosing the topology κ over the topology of convergence on precompact sets is the desired duality $(Y'_\kappa)' = Y$ by the Mackey–Arens theorem ([31, Chap. 8.5, Thm. 5]). On the other hand, in the above example the mapping $f \in X\varepsilon Y$ may fail to be continuous for general LCS Y. For quasi-complete spaces, and those are the only spaces we consider, both topologies coincide and thus the question does not appear.

2. The question for which LCS X and spaces of functions $\mathcal{F}(\Omega)$ defined on some set Ω it still holds that $\mathcal{F}(\Omega; X) = \mathcal{F}(\Omega)\varepsilon X$ was intensively studied in [38].

3. We shall also demonstrate the above mentioned symmetry of the ε-product. For $f \in C(\Omega; Y) = C(\Omega)\varepsilon Y = Y\varepsilon C(\Omega)$ one can interpret $\langle y', f\rangle$ as application of the linear operator f on the element y' yielding a function in $C(\Omega)$ (cf. first equality) or one can interpret this as a composition of linear operators yielding an element in $\big(C(\Omega)'_\kappa\big)' = C(\Omega)$ by the Mackey–Arens theorem which corresponds to the second equality.

The concept of integration of vector-valued functions can be generalised by considering the so-called ε-product of operators. Before we come to the definition, it should be pointed out that for a continuous linear operator $A \in \mathcal{L}(X, Y)$ the dual operator A' is continuous from Y'_κ to X'_κ.

Definition 3.1.5. Let X_1 and X_2 as well as Y_1 and Y_2 be four LCS and assume continuous linear operators $A \in \mathcal{L}(X_1, X_2)$ and $B \in \mathcal{L}(Y_1, Y_2)$ to be given. We define the *ε-product* $A\varepsilon B \in \mathcal{L}(X_1\varepsilon Y_1, X_2\varepsilon Y_2)$ to be the linear continuous operator given by

$$X_1\varepsilon Y_1 \ni u \mapsto AuB' \in X_2\varepsilon Y_2$$

where Y'_1 and Y'_2 are equipped with the topology κ of uniform convergence on compacts in the LCS Y_1 and Y_2, respectively.

We will not need the general definition of an ε-product of two operators but a rather special case of it. Namely, let $\mathcal{F}(\Omega)$ be any locally convex space of functions defined on some set Ω and let μ be a continuous functional on $\mathcal{F}(\Omega)$. Every such functional gives rise to a mapping, we will also denote it by μ, from $\mathcal{F}(\Omega)\varepsilon X$ to X given by $\mu\varepsilon 1_X$ where we used $\mathbb{C}\varepsilon X = \mathcal{L}_{ec}(X'_\kappa, \mathbb{C}) = X_{ec} = X$.

Example 3.1.6. *Let $\mu \in M_b\big([0,\infty)\big) = C_0\big([0,\infty)\big)'$. For a Banach space X consider the space $C_0\big([0,\infty); X\big)$ equipped with its usual topology of uniform convergence on $[0,\infty)$. Then $C_0\big([0,\infty); X\big)$ is also a Banach space, it holds that $C_0\big([0,\infty); X\big) = C_0\big([0,\infty)\big)\varepsilon X$ ([7, Chap. 2, 4. Thm]), and the map induced by μ is simply*

$$C_0\big([0,\infty); X\big) \ni f \mapsto \int_{[0,\infty)} f(t)\mu(\mathrm{d}t).$$

Let us collect some elementary properties which will turn out to be useful. To begin with, we need a tiny generalisation of [37, §34, (3)].

Lemma 3.1.7. *Let X and Y be two LCS, and let $A : \mathcal{D}(A) \to Y$ be a densely defined linear operator with domain $\mathcal{D}(A) \subseteq X$. Then A is closable if and only if $A' : \mathcal{D}(A') \to X'$ is densely defined in Y' w.r.t. the weak topology $\sigma(Y', Y)$. Moreover one has $\overline{A} = (A')'$.*

Proof. The proof is essentially the same as for closed operators in Hilbert spaces and their adjoints with orthogonality being replaced by an application of the Hahn–Banach theorem. So assume that A is closable and, by contradiction, that $\mathcal{D}(A')$ is not dense. Then there is $y \in \mathcal{D}(A')^\circ \backslash \{0\}$. By closability of A considered as a subset in $X \times Y$, we can find an element $x' \oplus y' \in X' \oplus Y' = (X \times Y)'$ which vanishes on $\overline{A}$ and such that $\langle x' \oplus y', (0, y)\rangle = 1$. But $\forall x \in \mathcal{D}(A) : \ 0 = \langle x' \oplus y', (x, Ax)\rangle = \langle x', x\rangle + \langle y', Ax\rangle$ means $y' \in \mathcal{D}(A')$ and $A'y' = -x'$. This contradicts $\langle y', y\rangle = \langle x' \oplus y', (0, y)\rangle = 1$ and shows the claimed density of $\mathcal{D}(A')$. Conversely, if $\mathcal{D}(A')$ is dense, we may define $A'' := (A')' : (X'_\sigma)' = X \to (Y'_\sigma)' = Y$ where we used the Mackey–Arens theorem. We shall show $A'' = \overline{A}$. One checks that A'' extends A. Furthermore, A'' is weakly closed as it is a dual operator and, moreover, it is a subspace of $X \times Y$. Hence, it is convex. This implies that A'' is closed in $X \times Y$ which gives $\overline{A} \subseteq A''$.

Let us finally show $A'' \subseteq A^{\circ\circ} = \overline{A}$ where the equality used the Bipolar theorem [31, Sect. 8.2, Thm. 2]. For this define $J : X \times Y \to Y \times X$ by $J(x, y) := (-y, x)$ and $K : Y \times X \to X \times Y$ by $K(y, x) := (x, -y)$. The dual mapping to J is given by $J'(y' \oplus x') = x' \oplus (-y')$. Then $(x, y) \in A''$ implies $J(x, y) \in (A')^\circ$ which means $A'' \subseteq K\big((A')^\circ\big)$ since $KJ = 1_{X\times Y}$. For what remains, one checks $J'(A') = A^\circ$ as well as $JK = 1_{Y\times X}$ and, using this, $K\big((A')^\circ\big) \subseteq \big(J'(A')\big)^\circ = A^{\circ\circ}$.

$\square$

Lemma 3.1.8. *Let $\mathcal{F}(\Omega)$ be any locally convex space of functions defined on some set Ω. Furthermore, let X and Y be two general LCS, $A \in \mathcal{C}(X,Y)$ and $\mu \in \mathcal{F}(\Omega)'$. Let $f \in \mathcal{F}(\Omega)\varepsilon X = X\varepsilon\mathcal{F}(\Omega)$ taking values in $\mathcal{D}(A)$ such that $Af \in \mathcal{F}(\Omega)\varepsilon Y$. Then $\mu\varepsilon 1_X f \in \mathcal{D}(A)$ and $A\mu\varepsilon 1_X f = \mu\varepsilon 1_Y Af$.*

Proof. Define $D := \overline{\mathcal{D}(A)}$ which is a closed subspace of X. The operator A gives rise to an element, denoted with the same symbol, $A \in \mathcal{C}(D,Y)$ which is densely defined and thus has a dual operator $A' : \mathcal{D}(A') \to D'$ where $\mathcal{D}(A')$ is a subspace of Y'. Choose $y' \in \mathcal{D}(A')$. Then

$$\langle y', \mu\varepsilon 1_Y Af\rangle = \big\langle \mu, \langle y', Af\rangle\big\rangle = \big\langle y', \langle \mu, Af\rangle\big\rangle = \big\langle 1_{X'}A'y', \langle \mu, f\rangle\big\rangle = \langle A'y', \mu\varepsilon 1_X f\rangle$$

where we used the definition of the ε-product of operators for the first equality, the identification $Af \in \mathcal{F}(\Omega)\varepsilon Y = Y\varepsilon\mathcal{F}(\Omega)$ for the second one, transposed A' and added an identity, and finally reversed the definition of the ε-product for the last equality sign. We conclude $\mu\varepsilon 1_X f \in \mathcal{D}(A'') = \mathcal{D}(A)$ and $A\mu\varepsilon 1_X f = A''\mu\varepsilon 1_X f = \mu\varepsilon 1_Y Af$ by Lemma 3.1.7. The proof is finished. □

Example 3.1.9. *Let again be $\mu \in M_b\big([0,\infty)\big)$, X a Banach space, and $f \in C_0\big([0,\infty); X\big)$. Assume further that there is $A \in \mathcal{C}(X,Y)$, where Y is another Banach space, such that $\forall t \in [0,\infty) :\ f(t) \in \mathcal{D}(A)$ and $Af \in C_0\big([0,\infty); Y\big)$. Then*

$$A \int\limits_{[0,\infty)} f(t)\mu(\mathrm{d}t) = \int\limits_{[0,\infty)} Af(t)\mu(\mathrm{d}t)$$

holds which is usually called Hille's theorem.

Corollary 3.1.10. *Let $\mathcal{F}(\Omega)$ be again any locally convex space of functions defined on some set Ω, $f \in \mathcal{F}(\Omega)\varepsilon X$, and $x' \in X'$. Then*

$$\langle x', \mu\varepsilon 1_X f\rangle = \big\langle \mu, \langle x', f\rangle\big\rangle.$$

Example 3.1.11. *Let X be a LCS and $f \in C\big(\Omega; X\big)$ where Ω is a locally compact space. Moreover, let $\mu \in M(\Omega)$ be a compactly supported measure. Finally, let $\|\cdot\|_p \in \mathcal{P}_X$. In this situation one has*

$$\Big\|\int\limits_\Omega f(t)\mu(\mathrm{d}t)\Big\|_p = \Big|\Big\langle x', \int\limits_\Omega f(t)\mu(\mathrm{d}t)\Big\rangle\Big| \le C\int\limits_\Omega \|f(t)\|_q\, |\mu|\,(\mathrm{d}t)$$

where $x' \in X'$ was chosen suitably, $C > 0$ is a constant and $\|\cdot\|_q$ is another continuous *seminorm.*

3.2. Hille–Phillips–Schwartz calculus

The last example showed how the Hille–Phillips calculus fits in the here presented more general framework. We are now going to extend the constructions towards continuous functionals on $C_0^\infty\big([0,\infty)\big) = \{f \in C^\infty\big((0,\infty)\big) \mid \forall n \in \mathbb{N}_0 : \, f^{(n)} \in C_0\big([0,\infty)\big)\}$ which we are going to equip with the usual family of seminorms given by suprema taken over the halfline $[0,\infty)$ and the order $k \in \mathbb{N}_0$ of differentiability up to a certain maximal order n. That means for $n \in \mathbb{N}_0$, the continuous seminorms are of the form

$$\|f\|_n := \max_{0\le k\le n} \sup_{t\in[0,\infty)} \left|f^{(k)}(t)\right|.$$

This turnes $C_0^\infty\big([0,\infty)\big)$ into a Fréchet space.

Lemma 3.2.1. *Let D be the ordinary derivative on $C_0^\infty\big([0,\infty)\big)$, i.e., $Df := f'$. Then D is a continuous linear operator on $C_0^\infty\big([0,\infty)\big)$. For given measures $\mu_0,\dots,\mu_n \in M_b\big([0,\infty)\big)$, the linear form $\mu := \sum_{k=0}^n (D^k)'\mu_k$ is continuous on $C_0^\infty\big([0,\infty)\big)$. Conversely, every $\mu \in C_0^\infty\big([0,\infty)\big)'$ is of this form.*

Proof. For the first statement it is enough to consider the already mentioned system of seminorms. One finds $\|Df\|_n \le \|f\|_{n+1}$ which shows the continuity of D.

To see that μ, defined above, is continuous we directly calculate

$$|\langle\mu,f\rangle| = \left|\Big\langle\sum_{k=0}^n (D^k)'\mu_k, f\Big\rangle\right| \le \sum_{k=0}^n \int_{[0,\infty)} \left|f^{(k)}(t)\right| |\mu_k|\,(\mathrm{d}t) \le \sum_{k=0}^n \|\mu_k\|\cdot\|f\|_n. \tag{3.2.1}$$

For the final assertion consider for $n \in \mathbb{N}_0$ the Banach space $C_0^n\big([0,\infty)\big) := \{f \in C^n\big((0,\infty)\big) \mid \forall k \in \{0,\dots,n\} : \, f^{(k)} \in C_0\big([0,\infty)\big)\}$. One naturally can consider this Banach space as a closed subspace of $C_0\big([0,\infty);\mathbb{C}^{n+1}\big) = \prod_{k=0}^n C_0\big([0,\infty)\big)$, namely $U := \{g = (g_0,\dots,g_n) \in C_0\big([0,\infty);\mathbb{C}^{n+1}\big) \mid \forall k \in \{0,\dots,n\} : \, g_k = g_0^{(k)}\}$. Therefore, every continuous functional on U can be extended to a continuous functional on $\oplus_{k=0}^n M_b\big([0,\infty)\big)$, the dual space of $C_0\big([0,\infty);\mathbb{C}^{n+1}\big)$. More explicitly, for every $\mu \in C_0^n\big([0,\infty)\big)'$ we can find measures $\mu_0,\dots,\mu_n$ such that

$$\forall f \in C_0^n\big([0,\infty)\big) : \, \langle\mu,f\rangle = \sum_{k=0}^n \int_{[0,\infty)} f^{(k)}(t)\mu_k(\mathrm{d}t).$$

Since $C_0^\infty\big([0,\infty)\big)$ is the reduced projective limit of the spaces $C_0^n\big([0,\infty)\big)$, its dual is the inductive limit of the dual spaces ([31, Sect. 8.8, Prop. 7]).

$\square$

Remark 3.2.2. As indicated by the above proof, one may wonder whether one can use the norm

$$||\sum_{k=0}^{n}(D^k)'\mu_k|| := \sum_{k=0}^{n}\|\mu_k\|$$

on $C_0^n\big([0,\infty)\big)'$. It is stronger than the usual Banach space norm of $C_0^n\big([0,\infty)\big)'$ what can be seen by inspecting again Inequality (3.2.1) for arbitrary $f \in C_0^n\big([0,\infty)\big)$. Also, it is the restriction of the norm

$$||\sum_{k=0}^{n}\mu_k|| := \sum_{k=0}^{n}\|\mu_k\|$$

defined on $\oplus_{k=0}^n M_b\big([0,\infty)\big)$ to the subspace $C_0^n\big([0,\infty)\big)'$. Hence, equivalence of the norms follows now as usual from the bounded inverse theorem and we could use this alternative norm as well. We shall also do this in what follows since it will prove to be of advantage when we will consider $M_b\big([0,\infty)\big)$ as an algebra and not just as a vector space.

For $s \geq 0$ let us denote by τ_s the *shift operator* on $C_0^\infty\big([0,\infty)\big)$ which maps $f(\cdot) \mapsto f(\cdot+s)$.

Lemma 3.2.3. *Let* $f \in C_0^\infty\big([0,\infty)\big)$, $\mu \in M_b\big([0,\infty)\big)$, $s \geq 0$, *and* $l \in \mathbb{N}_0$. *Then*

$$\tau_s f \in C_0^\infty\big([0,\infty)\big) \quad \textit{and} \quad \big([0,\infty) \ni s \mapsto \langle (D^l)'\mu, \tau_s f\rangle\big) \in C_0^\infty\big([0,\infty)\big).$$

Proof. We have $(\tau_s f)(t) = f(s+t)$. Therefore, $\tau_s f$ is again infinitely often continuously differentiable and, for $n \in \mathbb{N}_0$, one gets $D^n\tau_s f = \tau_s D^n f$ which establishes the first part of the claim. For the second part consider the function $g : [0,\infty) \to \mathbb{C}$ defined by

$$s \mapsto g(s) := \int\limits_{[0,\infty)} f^{(l)}(s+t)\mu(\mathrm{d}t).$$

Note that $g = \big(s \mapsto \langle (D^l)'\mu, \tau_s f\rangle\big)$. By standard theorems concerning continuity and differentiability of parameter integrals, we have that $g \in C^\infty\big([0,\infty)\big)$. Let again $n \in \mathbb{N}_0$ be a given natural number or 0. If $\mu = 0$, it follows that $g = 0$ and the statement is established. So let $\mu \neq 0$. In order to show that $D^n g$ is in $C_0\big([0,\infty)\big)$ in this case, for given $\varepsilon > 0$, let $s_0 \geq 0$ be such that $\left|f^{(l+n)}(t)\right| < \frac{\varepsilon}{|\mu|([0,\infty))}$ for $t \geq s_0$. Then

$$\forall s \geq s_0 : \left|g^{(n)}(s)\right| \leq \int\limits_{[0,\infty)} \left|f^{(l+n)}(s+t)\right| |\mu|(\mathrm{d}t) \leq \frac{\varepsilon}{|\mu|\big([0,\infty)\big)} \cdot |\mu|\big([0,\infty)\big) = \varepsilon$$

which finishes the proof. $\square$

Let $f \in C_0\big([0,\infty)\big)$. For two measures $\mu,\ \nu \in M_b\big([0,\infty)\big)$, one commonly defines the *convolution* $\mu * \nu \in M_b\big([0,\infty)\big)$ to be the measure $\mu * \nu \in M_b\big([0,\infty)\big)$ given by

$$\langle \mu * \nu, f\rangle = \big\langle \mu, s \mapsto \langle \nu, \tau_s f\rangle\big\rangle = \int\limits_{[0,\infty)} \int\limits_{[0,\infty)} f(s+t)\mu(\mathrm{d}t)\nu(\mathrm{d}s).$$

Let $n \in \mathbb{N}_0$ and assume now $f \in C_0^\infty\big([0,\infty)\big)$. The observation $\langle (D^n)'(\mu * \nu), f\rangle = \big\langle \mu, s \mapsto \langle (D^n)'\nu, f\rangle\big\rangle$ suggests the following extension of the convolution to the dual space of $C_0^\infty\big([0,\infty)\big)$.

Definition 3.2.4. Let $f \in C_0^\infty\big([0,\infty)\big)$, $\mu,\ \nu \in M_b\big([0,\infty)\big)$, and $n,\ m \in \mathbb{N}_0$. We define the *convolution* between the functionals $(D^n)'\mu,\ (D^m)'\nu \in C_0^\infty\big([0,\infty)\big)'$ to be the functional $(D^n)'\mu * (D^m)'\nu \in C_0^\infty\big([0,\infty)\big)'$ given by

$$\langle (D^n)'\mu * (D^m)'\nu, f\rangle := \big\langle (D^n)'\mu, s \mapsto \langle (D^m)'\nu, \tau_s f\rangle\big\rangle$$

and extend this on the whole dual space of $C_0^\infty\big([0,\infty)\big)$ by linearity w.r.t. the representation given in Lemma 3.2.1. By this we mean

$$\left(\sum_{k=0}^{n}(D^k)'\mu_k\right) * \left(\sum_{l=0}^{m}(D^l)'\nu_l\right) = \sum_{k=0}^{n}\sum_{l=0}^{m}(D^k)'\mu_k * (D^l)'\nu_l.$$

The so-defined convolution shows the expected behaviour when D is applied to a product.

Lemma 3.2.5. *Let $\mu,\ \nu \in C_0^\infty\big([0,\infty)\big)'$, $f \in C_0^\infty\big([0,\infty)\big)$, and $l \in \mathbb{N}_0$. Then,*

1. $\forall s \in [0,\infty) :\ \langle (D^l)'\mu, \tau_s f\rangle = D^l\big(s \mapsto \langle \mu, \tau_s f\rangle\big)$.
2. $(D^l)'(\mu * \nu) = \mu * (D^l)'\nu = (D^l)'\mu * \nu$.

Proof.

1. By Lemma 3.2.1, μ has a representation $\mu = \sum_{k=0}^{n}(D^k)'\mu_k$. We calculate

$$\begin{aligned}
\langle (D^l)'\mu, \tau_s f\rangle &= \sum_{k=0}^{n} \int\limits_{[0,\infty)} \frac{\mathrm{d}^l}{\mathrm{d}s^l} f^{(k)}(s+t)\mu_k(\mathrm{d}t)\\
&= \frac{\mathrm{d}^l}{\mathrm{d}s^l}\sum_{k=0}^{n} \int\limits_{[0,\infty)} f^{(k)}(s+t)\mu_k(\mathrm{d}t)\\
&= D^l\langle \mu, \tau_s f\rangle
\end{aligned}$$

which works since we are allowed to exchange differentiation and integration.

2. Using part 1, we calculate again

$$\begin{aligned}\langle (D^l)'(\mu * \nu), f\rangle &= \big\langle \mu, s \mapsto \langle (D^l)'\nu, \tau_s f\rangle\big\rangle \\ &= \langle \mu * (D^l)'\nu, f\rangle = \big\langle \mu, s \mapsto D^l\langle \nu, \tau_s f\rangle\big\rangle \\ &= \langle (D^l)'\mu * \nu, f\rangle\end{aligned}$$

which shows the desired result.

□

The next result is a general one which will tell us that $C_0^\infty\big([0,\infty)\big)'$ is a locally convex algebra when equipped with $*$.

Lemma 3.2.6. *Let $(X_n)_{n\in\mathbb{N}}$ be an inductive spectrum with inductive limit* $\operatorname{ind}_{n\in\mathbb{N}} X_n =: X$ *and assume that there is a bilinear, continuous mapping $* : X_n \times X_k \to X_{n+k}$. This mapping can be extended to a bilinear, continuous mapping, again denoted by $*$, from $X \times X$ to X given by $(L_n x_n, L_k x_k) \mapsto L_{n+k}(x_n * x_k)$. This additional structure makes X a locally convex algebra which is also graded, i.e., $L_n(X_n) * L_k(X_k) \subseteq L_{n+k}(X_{n+k})$.*

Proof. By assumption the extended mapping $*$ is still bilinear and the grading also directly follows from the properties of the original $*$. It remains to show its continuity. For this let $W \in \mathcal{U}_X$ be absolutely convex. We need to find U, $V \in \mathcal{U}_X$ such that $U * V \subseteq W$. Choose l, $m \in \mathbb{N}$. It holds that $W_{l,m} := L_{l+m}^{-1}(W) \in \mathcal{U}_{X_{l+m}}$. By continuity of the original mapping $*$, there are 0-neighborhoods $U_l \in \mathcal{U}_{X_l}$ and $V_m \in \mathcal{U}_{X_k}$ with $U_l * V_m \subseteq W_{l,m}$. Define $U := \operatorname{acx} \bigcup_l L_l(U_l)$ and $V := \operatorname{acx} \bigcup_m L_m(V_m)$. In the proof of Proposition 2.0.15 it was shown that U and V are 0-neighbourhoods. We calculate

$$U * V \subseteq \operatorname{acx} \bigcup_{n,k} \underbrace{L_n(U_n) * L_k(V_k)}_{\subseteq L_{n+k}(W_{n,k}) \subseteq W} \subseteq W.$$

□

Corollary 3.2.7. *The space $C_0^\infty\big([0,\infty)\big)'$ equipped with $*$ given by convolution is a locally convex, graded algebra.*

Proof. This is an application of Lemma 3.2.6. For the necessary calculation concerning the continuity of $*$ also pay attention to Remark 3.2.2. □

We now construct a functional calculus from this algebra. Before, note that every measure $\mu \in M_b\big([0,\infty)\big)$ extends naturally to a functional on the space $C_b\big([0,\infty)\big)$. Here however, one faces the problem that $C_b\big([0,\infty);X\big)$ with its standard Banach space topology cannot be considered as ε-product. The problem essentially is that for $f \in C_b\big([0,\infty);X\big)$ the orbit $\{f(t) \mid t \in [0,\infty)\}$ is not necessarily relatively compact in contrast to the case $f \in C_0\big([0,\infty);X\big)$. The following example illustrates this.

Example 3.2.8. *Let H be an infinite dimensional, separable Hilbert space with ONB $(e_n)_{n\in\mathbb{N}}$. Choose $f \in C_c\big((0,1)\big)$, $f \neq 0$ such that $0 \leq f \leq 1$. Extend this function to $\mathbb{R}$ by setting it 0 outside of $(0,1)$ and for $n \in \mathbb{N}$ define shifted versions $f_n : [0,\infty) \to \mathbb{R}$, $f_n(t) := f(t-n)$, $(t \geq 0)$. Finally define*

$$g : [0,\infty) \to H, \quad g(t) := \sum_{n=1}^{\infty} f_n(t)e_n.$$

It holds that $g \in C_b\big([0,\infty);H\big)$ but it is not contained in $C_b\big([0,\infty)\big)\varepsilon H$. To see this, assume by contradiction that this would be the case. It holds that $e_n \to 0$ in $\big(H, \sigma(H,H')\big)$. Actually even $e_n \to 0$ in (H,κ). However,

$$\sup_{t\geq 0} |(e_n|g(t)| = \|f\| > 0$$

and hence

$$\sup_{t\geq 0} |(e_n|g(t)| \nrightarrow 0 \text{ as } n \to \infty$$

which contradicts our assumption.

A possible way out is coarsening the Banach space topology towards a so-called *mixed topology* as it was introduced in [66].

Definition 3.2.9. Let $h \in C_0\big([0,\infty)\big)$. For $f \in C_b\big([0,\infty)\big)$, define the seminorm $\|f\|_h := \sup_{s\in[0,\infty)} |h(s)f(s)|$. We define the *mixed topology* on $C_b\big([0,\infty)\big)$ to be the locally convex topology generated by all such seminorms.

Remark 3.2.10.

1. Mixed topologies got their name because they are indeed 'mixtures' of topologies. In the above case, our mixed topology is generated from the standard norm topology of $C_b\big([0,\infty)\big)$ in combination with the topology of uniform convergence on compacts, the so-called compact-open topology.

2. One main field in which mixed topologies are used is the theory of so-called bi-continuous semigroups, see for example [21, 40].

3. The characterisation used in Definition 3.2.9 is not obvious from the original definition used in [66] and can be found in [16, Proposition 3]. Especially in the context of C_b, the mixed topology is often referred to as *strict topology*.

If equipped with the mixed topology, the space $C_b\big([0,\infty);X\big)$ is an ε-product. Moreover, if $X=\mathbb{C}$, its dual space is actually given by $M_b\big([0,\infty)\big)$ ([12, Thm. 2]). These considerations also transfer to the projective limit $C_b^\infty\big([0,\infty);X\big)$, $C_b^\infty\big([0,\infty)\big)$ and its dual space. All this follows from [38, Thm. 14 (iii)]. Note that the assumption of quasi-completeness of X is essential in order to apply the mentioned theorem. It in turn implies the so-called convex compactness property which is needed. From now on we shall always consider $C_b\big([0,\infty)\big)$ (and similarly $C_b^n\big([0,\infty)\big)$ as well as $C_b^\infty\big([0,\infty)\big)$) as equipped with its mixed topology.

Definition 3.2.11. Let X be a LCS and $A\in\mathcal{L}(X)$ such that $-A$ generates an equibounded C_0-semigroup $\big(\mathrm{e}^{-At}\big)_{t\geq 0}$. Set $\mathscr{A}:=C_b^\infty\big([0,\infty)\big)'$ and identify the element $\mu\in\mathscr{A}:=C_b^\infty\big([0,\infty)\big)'$ with $\mu\varepsilon 1_X : C_b^\infty\big([0,\infty)\big)\varepsilon X=C_b^\infty\big([0,\infty);X\big)\to X$. We define, for $\mu=\sum_{k=0}^n (D^k)'\mu_k$ and $x\in X$, the mapping $\Phi:\mathscr{A}\to\mathcal{L}(X)$ by

$$\Phi(\mu)x:=\langle\mu,t\mapsto\mathrm{e}^{-At}x\rangle=\sum_{k=0}^n(-A)^k\int\limits_{[0,\infty)}\mathrm{e}^{-At}x\,\mu_k(\mathrm{d}t).$$

Finally, we equip $\mathcal{L}(X)$ with a topology, also denoted by β which, by comparison with the situation for dual spaces, also shall be called *strong topology* and whose seminorms are given by

$$\|T\|_{B,p}:=\sup_{x\in B}\|Tx\|_p$$

where $B\subset X$ is any bounded set and $\|\cdot\|_p\in\mathcal{P}_X$. The resulting LCS will be denoted by $\mathcal{L}(X)_\beta$.

Proposition 3.2.12. *Let X be an LCS and $A\in\mathcal{L}(X)$ such that $-A$ generates an equicontinuous C_0-semigroup $\big(\mathrm{e}^{-At}\big)_{t\geq 0}$. Then the triple $(X,\mathscr{A},\Phi)$ defines a continuous functional calculus.*

Proof. Let μ, $\nu\in C_b^\infty\big([0,\infty)\big)'$ be functionals with representations $\mu=\sum_{k=0}^n(D^k)'\mu_k$ and $\nu=\sum_{l=0}^m(D^l)'\nu_l$, respectively. Furthermore, let $x\in X$ and α, $\beta\in\mathbb{C}$. W.l.o.g. we may assume $n \geq m$ and introduce $\nu_{m+1} = \cdots = \nu_n := 0$. Then

$$\begin{aligned}\Phi(\alpha\mu + \beta\nu)x &= \sum_{k=0}^{n}(-A)^k \int\limits_{[0,\infty)} \mathrm{e}^{-At}x\,(\alpha\mu_k + \beta\nu_k)(\mathrm{d}t)\\ &= \alpha\sum_{k=0}^{n}(-A)^k \int\limits_{[0,\infty)} \mathrm{e}^{-At}x\,\mu_k(\mathrm{d}t) + \beta\sum_{l=0}^{m}(-A)^l \int\limits_{[0,\infty)} \mathrm{e}^{-At}x\,\nu_l(\mathrm{d}t)\\ &= \alpha\Phi(\mu)x + \beta\Phi(\nu)x\end{aligned}$$

shows linearity while the multiplicativity follows from

$$\begin{aligned}\Phi(\mu * \nu)x &= \sum_{k=0}^{n}\sum_{l=0}^{m}(-A)^k(-A)^l \int\limits_{[0,\infty)} \mathrm{e}^{-At}x\,\mu_k * \nu_l(\mathrm{d}t)\\ &= \sum_{k=0}^{n}(-A)^k \int\limits_{[0,\infty)} \mathrm{e}^{-At}\sum_{l=0}^{m}(-A)^l \int\limits_{[0,\infty)} \mathrm{e}^{-As}x\,\nu_l(\mathrm{d}s)\,\mu_k(\mathrm{d}t) = \Phi(\mu)\Phi(\nu)x.\end{aligned}$$

As for the continuity, we use the same criterion which we applied by now several times and check that the mapping Φ is already continuous from $C_b^n\big([0,\infty)\big)'$ to $\mathcal{L}(X)_\beta$. This follows from

$$\left\|\Phi(\mu)x\right\|_p \leq M\left\|x\right\|_q \sum_{k=0}^{n}\left\|\mu_k\right\|.$$

One can now take the supremum over all $x \in B$ where $B \subseteq X$ is a given bounded subset and the continuity follows. $\square$

The continuous calculus from Proposition 3.2.12 shall be called *Hille–Phillips–Schwartz calculus*.

The situation changes dramatically when we consider the more generic situation of A being a closed but discontinuous operator such that $-A$ generates an equicontinuous C_0-semigroup. In this situation the calculus restricted to $C_b\big([0,\infty)\big)'$ will still be continuous but we cannot give sense to $\Phi(\mu)$ if $\mu \notin C_b\big([0,\infty)\big)'$. However, note that there is always $m \in \mathbb{N}_0$ such that the given definition can be used to define an operator $\Phi(\mu)$ on $\mathcal{D}(A^m)$.

Lemma 3.2.13. *Let $\mu \in C_b^\infty\big([0,\infty)\big)'$ with representation $\mu = \sum_{k=0}^n (D^k)'\mu_k$, $m \in \mathbb{N}_0$, $m \geq n$. Then the operator $\Phi(\mu) : \mathcal{D}(A^m) \to X$, defined by*

$$\Phi(\mu)x := \langle \mu, t \mapsto \mathrm{e}^{-At}x\rangle = \sum_{k=0}^{n}(-A)^k \int\limits_{[0,\infty)} \mathrm{e}^{-At}x\,\mu_k(\mathrm{d}t)$$

is closable.

Proof. Take a net (x_α) in $\mathcal{D}(A^m)$ convergent towards 0 such that $(\Phi(\mu)x_\alpha)$ converges to $y \in X$. By continuity of the resolvent this implies

$$(1+A)^{-m}y = \lim_\alpha (1+A)^{-m}\Phi(\mu)x_\alpha = \lim_\alpha \sum_{k=0}^{n}(-A)^k(1+A)^{-m}\int\limits_{[0,\infty)} \mathrm{e}^{-At}x_\alpha\,\mu_k(\mathrm{d}t) = 0.$$

Hence, $y = 0$. This finishes the proof. $\square$

At first glance the extension of the calculus by a closure procedure seems to be a nice workaround but the following result should be taken into account.

Proposition 3.2.14. *It holds that*

$$\forall \mu \in C_b^n\big([0,\infty)\big)' \subseteq C_b^\infty\big([0,\infty)\big)',\, m \geq n : \, \mu * \tfrac{1}{(m-1)!}t^{m-1}e^{-t}\mathrm{d}t \in C_b\big([0,\infty)\big)'$$

and as a consequence

$$\overline{\Phi(\mu)}(1+A)^{-m} = \Phi(\mu)(1+A)^{-m} = \Phi(\mu * \tfrac{1}{(m-1)!}t^{m-1}e^{-t}\mathrm{d}t) \in \mathcal{L}(X).$$

Moreover, $(1+A)^m\Phi(\mu)(1+A)^{-m}$ *is a closed operator extending* $\overline{\Phi(\mu)}$*, i.e.,*

$$\forall \mu \in C_b^n\big([0,\infty)\big)' \subseteq C_b^\infty\big([0,\infty)\big)',\, m \geq n : \, \overline{\Phi(\mu)} \subseteq (1+A)^m\Phi(\mu)(1+A)^{-m}.$$

If in addition we have that $\overline{\mathcal{D}(A)} = X$*, we even get*

$$\forall \mu \in C_b^n\big([0,\infty)\big)' \subseteq C_b^\infty\big([0,\infty)\big)',\, m \geq n : \, \overline{\Phi(\mu)} = (1+A)^m\Phi(\mu)(1+A)^{-m}.$$

Proof. To begin with let $f \in C_0^k\big((0,\infty)\big)$ such that $f^{(l)} \in L^1\big((0,\infty)\big)$ for $l \in \{0,\ldots,k\}$. Then we have $(D^l)'\big(f\mathrm{d}t\big) = (-1)^l f^{(l)}\mathrm{d}t$ which follows from integration by parts. Having this in mind, one can apply Lemma 3.2.5 which yields

$$\begin{aligned}\mu * \frac{1}{(m-1)!}t^{m-1}e^{-t}\mathrm{d}t &= \sum_{k=0}^{n}\mu_k * (-1)^k D^k \frac{1}{(m-1)!}t^{m-1}e^{-t}\mathrm{d}t \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k}\mu_k * (-1)^l\binom{k}{l}\frac{1}{(m-1-l)!}t^{m-1-l}e^{-t}\mathrm{d}t\end{aligned}$$

which is a finite measure since it is the sum of convolutions of measures.

The first claimed equality is based on Laplace transform. It holds that

$$(\lambda + A)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{e}^{-At}x\,\mathrm{d}t$$

and differentiating w.r.t. the variable λ yields

$$(\lambda + A)^{-m}x = \frac{1}{(m-1)!}\int_0^\infty t^{m-1}\mathrm{e}^{-\lambda t}\mathrm{e}^{-At}x\,\mathrm{d}t.$$

Hence,

$$\begin{aligned}\Phi(\mu)(1 + A)^{-m}x &= \sum_{k=0}^{n}\int_{[0,\infty)} \mathrm{e}^{-At}\frac{1}{(m-1)!}\int_0^\infty s^{m-1}\mathrm{e}^{-s}\mathrm{e}^{-As}x\,\mathrm{d}s\,(D^k)'\mu_k(\mathrm{d}t)\\ &= \Phi(\mu * \tfrac{1}{(m-1)!}t^{m-1}e^{-t}\mathrm{d}t)x.\end{aligned}$$

Since $\mu * \frac{1}{(m-1)!}t^{m-1}e^{-t}\mathrm{d}t \in C_b\big([0,\infty)\big)'$, we conclude that we indeed have $\Phi(\mu)(1+A)^{-m} = \Phi(\mu * \frac{1}{(m-1)!}t^{m-1}e^{-t}\mathrm{d}t) \in \mathcal{L}(X)$. Hence, the operator $(1 + A)^m\Phi(\mu)(1 + A)^{-m}$ is closed as composition of a continuous and a closed operator. Let now $x \in \mathcal{D}(\overline{\Phi(\mu)})$ for some $\mu \in C_b^n\big([0,\infty)\big)' \subseteq C_b^\infty\big([0,\infty)\big)'$ and choose $m \in \mathbb{N}_0$, $m \geq n$. We calculate

$$\overline{\Phi(\mu)}x = (1 + A)^m(1 + A)^{-m}\lim_\alpha \Phi(\mu)x_\alpha = (1 + A)^m\Phi(\mu)(1 + A)^{-m}x$$

where the second equality used the continuity of $(1+A)^{-m}$ and Lemma 3.1.8 applied to the continuous operator $(1 + A)^{-m}$. To establish equality in case of A being densely defined, we shall make use of Lemma 2.0.9. Applying it we get

$$\begin{aligned}(1 + A)^m\Phi(\mu)(1 + A)^{-m}x &= \lim_{\lambda\to\infty} \lambda^m(\lambda + A)^{-m}(1 + A)^m\Phi(\mu)(1 + A)^{-m}x\\ &= \lim_{\lambda\to\infty} \Phi(\mu)\lambda^m(\lambda + A)^{-m}x\\ &= \overline{\Phi(\mu)}x\end{aligned}$$

where the last equality took into account the fact that we have $\lambda^m(\lambda + A)^{-m}x \to x$ as $\lambda \to \infty$. The latter made use of the dense domain because otherwise it could not be guaranteed. The proof is finished □

From now on we shall simply write $\Phi(\mu)$ for the operator $(1 + A)^m\Phi(\mu)(1 + A)^{-m}$ where $\mu \in C_b^n\big([0,\infty)\big)' \subseteq C_b^\infty\big([0,\infty)\big)'$ and $m \in \mathbb{N}_0$, $m \geq n$. Besides the fact that this is operator has formally a bigger domain compared to $\overline{\Phi(\mu)}$, there is another fact which suggests that the expression $(1 + A)^m\Phi(\mu)(1 + A)^{-m}$ is a somehow 'better' extension of

Φ to $C_b^\infty\big([0,\infty)\big)'$ than $\mu \mapsto \overline{\Phi(\mu)}$. One may also want to generalise the homomorphism property included in Definition 3.0.1. We already sense that any extension will take values in the closed but typically discontinuous operators. Due to domain issues, a generalisation of the homomorphism property is not to be expected. We have the following though.

Proposition 3.2.15. *Let Φ be the mapping from above. Then it holds that*

1. $\forall \alpha, \beta \in \mathbb{C},\ \mu, \nu \in C_b^\infty\big([0,\infty)\big)' :\ \alpha\Phi(\mu) + \beta\Phi(\nu) \subseteq \Phi(\alpha\mu + \beta\nu)$.
2. $\forall \mu \in C_b\big([0,\infty)\big)',\ m \in \mathbb{N}_0 :\ \Phi(\mu)A^m \subseteq A^m\Phi(\mu)$.
3. $\forall \mu, \nu \in C_b^\infty\big([0,\infty)\big)' :\ \Phi(\mu)\Phi(\nu) \subseteq \Phi(\mu * \nu)$ *and*
$$\mathcal{D}(\Phi(\mu)\Phi(\nu)) = \mathcal{D}(\Phi(\nu)) \cap \mathcal{D}(\Phi(\mu * \nu)).$$

Proof.

1. Let $x \in \mathcal{D}(\Phi(\mu)) \cap \mathcal{D}(\Phi(\nu))$. Choose $m \in \mathbb{N}$ sufficiently large. Then

$$\begin{aligned}\alpha\Phi(\mu)x + \beta\Phi(\nu)x &= \alpha(1+A)^m\Phi(\mu)(1+A)^{-m}x + \beta(1+A)^m\Phi(\nu)(1+A)^{-m}x \\ &= (1+A)^m\Phi(\alpha\mu+\beta\nu)(1+A)^{-m}x \\ &= \Phi(\alpha\mu+\beta\nu)x.\end{aligned}$$

2. By the closedness of A^m and Hille's theorem, we have for $x \in \mathcal{D}(A^m)$

$$\Phi(\mu)A^m x = \int\limits_{[0,\infty)} \mathrm{e}^{-At}A^m x\mu(\mathrm{d}t) = A^m \int\limits_{[0,\infty)} \mathrm{e}^{-At}x\mu(\mathrm{d}t) = A^m\Phi(\mu)x.$$

3. Let $x \in \mathcal{D}(\Phi(\mu)\Phi(\nu))$ and choose $m, n \in \mathbb{N}$ sufficiently large. Then

$$\begin{aligned}\Phi(\mu)\Phi(\nu)x &= (1+A)^m \underbrace{\Phi(\mu)(1+A)^{-m}}_{\in\mathcal{L}(X)}(1+A)^n\Phi(\nu)(1+A)^{-n}x \\ &= (1+A)^{m+n}\Phi(\mu*\nu)(1+A)^{-m-n}x \\ &= \Phi(\mu*\nu)x.\end{aligned}$$

   The above calculation also shows the inclusion $\mathcal{D}(\Phi(\mu)\Phi(\nu)) \subseteq \mathcal{D}(\Phi(\nu)) \cap \mathcal{D}(\Phi(\mu*\nu))$. Conversely, assuming $x \in \mathcal{D}(\Phi(\nu)) \cap \mathcal{D}(\Phi(\mu*\nu))$ allows one to go through the same calculation backwards which establishes equality of the domains.

□

Remark 3.2.16.

1. The characterisation in Proposition 3.2.15 is used in [27] to define an unbounded functional calculus.

2. The used procedure to extend a given bounded calculus algebraically is introduced in a very general manner in [17, Definition 3.4] and refined in [26]. Following the terminology used in [26], one may say that every $\mu \in C_b^\infty\big([0,\infty)\big)'$ is anchored (a notation which will be explained in more detail later on) by the measure $\frac{1}{(m-1)!}t^{m-1}e^{-t}\mathrm{d}t \in M_b\big([0,\infty)\big)$ for sufficiently large $m \in \mathbb{N}$.

The above studied calculus was already considered in [52], [58], and [65] by using approximations. The equivalence of the approaches is contained in the next proposition.

Proposition 3.2.17. *Let $\mu \in C_b^\infty\big([0,\infty)\big)'$ considered as a functional on the subspace of all functions in $\mathcal{D}(\mathbb{R})$ whose support is contained in $[0,\infty)$ (a so-called summable distribution, $\mathcal{D}'_{L^1}\big([0,\infty)\big)$ denotes their set in the Schwartz terminology), (φ_k) a sequence in $\mathcal{D}(\mathbb{R})$ whose supports are contained in $[0,\infty)$ and which is convergent to δ_0 in $\mathcal{D}'(\mathbb{R})$ w.r.t. the weak topology, i.e., $\forall n \in \mathbb{N}_0,\ \psi \in \mathcal{D}(\mathbb{R}) : \lim_{k\to\infty}\langle(-1)^n\varphi_k^{(n)}\mathrm{d}t, \psi\rangle = \psi^{(n)}(0)$, and $x \in \mathcal{D}(\Phi(\mu))$. Then*

$$\lim_{k\to\infty} \Phi(\mu * \varphi_k \mathrm{d}t)x = \Phi(\mu)x.$$

Proof. First, note that every $\psi \in \mathcal{D}(\mathbb{R})$ gives rise to a function $\psi \in C_b^\infty\big([0,\infty)\big)$ by simply restricting to $[0,\infty)$. Second, by the support assumption on the φ_k one can consider the measures $\varphi_k\mathrm{d}t$ as functionals on $C_b^\infty\big([0,\infty)\big)'$ which is in correspondence with the above mentioned restriction procedure in the sense that

$$\langle\varphi_k\mathrm{d}t, \psi\rangle_{\mathcal{D}'(\mathbb{R})} = \langle\varphi_k\mathrm{d}t, \psi\rangle_{C_b^\infty([0,\infty))'}.$$

Third, the measures $\varphi_k\mathrm{d}t$ are, by smoothness of the functions φ_k, smoothing for all $\mu \in C_b^\infty\big([0,\infty)\big)'$ which means that $\varphi_k\mathrm{d}t * \mu \in C_b\big([0,\infty)\big)'$. Finally, assume $\mu = \sum_{l=0}^n (D^l)'\mu_l$

and choose $m \in \mathbb{N}$, $m \geq n$. Now one can calculate

$$
\begin{aligned}
\Phi(\mu)x &= (1+A)^m \sum_{l=0}^{n} \int\limits_{[0,\infty)} (-A)^l \mathrm{e}^{-As}(1+A)^{-m}x\,\mu_l(\mathrm{d}s) \\
&= (1+A)^m \sum_{l=0}^{n} \int\limits_{[0,\infty)} (-A)^l \lim_{k\to\infty} \int\limits_0^\infty \mathrm{e}^{-A(s+t)}(1+A)^{-m}x\,\varphi_k(t)\mathrm{d}t\,\mu_l(\mathrm{d}s) \\
&= (1+A)^m \lim_{k\to\infty} \int\limits_{[0,\infty)} \mathrm{e}^{-As}(1+A)^{-m}x\big(\mu * \varphi_k \mathrm{d}t\big)(\mathrm{d}s) \\
&= \lim_{k\to\infty} \Phi(\mu * \varphi_k \mathrm{d}t)x.
\end{aligned}
$$

□

The Laplace transform $L : C_b\big([0,\infty)\big)' \to \mathcal{H}(\mathbb{C}_{\mathrm{Re}>0})$, $\mu \mapsto (\lambda \mapsto \int_{[0,\infty)} \mathrm{e}^{-\lambda t}\,\mu(\mathrm{d}t))$ is an injective mapping, and its image is called *Laplace–Stieltjes algebra* and shall be denoted by $\mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})$. This is a subalgebra of the locally convex algebra $\mathcal{H}(\mathbb{C}_{\mathrm{Re}>0})$. We could also interpret the Hille–Phillips calculus as a mapping defined on $\mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})$ and in this situation one may define for $L\mu = f \in \mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})$ the notation $f(A) := \Phi(f) = \Phi(\mu)$ which also comes much closer to the interpretation of the calculus being the process of plugging in an operator A into a function f. From this point of view, the above presented extension may be seen as an extension of Φ from $\mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})$ to an algebra of the form $\{g \in \mathcal{H}(\mathbb{C}_{\mathrm{Re}>0}) \mid \exists f_0,\dots,f_n \in \mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})\, \forall \lambda \in \mathbb{C}_{\mathrm{Re}>0} : g(\lambda) = \sum_{k=0}^n (-\lambda)^k f_k(\lambda)\}$.

Proposition 3.2.18. *The mapping*

$$
L : C_b^\infty\big([0,\infty)\big)' \to \mathcal{H}(\mathbb{C}_{\mathrm{Re}>0}), \quad \mu \mapsto (\lambda \mapsto \langle \mu, t \mapsto \mathrm{e}^{-\lambda t}\rangle)
$$

is injective.

Proof. It is possible to define the Laplace transform L in a very general manner on all such distributions $\mu \in \mathscr{S}(\mathbb{R})'$ which are supported in $[0,\infty)$. Let us roughly sketch the details. One defines $L\mu \in \mathcal{H}(\mathbb{C}_{\mathrm{Re}>0})$ by

$$
(L\mu)(\lambda) := \langle \mu, t \mapsto \mathrm{e}^{-\lambda t}\phi(t)\rangle
$$

where $\phi \in C^\infty(\mathbb{R})$, $\phi(t) = 1$ for $t \in [0,\infty)$ and $\phi(t) = 0$ for $t \in (-\infty,-\varepsilon)$ for some $\varepsilon > 0$, cf. [64, Chap. 8.2] for the details. Basically ϕ ensures that $t \mapsto \mathrm{e}^{-\lambda t}\phi(t) \in \mathscr{S}(\mathbb{R})$ and one can show further that the definition is independent of the concrete choice of ϕ. One now checks that $C_b^\infty\big([0,\infty)\big)' \subseteq \mathscr{S}(\mathbb{R})'$ and the support condition is also met. So the mapping L under consideration is a restriction of L defined on $\mathscr{S}(\mathbb{R})'$ and the latter is already injective by [64, Thm. 8.2]. $\square$

Restricting the operator A under consideration further, the above mentioned more general definition of L on Schwartz distributions $\mathscr{S}(\mathbb{R})'$ supported in $[0,\infty)$ or even on Laplace-transformable distributions in $\mathcal{D}(\mathbb{R})'$ with the same exponential growth bound (they form a convolution algebra by [64, Chap. 8.3]) can be used to extend the calculus further. To be more concrete, every $\mu \in \mathscr{S}(\mathbb{R})'$ is of the form

$$\mu = \sum_{k=0}^{n} (D^k)' \mu_k$$

with measures μ_0 to μ_n each of them such that we can find a natural number $l_k \in \mathbb{N}$ with the property

$$\int\limits_{\mathbb{R}} \frac{1}{1+|t|^{l_k}} \, |\mu_k| \, (\mathrm{d}t) < \infty$$

(a consequence of $\mathscr{S}(\mathbb{R})$ being the projective limit of the spectrum discussed in Example 2.0.18 (c)). For a generator $-A$ of an equibounded semigroup and any $\varepsilon > 0$, we can consider the operator $-(A+\varepsilon)$ generating an exponentially stable semigroup. For generators of such exponentially stable semigroups the calculus extends to the algebra of elements in $\mathscr{S}(\mathbb{R})'$ supported in $[0,\infty)$ in an obvious way. The same considerations are true for Laplace-transformable distributions in $\mathcal{D}(\mathbb{R})'$ by considering $-(A+c)$ for some $c > 0$ sufficiently large.

The just discussed extension is not necessarily the largest possible extension. In order to explain the so-called *maximal extension*, let us quickly explain the concept of an *anchor set* already touched in Remark 3.2.16. We say that a function $f \in \mathcal{H}(\mathbb{C}_{\mathrm{Re}>0})$ is anchored in $\mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})$ if there is a subset $M \subseteq \mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})$, called an anchor set, such that $\bigcap_{g\in M} \mathcal{N}(\Phi(g)) = \{0\}$ and $gf \in \mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})$ for all $g \in M$. The set of all anchored elements forms a superalgebra of $\mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})$ on which the calculus Φ may be extended by means of

$$(x,y) \in \Phi_{\mathrm{ext}}(f) \quad \Leftrightarrow \quad \forall g \in M : \, \Phi(gf)x = \Phi(g)y.$$

This extension is the maximal possible extension of the calculus Φ. For more details on the matter, please see [26]. We will come back to this extension when studying the next calculus.

3.3. Stieltjes functional calculus

In this section we want to apply the so far introduced general framework to another functional calculus. We start by sketching the basic idea. Consider again the algebra $\mathcal{E}(S_\omega)$ but now for the 'extreme' case $\omega = \pi$. To be more precise, consider the algebra of functions $f \in \mathcal{H}^\infty\big(\mathbb{C} \setminus (-\infty, 0]\big)$ with the additional property that for all $-t \in (-\infty, 0)$ the limits $f(-t - i0) := \lim_{z \to -t, \operatorname{Im} z < 0} f(z)$ and $f(-t + i0) := \lim_{z \to -t, \operatorname{Im} z > 0} f(z)$ exist and that the difference $t \mapsto f(-t + i0) - f(-t - i0)$ is in $L^1\big((0, \infty), t^{-1}\mathrm{d}t\big)$. Furthermore, let us demand that $\lim_{R \to \infty} f(R\mathrm{e}^{it}) = 0$ for all $t \in (-\pi, \pi)$. A standard example for such a function would be $\lambda \mapsto \mathrm{e}^{-\lambda^\gamma}$ where $0 < \gamma < \frac{1}{2}$. An argument using a keyhole contour (see Figure 3.3.1) in combination with dominated convergence shows that

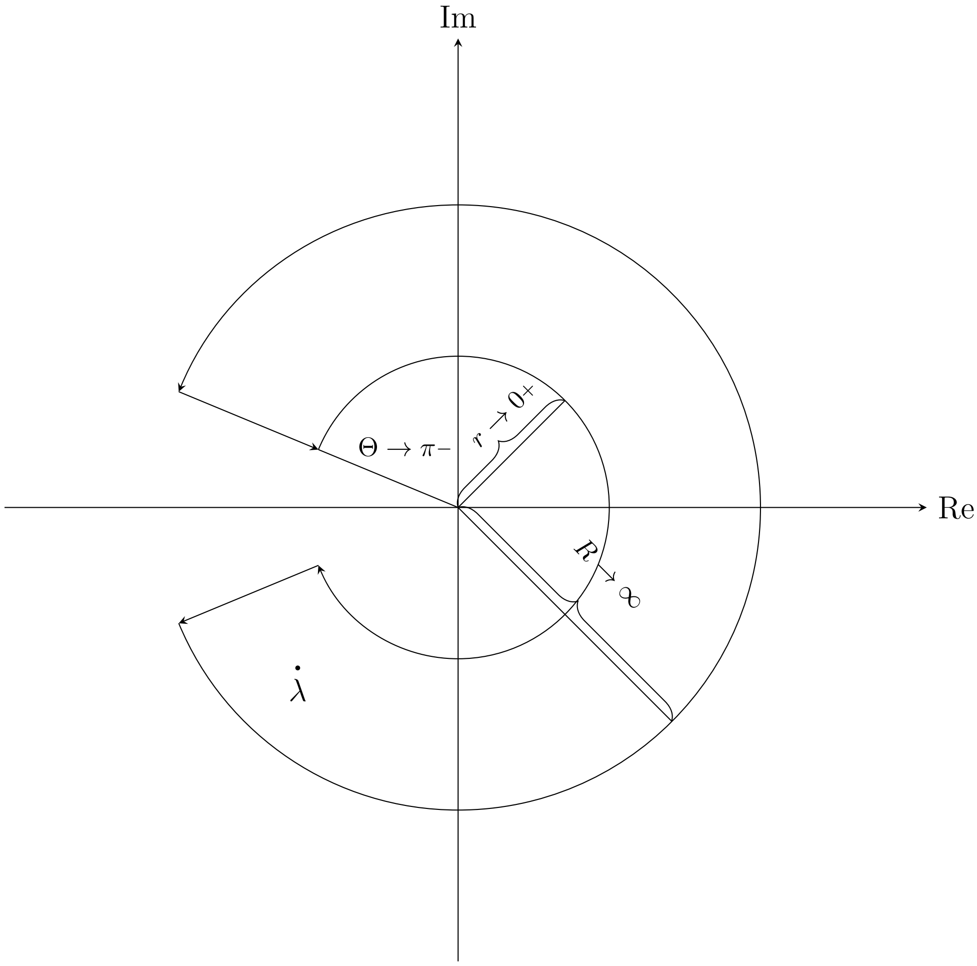

Figure 3.3.1.: Key hole contour for integration

$$\forall \lambda \in \mathbb{C} \setminus (-\infty, 0] : \ f(\lambda) = \frac{1}{2\pi i} \int\limits_0^\infty \frac{f(-t - i0) - f(-t + i0)}{t + \lambda} \, \mathrm{d}t.$$

Based on this formula one may try to set up a functional calculus by means of the equation

$$f(A)x = \frac{1}{2\pi i}\int_0^\infty \big(f(-t-i0) - f(-t+i0)\big)(t+A)^{-1}x\,\mathrm{d}t$$

or, having the methods in mind we already saw, by the more general approach

$$\Phi(\mu)x = \int_{(0,\infty)} (t+A)^{-1}x\,\mu(\mathrm{d}t)$$

where μ is a measure such that $t \mapsto t^{-1}$ is integrable. This integral transform is called a *Stieltjes transform* which was introduced by Stieltjes in [62] for the purpose of solving the Stieltjes moment problem ([57, Cmt. 2.5]).

We are now going to mimic the ideas already presented and will develop an even larger functional calculus. Such a Stieltjes functional calculus was apparently studied for the first time by Hirsch in [30]. There the author already introduced the right counterpart for the shift operator and the spaces of continuously differentiable functions we used before and therefore laid the foundation for a 'Stieltjes–convolution' explicitly studied in [61]. Further contributions to the calculus came from [43] (see also [44]) where it was used in view of its applicability to fractional powers) and from [28] where it already got a quite advanced appearance.

By now we already noted that the orbits of $t \mapsto (t+A)^{-1}$ is what is going to replace the orbits of $t \mapsto \mathrm{e}^{-At}$. This has advantages and disadvantages. As for the advantages, we already learnt in Proposition 2.0.3 that the orbits of the resolvent operators are smooth while for the orbits of a generic C_0-semigroup we can in general not hope for any better than continuity unless the smoothness of the considered element x is increased, i.e., we have additional information such as $x \in \mathcal{D}(A^m)$ for some $m \in \mathbb{N}$. However, the singularity at $t = 0$ becomes stronger by one order every time we differentiate and we therefore need to take this into account by introducing new spaces being the counterparts to the spaces $C_b^n\big([0,\infty)\big)$ and their projective limit $C_b^\infty\big([0,\infty)\big)$.

Definition 3.3.1. For $n \in \mathbb{N}_0$ we define

$$S^n := \{f \in C^n\big((0,\infty)\big) \mid \forall k \in \{0,\dots,n\} : s \mapsto s^{k+1}f^{(k)}(s) \in C_b\big((0,\infty)\big)\}$$

to be the space of weighted C^n-functions. Analogously, we define their projective limit S^∞ and we write $S^n(X)$ and $S^\infty(X)$, respectively, if the functions under consideration take values in an LCS X.

The spaces S^n replace the spaces $C_b^n\big([0,\infty)\big)$ from before. If X is an LCS and $x \in X$, a natural example of a function in $S^\infty(X)$ is $t \mapsto (t+A)^{-1}x$ for $A \in \mathcal{M}(X)$. The space S^1 already appeared in [30] where one also can find the suitable replacement for τ_s from before.

But first let us talk about the topology on $S := S^0$. For this simply note that $S \ni f \mapsto \big(s \mapsto sf(s)\big) \in C_b\big((0,\infty)\big)$ is an isomorphism and that for $C_b\big((0,\infty)\big)$, as for all spaces of bounded continuous functions on locally compact spaces, a mixed topology is available, see [16, Proposition 3].

Definition 3.3.2. Let $h \in C_0\big((0,\infty)\big)$. For $f \in S$ define the continuous seminorm $\|f\|_h := \sup_{s\in(0,\infty)} |h(s)sf(s)|$. We define the *mixed topology* on S to be the locally convex topology generated by all such seminorms.

Remark 3.3.3.

1. With the help of the above isomorphism and its dual isomorphism one also characterises S' as the space of all measures $\mu \in M\big((0,\infty)\big)$ of the form $\mu = s\nu$ for some $\nu \in M_b\big((0,\infty)\big)$. That means

$$S' = \{\mu \in M\big((0,\infty)\big) \mid \int_{(0,\infty)} \frac{1}{s} \, |\mu| \, (\mathrm{d}s) < \infty\}.$$

   As before, we equip this space with its natural Banach space topology and define $\|\mu\|_{S'} := \int_{(0,\infty)} \frac{1}{s} \, |\mu| \, (\mathrm{d}s)$.

2. At this point we already deviate from the strategy from before. Analogously to the situation we already encountered, we should have introduced spaces $\{f \in C\big((0,\infty)\big) \mid s \mapsto sf(s) \in C_0\big((0,\infty)\big)\}$. However, typically the orbits generated by the resolvents are not in such a space which is why, at this point, we would have to consider S with its mixed topology to overcome the problem that this space, equipped with its standard Banach space topology, cannot be represented as ε-product.

3. The projective limit S^∞ gets now its continuous seminorms in an obvious manner. To wit, let $n \in \mathbb{N}_0$ and $h = (h_0, \ldots, h_n) \in C_0\big((0,\infty); \mathbb{C}^{n+1}\big)$. Then a continuous seminorm is given by the expression

$$\|f\|_{n,h} := \max_{0\le k\le n} \sup_{s\in(0,\infty)} s^{k+1} \left|h_k(s) f^{(k)}(s)\right|.$$

The next lemma is of auxiliary character and will help to prove a result analogously to Lemma 3.2.1. It is actually a part of the proof of [12, Thm. 2] and can be found for the more general setting of a locally compact topological space there. It will be added for the readers convenience.

Lemma 3.3.4. *Let $\mu \in M_b\big((0,\infty)\big)$. Then there is $h \in C_0\big((0,\infty)\big)$ such that $0 < h \le 1$, $\frac{1}{h}$ is integrable w.r.t. the measure μ, and*

$$\|\mu\| \le \int\limits_{(0,\infty)} \frac{1}{h(s)} \, |\mu| \, (\mathrm{d}s) \le 4 \, \|\mu\| \, .$$

Moreover, we have $\forall \, 0 < s \le u \le t : \; h(t)h(s) \le h(u)$.

Proof. The statement is clear if $\mu = 0$. So w.l.o.g. we may assume $\mu \neq 0$. Let $(K_n)_{n \in \mathbb{N}_0}$ be defined by $K_0 := \varnothing$ and $K_n := [\frac{1}{n}, n]$. Then $K_n \subsetneq K_{n+1} \subsetneq (0,\infty)$ which means that (K_n) is a strictly increasing sequence of compacts such that $\bigcup_{n=1}^{\infty} K_n = (0,\infty)$. Define further the sequence $(a_n)_{n \in \mathbb{N}}$ by $a_n := |\mu| \, (K_n \setminus K_{n-1})$. Then $M := \|\mu\| = \sum_{n=1}^{\infty} a_n$. Assume that the measure μ has compact support. It follows that the sequence (a_n) has only finitely many elements different from 0 in this case. In this situation one can choose $h = 1$ on $\operatorname{supp}(\mu)$ and extend it to a function $h \in C_0\big((0,\infty)\big)$ subject to the above restrictions. If $\operatorname{supp}(\mu)$ is not compact, one is able to choose a strictly increasing sequence (n_k) in $\mathbb{N}_0$ such that $n_0 = 0$ and

$$\forall k \in \mathbb{N} : \; \frac{M}{2^k} \le \sum_{m=n_{k-1}+1}^{n_k} a_m \le \frac{M}{2^{k-1}}.$$

Define now a sequence (b_n) in $(0,\infty)$ by $b_n := \frac{1}{k}$ if $n \in \{n_{k-1}+1, \ldots, n_k\}$. This sequence has the properties that for all $n \in \mathbb{N}$ one has $b_n \le 1$, $\lim_{n \to \infty} b_n = 0$ and $\sum_{n=1}^{\infty} \frac{a_n}{b_n} \le 4M$. The latter follows from

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n} = \sum_{k=1}^{\infty} \sum_{m=n_{k-1}+1}^{n_k} k \cdot a_m \le M \sum_{k=1}^{\infty} \frac{k}{2^k} = 4M.$$

For $n \in \mathbb{N}$ define now

$$h_n(s) := \begin{cases} 1 & s \in K_n \\ n+1-s & s \in (n, n+1] \\ n(n+1)\big(s - \frac{1}{n+1}\big) & s \in [\frac{1}{n+1}, \frac{1}{n}) \\ 0 & \text{otherwise.} \end{cases}$$

For the so constructed functions one has $h_n \in C_0\big((0,\infty)\big)$, $h_n = 1$ on K_n, $0 \le h_n \le 1$ on $K_{n+1} \setminus K_n$, and $h_n = 0$ otherwise. Moreover, $1 \le t \le s$ implies $h_n(s) \le h_n(t)$ and

$s \leq t \leq 1$ implies also $h_n(s) \leq h_n(t)$. Define $c_n := b_n - b_{n+1}$. Note that (c_n) is summable with $\sum_{n=k}^{\infty} c_n = b_k$. Finally set $h := \sum_{n=1}^{\infty} c_n h_n$. Since all functions h_n are bounded by 1, the above series is convergent uniformly on $(0,\infty)$, and since also the sequence (b_n) is bounded by 1 so is the function h. Furthermore, for $s \in K_n \setminus K_{n-1}$ and $n \geq 2$ we have $b_n \leq h(s) = c_{n-1}h_{n-1}(s) + b_n \leq b_{n-1}$. This proves $h \in C_0\big((0,\infty)\big)$ and it further shows

$$\|\mu\| = \int\limits_{(0,\infty)} |\mu|\,(\mathrm{d}s) \leq \int\limits_{(0,\infty)} \frac{1}{h(s)}\,|\mu|\,(\mathrm{d}s) \leq \sum_{n=1}^{\infty} \int\limits_{K_n\setminus K_{n-1}} \frac{1}{b_n}\,|\mu|\,(\mathrm{d}s) \leq 4M = 4\,\|\mu\|\,.$$

For the last part let $0 < s \leq u \leq t$. Assume first $u \leq 1$. Then $h(u) \geq h(s) \geq h(s)h(t)$ since $h(t) \leq 1$. For the other case we have $h(u) \geq h(t) \geq h(t)h(s)$ and the claim is proven. $\square$

The space S^∞ is not invariant under differentiation. One can use weighted derivatives though.

Definition 3.3.5. Define for $n \in \mathbb{N}_0$ the operator $E_n :\ S^\infty \ni f \mapsto E_n f := (s \mapsto s^n f^{(n)}(s))$.

From the product rule one concludes for $n,k \in \mathbb{N}_0$

$$E_n E_k = \sum_{l=0}^{\min\{n,k\}} l! \binom{n}{l}\binom{k}{l} E_{n+k-l}.$$

In particular, $E_n E_k = E_k E_n$.

Completely analogously to Lemma 3.2.1 one proves the following result.

Lemma 3.3.6. *The operators E_n are continuous on S^∞. For given measures $\mu_0, \ldots, \mu_n \in S'$ the linear form $\mu := \sum_{k=0}^{n} E_k' \mu_k$ is continuous on S^∞. Conversely, every $\mu \in (S^\infty)'$ is of this form.*

Proof. Let $m \in \mathbb{N}_0$, $h = (h_0, \ldots, h_m) \in C_0\big((0,\infty); \mathbb{C}^{m+1}\big)$ and $n \in \mathbb{N}_0$. For $f \in S^\infty$ consider the seminorm $\|\cdot\|_{m,h}$ and the operator E_n. The claimed continuity of E_n follows from

$$\begin{aligned}\|E_n f\|_{m,h} &= \max_{0\leq k\leq m} \sup_{s\in(0,\infty)} |s h_k(s)(E_k E_n f)(s)| \\ &\leq \max_{0\leq k\leq m} \sum_{l=0}^{\min\{k,n\}} l! \binom{n}{l}\binom{k}{l} \sup_{s\in(0,\infty)} \underbrace{|s h_k(s)(E_{n+k-l} f)(s)|}_{\leq \|f\|_{n+m,(\tilde{h},h)}}\end{aligned}$$

for some $\tilde{h} \in C_0\big((0,\infty);\mathbb{C}^n\big)$. The functional μ is continuous since

$$\begin{aligned}
|\langle \mu, f\rangle| &= \left|\Big\langle \sum_{k=0}^{n} E_k'\mu_k, f\Big\rangle\right| \le \sum_{k=0}^{n} |\langle \mu_k, E_k f\rangle| \le \sum_{k=0}^{n} \int\limits_{(0,\infty)} s^{k+1}\left|f^{(k)}(t)\right| \frac{1}{s}\,|\mu_k|\,(\mathrm{d}t) \\
&\le \|f\|_{n,h} \cdot \sum_{k=0}^{n} \int\limits_{(0,\infty)} \frac{1}{sh_k(s)}\,|\mu_k|\,(\mathrm{d}s) \\
&\le \|f\|_{n,h} \cdot 2\sum_{k=0}^{n} \|\mu_k\|_{S'}
\end{aligned}$$

where we have chosen $h = (h_0, \ldots, h_n) \in C_0\big((0,\infty);\mathbb{C}^{n+1}\big)$ accordingly to Lemma 3.3.4.

The fact that all continuous functionals are of this form follows by a similar argument as already presented in the proof of Lemma 3.2.1. $\square$

We continue with the already announced generalisation of the shift operator τ_s.

Definition 3.3.7. Let $s > 0$. We define the operator δ_s on S^∞, $f \mapsto \delta_s f$ by $(\delta_s f)(t) := -\int_0^1 f'(xs + (1-x)t)\mathrm{d}x$.

Lemma 3.3.8. *Let $f \in S^\infty$, $\mu \in S'$, $s > 0$, and $n \in \mathbb{N}_0$. Then*

$$\delta_s f \in S^\infty \quad \textit{and} \quad (0,\infty) \ni s \mapsto \langle E_n'\mu, \delta_s f\rangle \in S^\infty.$$

Proof. Let $t > 0$. By differentiation under the integral sign one has $\delta_s f \in C^\infty\big((0,\infty)\big)$ and

$$(\delta_s f)^{(n)}(t) = -\int\limits_0^1 (1-x)^n f^{(n+1)}\big(xs + (1-x)t\big)\mathrm{d}x.$$

With the same argument one can also see that $s \mapsto (\delta_s f)(t) \in C^\infty\big((0,\infty)\big)$. Let $k \in \mathbb{N}_0$

and set $\|f\|_{n+k+1} := \sup_{s>0} \left|s^{n+k+2} f^{(n+k+1)}(s)\right|$. We calculate

$$\begin{aligned}
\left|\Big(\frac{\partial^{n+k}}{\partial s^k \partial t^n}\delta_s f\Big)(t)\right| &\leq \|f\|_{n+k+1} \int_0^1 \frac{x^k(1-x)^n}{(xs+(1-x)t)^{n+k+2}} \mathrm{d}x \\
&= \|f\|_{n+k+1} \frac{\partial^{n+k}}{\partial s^k \partial t^n} \frac{(-1)^{k+n}}{(k+n+1)!} \int_0^1 \frac{1}{(xs+(1-x)t)^2} \mathrm{d}x \\
&= \|f\|_{n+k+1} \frac{\partial^{n+k}}{\partial s^k \partial t^n} \frac{(-1)^{k+n}}{(k+n+1)!} \frac{1}{st} \\
&= \|f\|_{n+k+1} \frac{k! \cdot n!}{(k+n+1)!} \cdot \frac{1}{s^{k+1} \cdot t^{n+1}}.
\end{aligned}$$

Choosing $k = 0$ one sees $\delta_s f \in S^\infty$. That is the first part of the claim.

As for the second part, the same argument as above together with the representation

$$\langle E_n' \mu, \delta_s f \rangle = - \int_{(0,\infty)} t^n \int_0^1 (1-x)^n f^{(n+1)}(xs+(1-x)t) \mathrm{d}x \mu(\mathrm{d}t)$$

yields differentiability of $s \mapsto \langle E_n' \mu, \delta_s f \rangle$ and we finally estimate

$$\begin{aligned}
\left|s^{k+1} \frac{\mathrm{d}^k}{\mathrm{d}s^k} \langle E_n' \mu, \delta_s f \rangle\right| &\leq \|f\|_{n+k+1} \int_{(0,\infty)} s^{k+1} t^n \int_0^1 \frac{(1-x)^n x^k}{(xs+(1-x)t)^{n+k+2}} \mathrm{d}x \, |\mu| (\mathrm{d}t) \\
&\leq \frac{k! \cdot n!}{(n+k+1)!} \|f\|_{n+k+1} \cdot \|\mu\|_{S'}
\end{aligned}$$

which completes the entire proof. □

Remark 3.3.9.

1. A short calculation reveals

$$(\delta_s f)(t) = \begin{cases} -\frac{f(s)-f(t)}{s-t}, & s \neq t \\ -f'(t), & s = t. \end{cases}$$

   This is also the definition used in [30].

2. Let $a \geq 0$ and $f \in S$ be defined by $f(t) := \frac{1}{a+t}$. The definition of δ_s is inspired by the identity $(\delta_s f)(t) = f(s) \cdot f(t)$ which is the analog to τ_s applied to $t \mapsto \mathrm{e}^{-at}$.

Lemma 3.3.8 is the key to render the following definition meaningful.

Definition 3.3.10. Let $f \in S^\infty$, μ, $\nu \in S'$, and n, $m \in \mathbb{N}_0$. We define the *Stieltjes convolution* of the measures μ and ν by

$$\langle \mu * \nu, f\rangle := \big\langle \mu, s \mapsto \langle \nu, \delta_s f\rangle\big\rangle$$

or, more generally,

$$\langle E_n'\mu * E_m'\nu, f\rangle := \big\langle E_n'\mu, s \mapsto \langle E_m'\nu, \delta_s f\rangle\big\rangle$$

and extend again to the whole of $(S^\infty)'$ by linearity. So,

$$\left(\sum_{k=0}^{n} E_k'\mu_k\right) * \left(\sum_{l=0}^{m} E_l'\nu_l\right) = \sum_{k=0}^{n}\sum_{l=0}^{m} E_k'\mu_k * E_l'\nu_l.$$

A short calculation, for simplicity just performed for one summand of the above sum, yields

$$\langle E_k'\mu_k * E_l'\nu_l, f\rangle = -\int\limits_{(0,\infty)}\int\limits_{(0,\infty)} s^k t^l \int\limits_0^1 x^k(1-x)^l f^{(k+l+1)}(xs+(1-x)t)\mathrm{d}x\mu_k(\mathrm{d}s)\nu_l(\mathrm{d}t).$$

By the substitution $y := 1-x$ in the inner integral, one sees that $E_k'\mu_k * E_l'\nu_l = E_l'\nu_l * E_k'\mu_k$.

Modifying the proof of Lemma 3.2.6 slightly yields the tiny but useful generalisation.

Lemma 3.3.11. *Let $(X_n)_{n\in\mathbb{N}}$ be an inductive spectrum with inductive limit $\operatorname{ind}_{n\in\mathbb{N}} X_n =: X$ and assume that there is a bilinear, continuous mapping $* : X_n \times X_k \to X_r$ with $r \geq n+k$. This mapping can be extended to a bilinear, continuous mapping, again denoted by $*$, from $X \times X$ to X given by $(L_n x_n, L_k x_k) \mapsto L_r(x_n * x_k)$. This additional structure makes X a locally convex algebra.*

Remark 3.3.12. Compared to the situation from before, the grading of the resulting limit is lost. Or one may say it holds in a certain generalised sense. The author is not aware of other appearances of such a generalised grading in other contexts.

Lemma 3.3.13. *Let $\mu = \sum_{k=0}^n E_k'\mu_k \in (S^n)' \subseteq (S^\infty)'$ and $\nu = \sum_{l=0}^m E_l'\nu_l \in (S^m)' \subseteq (S^\infty)'$ with measures $\mu_0, \ldots, \mu_n, \nu_0, \ldots, \nu_m \in S'$. Then $\mu * \nu \in (S^{n+m+1})'$ and the mapping $* : (S^n)' \times (S^m)' \to (S^{n+m+1})'$ is continuous. Consequently, $(S^\infty)'$ is a locally convex algebra.*

Proof. Let $f \in S^\infty$. Choose for the measures $\mu_0, \ldots, \mu_n$ a function $h_1 \in C_0\big((0,\infty);\mathbb{C}^{n+1}\big)$ accordingly to Lemma 3.3.4 and similar for $\nu_0, \ldots, \nu_m$ a function $h_2 \in C_0\big((0,\infty);\mathbb{C}^{m+1}\big)$. Use these functions to define functions $h_{k+l+1} := (h_1)_k \vee (h_2)_l$, $0 \leq k \leq n$, $0 \leq l \leq m$ where $(h_1)_k \vee (h_2)_l$ denotes the maximum of the two functions. It holds that $h_{k+l+1} \in C_0\big((0,\infty)\big)$,

$0 < h_{k+l+1} \leq 1$ and $h_{k+l+1}(u) \geq h_{k+l+1}(s)h_{k+l+1}(t)$ for all $0 < s \leq u \leq t$, $0 \leq k \leq n$, and $0 \leq l \leq m$. Finally set $h := (h_0, h_1, \ldots, h_{n+m+1})$ where $h_0 \in C_0\big((0,\infty)\big)$ is actually arbitrary and estimate

$$\begin{aligned}
&|\langle \mu * \nu, f\rangle| \\
\leq &\sum_{k=0}^{n}\sum_{l=0}^{m} \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} s^k t^l \int\limits_0^1 x^k(1-x)^l \left|f^{(k+l+1)}\big(xs+(1-x)t\big)\right| \mathrm{d}x \, |\nu_l|\,(\mathrm{d}t)\,|\mu_k|\,(\mathrm{d}s) \\
\leq &\|f\|_{n+m+1,h} \sum_{k=0}^{n}\sum_{l=0}^{m} \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \int\limits_0^1 \frac{s^k t^l x^k (1-x)^l \mathrm{d}x \, |\nu_l|\,(\mathrm{d}t)\,|\mu_k|\,(\mathrm{d}s)}{\big(xs+(1-x)t\big)^{k+l+2} h_{k+l+1}\big(xs+(1-x)t\big)} \\
\leq &\|f\|_{n+m+1,h} \sum_{k=0}^{n}\sum_{l=0}^{m} \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \frac{k!\cdot l!}{(k+l+1)!} \frac{|\nu_l|\,(\mathrm{d}t)\,|\mu_k|\,(\mathrm{d}s)}{s h_{k+l+1}(s)\cdot t h_{k+l+1}(t)} \\
\leq &\|f\|_{n+m+1,h} \sum_{k=0}^{n}\sum_{l=0}^{m} \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \frac{k!\cdot l!}{(k+l+1)!} \frac{|\nu_l|\,(\mathrm{d}t)\,|\mu_k|\,(\mathrm{d}s)}{s (h_2)_l(s)\cdot t (h_1)_k(t)} \\
\leq &4n!\cdot m!\,\|f\|_{n+m+1,h} \sum_{k=0}^{n} \|\mu_k\|_{S'} \sum_{l=0}^{m} \|\nu_l\|_{S'}
\end{aligned}$$

which shows the result.

□

Remark 3.3.14. Alternatively, we could also have considered the space $\widetilde{S^\infty} := \{f \in C^\infty\big((0,\infty)\big) \mid \forall k \in \mathbb{N} : s \mapsto s^{k+1} f^{(k)}(s) \in C_0\big((0,\infty)\big\}$ equipped with the locally convex topology induced by the Banach space topology of $\{f \in C\big((0,\infty)\big) \mid s \mapsto sf(s) \in C_0\big((0,\infty)\big\}$. The space $\widetilde{S^\infty}$ has the same continuous functionals as S^∞ by the general theory of mixed topologies.

Definition 3.3.15. Let X be a LCS and $A \in \mathcal{M}(X)$. Set $\mathscr{A} := (S^\infty)'$ and identify the element $\mu = \sum_{k=0}^n E_k' \mu_k \in \mathscr{A}$ with $\mu \varepsilon 1_X : S^\infty \varepsilon X = S^\infty(X) \to X$. We define $\Phi : \mathscr{A} \to \mathcal{L}(X)$ by

$$\Phi(\mu)x := \langle \mu, t \mapsto (t+A)^{-1}x\rangle = \sum_{k=0}^{n} (-1)^k k! \int\limits_{(0,\infty)} t^k (t+A)^{-k-1} x \, \mu_k(\mathrm{d}t) \quad (x \in X).$$

Finally, we equip again $\mathcal{L}(X)$ with the strong topology β, cf. Definition 3.2.11.

Proposition 3.3.16. *For a LCS X and $A \in \mathcal{M}(X)$, the triple $(X, \mathscr{A}, \Phi)$ defines a bounded functional calculus.*

Proof. The proof will be almost the same as the one for Proposition 3.2.12. Let μ, $\nu \in (S^\infty)'$ with representations $\mu = \sum_{k=0}^n E_k' \mu_k$ and $\nu = \sum_{l=0}^m E_l' \nu_l$, respectively. Furthermore, let $x \in X$ and α, $\beta \in \mathbb{C}$. W.l.o.g. we may again assume $n \geq m$ and define $\nu_{m+1} = \cdots = \nu_n := 0$. Then

$$\begin{aligned}\Phi(\alpha\mu + \beta\nu)x &= \sum_{k=0}^n \sum_{k=0}^n \int\limits_{(0,\infty)} t^k (t+A)^{-k-1} x\, (\alpha\mu_k + \beta\nu_k)(\mathrm{d}t) \\ &= \alpha \sum_{k=0}^n \int\limits_{(0,\infty)} t^k (t+A)^{-k-1} x\, \mu_k(\mathrm{d}t) + \beta \sum_{l=0}^m \int\limits_{(0,\infty)} t^k (t+A)^{-k-1} x\, \nu_l(\mathrm{d}t) \\ &= \alpha\Phi(\mu)x + \beta\Phi(\nu)x\end{aligned}$$

again implies linearity while the multiplicativity follows this time from

$$\begin{aligned}\Phi(\mu * \nu)x &= \sum_{k=0}^n \sum_{l=0}^m \Big\langle \mu_k, s \mapsto s^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} (s+A)^{-1} \langle \nu_l, t \mapsto t^l \frac{\mathrm{d}^l}{\mathrm{d}t^l} (t+A)^{-1} x \rangle \Big\rangle \\ &= \sum_{k=0}^n (-1)^k k! \int\limits_{(0,\infty)} s^k (s+A)^{-1-k} \cdot \sum_{l=0}^m (-1)^l l! \int\limits_{(0,\infty)} t^l (t+A)^{-1-l} x\, \nu_l(\mathrm{d}t) \mu_k(\mathrm{d}s) \\ &= \Phi(\mu)\Phi(\nu)x.\end{aligned}$$

For the continuity we use again the criterion as before and check that Φ is already continuous from $(S^n)'$ to $\mathcal{L}(X)_\beta$. This is true since for every $\|\cdot\|_p \in \mathcal{P}_X$, by the equicontinuity of $\big(t(t+A)^{-1}\big)_{t>0}$, we find $\|\cdot\|_q \in \mathcal{P}_X$ and $M > 0$ such that

$$\|\Phi(\mu)x\|_p \leq n! M \|x\|_q \sum_{k=0}^n \|\mu_k\|_{S'}. \tag{3.3.1}$$

The proof is finished. □

Remark 3.3.17. One can combine the polynomial factors coming from the operators E_n and combine them with the corresponding measures. Then one can also write

$$\Phi(\mu)x = \sum_{k=0}^n (-1)^k k! \int\limits_{(0,\infty)} (A+t)^{-k-1} x\, \mu_k(\mathrm{d}t)$$

with measures $\mu_0, \ldots, \mu_n$ such that $\int_{(0,\infty)} t^{-k-1} |\mu_k| (\mathrm{d}t) < \infty$. This is the calculus as it was derived in [28].

In contrast to before, where it was quite clear in which direction one has to extend the calculus, the so far introduced *Stieltjes calculus* is generically bounded and no link towards the smoothness of the considered element $x \in X$ seems to appear. But for $A \in \mathcal{C}(X) \setminus \mathcal{L}(X)$ one surely does not expect the operator 'A^α', an object we finally aim to define, to be a continuous operator. In order to include it, we change now our perspective and identify the algebra $(S^\infty)'$ with its image

$$\mathcal{S} := \{f \in \mathcal{H}\big(\mathbb{C} \setminus (-\infty, 0]\big) \mid \exists \mu \in (S^\infty)' : f = \Phi(\mu)\}$$

which is a subalgebra of $\mathcal{H}\big(\mathbb{C} \setminus (-\infty, 0]\big)$ called the *Stieltjes algebra* which is indeed an algebra since Proposition 3.3.16 shows that the considered mapping is linear and multiplicative. In order to justify such an identification, we need to establish the injectivity of the mapping

$$\mathcal{S} : (S^\infty)' \to \mathcal{H}\big(\mathbb{C} \setminus (-\infty, 0]\big), \quad \mu \mapsto (\lambda \mapsto \langle \mu, t \mapsto \tfrac{1}{\lambda+t} \rangle).$$

Proposition 3.3.18. *The mapping*

$$\mathcal{S} : (S^\infty)' \to \mathcal{H}\big(\mathbb{C} \setminus (-\infty, 0]\big), \quad \mu \mapsto (\lambda \mapsto \langle \mu, t \mapsto \tfrac{1}{\lambda+t} \rangle)$$

is injective.

Proof. Let $\mu \in (S^\infty)'$ be such that $(\lambda \mapsto \langle \mu, t \mapsto \frac{1}{\lambda+t} \rangle) = 0$. We need to show that $\mu = 0$. We will make use of what we already know about the Laplace transform (see the proof of Proposition 3.2.18). One can consider μ as above as an element in $\mathscr{S}(\mathbb{R})'$ which admits a Laplace transform. The Laplace transform is again in $\mathscr{S}(\mathbb{R})'$ (extend to $(-\infty, 0]$ by 0) since for $f \in \mathscr{S}(\mathbb{R})$ one has

$$\begin{aligned}
\left| \int_0^\infty (L\mu)(\lambda) f(\lambda) \mathrm{d}\lambda \right| &= \left| \sum_{k=0}^n \int_0^\infty \int_{(0,\infty)} (-1)^k \lambda^k s^k \mathrm{e}^{-\lambda s} \mu_k(\mathrm{d}s) f(\lambda) \mathrm{d}\lambda \right| \\
&= \left| \sum_{k=0}^n \int_0^\infty \int_{(0,\infty)} \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \mathrm{e}^{-\lambda s} \mu_k(\mathrm{d}s) \lambda^k f(\lambda) \mathrm{d}\lambda \right| \\
&\le \max_{0 \le k \le n} \sup_{\lambda \in \mathbb{R}} \left| \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \lambda^k f(\lambda) \right| \sum_{k=0}^n \int_{(0,\infty)} \int_0^\infty \mathrm{e}^{-\lambda s} \mathrm{d}\lambda \, |\mu_k| \, (\mathrm{d}s) \\
&\le \max_{0 \le k \le n} \sup_{\lambda \in \mathbb{R}} \left| \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \lambda^k f(\lambda) \right| \sum_{k=0}^n \|\mu_k\|_{S'} .
\end{aligned}$$

Furthermore, it is supported on $[0,\infty)$. Taking Laplace transform of the Laplace transform of μ gives

$$\begin{aligned}(L^2\mu)(\lambda) &= \int_0^\infty (L\mu)(s)\mathrm{e}^{-\lambda s}\mathrm{d}s &&= \int_0^\infty \langle \mu, t\mapsto \mathrm{e}^{-st}\rangle \mathrm{e}^{-\lambda s}\mathrm{d}s \\ &= \langle \mu, \int_0^\infty \mathrm{e}^{-s(t+\lambda)}\mathrm{d}s\rangle &&= \langle \mu, t\mapsto \frac{1}{\lambda+t}\rangle \\ &= (S\mu)(\lambda) &&= 0\end{aligned}$$

Since the Laplace transform is injective, we conclude $\mu = 0$. $\square$

Functions like $\lambda \mapsto \lambda^\alpha$ for $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$ and $\lambda \mapsto (\lambda+t)^{-\alpha}$ for $\alpha \in \mathbb{C}$ and $t > 0$ are not contained in the Stieltjes algebra $\mathcal{S}$ but in the next chapter we will see that they are anchored, and therefore they are accessible by the maximal extension of the so far constructed calculus.

4. Fractional Powers of Linear Operators

In this chapter we finally define A^α for a non-negative operator A and $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$. The entire chapter and its content is this time inspired by [44, Chap. 4] where a well-working functional calculus (not perfectly in the sense of our definition but this is only a small drawback) for locally convex spaces is established.

4.1. Fractional powers of non-negative operators

Let us begin by studying some special functions contained in the Stieltjes algebra.

Lemma 4.1.1. *Let $\gamma \in (0, \frac{1}{2})$, $0 \leq \phi < (\frac{1}{2} - \gamma)\pi$, $z \in S_\phi \setminus \{0\}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, and $\alpha \in \mathbb{C}_{\mathrm{Re}>-1}$. Then we have*

1. $\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma} = \frac{1}{\pi} \int\limits_0^\infty t^\alpha \mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \sin\big(zt^\gamma \sin(\gamma\pi) - \alpha\pi\big) \frac{1}{t+\lambda} \mathrm{d}t,$

2. $0 = \frac{1}{\pi} \int\limits_0^\infty t^\alpha \mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \sin\big(zt^\gamma \sin(\gamma\pi) - \alpha\pi\big) \mathrm{d}t.$

Proof. The proof of both equalities is essentially nothing but the strategy already explained at the beginning of Section 3.3 and will be omitted. $\square$

For later use, we shall also study measures of the above kind in the special case $\alpha = 0$.

Lemma 4.1.2. *Let $\gamma \in (0, \frac{1}{2})$, $0 \leq \phi < (\frac{1}{2} - \gamma)\pi$ and $z \in S_\phi \setminus \{0\}$. Then*

$$\mu_z := \frac{1}{\pi} \mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \sin\big(zt^\gamma \sin(\gamma\pi)\big) \mathrm{d}t \in S' \subset (S^\infty)'.$$

Moreover, for every $0 \leq \phi < (\frac{1}{2} - \gamma)\pi$ there is $C > 0$ such that $\sup_{z \in S_\phi} \|\mu_z\|_{S'} \leq C$.

Proof. The first part is just Lemma 4.1.1 for the special case $\alpha = 0$. For the uniform boundedness, we start with a tiny observation. Assume that $\operatorname{Im} z \geq 0$. Then

$$\begin{aligned}\left|\mathrm{e}^{izt^\gamma \sin(\gamma\pi)} - \mathrm{e}^{-izt^\gamma \sin(\gamma\pi)}\right| &\leq \left|\mathrm{e}^{-izt^\gamma \sin(\gamma\pi)}\right| \cdot \Big|\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{2izs^\gamma \sin(\gamma\pi)}\mathrm{d}s\Big| \\ &\leq \mathrm{e}^{\operatorname{Im} z\cdot t^\gamma \sin(\gamma\pi)} \cdot 2\,|z|\sin(\gamma\pi)t^\gamma\end{aligned}$$

holds. Using this we calculate for all $z \in S_\phi$ with $\operatorname{Im} z \geq 0$

$$\begin{aligned}\|\mu_z\|_{S'} &= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{-\operatorname{Re} z\cdot t^\gamma \cos(\gamma\pi)}\left|\mathrm{e}^{izt^\gamma \sin(\gamma\pi)} - \mathrm{e}^{-izt^\gamma \sin(\gamma\pi)}\right|\frac{1}{t}\mathrm{d}t \\ &\leq \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\operatorname{Re} z\cdot t^\gamma \cos(\gamma\pi)}\mathrm{e}^{\operatorname{Im} z\cdot t^\gamma \sin(\gamma\pi)} \cdot |z|\sin(\gamma\pi)t^{\gamma-1}\mathrm{d}t.\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}\operatorname{Im} z \cdot \sin(\gamma\pi) &= \operatorname{Re} z \cdot \tan\big(\arg(z)\big)\cos\big((\tfrac{1}{2}-\gamma)\pi\big) \\ &\leq \operatorname{Re} z \cdot \sin(\phi) \cdot \underbrace{\frac{\cos\big((\frac{1}{2}-\gamma)\pi)\big)}{\cos(\phi)}}_{=:\varepsilon<1} \\ &\leq \operatorname{Re} z \cdot \varepsilon \cdot \cos(\gamma\pi)\end{aligned}$$

and therefore

$$\begin{aligned}\|\mu_z\|_{S'} &\leq \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-(1-\varepsilon)\operatorname{Re} z\cdot t^\gamma \cos(\gamma\pi)} \cdot |z|\sin(\gamma\pi)t^{\gamma-1}\mathrm{d}t \\ &\leq \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-(1-\varepsilon)\operatorname{Re} z\cdot t^\gamma \cos(\gamma\pi)} \cdot \frac{\operatorname{Re} z\cdot\cos(\gamma\pi)}{\cos(\phi)}\tan(\gamma\pi)t^{\gamma-1}\mathrm{d}t \\ &= \frac{\tan(\gamma\pi)}{\gamma\pi\cdot\cos(\phi)}\int_0^\infty \mathrm{e}^{-(1-\varepsilon)s}\mathrm{d}s \\ &= \frac{\tan(\gamma\pi)}{\gamma\pi\cdot\cos(\phi)\cdot(1-\varepsilon)}\end{aligned}\tag{4.1.1}$$

which establishes the claim for all $z \in S_\phi$ with $\operatorname{Im} z \geq 0$. The reasoning for $\operatorname{Im} z < 0$ is similar. □

Remark 4.1.3. Since $\gamma > 0$, we can send $\lambda \to 0$ in a proper subsector of $\mathbb{C} \setminus (-\infty, 0]$ and get with Lemma 4.1.1 $\int_{(0,\infty)} \frac{1}{t}\mu_z(\mathrm{d}t) = 1$.

Definition 4.1.4. Let X be a LCS and $A \in \mathcal{M}(X)$. Moreover, let $\gamma \in (0, \frac{1}{2})$, $0 \leq \phi < (\frac{1}{2} - \gamma)\pi$, and $z \in S_\phi$. Define the linear operator

$$X \ni x \mapsto \mathrm{e}^{-zA^\gamma}x := \Phi(\mu_z)x = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \sin\bigl(zt^\gamma \sin(\gamma\pi)\bigr)(t+A)^{-1}x \,\mathrm{d}t \qquad (4.1.2)$$

where μ_z is the measure from Lemma 4.1.2 and Φ is the Stieltjes calculus given in Definition 3.3.15.

Remark 4.1.5. The used notation already suggests that the generator of the semigroup will be the fractional power $-A^\gamma$. This result will be established to some extend in Section 4.2 once we introduced fractional powers. However, the definition of the fractional power will use the semigroup which is why it got introduced here. Alternatively, we could approach fractional powers via their resolvents which can be defined within the Stieltjes calculus. The author felt it to be more comfortable though to use the regularisation via semigroups.

The next proposition collects some basic properties of the family $(\mathrm{e}^{-zA^\gamma})_{z\in S_\phi}$. Before, let us generalise the notion of a *holomorphic*, also called *analytic*, semigroup. We shall call a semigroup $(\mathrm{e}^{-zA})_{z\in S_\omega}$ holomorphic if the mapping $\mathring{S}_\omega \ni z \mapsto \mathrm{e}^{-zA} \in \mathcal{L}(X)_\beta$ is holomorphic. Furthermore, let us introduce the space $\mathcal{D}(A^\infty) := \bigcap_{k\in\mathbb{N}} \mathcal{D}(A^k)$.

Proposition 4.1.6. *Let $\gamma \in (0, \frac{1}{2})$ and $0 \leq \phi < (\frac{1}{2} - \gamma)\pi$. The family $(\mathrm{e}^{-zA^\gamma})_{z\in S_\phi}$ has the following properties:*

1. *The family $(\mathrm{e}^{-zA^\gamma})_{z\in S_\phi}$ is an equicontinuous semigroup of linear operators.*
2. *The semigroup is holomorphic.*
3. *The semigroup is smoothing, i.e., $\forall z \in S_\phi \setminus \{0\} : \mathcal{R}(\mathrm{e}^{-zA^\gamma}) \subseteq \mathcal{D}(A^\infty)$.*
4. *The semigroup leaves $D := \overline{\mathcal{D}(A)}$ invariant and it is a holomorphic C_0-semigroup of angle ϕ if restricted to it.*

Proof.

1. The semigroup property follows from elementary algebra which yields for $z, w \in S_\phi$

$$\mathrm{e}^{-zA^\gamma}\mathrm{e}^{-wA^\gamma} = \Phi(a \mapsto \mathrm{e}^{-za^\gamma})\Phi(a \mapsto \mathrm{e}^{-wa^\gamma}) = \Phi(a \mapsto \mathrm{e}^{-(z+w)a^\gamma}) = \mathrm{e}^{-(z+w)A^\gamma}.$$

It is equicontinuous on every sector of S_ϕ with $0 \leq \phi < (\frac{1}{2}-\gamma)\pi$ because of Inequality (3.3.1) together with Lemma 4.1.2.

2. Differentiating the density $z \mapsto \frac{1}{\pi}\mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \sin\big(zt^\gamma \sin(\gamma\pi)\big)$ w.r.t. the variable $z \in \mathring{S}_\phi$ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{\pi}\mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \sin\big(zt^\gamma \sin(\gamma\pi)\big) &= \frac{-1}{\pi}\mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \sin\big(zt^\gamma \sin(\gamma\pi)\big)t^\gamma\cos(\gamma\pi)\\ &\quad+\frac{1}{\pi}\mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \cos\big(zt^\gamma \sin(\gamma\pi)\big)t^\gamma\sin(\gamma\pi)\end{aligned}$$

and one checks that both summands, considered as densities for the Lebesgue measure, give rise to measures in S'. Hence, by standard theorems on differentiating under the integral sign, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^{-zA^\gamma}x &= \int\limits_{(0,\infty)} \frac{-1}{\pi}\mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \sin\big(zt^\gamma \sin(\gamma\pi)\big)t^\gamma\cos(\gamma\pi)(t+A)^{-1}x\,\mathrm{d}t\\ &\quad+\int\limits_{(0,\infty)} \frac{1}{\pi}\mathrm{e}^{-zt^\gamma \cos(\gamma\pi)} \cos\big(zt^\gamma \sin(\gamma\pi)\big)t^\gamma\sin(\gamma\pi)(t+A)^{-1}x\,\mathrm{d}t\end{aligned}$$

3. First note that if $n \in \mathbb{N}_0$ and $x \in \mathcal{D}(A^n)$ one has $\mathrm{e}^{-zA^\gamma}x \in \mathcal{D}(A^{n+1})$ and, if $n \in \mathbb{N}$, $A\mathrm{e}^{-zA^\gamma}x = \mathrm{e}^{-zA^\gamma}Ax$. This follows from Lemma 3.1.8. That is, every application of the semigroup increases the smoothness of the considered element by lifting it from $\mathcal{D}(A^n)$ to $\mathcal{D}(A^{n+1})$. But now we can simply use the semigroup law and get for any $z \in S_\phi \setminus \{0\}$, $x \in X$, and $n \in \mathbb{N}$

$$\mathrm{e}^{-zA^\gamma}x = \big(\mathrm{e}^{-\frac{z}{n}A^\gamma}\big)^n x \in \mathcal{D}(A^n).$$

By arbitraryness of $n \in \mathbb{N}$ we conclude $\mathrm{e}^{-zA^\gamma}x \in \mathcal{D}(A^\infty)$.

4. By part 3 the semigroup maps even into $\mathcal{D}(A^\infty) \subseteq D$. It remains to show the strong continuity. We copy and adapt the strategy from [44, Thm. 5.5.1] and combine it with our estimates from the proof of Lemma 4.1.2. So let $\|\cdot\|_p \in \mathcal{P}_X$ and $x \in D$. There is $C > 0$ and $\|\cdot\|_q \in \mathcal{P}_X$ such that

$$\begin{aligned}&\big\|\mathrm{e}^{-zA^\gamma}x - \mathrm{e}^{-z}x\big\|_p\\ \leq&\frac{C}{\pi}\int\limits_0^\infty \mathrm{e}^{-\operatorname{Re} z\cdot t^\gamma\cdot\cos(\gamma\pi)}\frac{\big|\sin\big(zt^\gamma\sin(\gamma\pi)\big)\big|}{t}\cdot\underbrace{t\,\big\|(t+A)^{-1}x-(t+1)^{-1}x\big\|_q}_{=:H(t)}\,\mathrm{d}t.\end{aligned}$$

Since $x \in D$, one can tell that $\lim_{t\to\infty} H(t) = 0$, cf. Lemma 2.0.9. Now we use the same trick as in [44, Thm. 5.5.1] and combine it with our estimate (4.1.1). That is, we choose any $s > 0$ and use that there are $M, \varepsilon > 0$ and yet another $\|\cdot\|_r \in \mathcal{P}_X$ such that

$$\begin{aligned}\left\|\mathrm{e}^{-zA^\gamma}x - \mathrm{e}^{-z}x\right\|_p \leq & \frac{C}{\gamma\pi}\sup_{t>s} H(t) \cdot \frac{\tan(\gamma\pi)}{\cos(\phi)\cdot(1-\varepsilon)} \\ & + \frac{C}{\gamma\pi}(M+1)\|x\|_r \cdot \frac{\tan(\gamma\pi)}{\cos(\phi)\cdot(1-\varepsilon)} \cdot \left(1 - \mathrm{e}^{-(1-\varepsilon)\operatorname{Re} z\cdot s^\gamma\cdot\cos(\gamma\pi)}\right).\end{aligned}$$

Choosing $s > 0$ suffienctly large, the first term becomes arbitrary small while the second term converges to 0 as $z \to 0$ in S_ϕ for fixed $s > 0$. This shows the claim. □

Remark 4.1.7. With some effort further properties, also already concerning fractional powers, could be proved at this point. We refrain from doing so at this stage since later on a unified definition for A^α, $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$, will be given and used to prove properties.

Let us deduce two corollaries.

Corollary 4.1.8. *It even holds that* $D = \overline{\mathcal{D}(A^\infty)}$.

Corollary 4.1.9. *The operator* e^{-zA^γ} *is injective.*

Proof. Let $z \in \mathbb{C}$, $z \neq 0$, $|\arg(z)| < (\frac{1}{2} - \gamma)\pi$, and $x \in \mathcal{N}(\mathrm{e}^{-zA^\gamma})$. By standard results on semigroups, we have

$$\forall n \in \mathbb{N}_0 : \frac{\mathrm{d}^n}{\mathrm{d}z^n}\mathrm{e}^{-zA^\gamma}x = (-A^\gamma)^n \mathrm{e}^{-zA^\gamma}x = 0.$$

Since the domain of the function $z \mapsto \mathrm{e}^{-zA^\gamma}x$ is connected, it must be constantly 0. We finish our conclusions by

$$(1+A)^{-1}x = \lim_{z\to 0,\, z\in S_\phi} \mathrm{e}^{-zA^\gamma}(1+A)^{-1}x = 0$$

which means $x = 0$ as was to be shown. □

We finally arrived at the point to define fractional powers. Before, let us agree on the suggestive notation $\mathrm{e}^{zA^\gamma} = \left(\mathrm{e}^{-zA^\gamma}\right)^{-1}$ for the inverse of the semigroup operators. Furthermore, let us relax the existing notation to $\Phi(\lambda \mapsto f(\lambda)) =: f(A)$ and identify f with $f(\lambda)$ for certain functions such as powers and the exponential. For example, this agreement would imply $\mathrm{e}^{-zA^\gamma} = (\mathrm{e}^{-z\lambda^\gamma})(A)$. The variable λ will always designate the position of the operator which we plug in the function.

Lemma 4.1.10. *Let $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$ be fixed and consider for $\gamma \in (0, \frac{1}{2})$, $0 \leq \phi < (\frac{1}{2} - \gamma)\pi$ and $z \in S_\phi \setminus \{0\}$ the operators*

$$B_{\gamma,z} := \mathrm{e}^{zA^\gamma}(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma})(A)$$

where we used Lemma 4.1.1 for the definition of the term $(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma})(A)$. Then

$$\forall \gamma, \delta \in (0, \tfrac{1}{2}),\ 0 \leq \phi < \min\{(\tfrac{1}{2} - \gamma)\pi, (\tfrac{1}{2} - \delta)\pi\},\ z, w \in S_\phi \setminus \{0\} :\ B_{\gamma,z} = B_{\delta,w}$$

Proof. Let us show one inclusion. The other one follows in the same manner. So let $x \in \mathcal{D}(B_{\gamma,z})$. By definition this means

$$\mathrm{e}^{-zA^\gamma} B_{\gamma,z} x = (\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma})(A)x$$

which in turn implies

$$\begin{aligned}
\mathrm{e}^{-zA^\gamma} \mathrm{e}^{-wA^\delta} B_{\gamma,z} x &= \mathrm{e}^{-wA^\delta} \mathrm{e}^{-zA^\gamma} B_{\gamma,z} x \\
&= \mathrm{e}^{-wA^\delta} (\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma})(A)x \\
&= (\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma} \mathrm{e}^{-w\lambda^\delta})(A)x \\
&= \mathrm{e}^{-zA^\gamma} (\lambda^\alpha \mathrm{e}^{-w\lambda^\delta})(A)x.
\end{aligned}$$

Hence,

$$\mathrm{e}^{-wA^\delta} B_{\gamma,z} x = (\lambda^\alpha \mathrm{e}^{-w\lambda^\delta})(A)x.$$

But the last equality tells us

$$x \in \mathcal{D}(B_{\delta,w}) \quad \text{and} \quad B_{\delta,w} x = B_{\gamma,z} x,$$

i.e., $B_{\gamma,z} \subseteq B_{\delta,w}$. $\square$

Based on the last lemma we can give the following definition.

Definition 4.1.11. Let X be a LCS, $A \in \mathcal{M}(X)$ and $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$. We define the *fractional power* A^α of the operator A to be the closed operator

$$A^\alpha := \mathrm{e}^{zA^\gamma} \big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)$$

where $\gamma \in (0, \frac{1}{2})$, $0 \leq \phi < (\frac{1}{2} - \gamma)\pi$, and $z \in S_\phi \setminus \{0\}$ are arbitrary.

The reader should note that the given definition implies

$$\forall x \in \mathcal{D}(A^\alpha) :\ \mathrm{e}^{-zA^\gamma} A^\alpha x = \big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)x.$$

Remark 4.1.12. To the best of the authors knowledge nobody yet tried using the semigroups of the fractional powers as regularisers. This choice is motivated by a need but also by an advantage. As for the need, one simply cannot use resolvents (at least those of the original operator A) because they have too strong singularities on the negative real axis. When altering the paths of integration, these singularities will prevent us from finding convergent integral expressions. On the bright side, the exponential regularisers drop of faster than polynomials which is why they allow to anchor even more functions.

We will now introduce another integral representation which is a small generalisation of the one originally used by Balakrishnan in [5] to introduce fractional powers of non-negative operators, also cf. [44, Prop. 3.1.3].

Proposition 4.1.13. *Let $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$ and $n \in \mathbb{N}$ such that $n > \operatorname{Re}\alpha$. Moreover, let $x \in \mathcal{D}(A^n)$. Then $x \in \mathcal{D}(A^\alpha)$ and the Balakrishnan formula holds which reads*

$$A^\alpha x = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty t^{\alpha-1} A^n (t+A)^{-n} x \, \mathrm{d}t. \tag{4.1.3}$$

Proof. As before, choose $\gamma \in (0, \frac{1}{2})$ and $0 \le \phi < (\frac{1}{2} - \gamma)\pi$, and consider $z \in S_\phi \setminus \{0\}$. Let $\varepsilon > 0$. Then $z + \varepsilon \in S_\phi \setminus \{0\}$ and, using Lemma 4.1.1, dominated convergence yields

$$\forall x \in X : \lim_{\varepsilon \to 0+} \big(\lambda^\alpha \mathrm{e}^{-(z+\varepsilon)\lambda^\gamma}\big)(A)x = \big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)x.$$

Another thing which follows from the representation in Lemma 4.1.1 is that the mapping $\alpha \mapsto \big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)x$ is holomorphic in the open right halfplane $\mathbb{C}_{\mathrm{Re}>0}$.

A calculation gives

$$\begin{aligned}
\big(\lambda^\alpha \mathrm{e}^{-(z+\varepsilon)\lambda^\gamma}\big)(A)x &= \mathrm{e}^{-zA^\gamma} \big(\lambda^\alpha \mathrm{e}^{-\varepsilon\lambda^\gamma}\big)(A)x \\
&= \mathrm{e}^{-zA^\gamma} \cdot \frac{1}{\pi} \int_0^\infty t^\alpha \mathrm{e}^{-\varepsilon t^\gamma \cos(\gamma\pi)} \sin\big(\varepsilon t^\gamma \sin(\gamma\pi) - \alpha\pi\big)(t+A)^{-1} x \, \mathrm{d}t \\
&= \frac{1}{\pi} \int_0^\infty t^\alpha \mathrm{e}^{-\varepsilon t^\gamma \cos(\gamma\pi)} \sin\big(\varepsilon t^\gamma \sin(\gamma\pi) - \alpha\pi\big)(t+A)^{-1} \mathrm{e}^{-zA^\gamma} x \, \mathrm{d}t.
\end{aligned}$$

Assume now for the moment even $n - 1 < \operatorname{Re}\alpha < n$. Under this hypothesis a repeated application of Lemma 4.1.1 gives us

$$\begin{aligned}
\big(\lambda^\alpha \mathrm{e}^{-(z+\varepsilon)\lambda^\gamma}\big)(A)x &= \frac{1}{\pi}\int_0^\infty t^{\alpha-1}\mathrm{e}^{-\varepsilon t^\gamma\cos(\gamma\pi)}\sin\big(\varepsilon t^\gamma \sin(\gamma\pi)-\alpha\pi\big)t(t+A)^{-1}\mathrm{e}^{-zA^\gamma}x\,\mathrm{d}t\\
&= \frac{1}{\pi}\int_0^\infty t^{\alpha-1}\mathrm{e}^{-\varepsilon t^\gamma\cos(\gamma\pi)}\sin\big(\varepsilon t^\gamma \sin(\gamma\pi)-\alpha\pi\big)\mathrm{e}^{-zA^\gamma}(x-A(t+A)^{-1}x)\,\mathrm{d}t\\
&= \frac{-1}{\pi}\int_0^\infty t^{\alpha-1}\mathrm{e}^{-\varepsilon t^\gamma\cos(\gamma\pi)}\sin\big(\varepsilon t^\gamma \sin(\gamma\pi)-\alpha\pi\big)A(t+A)^{-1}\mathrm{e}^{-zA^\gamma}x\,\mathrm{d}t\\
&= \frac{(-1)^n}{\pi}\int_0^\infty t^{\alpha-n}\mathrm{e}^{-\varepsilon t^\gamma\cos(\gamma\pi)}\sin\big(\varepsilon t^\gamma \sin(\gamma\pi)-\alpha\pi\big)(t+A)^{-1}A^n\mathrm{e}^{-zA^\gamma}x\,\mathrm{d}t.
\end{aligned}$$

We now may send $\varepsilon \to 0+$ and conclude

$$\big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)x = \frac{(-1)^{n-1}\sin(\alpha\pi)}{\pi}\int_0^\infty t^{\alpha-n}(t+A)^{-1}A^n\mathrm{e}^{-zA^\gamma}x\,\mathrm{d}t.$$

Note that existence of the appearing integral in the last line follows from $A \in \mathcal{M}(X)$ and the assumption on $\operatorname{Re}\alpha$. In order to verify this, the integral needs to split at any number $R > 0$ and both integrals can then be discussed separately, also cf. [44, Def. 3.1.1]. The argument will be discussed in the next proposition anyway and shall not be our major concern at this point. To finish the proof of the proposition, note that integration by parts is possible and repeating this process again $n-1$ times gives

$$\begin{aligned}
\big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)x &= \frac{(n-1)!(-1)^{n-1}\sin(\alpha\pi)}{\pi(\alpha-n+1)(\alpha-n+2)\dots(\alpha-1)}\int_0^\infty t^{\alpha-1}(t+A)^{-n}A^n\mathrm{e}^{-zA^\gamma}x\,\mathrm{d}t\\
&= \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^\infty t^{\alpha-1}(t+A)^{-n}A^n\mathrm{e}^{-zA^\gamma}x\,\mathrm{d}t. \qquad (4.1.4)
\end{aligned}$$

Note that we used Euler's formula of complements

$$\frac{\sin(\alpha\pi)}{\pi} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$$

([1, 5.5.1]) for the prefactors. The expression on the right-hand side of Equation (4.1.4), considered as a function of α, is holomorphic on the entire strip $\mathbb{C}_{0<\mathrm{Re}<n}$ and it coincides with the left-hand side for $n-1 < \operatorname{Re}\alpha < n$. Hence, the assumption $n-1 < \operatorname{Re}\alpha < n$

can be dropped and the proposition is established. $\square$

As a direct application of the last proposition we get $\mathrm{e}^{-zA^\gamma}x \in \mathcal{D}(A^\alpha)$ and

$$A^\alpha \mathrm{e}^{-zA^\gamma}x = \mathrm{e}^{zA^\gamma}\big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)\mathrm{e}^{-zA^\gamma}x = \big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)x.$$

In particular $A^\alpha \mathrm{e}^{-zA^\gamma} \in \mathcal{L}(X)$.

The next proposition is commonly called Moment Inequality and can be used to see, among other things of interest, that the integral representation given by the Balakrishnan formula defines a closable operator. It also reminds one of the connections between the theory of fractional powers and interpolation theory and is therefore of independent interest.

Proposition 4.1.14. *Let $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$ and $n \in \mathbb{N}$ such that $n > \operatorname{Re}\alpha$. Then for every $\|\cdot\|_p \in \mathcal{P}_X$ there are $C > 0$ and $\|\cdot\|_q \in \mathcal{P}_X$ such that*

$$\forall x \in \mathcal{D}(A^n) : \; \|A^\alpha x\|_p \le C \|x\|_q^{1-\frac{\operatorname{Re}\alpha}{n}} \cdot \|A^n x\|_q^{\frac{\operatorname{Re}\alpha}{n}} .$$

Proof. The proof is the same as in the situation of X being a Banach space. So let $\|\cdot\|_p$ be given. Choose any $R > 0$. There are constants $C_{n,p}$ and $D_{n,p}$ as well as continuous seminorms $\|\cdot\|_{q_1}$ and $\|\cdot\|_{q_2}$ such that

$$\sup_{t>0} \left\|A^n(t+A)^{-n}x\right\|_p \le C_{n,p}\|x\|_{q_1} \quad \text{and} \quad \sup_{t>0}\left\|t^n(t+A)^{-n}x\right\|_p \le D_{n,p}\|x\|_{q_2}$$

because of the equicontinuity of the families $\big(A(t+A)^{-1}\big)_{t>0}$ and $\big(t(t+A)^{-1}\big)_{t>0}$. Choose $\|\cdot\|_q \in \mathcal{P}_X$ such that $\|\cdot\|_q \ge \max\{\|\cdot\|_{q_1}, \|\cdot\|_{q_2}\}$ which is possible since the system is assumed to be directed. Then there is $C > 0$ such that for all $R > 0$

$$\begin{aligned}\|A^\alpha x\|_p &\le \frac{\Gamma(n)}{|\Gamma(\alpha)\Gamma(n-\alpha)|} \cdot \Big\{C_{n,p}\|x\|_q \int_0^R t^{\operatorname{Re}\alpha - 1}\mathrm{d}t + D_{n,p}\|A^n x\|_q \int_R^\infty t^{\operatorname{Re}\alpha-1-n}\mathrm{d}t\Big\}\\ &\le C\Big(R^{\operatorname{Re}\alpha}\|x\|_q + R^{\operatorname{Re}\alpha - n}\|A^n x\|_q\Big).\end{aligned}$$

If $\|x\|_q = 0$, one could send $R \to \infty$ and gets that $\|A^\alpha x\|_p = 0$ as well. So in this situation the statement is true for any $C > 0$. Otherwise, the derived expression has a minimum for $R = \Big(\frac{\|A^n x\|_q}{\|x\|_q}\Big)^{\frac{1}{n}}$ and the statement follows as well. $\square$

Corollary 4.1.15. *Let $t > 0$ and $n \in \mathbb{N}$ such that $n > \operatorname{Re}\alpha$. Then $A^\alpha(t+A)^{-n}$ is continuous.*

Proof. This directly follows from the Moment inequality since we get for any $\|\cdot\|_p \in \mathcal{P}_X$

$$\left\|A^\alpha(t+A)^{-n}x\right\|_p \leq C\left\|(t+A)^{-n}x\right\|_q^{1-\frac{\operatorname{Re}\alpha}{n}} \cdot \left\|A^n(t+A)^{-n}x\right\|_q^{\frac{\operatorname{Re}\alpha}{n}}.$$

for some suitable $\|\cdot\|_q \in \mathcal{P}_X$. ☐

For later use let us define the *Balakrishnan operator*, denoted by J^α with domain $\mathcal{D}(J^\alpha) := \mathcal{D}(A^n)$, $n \in \mathbb{N}$, $n > \operatorname{Re}\alpha$, to be the restriction

$$J^\alpha := A^\alpha|_{\mathcal{D}(A^n)}$$

Let us now talk about properties we expect fractional powers to have. In order to prove a first one, namely a power law and in particular the fact that the domains of fractional powers are nested, we will need the following lemma which says that the operator $A(t+A)^{-1}$ does not influence the 'smoothness' of an element $x \in X$.

Lemma 4.1.16. *Let $t > 0$, $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then we have*

$$A^n(t+A)^{-n}x \in \mathcal{D}(A^m) \Leftrightarrow x \in \mathcal{D}(A^m).$$

Proof. Assume $A^n(t+A)^{-n}x \in \mathcal{D}(A^m)$. This means in particular that $A^k(t+A)^{-n}x \in \mathcal{D}(A^m)$ for $k \in \{0,\dots,n\}$. Having this in mind we conclude

$$x = (A+t)^n(t+A)^{-n}x = \sum_{k=0}^{n}\binom{n}{k}t^{n-k}\underbrace{A^k(t+A)^{-n}x}_{\in\mathcal{D}(A^m)} \in \mathcal{D}(A^m).$$

Conversely, if $x \in \mathcal{D}(A^m)$, the bounded operator $A^n(t+A)^{-n}$ commutes with all resolvent operators and we therefore get with $y = (1+A)^m x$

$$A^n(t+A)^{-n}x = A^n(t+A)^{-n}(1+A)^{-m}y = (1+A)^{-m}A^n(t+A)^{-n}y \in \mathcal{D}(A^m).$$

☐

Another ingredient we need is the fact that we might not use resolvents to introduce A^α but they can be used to characterise it.

Lemma 4.1.17. *Let $t > 0$ and $n \in \mathbb{N}$ such that $n > \operatorname{Re}\alpha$. Then $x \in \mathcal{D}(A^\alpha)$ if and only if $A^\alpha(t+A)^{-n}x \in \mathcal{D}(A^n)$. In this situation $(t+A)^n A^\alpha (t+A)^{-n}x = A^\alpha x$*

Proof. Using the fact that $A^\alpha \mathrm{e}^{-zA^\gamma}$ is a continuous operator which commutes with resolvents and which equals $\mathrm{e}^{-zA^\gamma}A^\alpha$ on $\mathcal{D}(A^\alpha)$, one establishes $(t+A)^{-n}A^\alpha \subseteq A^\alpha(t+A)^{-n}$. This being said let $x \in \mathcal{D}(A^\alpha)$. Then

$$A^\alpha x = (t+A)^n(t+A)^{-n}A^\alpha x = (t+A)^n \underbrace{A^\alpha(t+A)^{-n}x}_{\in\mathcal{D}(A^n)}$$

while conversely we can define $y := (t+A)^n A^\alpha (t+A)^{-n}x$ and find, bringing $(t+A)^n$ on the left-hand side and applying e^{-zA^γ},

$$(t+A)^{-n}A^\alpha \mathrm{e}^{-zA^\gamma}x = (t+A)^{-n}\mathrm{e}^{-zA^\gamma}y,$$

i.e., $y = A^\alpha x$. □

We technically did not show yet the exptected fact that

$$\forall k \in \mathbb{N}_0 : \ \mathrm{e}^{zA^\gamma}\big(\lambda^k \mathrm{e}^{-z\lambda^\gamma}\big)(A) = A^k$$

where we interpret the right-hand side in its usual meaning as k-folded application of A.

Corollary 4.1.18. *It holds that*

$$\forall k \in \mathbb{N}_0 : \ \mathrm{e}^{zA^\gamma}\big(\lambda^k \mathrm{e}^{-z\lambda^\gamma}\big)(A) = A^k.$$

Proof. Let $x \in X$ and $k \in \mathbb{N}_0$. If $k \in \mathbb{N}$, we calculate, using $t(t+A)^{-1} = 1 - A(t+A)^{-1}$ and Lemma 4.1.1 k times,

$$\begin{aligned}
\big(\lambda^k \mathrm{e}^{-z\lambda^\gamma}\big)(A)x &= \frac{1}{\pi}\int_0^\infty t^k \mathrm{e}^{-zt^\gamma\cos(\gamma\pi)}\sin\big(zt^\gamma\sin(\gamma\pi) - k\pi\big)(t+A)^{-1}x\,\mathrm{d}t \\
&= \frac{1}{\pi}A\int_0^\infty t^{k-1}\mathrm{e}^{-zt^\gamma\cos(\gamma\pi)}\sin\big(zt^\gamma\sin(\gamma\pi) - (k-1)\pi\big)(t+A)^{-1}x\,\mathrm{d}t \\
&= \frac{1}{\pi}A^k\int_0^\infty \mathrm{e}^{-zt^\gamma\cos(\gamma\pi)}\sin\big(zt^\gamma\sin(\gamma\pi)\big)(t+A)^{-1}x\,\mathrm{d}t \\
&= A^k \mathrm{e}^{-zA^\gamma}x.
\end{aligned}$$

Let now $x \in \mathcal{D}(A^k)$. Then

$$\big(\lambda^k \mathrm{e}^{-z\lambda^\gamma}\big)(A)x = A^k \mathrm{e}^{-zA^\gamma} x = \mathrm{e}^{-zA^\gamma} A^k x,$$

i.e., $A^k \subseteq \mathrm{e}^{zA^\gamma}\big(\lambda^k \mathrm{e}^{-z\lambda^\gamma}\big)(A)$ while conversely we use Lemma 4.1.17 and get

$$\mathrm{e}^{zA^\gamma}\big(\lambda^k \mathrm{e}^{-z\lambda^\gamma}\big)(A)x = (t+A)^n \mathrm{e}^{zA^\gamma} A^k \mathrm{e}^{-zA^\gamma}(t+A)^{-n}x = (t+A)^n A^k (t+A)^{-n}x = A^k x$$

which proves the claim. $\square$

Proposition 4.1.19. *Let $A \in \mathcal{M}(X)$ and $\alpha, \beta \in \mathbb{C}_{\mathrm{Re}>0}$. Then $A^{\alpha+\beta} = A^\alpha A^\beta$. In particular, $\mathcal{D}(A^\alpha) \subseteq \mathcal{D}(A^\beta)$ whenever $\operatorname{Re}\alpha > \operatorname{Re}\beta$.*

Proof. From the general properties of the functional calculus the inclusion $A^\alpha A^\beta \subseteq A^{\alpha+\beta}$ follows. The converse inclusion is of interest. So let $x \in \mathcal{D}(A^{\alpha+\beta})$. Choose $n \in \mathbb{N}$, $n > \max\{\operatorname{Re}\alpha, \operatorname{Re}\beta\}$. Then

$$A^n(1+A)^{-n}A^\beta(1+A)^{-2n}x = A^{n-\alpha}(1+A)^{-n}A^{\beta+\alpha}(1+A)^{-2n}x \in \mathcal{D}(A^{2n})$$

which, by Lemma 4.1.16, yields $A^\beta(1+A)^{-2n}x \in \mathcal{D}(A^{2n})$, i.e., $x \in \mathcal{D}(A^\beta)$. Furthermore, $A^\beta(1+A)^{-2n}x \in \mathcal{D}(A^{2n}) \subseteq \mathcal{D}(A^\alpha)$ and

$$A^\alpha A^\beta(1+A)^{-2n}x = A^\alpha(1+A)^{-2n}A^\beta x.$$

Therefore, $(1+A)^{-2n}x \in \mathcal{D}(A^\alpha A^\beta) \subseteq \mathcal{D}(A^{\alpha+\beta})$ and

$$A^\alpha A^\beta(1+A)^{-2n}x = A^{\alpha+\beta}(1+A)^{-2n}x = (1+A)^{-2n}A^{\alpha+\beta}x$$

where the last equality used again $x \in \mathcal{D}(A^{\alpha+\beta})$. Putting things together we have

$$A^\alpha(1+A)^{-2n}A^\beta x = (1+A)^{-2n}A^{\alpha+\beta}x \in \mathcal{D}(A^{2n})$$

which means $A^\beta x \in \mathcal{D}(A^\alpha)$ and $A^\alpha A^\beta x = A^{\alpha+\beta}x$. $\square$

The reader may note that another consequence of the nested domain property is an interpolation of Lemma 2.0.9. We namely even have for all $\alpha \in \mathbb{C}_{\mathrm{Re}>0} : \overline{\mathcal{D}(A^\alpha)} = D$.

Let us come to another application of Lemma 4.1.17. In fact we can actually use it to show that fractional powers defined by the calculus introduced in this work are the same as fractional powers introduced by the sectorial functional calculus as it is discussed thoroughly in [25]. The basic ideas from the Banach space theory can be readily transferred to the locally convex setting.

Proposition 4.1.20. *Let $A \in \mathcal{S}(X)$ and $\phi \in [0,\pi)$ such that $\sigma(A) \subseteq S_\phi$. Choose $n \in \mathbb{N}$, $n > \operatorname{Re}\alpha$ and $\theta \in (\phi,\pi)$. Then we define*

$$\big(\lambda^\alpha(1+\lambda)^{-n}\big)(A)x := \frac{1}{2\pi i}\int\limits_{\partial S_\theta} \frac{w^\alpha}{(1+w)^n}(w-A)^{-1}x\,\mathrm{d}w$$

where the boundary ∂S_θ is orientated from $+i\infty$ towards $-i\infty$. One has $\big(\lambda^\alpha(1+\lambda)^{-n}\big)(A)x \in \mathcal{D}(A^n)$ if and only if $x \in \mathcal{D}(A^\alpha)$. In this situation,

$$(1+A)^n\big(\lambda^\alpha(1+\lambda)^{-n}\big)(A)x = A^\alpha x.$$

Proof. A minor generalisation of Proposition 2.0.3 yields that for any $y \in X$ we have $w \mapsto (w-A)^{-1}y \in \mathcal{H}(\mathbb{C}\setminus S_\phi\,; X)$. Let $x \in X$ and choose numbers $r, R > 0$, $\psi, \theta \in (\phi,\pi)$, $\psi > \theta$, $\gamma \in (0,\frac{1}{2})$, and $z \in \mathbb{C}\setminus\{0\}$ with $|\arg(z)| < (\frac{1}{2}-\gamma)\pi$. By holomorphy

$$0 = \frac{1}{2\pi i}\oint\limits_{C} w^\alpha \mathrm{e}^{-zw^\gamma}(w-A)^{-1}x\,\mathrm{d}w$$

with the cycle (a formal integer combination of a finite number of paths) C be given as indicated in Figure 4.1.1.

Since this result does not depend on the chosen parameters, we can take the limits already indicated in Figure 4.1.1 and find

$$\big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)x = \frac{1}{2\pi i}\int\limits_{\partial S_\theta} w^\alpha \mathrm{e}^{-zw^\gamma}(w-A)^{-1}x\,\mathrm{d}w.$$

One may multiply this equation with $(1+A)^{-n}$ from the left and use the resolvent identity

$$(1+A)^{-1}(w-A)^{-1} = (1+w)^{-1}\big((w-A)^{-1} + (1+A)^{-1}\big)$$

n times under the integral sign on the right-hand side (the integral for the second summand

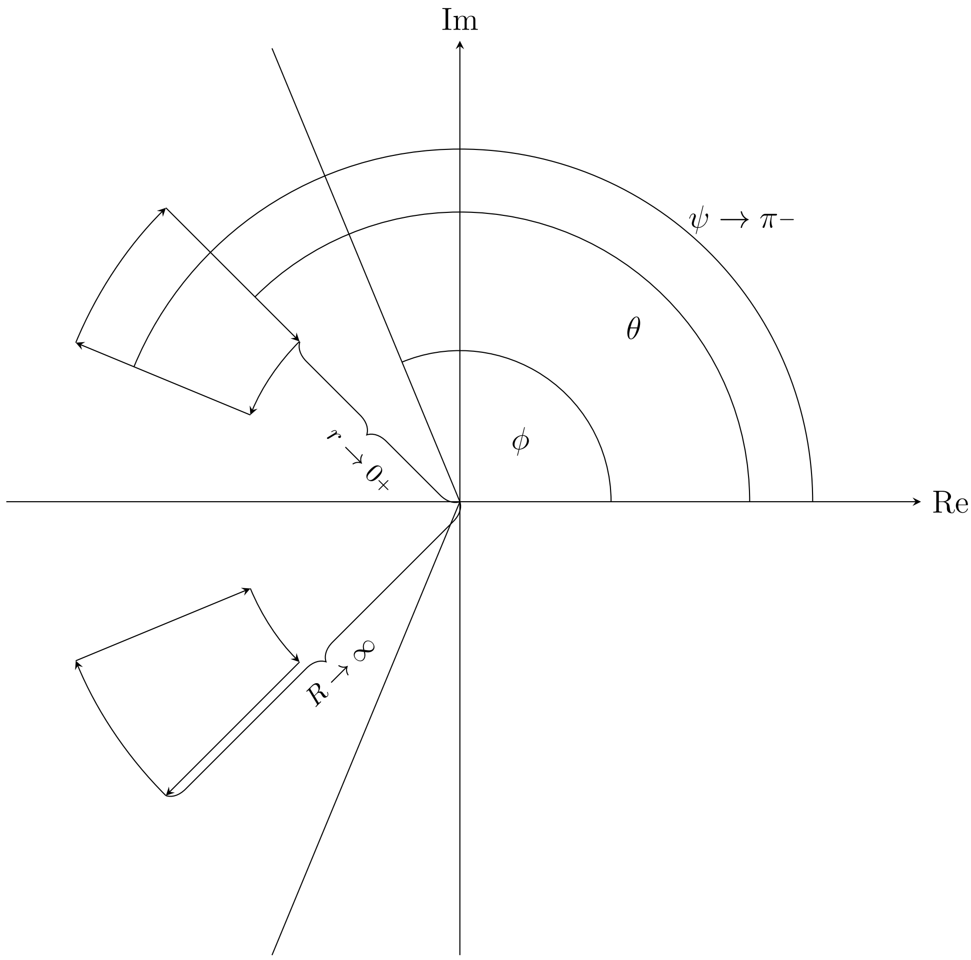

Figure 4.1.1.: Integration cycle C

is always 0) to obtain

$$(1+A)^{-n}\big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)x = \big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)(1+A)^{-n}x = \mathrm{e}^{-zA^\gamma}A^\alpha(1+A)^{-n}x$$
$$= \frac{1}{2\pi i}\int\limits_{\partial S_\theta} \frac{w^\alpha}{(1+w)^n}\mathrm{e}^{-zw^\gamma}(w-A)^{-1}x\,\mathrm{d}w.$$

We may send $z \to 0$ which gives

$$A^\alpha(1+A)^{-n}x = \big(\lambda^\alpha(1+\lambda)^{-n}\big)(A)x.$$

The claimed equivalence follows now from this together with Lemma 4.1.17. $\square$

Remark 4.1.21. For later use we shall also add here that the operator $-A^{\frac{1}{2}}$ generates a semigroup given by

$$\mathrm{e}^{-sA^{\frac{1}{2}}} = \frac{2}{\pi s^2} \int_0^\infty \big(\sin(s\sqrt{t}) - s\sqrt{t}\cos(s\sqrt{t})\big)(t+A)^{-2}x\,\mathrm{d}t$$

where $x \in X$ and $s > 0$ (see [44, Thm. 5.5.2]). In contrast to the semigroups generated by the lower powers, this semigroup is in general not holomorphic unless A is properly sectorial of some angle $\omega \in [0, \pi)$. In this case $A^{\frac{1}{2}} \in \mathcal{S}_{\frac{\omega}{2}}$ which can be seen from the sectorial calculus of A (cf. Proposition 4.1.20). This result will play a role in the first part of the next chapter.

Our next goal is to establish the main reason why Balakrishnan's original closure definition of A^α does not suffice in general. We start with a lemma.

Lemma 4.1.22. *Let $x \in \mathcal{D}(A^\alpha)$ and assume $y := A^\alpha x \in D$. Then $x \in \mathcal{D}\big((A_D)^\alpha\big)$ and $(A_D)^\alpha x = y$.*

Proof. By Lemma 2.0.24 the operator A_D is non-negative. Hence, an expression like $(A_D)^\alpha$ makes sense. For every element $x \in D$ and any $t > 0$, we have $(t+A)^{-1}x = (t+A_D)^{-1}x$ and for this reason the bounded Stieltjes calculi of A and A_D agree on the space D. In particular,

$$\mathrm{e}^{-z(A_D)^\gamma} y = \mathrm{e}^{-zA^\gamma} y = \big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A)x = \big(\lambda^\alpha \mathrm{e}^{-z\lambda^\gamma}\big)(A_D)x.$$

So $x \in \mathcal{D}\big((A_D)^\alpha\big)$ and $(A_D)^\alpha x = y$. □

The lemma actually even shows $(A_D)^\alpha = A_D^\alpha$ where the expression on the right-hand side means the part of the operator A^α in the subspace D. In the following we will use the notation without parentheses.

Proposition 4.1.23. *The operator J^α is closable and $\overline{J^\alpha} = A_D^\alpha$.*

Proof. Let $n \in \mathbb{N}$, $n > \operatorname{Re}\alpha$. The first part follows since we already know that $J^\alpha x = A^\alpha_{|\mathcal{D}(A^n)} x$ and A^α is closed by construction. So J^α is closable as restriction of a closed operator.

Let $x \in \mathcal{D}(\overline{J^\alpha})$ and let (x_κ) in $\mathcal{D}(A^n) \subseteq D$ be a net convergent towards $x \in D$ such that $J^\alpha x_\kappa = A^\alpha x_\kappa \to A^\alpha x \in D$ (the last statement follows since $A^\alpha x_\kappa \in \mathcal{D}(A^{n-\alpha}) \subseteq D$). By Lemma 4.1.22 we can see $\overline{J^\alpha} \subseteq A_D^\alpha$.

Conversely, let $x \in \mathcal{D}(A_D^\alpha)$. Define $x_t := t^n(t+A)^{-n}x \in \mathcal{D}(A^n)$ where $n > \operatorname{Re}\alpha$. Then, by Lemma 2.0.9,

$$x_t \to x \quad \text{and} \quad J^\alpha x_t = A_D^\alpha t^n(t+A)^{-n}x = t^n(t+A)^{-n}A_D^\alpha x \to A_D^\alpha x$$

as $t \to \infty$ which proves the claim. $\square$

The next proposition will be devoted to a bunch of standard properties one expects fractional powers to have.

Proposition 4.1.24. *Let $A \in \mathcal{M}(X)$ and $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$. Then*

1. *$A \in \mathcal{L}(X)$ implies $A^\alpha \in \mathcal{L}(X)$.*
2. *Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}_{0<\mathrm{Re}<n}$. Then*

$$\mathbb{C}_{0<\mathrm{Re}<n} \ni \alpha \mapsto A^\alpha x \in X$$

   is holomorphic for all $x \in \mathcal{D}(A^n)$.

3. *One has $\mathcal{N}(A) = \mathcal{N}(A^\alpha)$.*

Proof.

1. This follows from the Moment Inequality.
2. This follows from the Balakrishnan formula.
3. Let $x \in \mathcal{N}(A^\alpha)$. By Proposition 4.1.19 we have $A^n = A^{n-\alpha}A^\alpha$ for some $n \in \mathbb{N}$, $n > \operatorname{Re}\alpha$ which implies $x \in \mathcal{N}(A^n) = \mathcal{N}(A)$ by Corollary 2.0.10.

   Conversely, let $x \in \mathcal{N}(A)$. The strategy from above only works for $\alpha \in \mathbb{C}_{\mathrm{Re}>1}$. In order to include all $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$ we note that $x \in \mathcal{N}(A^n) \subseteq \mathcal{D}(A^n)$ for some $n \in \mathbb{N}$, $n > \operatorname{Re}\alpha$. Hence,

$$A^\alpha x = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty t^{\alpha-1} A^n (t+A)^{-n} x \,\mathrm{d}t = 0$$

   which shows the missing inclusion. $\square$

At this point we could elaborate further on many, many aspects of the theory of fractional powers one would expect to hold in LCS as well. Just to mention some, we could think about $(A^{-1})^\alpha = (A^\alpha)^{-1}$ for injective operators A as suggested by Lemma 2.0.8 and Proposition 4.1.24, one could establish that the operators A^γ, $\gamma \in (0, \frac{1}{2})$ are really the generators of the before introduced semigroup (a partial answer to this question is going to be given in the next section), and one could think about the expected second power law $(A^\delta)^\alpha = A^{\delta\alpha}$. All these questions would go beyond the scope of the work at hand but are noteworthy to be considered in future works.

4.2. Fractional powers with $0 < \alpha < 1$ for generators of equicontinuous C_0-Semigroups

There are well known formulas to describe fractional powers of the strictly smaller class of operators whose negatives generate equibounded C_0-semigroups. One can see this by combining the Balakrishnan formula with the fact that the resolvents of such operators are given by Laplace transformation of the corresponding semigroups, cf. [44, Prop. 3.2.1]. Actually, they were the first class of operators in general Banach spaces for which people studied fractional powers. In this section however, we will concentrate on the special case of $0 < \alpha < 1$. For these values of the power α, the function $z \mapsto z^\alpha$ belongs to a special class of functions and this fact yields yet another way to describe the corresponding fractional powers. To be more precise, in this situation one can use subordination.

Definition 4.2.1 (Bernstein function)**.** Let $f\colon (0,\infty) \to [0,\infty)$. Then f is called a *Bernstein function* provided $f \in C^\infty\big((0,\infty)\big)$ and $(-1)^{k-1} f^{(k)} \geq 0$ for all $k \in \mathbb{N}$.

Bernstein functions appear in a vast number of fields such as probability theory, harmonic analysis, complex analysis and operator theory under different names, e.g., Laplace exponents, negative definite functions or Pick, Nevanlinna or Herglotz functions (complete Bernstein functions, cf. [57]). It will not cause any greater difficulty which is why the following results, taken from [39], shall be presented for general Bernstein functions. As a demonstration example, we shall return to $z \mapsto z^\alpha$ various times. A first fact which we will need is that Bernstein functions allow for a characterisation by two numbers and a measure with a growth property.

Proposition 4.2.2 ([57, Thm. 3.2])**.** *Let $f\colon (0,\infty) \to [0,\infty)$. The following are equivalent.*

1. *f is a Bernstein function.*
2. *There exist constants $a, b \geq 0$ and a positive Radon measure μ on $(0,\infty)$ having the growth property $\int_{(0,\infty)} 1 \wedge t \, \mu(\mathrm{d}t) < \infty$ such that*

$$f(\lambda) = a + b\lambda + \int\limits_{(0,\infty)} \left(1 - \mathrm{e}^{-\lambda t}\right) \mu(\mathrm{d}t) \quad (\lambda > 0). \tag{4.2.1}$$

The representation of f in (4.2.1) is called *Lévy–Khintchine representation*. The function f determines the two numbers a, b and the measure μ in the Lévy–Khintchine representation uniquely and the triple (a, b, μ) is called *Lévy triplet* for f.

Every Bernstein function admits a continuous extension to $[0,\infty)$ since by applying dominated convergence to the representation formula (4.2.1) one gets $f(0+) = a$.

Example 4.2.3. *Let $\alpha \in (0,1)$ and $f\colon (0,\infty) \to [0,\infty)$ be defined by $f(\lambda) := \lambda^\alpha$. Then f is a Bernstein function with Lévy triplet $(0,0,\mu)$ and $\mu \in M\big((0,\infty)\big)$ given by*

$$\mu(A) := \frac{-1}{\Gamma(-\alpha)} \int\limits_A t^{-1-\alpha}\,\mathrm{d}t.$$

Hence,

$$\lambda^\alpha = \frac{-1}{\Gamma(-\alpha)} \int\limits_{(0,\infty)} \big(1 - e^{-\lambda t}\big) t^{-1-\alpha} \mathrm{d}t.$$

Let us now turn to a concept closely related to Bernstein functions.

Definition 4.2.4. Let $(\mu_t)_{t\geq 0}$ be a family in $M\big([0,\infty)\big)$. Then (μ_t) is called

1. a family of *sub-probability measures* if $\forall t \in [0,\infty):\ \mu_t\big([0,\infty)\big) \leq 1$,
2. *vaguely continuous* (at $s \geq 0$ with limit μ_s) if

$$\forall f \in C_c\big([0,\infty)\big):\ \lim_{t\to s} \int\limits_{[0,\infty)} f(\lambda)\,\mu_t(\mathrm{d}\lambda) = \int\limits_{[0,\infty)} f(\lambda)\,\mu_s(\mathrm{d}\lambda)$$

3. *convolution semigroup* if $\mu_0 = \delta_0$ and $\forall s,t \in [0,\infty):\ \mu_t * \mu_s = \mu_{t+s}$.

Remark 4.2.5.

1. A vaguely continuous convolution semigroup (μ_t) of sub-probability measures is also *weakly continuous*, i.e., Definition 4.2.4 actually holds for all $f \in C_b\big([0,\infty)\big)$. In order to see this choose $f \in C_c\big([0,\infty)\big)$ such that $0 \leq f \leq 1$ and $f = 1$ in a neighbourhood of 0. Then

$$1 \geq \mu_t\big([0,\infty)\big) \geq \int\limits_{[0,\infty)} f\mu_t \to f(0) = 1 \text{ as } t \to 0+.$$

So $\mu_t\big([0,\infty)\big) \to 1$ as $t \to 0+$. The same argument works to show that for every $c > 0$ one has $\mu_t\big([0,c)\big) \to 1$ as $t \to 0+$. Therefore also $\mu_t\big([c,\infty)\big) \to 0$ as $t \to 0+$. Now weak continuity in 0 follows from ($g \in C_b\big([0,\infty)\big)$, $fC_c\big([0,\infty)\big)$ as above with $f = 1$ on $[0,\varepsilon)$ for some $\varepsilon > 0$)

$$\Big|\int\limits_{[0,\infty)} g\mu_t - g(0)\Big| \leq \Big|\int\limits_{[0,\infty)} gf\mu_t - g(0)\Big| + \|f\|\,\mu_t\big([\varepsilon,\infty)\big) \to 0$$

as $t \to 0+$.

2. One can use a vaguely continuous convolution semigroup of sub-probability measures (μ_t) to define a C_0-semigroup $(\mathrm{e}^{-tA})_{t\geq 0}$ on $C_{bu}\big([0,\infty)\big)$ (bounded uniformly continuous functions) via

$$(\mathrm{e}^{-tA}f)(x) := \int\limits_{[0,\infty)} f(x+y)\mu_t(\mathrm{d}y).$$

   In particular, the convolution semigroup is actually vaguely continuous in every $t \in [0,\infty)$. We shall briefly sketch the proof. For given $f \in C_{bu}\big([0,\infty)\big)$ and $\delta > 0$, chosen suffienctly small, we estimate for all $x \in [0,\infty)$

$$\begin{aligned}\Big|\int_{[0,\infty)} f(x+y)\mu_t(\mathrm{d}y) - f(x)\Big| \leq \Big|\int_{[0,\delta)} f(x+y) - f(x)\mu_t(\mathrm{d}y)\Big| \\ + 2\,\|f\|\,\mu_t\big([\delta,\infty)\big) \\ + |f(x)|\,\big(1-\mu_t\big([0,\infty)\big)\end{aligned}$$

   which shows the desired result. One can also establish strong continuity in $C_b\big([0,\infty)\big)$ if one is willing to coarse its Banach space topology to the strict topology which is a mixture of its original Banach space topology with the topology of uniform convergence on compacts.

Every Bernstein function is naturally associated to a vaguely continuous convolution semigroup of sub-probability measures $(\mu_t)_{t\geq 0}$ and vice versa.

Proposition 4.2.6 ([57, Thm. 5.2])**.** *Let $(\mu_t)_{t\geq 0}$ be a vaguely continuous convolution semigroup of sub-probability measures on $[0,\infty)$. Then there exists a unique Bernstein function f such that for all $t \geq 0$ the Laplace transform of μ_t is given by*

$$L(\mu_t) = \mathrm{e}^{-tf}.$$

Conversely, given any Bernstein function f, there exists a unique vaguely continuous convolution semigroup of sub-probability measures (μ_t) on $[0,\infty)$ such that the above equation holds.

By the above proposition, we obtain that the sub-probability measures μ_t are probability measures if and only if $f(0+) = 0$, since

$$\mu_t\big([0,\infty)\big) = \lim_{\lambda\to 0+} \int\limits_{[0,\infty)} \mathrm{e}^{-\lambda s}\mu_t(\mathrm{d}s) = \lim_{\lambda\to 0+} \mathrm{e}^{-tf(\lambda)} = \mathrm{e}^{-tf(0+)}. \tag{4.2.2}$$

Example 4.2.7. *Let $\alpha \in (0,1)$ and $f\colon (0,\infty) \to [0,\infty)$ be defined by $f(\lambda) := \lambda^\alpha$ and let (μ_t) be the corresponding family of sub-probability measures. Then, for $t > 0$, the measure μ_t has a density g_t w.r.t. the Lebesgue measure given by*

$$g_t(s) = \frac{1}{2\pi i}\int_{\gamma_\Theta} \mathrm{e}^{-tw^\alpha}\mathrm{e}^{sw}\mathrm{d}w \quad (s > 0), \tag{4.2.3}$$

where $\gamma_\Theta = \gamma_\Theta^+ \cup \gamma_\Theta^-$ is parametrised by

$$\gamma_\Theta^-(r) := -r\mathrm{e}^{-i\Theta} \quad (r \in (-\infty,0)), \qquad \gamma_\Theta^+(r) := r\mathrm{e}^{i\Theta} \quad (r \in (0,\infty))$$

and $\Theta \in [\pi/2,\pi]$ is arbitrary, see Figure 4.2.1 for one possible choice.

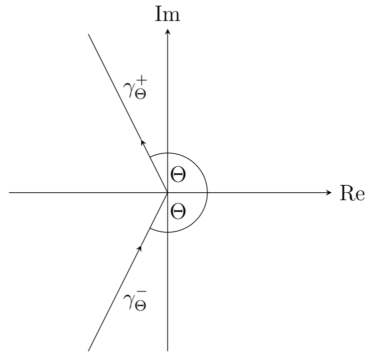

Figure 4.2.1.: Possible path of integration

For $\alpha = \frac{1}{2}$ one can explicitly calculate the integral and finds (see [69, p. 259–268] for details)

$$g_t(s) = \frac{t\mathrm{e}^{-\frac{t^2}{4s}}}{2\sqrt{\pi}s^{\frac{3}{2}}} \quad (s > 0).$$

Analogously to the case of bounded C_0-semigroups on Banach spaces (see [57, Proposition 13.1]) we can construct a new equicontinuous C_0-semigroup from an existing one using a vaguely continuous semigroup (μ_t) of sub-probability measures.

Proposition 4.2.8. *Let X be a LCS, (e^{-tA}) be an equicontinuous C_0-semigroup on X and (μ_t) be a vaguely continuous convolution semigroup of sub-probability measures with* *associated Bernstein function f. For $t \geq 0$ define* $\mathrm{e}^{-tf(A)}\colon X \to X$ *by*

$$x \mapsto \mathrm{e}^{-tf(A)}x := \int\limits_{[0,\infty)} \mathrm{e}^{-sA}x \, \mu_t(\mathrm{d}s). \tag{4.2.4}$$

Then $(\mathrm{e}^{-tf(A)})_{t\geq 0}$ *is an equicontinuous C_0-semigroup on X.*

We will call $(\mathrm{e}^{-tf(A)})$ the *subordinated semigroup to* (e^{-tA}) *w.r.t. f.*

Proof. Let $t \geq 0$. The linearity of $\mathrm{e}^{-tf(A)}$ is evident. The semigroup property of $(\mathrm{e}^{-tf(A)})$ is inherited from the semigroup property of (μ_t) and of (e^{-sA}). Indeed, let $s, t \geq 0$. For $x \in X$, $x' \in X'$ we have

$$\begin{aligned}
\langle x', \mathrm{e}^{-tf(A)}\mathrm{e}^{-sf(A)}x\rangle &= \int\limits_{[0,\infty)} \int\limits_{[0,\infty)} \langle x', \mathrm{e}^{-uA}\mathrm{e}^{-vA}x\rangle \mu_s(\mathrm{d}u)\mu_t(\mathrm{d}v) \\
&= \int\limits_{[0,\infty)} \int\limits_{[0,\infty)} \langle x', \mathrm{e}^{-(u+v)A}x\rangle \mu_s(\mathrm{d}u)\mu_t(\mathrm{d}v) = \int\limits_{[0,\infty)} \langle x', \mathrm{e}^{-wA}x\rangle \mu_s * \mu_t(\mathrm{d}w) \\
&= \int\limits_{[0,\infty)} \langle x', \mathrm{e}^{-wA}x\rangle \mu_{s+t}(\mathrm{d}w) = \langle x', \mathrm{e}^{-(t+s)f(A)}x\rangle.
\end{aligned}$$

For the strong continuity of $(\mathrm{e}^{-tf(A)})$, let $x \in X$. We estimate

$$\left\|\mathrm{e}^{-tf(A)}x - x\right\|_p \leq \int\limits_{[0,\infty)} \left\|\mathrm{e}^{-sA}x - x\right\|_q \mu_t(\mathrm{d}s) \to 0,$$

since $\left(s \mapsto \left\|\mathrm{e}^{-sA}x - x\right\|_q\right) \in C_b\big([0,\infty)\big)$ with value 0 at $s = 0$ and $\mu_t \to \delta_0$ weakly.

It remains to show that $(\mathrm{e}^{-tf(A)})$ is equicontinuous. This follows from

$$\left\|\mathrm{e}^{-tf(A)}x\right\|_p \leq \int_{[0,\infty)} \left\|\mathrm{e}^{-sA}x\right\|_q \mu_t(\mathrm{d}s) \leq C \left\|x\right\|_r$$

where we used the equicontinuity of (e^{-sA}) as well as the uniform boundedness (namely by 1) of the family (μ_t). □

Our next goal is to represent the generator $f(A)$ of a subordinated semigroup $(\mathrm{e}^{-tf(A)})$ for a given Bernstein function f as it was performed in [57, Eq. (13.10)] for Banach spaces.

Proposition 4.2.9. *Let f be a Bernstein function with Lévy triplet (a, b, μ) and $x \in \mathcal{D}(A)$. Then the function*

$$(0,\infty) \ni s \mapsto x - \mathrm{e}^{-sA}x \in X$$

is integrable in the sense that

$$\int_{(0,\infty)} (x - \mathrm{e}^{-tA}x)\mu(\mathrm{d}t) = \int_{(0,1)} \frac{(x - \mathrm{e}^{-tA}x)}{t} t\mu(\mathrm{d}t) + \int_{[1,\infty)} (x - \mathrm{e}^{-tA}x)\mu(\mathrm{d}t) \tag{4.2.5}$$

Proof. Note that the measures $t\mu$ on $(0,1)$ and μ on $[1,\infty)$ are both bounded. Since $x \in \mathcal{D}(A)$, the mapping $s \mapsto \mathrm{e}^{-sA}x$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-sA}x = -A\mathrm{e}^{-sA}x = -\mathrm{e}^{-sA}Ax \quad (s \geq 0).$$

Hence, we get for a continuous seminorm $\|\cdot\|_p$

$$\left\|x - \mathrm{e}^{-tA}x\right\|_p \leq \int_0^t \left\|\mathrm{e}^{-sA}Ax\right\|_q \mathrm{d}s \leq C\left\|Ax\right\|_r \cdot t \quad (t \geq 0)$$

and which is what one needs to show that the first summand in (4.2.5) makes sense. For the second summand, we can use the equicontinuity of (e^{-tA}) and get

$$\left\|x - \mathrm{e}^{-tA}x\right\|_p \leq (C+1)\left\|x\right\|_q.$$

□

Theorem 4.2.10. *Let X be a LCS, (e^{-tA}) be an equicontinuous C_0-semigroup on X with generator $-A$ and f be a Bernstein function with Lévy triplet (a,b,μ). In this situation we have for all $x \in \mathcal{D}(A)$ that $x \in \mathcal{D}(f(A))$ and*

$$f(A)x = ax + bAx + \int_{(0,\infty)} \left(x - \mathrm{e}^{-tA}x\right)\mu(\mathrm{d}t). \tag{4.2.6}$$

Remark 4.2.11. For Banach spaces X this result is due to Phillips [53].

Proof. We adapt the proof of [57, Thm. 13.6] to our context. Let (a,b,μ) be the Lévy triplet for f and (μ_t) the associated family of measures.

(i) Let us first assume that $f(0+) = a = 0$, i.e., (μ_t) is actually a family of probability measures. Recall that

$$\mathrm{e}^{-tf(\lambda)} = \int_{[0,\infty)} \mathrm{e}^{-\lambda s}\mu_t(\mathrm{d}s)$$

and define

$$f_n(\lambda) := \int\limits_{(0,\infty)} (1 - \mathrm{e}^{-\lambda s})\, n\mu_{\frac{1}{n}}(\mathrm{d}s).$$

It follows that

$$f_n(\lambda) = \frac{1 - \mathrm{e}^{-\frac{1}{n}f(\lambda)}}{\frac{1}{n}} \to f(\lambda) \text{ as } n \to \infty \tag{4.2.7}$$

for all $\lambda > 0$, i.e., (f_n) is a sequence of Bernstein functions converging pointwise to f. Using [57, Coro. 3.9], we obtain

$$\lim_{n\to\infty} n\mu_{\frac{1}{n}} = \mu \text{ vaguely on } (0,\infty), \tag{4.2.8}$$

$$\lim_{\substack{C\to\infty\\ \mu(\{C\})=0}} \lim_{n\to\infty} n\mu_{\frac{1}{n}}\big([C,\infty)\big) = 0, \tag{4.2.9}$$

$$\lim_{\substack{c\to 0+\\ \mu(\{c\})=0}} \lim_{n\to\infty} \int\limits_{(0,c)} t\, n\mu_{\frac{1}{n}}(\mathrm{d}t) = b. \tag{4.2.10}$$

Let $x \in \mathcal{D}(A)$ and $x' \in X'$. Furthermore, choose $c, C > 0$ such that $\mu(\{c, C\}) = 0$, i.e., c and C are points of continuity of the measure μ. Note that $\big(t \mapsto \langle x', x - \mathrm{e}^{-tA}x\rangle\big) \in C_b\big([c,C)\big)$. Using the fact that c and C are points of continuity, one can show that $n\mu_{\frac{1}{n}}\big([c,C)\big) \to \mu\big([c,C)\big)$ by approximating $\mathbb{1}_{[c,C)}$ from below and from above by functions f, $g \in C_c\big((0,\infty)\big)$ with supports contained in $[c+\varepsilon, C-\varepsilon]$ and $[c-\varepsilon, C+\varepsilon]$, respectively, for some $\varepsilon > 0$ chosen sufficiently small. Moreover, essentially the same kind of argument gives $n\mu_{\frac{1}{n}}|_{[c,C)} \to \mu|_{[c,C)}$ vaguely. Using again [57, Thm. A.4], we therefore have $n\mu_{\frac{1}{n}}|_{[c,C)} \to \mu|_{[c,C)}$ weakly. Hence,

$$\lim_{n\to\infty} \int\limits_{[c,C)} \langle x', x - \mathrm{e}^{-tA}x\rangle\, n\mu_{\frac{1}{n}}(\mathrm{d}t) = \int\limits_{[c,C)} \langle x', x - \mathrm{e}^{-tA}x\rangle\, \mu(\mathrm{d}t).$$

By dominated convergence we obtain

$$\lim_{\substack{c\to 0+\\ C\to\infty\\ \mu(\{c,C\})=0}} \lim_{n\to\infty} \int\limits_{[c,C)} \langle x', x - \mathrm{e}^{-tA}x\rangle\, n\mu_{\frac{1}{n}}(\mathrm{d}t) = \int\limits_{(0,\infty)} \langle x', x - \mathrm{e}^{-tA}x\rangle\, \mu(\mathrm{d}t). \tag{4.2.11}$$

Let us keep this result in mind for a moment and consider for $c > 0$

$$\int\limits_{(0,c)} \langle x', x - \mathrm{e}^{-tA}x\rangle \, n\mu_{\frac{1}{n}}(\mathrm{d}t) = \int\limits_{(0,c)} \int\limits_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \, \mathrm{d}s \, n\mu_{\frac{1}{n}}(\mathrm{d}t) + \int\limits_{(0,c)} t\langle x', Ax\rangle \, n\mu_{\frac{1}{n}}(\mathrm{d}t).$$

Note that $\big((0,c) \ni t \mapsto \int_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \, \mathrm{d}s\big) \in C_b\big((0,c)\big)$. Let $\phi \in C_c\big((0,c)\big)$. One gets $\big(t \mapsto \phi(t) \int_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \, \mathrm{d}s\big) \in C_c\big((0,c)\big)$ which results in

$$\int\limits_{(0,c)} \phi(t) \int\limits_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \mathrm{d}s \, n\mu_{\frac{1}{n}}(\mathrm{d}t) \to \int\limits_{(0,c)} \phi(t) \int\limits_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \mathrm{d}s \mu(\mathrm{d}t)$$

i.e., the sequence $\Big(\int_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \, \mathrm{d}s \, n\mu_{\frac{1}{n}}\Big)_{n\in\mathbb{N}}$ converges vaguely on the interval $(0,c)$ to the measure $\int_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \, \mathrm{d}s \, \mu$. Moreover,

$$\int\limits_{(0,c)} \int_0^t \big|\langle x', \mathrm{e}^{-sA}Ax - Ax\rangle\big| \, \mathrm{d}s\mu(\mathrm{d}t) \le \sup_{s\in(0,c)} \big|\langle x', \mathrm{e}^{-sA}Ax - Ax\rangle\big| \int\limits_{(0,c)} t\mu(\mathrm{d}t)$$

which says that $\int_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \, \mathrm{d}s \, \mu$ is a finite measure. If $\mu(\{c\}) = 0$, we thus obtain that the convergence is even weak, i.e., for all $f \in C_b\big((0,c)\big)$. In particular, for such c we have

$$\lim_{n\to\infty} \int\limits_{(0,c)} \int\limits_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \, \mathrm{d}s \, n\mu_{\frac{1}{n}}(\mathrm{d}t) = \int\limits_{(0,c)} \int\limits_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \, \mathrm{d}s\mu(\mathrm{d}t),$$

and therefore

$$\lim_{\substack{c\to 0+\\ \mu(\{c\})=0}} \lim_{n\to\infty} \int\limits_{(0,c)} \int\limits_0^t \langle x', \mathrm{e}^{-sA}Ax - Ax\rangle \, \mathrm{d}s \, n\mu_{\frac{1}{n}}(\mathrm{d}t) = 0. \tag{4.2.12}$$

Finally,

$$\lim_{\substack{c\to 0+\\ \mu(\{c\})=0}} \lim_{n\to\infty} \int\limits_{(0,c)} t\langle x', Ax\rangle \, n\mu_{\frac{1}{n}}(\mathrm{d}t) = b \, \langle x', Ax\rangle \tag{4.2.13}$$

by (4.2.10).

Putting together Equation (4.2.12) and Equation (4.2.13), we get as a second intermediate result

$$\lim_{\substack{c\to 0+\\ \mu(\{c\})=0}} \lim_{n\to\infty} \int\limits_{(0,c)} \langle x', x - \mathrm{e}^{-tA}x\rangle \, n\mu_{\frac{1}{n}}(\mathrm{d}t) = b\,\langle x', Ax\rangle. \tag{4.2.14}$$

Before we can summarise everything, we need to discuss the region $[C,\infty)$ as well. Luckily the argument is rather short in this case since (e^{-tA}) is equicontinuous and, by (4.2.9), we obtain

$$\lim_{\substack{C\to \infty\\ \mu(\{C\})=0}} \lim_{n\to\infty} \int\limits_{[C,\infty)} \langle x', x - \mathrm{e}^{-tA}x\rangle n\mu_{\frac{1}{n}}(\mathrm{d}t) = 0. \tag{4.2.15}$$

We are ready for the main argument. We are now going to consider the integral expression in Equation (4.2.6) and split it in the three regions $(0,c)$, $[c,C)$, and $[C,\infty)$. This will enable us to bring in the intermediate results from Equation (4.2.14) for $(0,c)$, Equation (4.2.11) for $[c,C)$, and Equation (4.2.15) for $[C,\infty)$. So, one has

$$\begin{aligned}
\Big\langle x', bAx + \int\limits_{(0,\infty)} (x - \mathrm{e}^{-tA}x)\,\mu(\mathrm{d}t)\Big\rangle &= \lim_{\substack{c\to 0+\\ \mu(\{c\})=0}} \lim_{n\to\infty} \int\limits_{(0,c)} \langle x', x - \mathrm{e}^{-tA}x\rangle \, n\mu_{\frac{1}{n}}(\mathrm{d}t)\\
&\quad + \lim_{\substack{c\to 0+\\ C\to\infty\\ \mu(\{c,C\})=0}} \lim_{n\to\infty} \int\limits_{[c,C)} \langle x', x - \mathrm{e}^{-tA}x\rangle \, n\mu_{\frac{1}{n}}(\mathrm{d}t)\\
&\quad + \lim_{\substack{C\to \infty\\ \mu(\{C\})=0}} \lim_{n\to\infty} \int\limits_{[C,\infty)} \langle x', x - \mathrm{e}^{-tA}x\rangle \, n\mu_{\frac{1}{n}}(\mathrm{d}t)\\
&= \lim_{\substack{c\to 0+\\ C\to\infty\\ \mu(\{c,C\})=0}} \lim_{n\to\infty} \int\limits_{(0,\infty)} \langle x', x - \mathrm{e}^{-tA}x\rangle \, n\mu_{\frac{1}{n}}(\mathrm{d}t)\\
&= \lim_{n\to\infty} \langle x', n(x - \mathrm{e}^{\frac{1}{n}f(A)}x)\rangle.
\end{aligned}$$

where the last equality follows similar to its scalar counterpart in Equation (4.2.7). The result tells us that the element $x \in \mathcal{D}(A)$ is apparently also in the domain of the weak closure of the operator $f(A)$. However, by general considerations weak and ordinary closure are the same and $f(A)$ is closed since $-f(A)$ is the generator of a C_0-semigroup. We conclude $x \in \mathcal{D}(f(A))$ as well as the representation formula

$$f(A)x = bAx + \int\limits_{(0,\infty)} \big(x - \mathrm{e}^{-tA}x\big)\mu(\mathrm{d}t) \quad (x \in \mathcal{D}(A)).$$

(ii) For the general case $f(0+) = a \geq 0$ consider $h := f - a$. Then h is a Bernstein function with $h(0+) = 0$. Let (ν_t) be the associated family of sub-probability measures. The family (μ_t), given by $\mu_t = \mathrm{e}^{-ta}\nu_t$ $(t \geq 0)$, is the family of measures associated to f.

Thus, for $t \geq 0$ and $x \in X$, we have

$$\mathrm{e}^{-tf(A)}x = \int\limits_{[0,\infty)} \mathrm{e}^{-sA}x\, \mu_t(\mathrm{d}s) = \int\limits_{[0,\infty)} \mathrm{e}^{-sA}x\, \mathrm{e}^{-ta}\nu_t(\mathrm{d}s) = \mathrm{e}^{-ta}\mathrm{e}^{-th(A)}x,$$

i.e., $(\mathrm{e}^{-tf(A)})$ is a rescaling of $(\mathrm{e}^{-th(A)})$. Analogously to the case of C_0-semigroups on Banach spaces, one proves that this implies $f(A) = h(A) + a$. Thus the general case follows from (i). $\square$

Corollary 4.2.12. *We have that* $\overline{f(A)_{|\mathcal{D}(A)}} = f(A)$.

Proof. The domain $\mathcal{D}(A)$ is dense in X since $-A$ is a generator of a C_0-semigroup, by Theorem 4.2.10 we have $\mathcal{D}(A) \subseteq \mathcal{D}(f(A))$. Moreover, the semigroup $(\mathrm{e}^{-tf(A)})$ commutes with resolvent operators $(\lambda + A)^{-1}$ which means that the semigroup leaves $\mathcal{D}(A)$ invariant. Thus, as in the case of C_0-semigroups on Banach spaces, we conclude that $\mathcal{D}(A)$ is a core for $f(A)$ (see e.g. [2, Prop. 1.14] for the case of C_0-semigroups on Banach spaces). $\square$

It remains to answer the question whether the generator of the subordinated semigroup is the fractional power assuming we use the Bernstein function $f(\lambda) = \lambda^\alpha$ for the subordination process. Indeed, this is the case.

Theorem 4.2.13. *Let $-A$ be the generator of an equicontinuous C_0-semigroup in an LCS X, $\alpha \in (0,1)$ and $f(\lambda) = \lambda^\alpha$. Then $A^\alpha = f(A)$.*

Proof. The theorem is a consequence of Laplace transform which reads for $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$ and $x \in X$

$$(t + A)^{-1}x = \int\limits_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-sA}x\, \mathrm{d}s.$$

Let $x \in \mathcal{D}(A)$. Using the representation of the resolvent as Laplace transform of the generated semigroup in Equation (4.1.3) gives

$$\begin{aligned}
A^\alpha x &= \frac{\sin(\alpha\pi)}{\pi} \int\limits_0^\infty t^{\alpha-1} A(t+A)^{-1} x \,\mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int\limits_0^\infty t^{\alpha-1} \int\limits_0^\infty (-\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{-sA} x) \cdot \mathrm{e}^{-st} \mathrm{d}s \,\mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)(-\alpha)\Gamma(-\alpha)} \int\limits_0^\infty (x - \mathrm{e}^{-sA} x) \int\limits_0^\infty t^{\alpha+1-1} \mathrm{e}^{-st} \mathrm{d}t \,\mathrm{d}s \\
&= \frac{-1}{\Gamma(-\alpha)} \int\limits_0^\infty \big(x - \mathrm{e}^{-sA} x\big) s^{-1-\alpha} \mathrm{d}s \\
&= f(A)x
\end{aligned}$$

where the last equality used Example 4.2.3 and Theorem 4.2.10. The domain $\mathcal{D}(A)$ is a core for $f(A)$ by Corollary 4.2.12. Moreover it is dense since $-A$ generates a C_0-semigroup. Together with Proposition 4.1.23 it is therefore also a core for A^α. Taking closures thus implies $f(A) = A^\alpha$. □

Remark 4.2.14. Let us finally answer an open question from Remark 4.1.5. Namely one may use the representation of the resolvent as Laplace transform in Equation (4.1.2) and finds it to be in agreement with Equation (4.2.3) for the special case $\Theta = \pi$. Thus, by Theorem 4.2.13, the family defined in Equation (4.1.2) is really the semigroup generated by the negative fractional power $-A^\gamma$ at least under the hypothesis that $-A$ itself generates a C_0-semigroup.

5. The Caffarelli-Silvestre Extension Problem

In this chapter we shall deal with what is either called Caffarelli-Silvestre Extension problem or, alternatively, harmonic extension problem. Before introducing it in more detail, we shall briefly make some remarks on its history. Its first and rather indirect appearance dates back to the short paper [51] from 1968. The authors actually considered stochastic processes subordinated to a Brownian motion and studied methods to describe them by trace processes taking place in the original space extended by an 'additional dimension'. To be more concrete, the authors studied for $\alpha \in (0,1)$ the one-dimensional process generated by the operator

$$S_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{1-2\alpha}{t}\frac{\mathrm{d}}{\mathrm{d}t}.$$

Formally one can think of this process as the magnitude of a Brownian motion in $2-2\alpha$ dimensions since for an n-dimensional Brownian motion (B_t) the process (J_t) given by $J_t := \|B_t\|$, called a Bessel process of order n, is generated by the operator

$$S_{\frac{2-n}{2}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{n-1}{t}\frac{\mathrm{d}}{\mathrm{d}t}$$

(sometimes with an additional factor 1/2). Using the local time of the Bessel process, the authors constructed a time change for an independent Brownian motion which yields the subordinated processes commonly called α-stable Lévy process. The fact that this observation is actually independent of the special example of a Brownian motion seems to be noticed for the first time in [52].

The approach was rediscovered 40 years later from the PDE point of view in the influential paper of Caffarelli and Silvestre [13] where the authors described fractional powers of the Laplacian by means of taking traces of functions solving the PDE

$$\begin{aligned} \partial_t^2 u(t,x) + \frac{1-2\alpha}{t}\partial_t u(t,x) &= -\Delta_x u(t,x), && (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) &= f(x), && x \in \mathbb{R}^n, \end{aligned} \tag{5.0.1}$$

with $\alpha \in (0,1)$ being the fractional power. Formally, one could interpret solutions to (5.0.1) as harmonic functions defined on $\mathbb{R}^n \times \mathbb{R}^{2-2\alpha}$. In this case the Equation (5.0.1) is nothing but the usual Laplacian applied to a function v of the special form

$$v : \mathbb{R}^n \times \mathbb{R}^{2-2\alpha} \to \mathbb{R}, \quad v(x,y) = u\left(\|y\|, x\right),$$

with a suitable function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. So it just depends on the norm of the additional $2 - 2\alpha$ coordinates. This may be the reason why the technique is sometimes also referred to as harmonic extension problem. The here sketched idea is used nicely in [13] to obtain results analogously to the ones available for the Laplacian in $\mathbb{R}^n$.

Using a solution u to (5.0.1), one can then calculate $(-\Delta_x)^\alpha$ as

$$c_\alpha\big((-\Delta_x)^\alpha f\big)(x) = -\lim_{t \to 0+} t^{1-2\alpha}\partial_t u(t,x), \qquad x \in \mathbb{R}^n,$$

with an explicitly known constant c_α which was calculated for the first time in [63].

The relation between the PDE and the stochastic setting is that Brownian motions are generated by the Laplacian while the processes subordinated to the Brownian motion are generated by the fractional powers of the Laplacian for $\alpha \in (0,1)$ (in stochastics the parameter 2α is typically used which is why it ranges there from 0 to 2).

One may generalise Equation (5.0.1) by replacing $-\Delta$ with a more general sectorial operator A in some Banach space X or even a non-negative operator in some LCS and furthermore consider $\alpha \in \mathbb{C}_{0<\mathrm{Re}<1}$. Now (5.0.1) becomes

$$u''(t) + \frac{1-2\alpha}{t}u'(t) = Au(t), \quad t \in (0,\infty), \tag{5.0.2}$$

$$u(0) = x, \tag{5.0.3}$$

i.e., a linear ODE in the space X with initial datum $x \in X$ which degenerates for $t = 0$ (unless $\alpha = \frac{1}{2}$) and which seems to be 'incomplete' since no initial condition for u' is given. The case $\alpha = \frac{1}{2}$ was already studied in [5] for Banach spaces with the result that under the additional assumption of u being bounded the problem has the unique solution given by

$$u(t) = \mathrm{e}^{-t\sqrt{A}}x,$$

i.e., is given by the C_0-semigroup generated by $\sqrt{A}$. We did not deal with $\sqrt{A}$ but one can show that its negative always generates a semigroup which is in general not analytic unless A is even sectorial.

It was then noticed in [63] that if the considered Banach space X is one-dimensional, Equation (5.0.2) is another form of Bessel's differential equation and integral representations of its solutions provide solutions to (5.0.2) by interpreting them via a functional calculus.

Assuming u to be a bounded solution to (5.0.2), in the one-dimensional case it follows that u is unique and that

$$-\lim_{t\to 0+} t^{1-2\alpha}u'(t) = c_\alpha A^\alpha x. \tag{5.0.4}$$

The expression on the left is usually referred to as a generalised Neumann boundary condition (namely the evalutation of the weighted negative derivate of u at $t = 0$). Following this terminology one calls $u(0) = x$ a Dirichlet boundary condition and therefore thinks of the limit in (5.0.4) as generalised Dirichlet-to-Neumann operator, an operator mapping the Dirichlet-data of a solution on its corresponding Neumann-data, which is given by the fractional power of the operator A appearing in the ODE. The question arises whether the solution u is also unique in general LCS (that would mean that the Dirichlet-to-Neumann operator can be defined) and whether this operator is up to a constant still given by a fractional power.

The entire topic got studied in a couple of works since 2007 starting with Hilbert space approaches and their generalisations to Banach spaces, see [3, 23, 46, 48, 47, 49, 63].

While existence of a solution can usually be tackled by means of functional calculus methods, it turned out that the discussion of uniqueness of the solution is slightly more challenging. In the following we shall deal with both points in a LCS X but under the slightly more restrictive assumption on A being a 'proper' sectorial operator rather than solely being non-negative. For Banach spaces this is no restriction as there both classes coincide.

5.1. Problem and preliminaries

For the entire section let X be a LCS with fundamental system of seminorms $\mathcal{P}_X$ and $A \in \mathcal{S}_\omega(X)$, $\omega \in [0, \pi)$, a densely defined, sectorial operator and $\alpha \in \mathbb{C}_{0<\mathrm{Re}<1}$.

Definition 5.1.1. A function $u \in C_b\big([0,\infty); X\big)$ such that we have for all $\phi \in \mathcal{D}\big((0,\infty)\big)$

$$\int_0^\infty u(t)\phi(t)\mathrm{d}t \in \mathcal{D}(A), \int_0^\infty u(t)\frac{\mathrm{d}}{\mathrm{d}t}t^{1-2\alpha}\frac{\mathrm{d}}{\mathrm{d}t}t^{2\alpha-1}\phi(t)\mathrm{d}t = A\int_0^\infty u(t)\phi(t)\mathrm{d}t \text{ and } u(0) = x,$$

where $x \in X$ is given, shall be called a *weak solution of the Caffarelli-Silvestre Problem*. A weak solution u of the Caffarelli-Silvestre Problem will be called *strong solution* if even

$u \in C_b\big([0,\infty);\mathcal{D}(A)\big)$. In this situation one has

$$A\int_0^\infty u(t)\phi(t)\mathrm{d}t = \int_0^\infty Au(t)\phi(t)\mathrm{d}t,$$

i.e.,

$$Au = t^{2\alpha-1}Dt^{1-2\alpha}Du \quad \text{in} \quad \mathcal{D}'\big((0,\infty);X\big) := \mathcal{D}'\big((0,\infty)\big)\varepsilon X,$$

i.e., the functions correspond as vector-valued distributions. Furthermore, $Au \in C_b\big([0,\infty);X\big)$. Hence, $u \in C^2\big((0,\infty);X\big)$ and therefore

$$\forall t > 0 : \; t^{2\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t}t^{1-2\alpha}u'(t) = u''(t) + \frac{1-2\alpha}{t}u'(t) = Au(t). \tag{5.1.1}$$

Remark 5.1.2.

1. The here presented definition of a solution differs and is actually more general than the one in [48]. The changes are a result of the discussion in [49].

2. The assumption of a dense domain of the operator A is as usual pure convenience. Assume A for the moment to be non-densely defined and set $D := \overline{\mathcal{D}(A)}$. Since we are seeking continuous solutions of the above equation, no solution for $x \in X \setminus D$ can be expected even though the solution operators will be defined on the full space. The differential equation will still be solved but no continuity at $t = 0$ will hold. This effect can already be observed for the much simpler case of an ordinary Cauchy problem
$$u'(t) = Au(t), \quad u(0) = x.$$

5.2. Uniqueness of a solution

In this section we will begin to discuss the uniqueness of possible weak solutions to the Caffarelli-Silvestre Problem. We adopt the assumptions and notation from Section 5.1. Because A is sectorial, we may choose any $z \in \mathbb{C} \setminus S_\omega$ and consider for a given weak solution u with $u(0) = x$ the function $v := (z-A)^{-1}u \in C_b\big([0,\infty);\mathcal{D}(A)\big)$. Then v is a strong solution with $v(0) = (z-A)^{-1}x$. The statement $u = 0$ is equivalent to $v = 0$ which is why we can concentrate on the uniquess of strong solutions to the Caffarelli-Silvestre Problem. Let us collect some properties of v.

Lemma 5.2.1. *Let v be a strong solution to the Caffarelli-Silvestre Problem with $v(0) = x \in \mathcal{D}(A)$. Then we have*

1. $y := \lim_{t\to 0+} -t^{1-2\alpha}v'(t)$ *exists,*
2. $\forall \|\cdot\|_p \in \mathcal{P}_X \,\exists C_1, C_2 > 0\, \forall t > 0: \, \|v'(t)\|_p \leq C_1 t^{2\operatorname{Re}\alpha - 1} + C_2 t,$
3. $\forall \|\cdot\|_p \in \mathcal{P}_X \,\exists C > 0\, \forall t > 0: \, \|v(t) - x\|_p \leq C(t^{2\operatorname{Re}\alpha} \wedge 1).$

Proof.

1. Consider Equation (5.1.1) and bring the prefractor $t^{2\alpha-1}$ on the right-hand side. Let $0 < t < s$. Integration from t to s gives

$$-t^{1-2\alpha}v'(t) = -s^{1-2\alpha}v'(s) + \int_t^s r^{1-2\alpha}Av(r)\mathrm{d}r.$$

   It is possible to send $t \to 0+$ since $r \mapsto r^{1-2\alpha}$ is locally integrable near $t = 0$ while the function Av is continuous.

2. Let $\|\cdot\|_p$ be a continuous seminorm. Using the same argument from the first part, we conclude

$$t^{1-2\operatorname{Re}\alpha}\|v'(t)\|_p = \left\|t^{1-2\alpha}v'(t)\right\|_p \leq \|y\|_p + \frac{C}{2-2\operatorname{Re}\alpha}t^{2-2\operatorname{Re}\alpha}.$$

   for some $C > 0$. This implies the statement.

3. The case $t \geq 1$ follows from boundedness of v. It remains to discuss $t < 1$. Since the limit $\lim_{t\to 0+} -t^{1-2\alpha}v'(t)$ exists, the function $t \mapsto t^{1-2\alpha}v'(t)$ has to be bounded in a neighborhood of $t = 0$ which in turn implies integrability of v' in a neighborhood of $t = 0$. Now the fundamental theorem yields

$$\|v(t) - x\|_p \leq \int_0^t t^{2\operatorname{Re}\alpha - 1}\left\|t^{1-2\alpha}v'(t)\right\|_q \mathrm{d}t \leq Ct^{2\operatorname{Re}\alpha}$$

   where $\|\cdot\|_q \in \mathcal{P}_X$ is another continuous seminorm. □

For both, the uniqueness and the existence part, we will be needing a certain generalisation (among many possible) of the exponential function. As preparation, note that the functions I_α and K_α from $\mathbb{C} \setminus (-\infty, 0]$ to $\mathbb{C}$ defined by

$$I_\alpha(t) := \left(\frac{t}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{t^{2k}}{4^k \cdot k! \cdot \Gamma(\alpha + k + 1)} \quad \text{and} \quad K_\alpha(t) := \frac{\pi}{2\sin(\alpha\pi)}\big(I_{-\alpha}(t) - I_\alpha(t)\big),$$

are called *modified Bessel functions* of first and second kind, respectively. The name origins from the fact that both functions solve

$$t^2 u''(t) + tu'(t) - (t^2 + \alpha^2)u(t) = 0,$$

the so–called modified Bessel equation.

Remark 5.2.2. From its definition it seems at first glance that K_α cannot be defined for $\alpha \in \mathbb{Z}$. However, one can extend the representation also to those values of α. A representation which works for all $\alpha \in \mathbb{C}_{\mathrm{Re}>-\frac{1}{2}}$, $t > 0$ is

$$K_\alpha(t) = \frac{\sqrt{\pi}}{\Gamma(\alpha + \frac{1}{2})} \left(\frac{t}{2}\right)^\alpha \int_1^\infty \mathrm{e}^{-ts}(s^2 - 1)^{\alpha - \frac{1}{2}} \mathrm{d}s \tag{5.2.1}$$

([1, 10.32.8]). Using integration by parts, one can derive integral representations from it which are even valid for larger halfplanes but we will not need this.

Definition 5.2.3. For $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$ we define the function $E_\alpha : \mathbb{C} \setminus (-\infty, 0) \to \mathbb{C}$ by

$$E_\alpha(t) := \frac{2}{\Gamma(\alpha)} \left(\frac{t}{2}\right)^\alpha K_\alpha(t) \tag{5.2.2}$$

Remark 5.2.4.

1. The function K_α decays exponentially as $t \to \infty$ but it diverges as $t^{-\operatorname{Re}\alpha}$ when $t \to 0+$ which is why the factor t^α appears in the above definition. Both limit behaviours stay true when the function is considered on a sector with opening angle strictly less than $\frac{\pi}{2}$.

2. The rather complicated α-dependent prefactor simply ensures $E_\alpha(0) = 1$, cf. [1, 10.30.2]

3. Note that $E_{\frac{1}{2}}(t) = \mathrm{e}^{-t}$.

Lemma 5.2.5. *Let $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$. Then we have for all $t \geq 0$ the representation*

$$E_\alpha(\lambda t) = \frac{\lambda^{2\alpha}}{\Gamma(2\alpha)} \int_t^\infty \mathrm{e}^{-\lambda s}(s^2 - t^2)^{\alpha - \frac{1}{2}} \mathrm{d}s$$

Consequently,

$$\forall \phi \in [0, \tfrac{\pi}{2})\, \exists C > 0\, \forall t > 0, \lambda \in S_\phi :\, |E_\alpha(\lambda t)| \leq C\mathrm{e}^{-\frac{1}{2}\operatorname{Re}\lambda \cdot t}.$$

Proof. The given representation is just the definition of E_α combined with the integral representation (5.2.1), the substitution $y := ts$ applied to

$$E_\alpha(\lambda t) = \frac{2\sqrt{\pi}}{\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})} \left(\frac{\lambda t}{2}\right)^{2\alpha} \int_1^\infty \mathrm{e}^{-\lambda ts}(s^2 - 1)^{\alpha - \frac{1}{2}} \mathrm{d}s,$$

and Legendre's duplication formula $\sqrt{\pi} \cdot \Gamma(2\alpha) = 2^{2\alpha - 1}\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})$. Because of

$$\begin{aligned} |E_\alpha(\lambda t)| &\leq D(|\lambda|\, t)^{2\operatorname{Re}\alpha} \mathrm{e}^{-\frac{1}{2}\operatorname{Re}\lambda t} \int_1^\infty \mathrm{e}^{-\frac{1}{2}\operatorname{Re}\lambda ts}(s^2 - 1)^{\operatorname{Re}\alpha - \frac{1}{2}} \mathrm{d}s \\ &= D\mathrm{e}^{-\frac{1}{2}\operatorname{Re}\lambda t} \left(\frac{|\lambda|}{\operatorname{Re}\lambda}\right)^{2\operatorname{Re}\alpha} \int_{\operatorname{Re}\lambda t}^\infty \mathrm{e}^{-\frac{1}{2}r}\left(r^2 - (\operatorname{Re}\lambda t)^2\right)^{\operatorname{Re}\alpha - \frac{1}{2}} \mathrm{d}r, \end{aligned}$$

D being a constant only depending on α, the entire discussion is basically just about the boundedness of the continuous function f given by

$$f(t) := \int_0^\infty \underbrace{\mathrm{e}^{-\frac{1}{2}s}(s^2 - t^2)^{\operatorname{Re}\alpha - \frac{1}{2}} \mathbb{1}_{(t,\infty)}(s)}_{=: g_t(s)} \mathrm{d}s. \tag{5.2.3}$$

Let us first consider the case $\operatorname{Re}\alpha \geq \frac{1}{2}$. In this case we have $g_0 \in L^1\big((0,\infty)\big)$, $g_0 \geq g_t$ and

$$\lim_{t \to 0+} g_t(s) = g_0(s) \quad \text{as well as} \quad \lim_{t \to \infty} g_t(s) = 0.$$

By dominated convergence f is bounded at $t = 0$ and $t = \infty$. Concerning $\operatorname{Re}\alpha < \frac{1}{2}$, one may substitute $y := s - t$ to get

$$f(t) = \int_0^\infty \underbrace{\mathrm{e}^{-\frac{1}{2}(y+t)} y^{\operatorname{Re}\alpha - \frac{1}{2}} (y+2t)^{\operatorname{Re}\alpha - \frac{1}{2}}}_{=:h_t(y)} \, \mathrm{d}y.$$

Discussing the family (h_t) analogously as (g_t) gives the same result. So the claim follows by choosing $C := \sup_{t>0} f(t)$. $\square$

Remark 5.2.6. Note that, for given values $t > 0$, $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$, the function $\alpha \mapsto E_\alpha(\lambda t)$ is analytic in the strip $\mathbb{C}_{0<\mathrm{Re}<1}$.

A second ingredient for the uniqueness proof will be a special operator.

Definition 5.2.7. On the Banach space $C_0\big([0,\infty)\big)$ we define the operator $T : \mathcal{D}(T) \to C_0\big([0,\infty)\big)$ by

$$(Tf)(t) := -\frac{1}{t} f'(t)$$

on the domain $\mathcal{D}(T) := \{f \in C_0\big([0,\infty)\big) \cap C^1\big((0,\infty)\big) \mid t \mapsto \frac{1}{t} f'(t) \in C_0\big([0,\infty)\big)\}$.

Remark 5.2.8.

1. For the definition of the operator we could have chosen L^p-spaces as well. However, this would have made the discussion slightly more difficult.

2. The importance of the operator comes from the equation

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^k z^\alpha K_\alpha(z) = (-1)^k z^{\alpha-k} K_{\alpha-k}(z) \tag{5.2.4}$$

([1, 10.29.4]) as we will see.

Proposition 5.2.9. *The operator T is injective with (unbounded) inverse. Both operators are densely defined. It is also sectorial of angle $\omega = \frac{\pi}{2}$ and m-accretive.*

Proof. The operator is injective since $Tf = 0$ implies that $f = c$ for some $c \in \mathbb{C}$ which got to be 0 since $f \in C_0\big([0,\infty)\big)$. With a direct calculation one confirms that for $\lambda \in C_{\mathrm{Re}>0}$ the resolvent $(\lambda + T)^{-1}$ is given by

$$\big((\lambda + T)^{-1} g\big)(t) = \mathrm{e}^{\frac{1}{2}\lambda t^2} \int_t^\infty s\mathrm{e}^{-\frac{1}{2}\lambda s^2} g(s) \mathrm{d}s.$$

The functions f_λ, defined by $f_\lambda(t) := \mathrm{e}^{-\frac{1}{2}\lambda t^2}$ with $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$, are eigenfunctions of T with eigenvalue λ. We conclude that $\sigma(T) = \mathbb{C}_{\mathrm{Re}\geq 0}$ while $\rho(T) = C_{\mathrm{Re}<0}$. Using the explicit representation of the resolvent, we can estimate

$$\left\|(\lambda + T)^{-1}\right\| \leq \frac{1}{\operatorname{Re}\lambda}$$

which implies sectoriality, $\omega = \frac{\pi}{2}$, and m-accretivity. Finally T^{-1} is unbounded because one has the representation

$$\big(T^{-1}\phi\big)(t) = \int_t^\infty s\phi(s)\mathrm{d}s$$

for all $\phi \in C_c\big([0,\infty)\big)$ which does not define a continuous operator. However, apparently T^{-1} has got $C_c\big([0,\infty)\big)$ in its domain which is why it is densely defined.

In order to show denseness of $\mathcal{D}(T)$ let us show that $\lambda(\lambda + T)^{-1}f \to f$ as $\lambda \to \infty$ for every $f \in C_0\big([0,\infty)\big)$. So let $f \in C_0\big([0,\infty)\big)$, $t \in [0,\infty)$, and $\varepsilon > 0$ be given. Choose $t_0 > 0$ such that $\sup_{t>t_0} |f(t)| < \frac{\varepsilon}{3}$. Furthermore, choose $\delta > 0$ with $|f(x) - f(y)| < \frac{\varepsilon}{3}$ whenever $|x - y| < \delta$. We estimate for $t \geq 0$

$$\begin{aligned}
\left|\big(\lambda(\lambda + T)^{-1}f\big)(t) - f(t)\right| &\leq \int_t^\infty \lambda s \mathrm{e}^{-\frac{1}{2}\lambda(s^2-t^2)}\, |f(s) - f(t)|\, \mathrm{d}s \\
&= \int_0^\infty \mathrm{e}^{-y} \left|f(\sqrt{2y\lambda^{-1} + t^2}) - f(t)\right| \mathrm{d}y \\
&= \int_0^{t_0} \mathrm{e}^{-y} \left|f(\sqrt{2y\lambda^{-1} + t^2}) - f(t)\right| \mathrm{d}y \\
&+ \int_{t_0}^\infty \mathrm{e}^{-y} \left|f(\sqrt{2y\lambda^{-1} + t^2}) - f(t)\right| \mathrm{d}y \\
&\leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3}
\end{aligned}$$

for $\lambda > \lambda_0$ where we have choosen λ_0 such that $\sqrt{2t_0\lambda_0^{-1}} < \delta$. This finishes the proof. $\square$

If we rewrite Equation (5.2.4) using T and the functions E_α we find for $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$, $\operatorname{Re}\alpha > k$

$$T^k E_\alpha = \frac{\Gamma(\alpha - k)}{2^k \Gamma(\alpha)} E_{\alpha - k}. \tag{5.2.5}$$

Furthermore, T admits fractional powers. At this point the reader may already sense that one could interpolate Equation (5.2.5) for $k \in \mathbb{C}_{0 \le k < \operatorname{Re}\alpha}$. This is actually true as we will see. The following lemma will be needed. Before the reader may recall that it holds that T^{-1} is non-negative by Lemma 2.0.8 (actually even sectorial of the same angle as T) and admits therefore fractional powers as well. It holds that $(T^{-1})^{\alpha} = T^{-\alpha}$. For Banach spaces this can be found in [25, Prop. 3.2.1] but in fact it actually remains also true in general LCS.

Lemma 5.2.10. *Let $f \in \mathcal{R}(T) = \mathcal{D}(T^{-1})$, $g \in \mathcal{D}(T)$, and $\alpha \in \mathbb{C}_{\mathrm{Re}>0}$ such that $\operatorname{Re}\alpha < 1$. Then we have for $t \ge 0$ the representations*

$$\big(T^{-\alpha} f\big)(t) = \frac{1}{2^{\alpha-1}\Gamma(\alpha)} \int_t^{\infty} s(s^2 - t^2)^{\alpha-1} f(s) \mathrm{d}s \tag{5.2.6}$$

and

$$\big(T^{\alpha} g\big)(t) = \frac{1}{2^{-\alpha-1}\Gamma(-\alpha)} \int_t^{\infty} s(s^2 - t^2)^{-\alpha-1} \big(g(s) - g(t)\big) \mathrm{d}s. \tag{5.2.7}$$

Proof. This result is just an application of the Balakrishnan formula (4.1.3). Note that $T^{-1}(\lambda + T^{-1})^{-1} = \frac{1}{\lambda}(\frac{1}{\lambda} + T)^{-1}$. Having this in mind and making use of the Balakrishnan representation, we can calculate

$$\begin{aligned}
\big(T^{-\alpha} f\big)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^{\infty} \lambda^{\alpha-1} \frac{1}{\lambda} \mathrm{e}^{\frac{t^2}{2\lambda}} \int_t^{\infty} s\mathrm{e}^{-\frac{s^2}{2\lambda}} f(s) \mathrm{d}s \mathrm{d}\lambda \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_t^{\infty} s f(s) \int_0^{\infty} \lambda^{\alpha-1} \mathrm{e}^{-\frac{s^2-t^2}{2\lambda}} \frac{\mathrm{d}\lambda}{\lambda} \mathrm{d}s \\
&= \frac{1}{2^{\alpha-1}\Gamma(\alpha)} \int_t^{\infty} s f(s) (s^2 - t^2)^{\alpha-1} \mathrm{d}s
\end{aligned}$$

as well as

$$
\begin{aligned}
\big(T^{\alpha}g\big)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int\limits_0^\infty \lambda^{\alpha-1}\big(-\frac{1}{t}D\big)\mathrm{e}^{\frac{1}{2}\lambda t^2}\int\limits_t^\infty s\mathrm{e}^{-\frac{1}{2}\lambda s^2}g(s)\mathrm{d}s\,\mathrm{d}\lambda \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int\limits_0^\infty \lambda^{\alpha-1}\cdot\lambda\int\limits_t^\infty s\mathrm{e}^{-\frac{1}{2}\lambda(s^2-t^2)}\big(g(t)-g(s)\big)\mathrm{d}s\,\mathrm{d}\lambda \\
&= \frac{1}{\Gamma(\alpha)(-\alpha)\Gamma(-\alpha)}\int\limits_t^\infty s\big(g(t)-g(s)\big)\int\limits_0^\infty \lambda^{\alpha}\mathrm{e}^{-\frac{1}{2}\lambda(s^2-t^2)}\mathrm{d}\lambda\,\mathrm{d}s \\
&= \frac{1}{2^{-\alpha-1}\Gamma(-\alpha)}\int\limits_t^\infty s(s^2-t^2)^{-\alpha-1}\big(g(s)-g(t)\big)\mathrm{d}s
\end{aligned}
$$

which shows the claim. □

Corollary 5.2.11. *Let $\alpha \in \mathbb{C}_{0<\mathrm{Re}<1}$ and $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$. Then $\big(t \mapsto \mathrm{e}^{-\lambda t}\big) \in \mathcal{D}(T^{\frac{1}{2}-\alpha})$ and*

$$
\forall t \in [0,\infty):\ E_\alpha(\lambda t) = \frac{\sqrt{\pi}\lambda^{2\alpha-1}}{2^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\big(T^{\frac{1}{2}-\alpha}(s\mapsto \mathrm{e}^{-\lambda s})\big)(t)
$$

as well as $\big(t \mapsto E_\alpha(\lambda t)\big) \in \mathcal{D}(T^{\alpha-\frac{1}{2}})$ and for all $t \in [0,\infty)$

$$
\frac{2^{\alpha-\frac{1}{2}}\Gamma(\alpha)}{\sqrt{\pi}\lambda^{2\alpha-1}}\Big(T^{\alpha-\frac{1}{2}}\big(s\mapsto E_\alpha(s\lambda)\big)\Big)(t) = \frac{\lambda\cdot\Gamma(1-\alpha)}{\sqrt{\pi}\cdot\Gamma(\frac{3}{2}-\alpha)}\int\limits_t^\infty E_{1-\alpha}(\lambda s)s^{2\alpha-1}(s^2-t^2)^{\frac{1}{2}-\alpha}\mathrm{d}s. \tag{5.2.8}
$$

Proof. Let us first consider the case $\mathrm{Re}\,\alpha > \frac{1}{2}$. Note that $\big(t \mapsto \mathrm{e}^{-\lambda t}\big) \in \mathcal{D}(T^{-1})$. Hence, $\big(t \mapsto \mathrm{e}^{-\lambda t}\big) \in \mathcal{D}(T^{\frac{1}{2}-\alpha})$ and using the first representation from Lemma 5.2.10 we calculate

$$
\begin{aligned}
\big(T^{\frac{1}{2}-\alpha}(s\mapsto \mathrm{e}^{-\lambda s})\big)(t) &= \frac{1}{2^{\alpha-\frac{3}{2}}\Gamma(\alpha-\frac{1}{2})}\int\limits_t^\infty s\mathrm{e}^{-\lambda s}(s^2-t^2)^{\alpha-\frac{3}{2}}\mathrm{d}s \\
&= \frac{\lambda}{2^{\alpha-\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}\int\limits_t^\infty \mathrm{e}^{-\lambda s}(s^2-t^2)^{\alpha-\frac{1}{2}}\mathrm{d}s
\end{aligned}
$$

where we used integration by parts which is possible due to $\mathrm{Re}\,\alpha > \frac{1}{2}$. Together with Lemma 5.2.5 the claim is shown in this situation.

Let now $\mathrm{Re}\,\alpha < \frac{1}{2}$. To begin with note that $\big(t \mapsto \mathrm{e}^{-\lambda t}\big) \notin \mathcal{D}(T)$. However, the formula

(5.2.7) still makes sense for this function. By this we mean that the appearing integral is convergent and integration by parts yields

$$\frac{1}{2^{\alpha-\frac{3}{2}}\Gamma(\alpha-\frac{1}{2})}\int_t^\infty s(s^2-t^2)^{\alpha-\frac{3}{2}}\big(\mathrm{e}^{-\lambda s}-\mathrm{e}^{-\lambda t}\big)\mathrm{d}s = \frac{\lambda}{2^{\alpha-\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}\int_t^\infty \mathrm{e}^{-\lambda s}(s^2-t^2)^{\alpha-\frac{1}{2}}\mathrm{d}s.$$

Tracing back the steps we undertook to derive (5.2.7) from the Balakrishnan formula, we conclude

$$\lim_{n\to\infty}\underbrace{\frac{\sqrt{\pi}\lambda^{2\alpha-1}}{2^{\alpha-\frac{1}{2}}\Gamma(\alpha)\Gamma(\frac{1}{2}-\alpha)\Gamma(\frac{1}{2}+\alpha)}\int_0^n \mu^{-\frac{1}{2}-\alpha}\cdot\mu\int_t^\infty s\mathrm{e}^{-\frac{1}{2}\mu(s^2-t^2)}\big(\mathrm{e}^{-\lambda t}-\mathrm{e}^{-\lambda s}\big)\mathrm{d}s\,\mathrm{d}\mu}_{=:f_n(t)} = E_\alpha(\lambda t),$$

i.e., (f_n) converges to $t\mapsto E_\alpha(\lambda t)$ pointwisely and boundedly. By dominated convergence the convergence is weakly in $C_0\big([0,\infty)\big)$. By [44, Thm. 3.1.10, Rem. 3.1.3] we obtain $\big(t\mapsto \mathrm{e}^{-\lambda t}\big)\in\mathcal{D}(T^{\frac{1}{2}-\alpha})$ and

$$\frac{\sqrt{\pi}\lambda^{2\alpha-1}}{2^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\big(T^{\frac{1}{2}-\alpha}(s\mapsto \mathrm{e}^{-\lambda s})\big)(t) = E_\alpha(\lambda t).$$

The validity for $\operatorname{Re}\alpha=\frac{1}{2}$ follows from [44, Prop. 7.1.5] since it implies that $\alpha\mapsto \big(T^{\frac{1}{2}-\alpha}(s\mapsto \mathrm{e}^{-\lambda s})(t)$ is holomorphic on the strip $0<\operatorname{Re}\alpha<1$ and on the parts $0<\operatorname{Re}\alpha<\frac{1}{2}$ as well as $\frac{1}{2}<\operatorname{Re}\alpha<1$ it coincides with the function $\alpha\mapsto E_\alpha(\lambda t)$ which in turn is holomorphic on the entire strip.

The so far presented argument also implies $\big(t\mapsto E_\alpha(\lambda t)\big)\in\mathcal{D}(T^{\alpha-\frac{1}{2}})$. The remaining integral representation follows similar to the two derivations from above where one additionally needs to use the identity

$$\frac{\mathrm{d}}{\mathrm{d}s}E_\alpha(\lambda s) = -\frac{2^{-\alpha}\Gamma(1-\alpha)}{2^{\alpha-1}\Gamma(\alpha)}\lambda^{2\alpha}s^{2\alpha-1}E_{1-\alpha}(\lambda s) \tag{5.2.9}$$

which in turn follows from Equation (5.2.4). This finishes the proof. □

Remark 5.2.12. Corollary 5.2.11 is nothing but a special case of the earlier announced relation how the different E_α are related to each other by the fractional powers of T. Reasoning as above one can actually show that the Balakrishnan representation of T^β, $\beta\in\mathbb{C}_{\mathrm{Re}<\operatorname{Re}\alpha}$ is applicable to $t\mapsto E_\alpha(\lambda t)$ which gives $\big(t\mapsto E_\alpha(\lambda t)\big)\in\mathcal{D}(T^\beta)$ and, using

Corollary 5.2.11, we obtain

$$\begin{aligned}T^{\beta}\big(t\mapsto E_{\alpha}(\lambda t)\big) &= \frac{\sqrt{\pi}\lambda^{2\alpha-1}}{2^{\alpha-\frac{1}{2}}\Gamma(\alpha)}T^{\beta}T^{\frac{1}{2}-\alpha}(t\mapsto \mathrm{e}^{-\lambda t}) = \frac{\sqrt{\pi}\lambda^{2\alpha-1}}{2^{\alpha-\frac{1}{2}}\Gamma(\alpha)}T^{\beta+\frac{1}{2}-\alpha}(t\mapsto \mathrm{e}^{-\lambda t})\\ &= \frac{\lambda^{2\beta}\Gamma(\alpha-\beta)}{2^{\beta}\Gamma(\alpha)}\big(t\mapsto E_{\alpha-\beta}(\lambda t)\big).\end{aligned}$$

Technically the thesis only covers the situation for $\operatorname{Re}\alpha > \frac{1}{2}$. However, the general Banach space theory suffices and the necessary result for the remaining values of α can be found in [44, 7.1.1].

We proceed by using the so far derived results to show the injectivity of an integral transform generalising the Laplace transformation.

Definition 5.2.13. Let $\alpha \in \mathbb{C}_{0<\operatorname{Re}\alpha<1}$. We define $L_{\alpha} : C_0\big([0,\infty)\big) \to \mathcal{H}(\mathbb{C}_{\operatorname{Re}>0})$ by

$$\big(L_{\alpha}f\big)(\lambda) := \int_0^{\infty} f(t)t^{1-2\alpha}E_{\alpha}(\lambda t)\mathrm{d}t.$$

Note that we already know that the just introduced integral transform is injective for $\alpha = \frac{1}{2}$ since in this case it is simply the Laplace transform. We shall show that this is actually true for all values $\alpha \in \mathbb{C}_{0<\operatorname{Re}<1}$.

Proposition 5.2.14. *The mapping L_{α} is injective for all $\alpha \in \mathbb{C}_{0<\operatorname{Re}<1}$.*

Proof. Let $f \in C_0\big([0,\infty)\big)$. Set $F : (0,\infty) \to \mathbb{C}$, $F(\lambda) := \big(L_{\alpha}f\big)(\lambda)$. Assume $F \in \mathcal{D}(T) \cap \mathcal{D}(T^{-1})$, i.e., F admits a continuous extension to $\lambda = 0$ and the extension, again denoted by F, is in the domain of T and T^{-1}. Let us first consider the case $\operatorname{Re}\alpha > \frac{1}{2}$. From the assumptions we conclude $F \in \mathcal{D}(T^{\alpha-\frac{1}{2}})$ and we calculate

$$\begin{aligned}&\big(T^{\alpha-\frac{1}{2}}F\big)(\lambda)\\ &= \frac{1}{2^{-\alpha-\frac{1}{2}}\Gamma(-\alpha+\frac{1}{2})}\int_{\lambda}^{\infty} s(s^2-\lambda^2)^{-\alpha-\frac{1}{2}}\int_0^{\infty} f(t)t^{1-2\alpha}\big(E_{\alpha}(st)-E_{\alpha}(\lambda t)\big)\mathrm{d}t\,\mathrm{d}s\\ &= \int_0^{\infty} f(t)t^{1-2\alpha}\Big(T^{\alpha-\frac{1}{2}}\big(s\mapsto E_{\alpha}(ts)\big)\Big)(\lambda)\mathrm{d}t\\ &= \frac{\sqrt{\pi}}{2^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\int_0^{\infty} f(t)\mathrm{e}^{-\lambda t}\mathrm{d}t.\end{aligned}$$

Let now $L_\alpha f = F = 0$ for a given $f \in C_0\big([0,\infty)\big)$. As a consequence we also have

$$\forall \lambda \in (0,\infty) : \int_0^\infty f(t) \mathrm{e}^{-\lambda t} \mathrm{d}t = 0$$

which implies the claim in this case since the Laplace transform is injective. The same reasoning also works for the case $\operatorname{Re}\alpha < \frac{1}{2}$. It remains the case $\operatorname{Re}\alpha = \frac{1}{2}$, $\operatorname{Im}\alpha \neq 0$. No direct integral representation is available in this situation which helps to verify exchangeability with the integral transform. However,

$$\begin{aligned}
0 &= \lim_{\varepsilon \to 0+} \big(T^{\alpha - \frac{1}{2} - \varepsilon} F\big)(\lambda) \\
&= \lim_{\varepsilon \to 0+} \int_0^\infty f(t) t^{1-2\alpha} \big(T^{\alpha - \frac{1}{2} - \varepsilon}\big(s \mapsto E_\alpha(ts)\big)\big)(\lambda) \mathrm{d}t \\
&= \lim_{\varepsilon \to 0+} \int_0^\infty f(t) \frac{t^{-2\varepsilon} \Gamma(\frac{1}{2} - \varepsilon)}{2^{\alpha - \frac{1}{2} + \varepsilon} \Gamma(\alpha)} E_{\frac{1}{2} + \varepsilon}(t\lambda) \mathrm{d}t \\
&= \frac{\sqrt{\pi}}{2^{\alpha - \frac{1}{2}} \Gamma(\alpha)} \int_0^\infty f(t) \mathrm{e}^{-\lambda t} \mathrm{d}t
\end{aligned}$$

where we used dominated convergence for the last equality. The necessary bound for applying it follows from the representation (5.2.8) valid in the entire strip $\mathbb{C}_{0<\mathrm{Re}<1}$. $\square$

With the injectivity result at hand, we can proceed towards our final goal and show that (weak and strong) solutions to the Caffarelli-Silvestre Problem are unique.

Theorem 5.2.15. *A solution of the Caffarelli-Silvestre problem is unique.*

Proof. By what has been said at the beginning of Section 5.2, it is enough to consider strong solutions since every weak solution in our sense gives rise to a uniquely determined strong solution. So let $v \in C^2\big((0,\infty); X\big) \cap C_b\big([0,\infty); \mathcal{D}(A)\big)$ be such that $v(0) = 0$ obtained from a weak solution u via regularisation with a resolvent, i.e., $v = (\varepsilon + A)^{-1} u$ for some $\varepsilon > 0$. We need to verify that $v = 0$. Apply the integral transformation L_α to

$$v''(t) + \frac{1 - 2\alpha}{t} v'(t) = t^{2\alpha - 1} \frac{\mathrm{d}}{\mathrm{d}t} t^{1-2\alpha} v'(t) = Av(t).$$

We define $y := -\lim_{t\to 0+} t^{1-2\alpha}v'(t)$ and calculate

$$\begin{aligned}
\int_0^\infty E_\alpha(\lambda t)\frac{\mathrm{d}}{\mathrm{d}t}t^{1-2\alpha}v'(t)\mathrm{d}t &= t^{1-2\alpha}v'(t)E_\alpha(\lambda t)\Big|_0^\infty - \int_0^\infty v'(t)\cdot t^{1-2\alpha}\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha(\lambda t)\mathrm{d}t \\
&= y - \int_0^\infty v'(t)\cdot t^{1-2\alpha}\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha(\lambda t)\mathrm{d}t
\end{aligned}$$

where we used integration by parts as well as Lemma 5.2.1 and Lemma 5.2.5 for the evaluation of the boundary terms. Also be aware that, again by Lemma 5.2.5 together with formula (5.2.9), the appearing integrand in the second line is in $L^1\big((0,\infty)\big)$. We may integrate one more time by parts to get

$$y - \int_0^\infty v'(t)\cdot t^{1-2\alpha}\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha(\lambda t)\mathrm{d}t = y + \int_0^\infty v(t)t^{1-2\alpha}\cdot t^{2\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t}t^{1-2\alpha}\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha(\lambda t)\mathrm{d}t$$

which is possible, yet another time, by formula (5.2.9), and Lemma 5.2.5 and Lemma 5.2.1. Since K_α is a solution to the modified Bessel equation, a calculation reveals

$$t^{2\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t}t^{1-2\alpha}\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha(\lambda t) = \lambda^2 E_\alpha(\lambda t). \tag{5.2.10}$$

For now let us set $f := L_\alpha v \in \mathcal{H}(\mathbb{C}_{\mathrm{Re}>0}; X)$ and, using our last finding, write

$$y + \lambda^2 f(\lambda) = Af(\lambda). \tag{5.2.11}$$

This equation implies

$$(\lambda+\sqrt{A})(\lambda-\sqrt{A})f(\lambda) = (\lambda^2 - A)f(\lambda) = -y. \tag{5.2.12}$$

Let us now define a function g on $\mathbb{C}\setminus\{0\}$ by

$$g(\lambda) = \begin{cases} -\lambda(\lambda-\sqrt{A})^{-1}y & \lambda\in\mathbb{C}\setminus S_{\frac{\omega}{2}}, \\ \lambda(\lambda+\sqrt{A})f(\lambda) & \lambda\in\mathbb{C}_{\mathrm{Re}>0}. \end{cases}$$

Note that both expressions are well defined holomorphic functions on the respective domains and that g is well defined because the functions coincide on the overlapping region $(\mathbb{C}\setminus S_{\frac{\omega}{2}})\cap\mathbb{C}_{\mathrm{Re}>0}$ as a consequence of Equation (5.2.12). For this reason g is holomorphic on $\mathbb{C}\setminus\{0\}$. Furthermore, $f(\lambda)\to 0$ as $\operatorname{Re}\lambda\to\infty$ in a fixed sector S_ϕ in the right half plane by dominated convergence and using the bound from Lemma 5.2.5. This behaviour carries over to $Af(\lambda)$ since the resolvent regularisation applied to u at the beginning of the proof commutes with applying our generalised Laplace transform. Again by Equation (5.2.12), we conclude that $\lambda^2 f(\lambda) \to -y$. From this we are able to tell that g got to be bounded on $\mathbb{C} \setminus B(0, R)$ for any $R > 0$ since $\lambda A^{\frac{1}{2}} f(\lambda) \to 0$ as $\operatorname{Re}\lambda \to \infty$ by Proposition 4.1.14. A standard estimate using the concrete form of the integral transform as well as again Lemma 5.2.5 shows the existence of some constant M (depending on a concretely chosen continuous seminorm $\|\cdot\|_p \in \mathcal{P}_X$) such that

$$\|f(\lambda)\|_p \leq M(\operatorname{Re}\lambda)^{2\operatorname{Re}\alpha - 2}\,\Gamma(2 - 2\operatorname{Re}\alpha). \tag{5.2.13}$$

This means that $\lambda \mapsto \lambda^2 g(\lambda)$ is bounded in a neighborhood of $\lambda = 0$ and admits therefore an entire extension, denoted by h, due to Riemanns theorem on removable singularities (which holds for holomorphic functions with values in locally convex spaces with the same proof as for the scalar-valued case based on the power series expansion, see [32, Satz 10.11 (e)] why the latter is true). Since $h(0) = 0$ (one may use a limit and approach $\lambda = 0$ within the region $\mathbb{C} \setminus S_{\frac{\omega}{2}}$), the function $\lambda \mapsto \lambda^{-1} h(\lambda)$ is still entire. Using this argument a second time, we conclude that the original function g actually already extends to an entire function. By Liouville's theorem ([32, p. 242]) we can tell that g is constant and by our findings from before we obtain $g(\lambda) = -y$. Taking into account the definition of g this means that $y \in \mathcal{N}(A)$ because $\lambda(\lambda - \sqrt{A})^{-1} y = y$ means $y \in \mathcal{N}(\sqrt{A})$ which in turn implies $y \in \mathcal{N}(A)$ by using Proposition 4.1.24. Considering yet another time the definition of g in the overlapping region $(\mathbb{C} \setminus S_{\frac{\omega}{2}}) \cap \mathbb{C}_{\operatorname{Re}>0}$, we obtain $f(\lambda) = -\lambda^{-2} y$. Putting this into (5.2.13) results in

$$\|y\|_p \leq M(\operatorname{Re}\lambda)^{2\operatorname{Re}\alpha}\,\Gamma(2 - 2\operatorname{Re}\alpha)$$

which is only possible for every continuous seminorm $\|\cdot\|_p \in \mathcal{P}_X$ if $y = 0$. Hence $f = 0$ which eventually implies $v = 0$ by Proposition 5.2.14. $\square$

The proof of Theorem 5.2.15 is, even though being similar in spirit, different from the uniquess proof presented in [48] which relied on growth properties. As a drawback it makes substantial use of the 'proper' sectoriality of A which is no loss of generality in a Banach space. The author is not sure whether the before mentioned proof can be generalised to LCS and would therefore possibly extend the uniquessness result to non-negative operators in LCS.

5.3. Existence of solutions

Let us come to the discussion of the existence of a solution to the Caffarelli-Silvestre Problem. Let again $\omega \in [0, \pi)$ and $A \in \mathcal{S}_\omega(X)$. We choose $\sigma \in [0, \frac{\pi-\omega}{2})$, $z \in S_\sigma$ and consider the 'germ' (a holomorphic function defined on a slightly larger sector $S_{\omega+\varepsilon}$ for some $\varepsilon > 0$ but suitably small) f_z given by $\lambda \mapsto f_z(\lambda) := E_\alpha(\sqrt{\lambda} z)$. This function is not contained in $\mathcal{E}_{sec}(S_\omega)$ (it decays at infinity but does not approach 0 at $\lambda = 0$; cf. Example 3.0.2 (c) for recalling the definition) but in an extended version of it. The major idea which enables us to plug in A into this function is the algebraic manipulation

$$f_z(\lambda) = f_z(\lambda) - \frac{f_z(0)}{1+z^2\lambda} + \frac{f_z(0)}{1+z^2\lambda}$$

for all $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. This is nothing but the sectorial calculus of the operator A known from the Banach space case, see [25]. It gives rise to the following definition.

Definition 5.3.1. Let everything be as above, fix $\theta \in (\omega, \pi - 2\sigma)$, and define the continuous linear operator $f_z(A) \in \mathcal{L}(X)$ by

$$f_z(A)x := \big(f_z - f_z(0) \cdot (1+z^2\lambda)^{-1}\big)(A)x + f_z(0)(1+z^2A)^{-1}x.$$

Here the expression $\big(f_z - f_z(0)(1+z^2\lambda)^{-1}\big)(A)x$ is given by the Cauchy integral

$$\big(f_z - f_z(0)(1+z^2\lambda)^{-1}\big)(A)x := \frac{1}{2\pi i} \int\limits_{\partial S_\theta} \Big(f_z(\lambda) - \frac{f_z(0)}{1+z^2\lambda}\Big)(\lambda - A)^{-1}x \,\mathrm{d}\lambda$$

where the path of integration is as usual orientated from $i\infty$ to $-i\infty$.

As in the case of Banach spaces, one shows that the definition does not depend on the angle θ entering in the integration path. What remains is to show that $z \mapsto f_z(A)x$ actually defines a strong solution to the Caffarelli-Silvestre problem for every $x \in X$.

Remark 5.3.2. The chosen definition to introduce the main object of interest differs from the approach in [48]. There, the here used definition also came into play but solely for the purpose of establishing continuity of the function $z \mapsto f_z(A)x$ at $z = 0$. However, the sectorial calculus approach is also suitable for deriving all other needed properties as we will see.

Lemma 5.3.3. *Let $\delta > 0$ and $x \in X$. Then*

$$\delta^2 A(\delta + A)^{-2}x = \frac{1}{2\pi i} \int\limits_{\partial S_\theta} \frac{\delta^2\lambda}{(\delta+\lambda)^2}(\lambda - A)^{-1}x \,\mathrm{d}\lambda$$

holds.

Proof. Choose $R > \delta$ and set $\Omega_R := (\mathbb{C} \setminus S_\theta) \cap B(0, R)$. Orientate the boundary of Ω_R clockwise. Cauchy's integral theorem and the residue theorem (the integrand has a second order pole within the contour) then give

$$\begin{aligned}
&\frac{1}{2\pi i} \int\limits_{\partial S_\theta} \frac{\delta^2 \lambda}{(\delta + \lambda)^2} (\lambda - A)^{-1} x \, \mathrm{d}\lambda \\
&= \lim_{R \to \infty} \frac{1}{2\pi i} \int\limits_{\partial \Omega_R} \frac{\delta^2 \lambda}{(\delta + \lambda)^2} (\lambda - A)^{-1} x \, \mathrm{d}\lambda \\
&= -\left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \delta^2 \lambda (\lambda - A)^{-1} x \right)\Big|_{\lambda = -\delta} = \delta^2 A (\delta + A)^{-2} x.
\end{aligned}$$

□

Remark 5.3.4. Note that the chosen proving strategy can be generalised to get integral represenations for all bounded operators of the form $A^n \Pi_{k=1}^m (\lambda_k - A)^{-1}$ where $n, m \in \mathbb{N}$, $m - n \geq 1$ and $\lambda_1, \ldots, \lambda_m \in \rho(A)$.

A second rather technical result is also needed.

Lemma 5.3.5. *Let $\omega \in [0, \pi)$ and $f \in \mathcal{E}(S_\omega)$. Then*

$$\sup_{-\omega \leq \delta \leq \omega} \int\limits_0^\infty \frac{|f(s\mathrm{e}^{i\delta})|}{s} \mathrm{d}s < \infty.$$

Proof. Let $f \in \mathcal{E}(S_\omega)$. A standard argument using Cauchy's integral formula and a limit shows that

$$\forall a \in S_\omega \setminus \{0\} : \, f(a) = \frac{1}{2\pi i} \int\limits_{\partial S_{\omega+\varepsilon}} \frac{f(z)}{z - a} \mathrm{d}z$$

for some $\varepsilon > 0$ but suitably small. Let $a \in S_\omega \setminus \{0\}$ and consider the function given by $g(z) = \mathrm{e}^{-(\ln z - \ln a)^2}$ defined on $\mathbb{C} \setminus (-\infty, 0]$. One has $g \in \mathcal{E}(S_\omega)$ and, since $g(a) = 1$,

$$f(a) = \frac{1}{2\pi i} \int\limits_{\partial S_{\omega+\varepsilon}} \frac{f(z)\mathrm{e}^{-(\ln z - \ln a)^2}}{z - a} \mathrm{d}z. \tag{5.3.1}$$

Let $\delta \in [-\omega, \omega]$. We have to estimate $\int_0^\infty \frac{|f(s\mathrm{e}^{i\delta})|}{s} \mathrm{d}s$. To do so, we replace the function in the integrand using equation (5.3.1) which, after an application of the triangle inequality, will yield the sum of two convergent integrals. In the following we just consider one summand corresponding to the integration path γ given by the parametrisation $z(t) = t\mathrm{e}^{i(\omega+\varepsilon)}$, $0 \le t \le \infty$. The other summand can be obtained by replacing ω with $-\omega$. We estimate

$$\int_0^\infty \frac{1}{s}\left|\frac{1}{2\pi i}\int_\gamma \frac{f(z(t))\mathrm{e}^{-(\ln z(t)-\ln s - i\delta)^2}}{z(t)-s\mathrm{e}^{i\delta}}\mathrm{d}z(t)\right|\mathrm{d}s \le \int_0^\infty \frac{1}{s}\frac{1}{2\pi}\int_0^\infty \frac{\left|f(t\mathrm{e}^{i(\omega+\varepsilon)})\right|\mathrm{e}^{-\ln^2(\frac{t}{s})}\mathrm{e}^{(\omega+\varepsilon-\delta)^2}}{\left|t\mathrm{e}^{i(\omega+\varepsilon)}-s\mathrm{e}^{i\delta}\right|}\mathrm{d}t\mathrm{d}s$$

For the appearing denominator one can use an orthogonal projection (in $\mathbb{C}$ considered as $\mathbb{R}^2$) and minimise with respect to δ to get

$$\left|t\mathrm{e}^{i(\omega+\varepsilon)}-s\mathrm{e}^{i\delta}\right| \ge t\sin(\varepsilon)$$

Hence,

$$\int_0^\infty \frac{1}{s}\left|\frac{1}{2\pi i}\int_\gamma \frac{f(z(t))\mathrm{e}^{-(\ln z(t)-\ln s - i\delta)^2}}{z(t)-s\mathrm{e}^{i\delta}}\mathrm{d}z(t)\right|\mathrm{d}s \le \frac{\mathrm{e}^{(2\omega+\varepsilon)^2}}{2\pi\sin(\varepsilon)}\int_0^\infty \frac{\left|f(t\mathrm{e}^{i(\omega+\varepsilon)})\right|}{t}\mathrm{d}t\int_0^\infty \frac{\mathrm{e}^{-\ln^2 x}}{x}\mathrm{d}x$$

where we applied the substitution $x = \frac{t}{s}$ in the integral with respect to s. This gives the result for one of the two appearing summands. The other summand is estimated similarly. ☐

As a direct corollary we obtain a uniform boundedness principle applicable for a common construction of families of bounded operators.

Corollary 5.3.6. *Let $\omega \in (0,\pi)$, $h \in \mathcal{E}(S_\omega)$, $\delta \in [0,\omega)$, and $\varepsilon > 0$ be such that $\delta + \varepsilon \le \omega$. Define a family of functions $(h_z)_{z\in S_\varepsilon}$ by $h_z(\lambda) := h(z\lambda)$. Then*

$$\sup_{\substack{z\in S_\varepsilon,\\ -\delta\le\alpha\le\delta}} \int_0^\infty \frac{|f_z(t\mathrm{e}^{i\alpha})|}{t}\mathrm{d}t < \infty.$$

Proof. Let $\alpha \in [-\delta,\delta]$. Since for all $z \in S_\varepsilon$ one has $|\arg(z)| \le \varepsilon$, we conclude that $|\alpha + \arg(z)| \le \omega$. Therefore,

$$\int_0^\infty \frac{|f_z(t\mathrm{e}^{i\alpha})|}{t}\mathrm{d}t \le \int_0^\infty \frac{\left|f(t\mathrm{e}^{i(\alpha+\arg(z))})\right|}{t}\mathrm{d}t \le \sup_{-\omega\le\gamma\le\omega}\int_0^\infty \frac{|f(t\mathrm{e}^{i\gamma})|}{t}\mathrm{d}t.$$

☐

We are now already in the position to prove the main result.

Theorem 5.3.7. *Let X be a LCS, $\omega \in [0,\pi)$ and $A \in \mathcal{S}_\omega(X)$. Furthermore, let $x \in X$ and $\sigma \in [0, \frac{\pi-\omega}{2})$. Define $u : S_\sigma \to X$ by $u(z) := f_z(A)x$. Then the following properties hold:*

1. $\forall z \in S_\sigma \setminus \{0\} : u(z) \in \mathcal{D}(A^\infty)$,
2. $u \in C_b(S_\sigma)$ *and* $u(0) = x$,
3. $u \in \mathcal{H}(\mathring{S}_\sigma)$ *and* $\forall z \in S_\sigma \setminus \{0\} : u''(z) + \frac{1-2\alpha}{z}u'(z) = Au(z)$.

Before we proceed with the proof, let us make some remarks.

Remark 5.3.8.

1. The first property of above's theorem says that the family $f_z(A)$ is smoothing. It maps the entire space into $\mathcal{D}(A^\infty)$. This is not totally surprising. If $\alpha = \frac{1}{2}$, one has $f_z(A) = \mathrm{e}^{-z\sqrt{A}}$. It is known that this is a holomorphic semigroup in Banach spaces and this result stays true in locally convex spaces for sectorial operators A.
2. The second property may be translated as $z \mapsto f_z(A)$ is strongly continuous. Again considering the special case $\alpha = \frac{1}{2}$, this is not surprising at all. As we will see the strong continuity holds on $\overline{\mathcal{D}(A)}$ which is X by the assumption that A is densely defined.
3. Since σ is arbitrary the last property yields that $z \mapsto f_z(A)x$ is holomorphic on $\mathring{S}_{\frac{\pi-\omega}{2}}$. This result can be strengthened to $z \mapsto A^k f_z(A)x$ is holomorphic for all $k \in \mathbb{N}$.

Proof.

1. Only the step $u(z) \in \mathcal{D}(A)$ will be somewhat difficult. The statement $u(z) \in \mathcal{D}(A^k)$ for $k \geq 2$ will follow in the same manner and actually much easier. Choose $\delta > 0$ and $\theta_1, \theta_2 \in (\omega, \pi)$, $\theta_2 > \theta_1$. We calculate

$$\begin{aligned}
&\delta^2 A(\delta + A)^{-2}u(z) \\
&= \frac{1}{(2\pi i)^2} \int\limits_{\partial S_{\theta_1}} \frac{\delta^2\lambda}{(\delta+\lambda)^2}(\lambda - A)^{-1}\,\mathrm{d}\lambda \cdot \int\limits_{\partial S_{\theta_2}} \Big(f_z(\lambda) - \frac{f_z(0)}{1+z^2\lambda}\Big)(\lambda - A)^{-1}x\,\mathrm{d}\lambda \\
&+ f_z(0)\delta^2 A(\delta + A)^{-2}(1+z^2A)^{-1}x
\end{aligned}$$

where we made us of Lemma 5.3.3. Using the resolvent identity and the holomorphicity of the integrands, the appearing integral can be simplified (a manifestation of the fact that this is actually the sectorial calculus which is multiplicative) and we get

$$\begin{aligned}\delta^2 A(\delta+A)^{-2}u(z) =& \frac{1}{2\pi i}\int\limits_{\partial S_{\theta_1}} \frac{\delta^2\lambda}{(\delta+\lambda)^2}\Big(f_z(\lambda)-\frac{f_z(0)}{1+z^2\lambda}\Big)(\lambda-A)^{-1}x\,\mathrm{d}\lambda\\ &+f_z(0)\delta^2 A(\delta+A)^{-2}(1+z^2A)^{-1}x\\ =&\frac{1}{2\pi i}\int\limits_{\partial S_{\theta_1}} \frac{\delta^2\lambda}{(\delta+\lambda)^2}f_z(\lambda)(\lambda-A)^{-1}x\,\mathrm{d}\lambda\end{aligned}$$

where for the last equality we made use of the fact that due to the additional factor in the integrand both summands yield convergent integrals and the second integral precisely cancels the last summand.

Now we are in the position to send $\delta\to\infty$ where the expression on the right-hand side is convergent by dominated convergence. By density of $\mathcal{D}(A)$ we also have $\delta^2(\delta+A)^{-2}u(z)\to u(z)$ (Lemma 2.0.9). Moreover A is closed. So $u(z)\in\mathcal{D}(A)$ and

$$Au(z)=\frac{1}{2\pi i}\int\limits_{\partial S_{\theta_1}}\lambda f_z(\lambda)(\lambda-A)^{-1}x\,\mathrm{d}\lambda.$$

Using this expression one can iterate the argument now.

2. The equation $u(0)=x$ holds by definition of $f_0(A)$.

   Continuity for $z\in S_\sigma\setminus\{0\}$ can be deduced from dominated convergence. Namely, one splits the integral under consideration in a part where λ is close to 0 and its complement. For the complement part one argues

   $$\left|f_z(\lambda)-\frac{f_z(0)}{1+z^2\lambda}\right|\le C\mathrm{e}^{-\frac{1}{2}\sqrt{|\lambda|}|z|\cos(\frac{\theta}{2}+\sigma)}+\left|\frac{1}{1+z^2\lambda}\right| \tag{5.3.2}$$

   where we used that $f_z(0)=1$ as well as Lemma 5.2.5 in combination with the definitions of f_z and $f_z(A)$. Note that f_z is a very well behaved function at infinity (it drops of exponentially).

   The second summand only gives the necessary regularisation at $\lambda=0$. So let us now discuss this part. Going back to the very definition, the following expansion for f_z can be written down:

   $$f_z(\lambda)=\frac{\Gamma(1-\alpha)}{2^\alpha}\sum_{k=0}^{\infty}\frac{1}{4^k\cdot k!}\left(\frac{\lambda^k z^{2k}}{2^{-\alpha}\Gamma(-\alpha+k+1)}-\frac{\lambda^{k+\alpha}z^{2k+2\alpha}}{2^\alpha\Gamma(\alpha+k+1)}\right). \tag{5.3.3}$$

Using it one sees the integrability of

$$\lambda \mapsto \Big(f_z(\lambda) - \frac{1}{1+z^2\lambda}\Big) \cdot \frac{1}{\lambda}$$

close to $\lambda = 0$ uniformly for all z in compact subsets of S_σ.

The argument requires modification for $z = 0$ since in this case Inequality (5.3.2) does not give an integrable majorant anymore. We shall show now the following:

$$x \in \overline{\mathcal{D}(A)} \;\Leftrightarrow\; \lim_{\substack{z\to 0\\ z\in S_\sigma}} f_z(A)x = x.$$

One implication is clear. Let $f_z(A)x \to x$. We already know that $f_z(A)x \in \mathcal{D}(A^\infty) \subseteq \mathcal{D}(A)$ and therefore $x \in \overline{\mathcal{D}(A)}$. The converse is more interesting. Consider the family $(h_z)_{z\in S_\sigma}$, $h_z\colon S_\omega \to \mathbb{C}$ defined by

$$h_z(\lambda) := \frac{f_z(\lambda)-1}{(1+\lambda)} = z^{2\alpha}\frac{f_z(\lambda)-1}{(z^2\lambda)^\alpha}\,\frac{\lambda^\alpha}{1+\lambda}.$$

The expansion (5.3.3) yields

$$\frac{f_1(\lambda)-1}{\lambda^\alpha} \to -\frac{\Gamma(1-\alpha)}{2^{2\alpha}\Gamma(\alpha+1)}.$$

as $\lambda \to 0$ in S_γ for every fixed $\gamma \in [0,\pi)$. As for large values of λ, the fraction $\frac{f_1(\lambda)-1}{\lambda^\alpha}$ is bounded on S_γ. It follows that $\frac{f_z(\lambda)-1}{(z^2\lambda)^\alpha}$ is uniformly bounded on $S_{\omega+\varepsilon}$ for $z \in S_\sigma$ and $\varepsilon > 0$ such that $\omega + 2\sigma + \varepsilon < \pi$. So the net (h_z) is in $\mathcal{E}(S_\omega)$ and one has

$$\int_0^\infty \frac{\left|h_z\left(s\mathrm{e}^{i(\omega+\varepsilon)}\right)\right|}{s}\mathrm{d}s \le C\,|z|^{2\operatorname{Re}\alpha} \sup_{|\delta|\le\omega+\varepsilon}\int_0^\infty \frac{s^{\operatorname{Re}\alpha}}{s\,|1+s\mathrm{e}^{i\delta}|}\mathrm{d}s$$

where $C > 0$ is independent of z. Hence,

$$\lim_{z\to 0}\sup_{|\delta|\le\omega+\varepsilon}\int_0^\infty \frac{\left|h_z\left(s\mathrm{e}^{i\delta}\right)\right|}{s}\mathrm{d}s = 0$$

and therefore, using that

$$\|h_z(A)x\|_p \le \frac{M}{\pi}\,\|x\|_q \sup_{|\delta|\le\omega+\varepsilon}\int_0^\infty \frac{\left|h_z\left(s\mathrm{e}^{i(\omega+\varepsilon)}\right)\right|}{s}\mathrm{d}s$$

where $M \geq 1$ is a constant, it follows that

$$(f_z(A)x - 1)(1+A)^{-1}x = h_z(A)x \to 0$$

in X as $z \to 0$ in S_σ which shows

$$\lim_{\substack{z \to 0 \\ z \in S_\sigma}} f_z(A)x = x$$

for $x \in \mathcal{D}(A)$. The final claim follows if we could show that $\big(f_z(A)\big)_{z \in S_\sigma}$ is equicontinuous which would imply that the limit behaviour holds for all $x \in \overline{\mathcal{D}(A)}$. Also, uniform boundedness of the solution was claimed anyway in the proposition and has not been shown yet. It follows from Corollary 5.3.6. On the one hand, one has

$$\begin{aligned}\left\|\big(f_z(\lambda) - (1+z^2\lambda)^{-1}\big)(A)x\right\|_p &\leq \sup_{\substack{z \in S_\sigma \\ |\delta| \leq \omega+\varepsilon}} \int_0^\infty \frac{\left|f_z\left(se^{i\delta}\right) - \frac{1}{1+z^2se^{i\delta}}\right|}{s} \mathrm{d}s \, \frac{M}{\pi} \|x\|_q \\ &\leq \sup_{|\delta| \leq \omega+2\sigma+\varepsilon} \int_0^\infty \frac{\left|f_1\left(se^{i\delta}\right) - \frac{1}{1+se^{i\delta}}\right|}{s} \mathrm{d}s \, \frac{M}{\pi} \|x\|_q\end{aligned}$$

while on the other hand

$$\left\|(1+z^2A)^{-1}x\right\|_p = \left\|z^{-2}(z^{-2}+A)^{-1}x\right\|_p \leq M\|x\|_r$$

by the sectoriality of A and, for $z \in S_\sigma \setminus \{0\}$, $z^{-2} \in S_{2\sigma} \setminus \{0\} \subsetneq \mathring{S}_{\pi-\omega} \subset \rho(-A)$.

3. It remains to show the differentiability of $z \mapsto u(z) = f_z(A)x$ as well as the fact that this function fulfills the differential equation

$$\forall z \in S_\sigma \setminus \{0\} : u''(z) + \frac{1-2\alpha}{z} u'(z) = Au(z).$$

So far we have not changed the parameter $\alpha \in \mathbb{C}_{0<\mathrm{Re}<1}$ during our considerations. This will change now. So let us write for this last part $f_{z,\alpha}$ instead of simply f_z. Let $\lambda \in S_{\omega+\varepsilon}$. A calculation confirms the equality

$$\frac{\mathrm{d}}{\mathrm{d}z} f_{z,\alpha}(\lambda) = -c_\alpha z^{2\alpha-1}\lambda^\alpha f_{z,1-\alpha}(\lambda) \tag{5.3.4}$$

for all $z \in S_\sigma \setminus \{0\}$ where $c_\alpha = \frac{\Gamma(1-\alpha)}{2^{2\alpha-1}\Gamma(\alpha)}$. Hence, $\lambda \mapsto \frac{\mathrm{d}}{\mathrm{d}z} f_z(\lambda) \in \mathcal{E}(S_\omega)$ for all

$z \in S_\sigma \setminus \{0\}$. Applying this equality a second time gives

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}z^2} f_z(\lambda) &= -(2\alpha-1)c_\alpha\lambda^\alpha z^{2\alpha-2} f_{z,1-\alpha}(\lambda) - c_\alpha\lambda^\alpha z^{2\alpha-1}(-c_{1-\alpha})z^{1-2\alpha}\lambda^{1-\alpha} f_{z,\alpha}(\lambda)\\ &= (1-2\alpha)c_\alpha\lambda^\alpha\frac{z^{2\alpha-1}}{z} f_{z,1-\alpha}(\lambda) + \lambda f_{z,\alpha}(\lambda).\end{aligned}$$

We also read of $\lambda \mapsto \frac{\mathrm{d}^2}{\mathrm{d}z^2} f_z(\lambda) \in \mathcal{E}(S_\omega)$ as well as $\frac{d^2}{\mathrm{d}z^2} f_z(\lambda) + \frac{1-2\alpha}{z}\frac{\mathrm{d}}{\mathrm{d}z} f_z(\lambda) = \lambda f_z(\lambda)$ (a fact which we technically already know from (5.2.9)). Finally,

$$\Big(\lambda \mapsto \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{1+z^2\lambda}\Big) = \Big(\lambda \mapsto \frac{-2z\lambda}{(1+z^2\lambda)^2}\Big) \in \mathcal{E}(S_\omega).$$

So, putting all pieces together, we conclude

$$\begin{aligned}u'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2\pi i}\int\limits_{\partial S_\theta}\Big(f_z(\lambda) - \frac{1}{1+z^2\lambda}\Big)(\lambda - A)^{-1}x\,\mathrm{d}\lambda + \frac{\mathrm{d}}{\mathrm{d}z}(1+z^2A)^{-1}x\\ &= \frac{1}{2\pi i}\int\limits_{\partial S_\theta}(-c_\alpha)z^{2\alpha-1}\lambda^\alpha f_{z,1-\alpha}(\lambda)(\lambda - A)^{-1}x\,\mathrm{d}\lambda\end{aligned}$$

as well as

$$u''(z) = \frac{1}{2\pi i}\int\limits_{\partial S_\theta}\Big((1-2\alpha)c_\alpha\lambda^\alpha\frac{z^{2\alpha-1}}{z} f_{z,1-\alpha}(\lambda) + \lambda f_{z,\alpha}(\lambda)\Big)(\lambda - A)^{-1}x\,\mathrm{d}\lambda$$

and therefore

$$\begin{aligned}&u''(z) + \frac{1-2\alpha}{z}u'(z) - Au(z)\\ =&\frac{1}{2\pi i}\int\limits_{\partial S_\theta}\Big(\tfrac{d^2}{\mathrm{d}z^2} f_z(\lambda) + \tfrac{1-2\alpha}{z}\frac{\mathrm{d}}{\mathrm{d}z} f_z(\lambda) - \lambda f_z(\lambda)\Big)(\lambda - A)^{-1}x\,\mathrm{d}\lambda\\ =&\,0\end{aligned}$$

which shows everything which was to be shown.

□

Remark 5.3.9. Equation (5.3.4) is the key to a description of fractional powers of the operator A by solutions of the Caffarelli-Silvestre Problem. Namely, from it one can deduce the equality

$$\lim_{z\to 0} -z^{1-2\alpha}\frac{\mathrm{d}}{\mathrm{d}z} f_z(A)x = c_\alpha A^\alpha x$$

for all $x \in \mathcal{D}(A^\alpha)$. The left-hand side is typically called a generalised Dirichlet-to-Neumann operator in analogy to other problems where Dirichlet boundary conditions get mapped onto the corresponding Neumann conditions. The interpretation becomes more clear when one just considers the domain $(0, \infty)$ with boundary 0. Following this interpreation, $u(0) = x$ should be called Dirichlet boundary condition. The expression $\lim_{t\to 0+} -t^{1-2\alpha}u'(t) = y$ may be seen as considering the negative first derivative at $t = 0$. The necessary scaling is the origin of the additional term 'generalised'. For more details and proofs (at least in the case of a Banach space) see [48].

We summarise the main finding of this last chapter in a theorem.

Theorem 5.3.10. *Let X be a Hausdorff quasi–complete LCS, $A \in \mathcal{S}_\omega(X)$, $\omega \in [0, \pi)$, a densely defined, sectorial operator, $\alpha \in \mathbb{C}_{0<\mathrm{Re}<1}$ a given parameter, and $x \in X$ a given vector. Then the Caffarelli-Silvestre problem*

$$u''(t) + \frac{1-2\alpha}{t}u'(t) = Au(t)\ (t > 0), \quad u(0) = x$$

has a unique solution u in the sense of Defintion 5.1.1. The function u is holomorphic on $\mathring{S}_{\frac{\pi-\omega}{2}}$ and continuous and bounded on S_σ for every $\sigma \in [0, \frac{\pi-\omega}{2})$.

A. Notions from Measure Theory

The first part of the appendix will quickly introduce the basic definitions concerning measure theory which we will occasionally need and which we will make use of without further mentioning.

The *two-point compactification of* $\mathbb{R}$ will be denoted by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. We use the convention $\forall x \in \mathbb{R} : x + \infty = \infty + x := \infty$ and $x - \infty = -\infty + x := -\infty$. We do not intend to define $\infty - \infty$. The *one-point compactification of* $\mathbb{C}$ will be denoted by $\overline{C} := \mathbb{C} \cup \{\infty\}$ and similarly to above we shall agree on $\forall z \in \mathbb{C} : z + \infty = \infty + z := \infty$

Definition A.0.1. Let Ω be a set and $\Sigma \subseteq 2^\Omega$ be a σ-algebra on Ω. A *real-valued measure* is a mapping $\nu : \Sigma \to \overline{\mathbb{R}}$ such that either its range $\mathcal{R}(\nu) \subseteq (-\infty, \infty]$ or $\mathcal{R}(\nu) \subseteq [-\infty, \infty)$, $\nu(\varnothing) = 0$, and for every sequence (A_n) in Σ of pairwise disjoint measurable sets it holds that

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n).$$

A real-valued measure μ is said to be *positive* if $\mathcal{R}(\mu) \subseteq [0, \infty]$ and *finite* if $\mathcal{R}(\mu) \subseteq \mathbb{R}$. A *complex measure* is a mapping $\mu : \Sigma \to \overline{\mathbb{C}}$ for which there are two real-valued measures ν_1 and ν_2 with the property that $\mu(A) = \nu_1(A) + i\nu_2(A)$ whenever $\nu_1(A), \nu_2(A) \in \mathbb{R}$ and $\mu(A) = \infty \in \overline{\mathbb{C}}$ if $\nu_1(A) \in \{-\infty, \infty\} \subset \overline{\mathbb{R}}$ or $\nu_2(A) \in \{-\infty, \infty\} \subset \overline{\mathbb{R}}$. A complex measure μ is finite if $\mathcal{R}(\mu) \subseteq \mathbb{C}$.

If μ is a complex measure, we can define an associated positive measure $|\mu|$ called its *total variation* via ($A \in \Sigma$)

$$|\mu|(A) := \sup \sum_{i=1}^{\infty} |\mu(A_i)|$$

where the supremum is taken over all countable partitions (A_i) in Σ of A. A measurable set N is called *null set* for the complex measure μ if $|\mu|(N) = 0$ (which implies $\mu(N) = 0$ but the converse is in general wrong unless μ is positive). If Λ is another set carrying a σ-algebra Ξ and $f : \Omega \to \Lambda$ is a measurable function, we call the class of all functions μ-a.e. equal to f an *essentially measurable function* which we also shall denote again by f. We write $L^0(\Omega, \Sigma, \mu; \Lambda, \Xi)$ for the set of all essentially measurable functions and we will drop the reference to the domain Ω, the σ-algebra Σ and / or the measure μ when no confusion can arise. The reference to the codomain Λ will not appear if $\Lambda = \mathbb{C}$ in which case we shall choose $\Xi = \mathcal{B}(\mathbb{C})$; the Borel σ-algebra of $\mathbb{C}$ (a similarly notation will be used for the Borel σ-algebra of more general topological spaces). Every $f \in L^0(\Omega, \Sigma, \mu)$ allows us to define a measure μ_f on $\mathcal{B}(\mathbb{C})$ called its *image measure* which is defined by $\mu_f(B) = \mu\big(f^{-1}(B)\big)$. For the concrete calculation of $f^{-1}(B)$ one can use any representative of f. The outcome will only differ by a μ–nullset which means that μ_f is well defined. Note that for every $A \in \mathcal{B}(\mathbb{C})$ one has $|\mu_f|(A) \leq |\mu|_f(A)$ and in general this inequality is strict. In particular, every null set of $|\mu|_f$ is also a null set for μ_f. One says that a complex measure μ is *absolutely continuous* w.r.t. the complex measure ν if both measures share the same domain of definition and if every ν-null set is also a μ-null set. One writes $\mu \ll \nu$. The reader should be aware that the above notation does not imply $|\nu|(A) \geq |\mu|(A)$ for all measurable A in a common domain. Coming back to our last observation we could rephrase that, in particular, the measure μ_f is absolutely continuous w.r.t. the measure $|\mu|_f$. The support of the measure $|\mu|_f$ will be called *essential range of* f and we are going to use the symbol $\mathcal{R}_{ess}(f)$ for it.

The set of *integrable functions* is

$$L^1(\Omega, \Sigma, \mu) = \{f \in L^0(\Omega) \mid \int\limits_\Omega |f(x)|\,|\mu|(\mathrm{d}x) < \infty\}.$$

Integration w.r.t. the complex measure $\mu = \nu_1 + i\nu_2$ is defined by

$$\int\limits_\Omega f(x)\mu(\mathrm{d}x) := \int\limits_\Omega f(x)\nu_1^+(\mathrm{d}x) - \int\limits_\Omega f(x)\nu_1^-(\mathrm{d}x) + i\int\limits_\Omega f(x)\nu_2^+(\mathrm{d}x) - i\int\limits_\Omega f(x)\nu_2^-(\mathrm{d}x)$$

where ν_k^+ and ν_k^- are a Hahn-Jordan decomposition of the real-valued measure ν_k, e.g., [11, Coro. 3.1.2]. Analogously to L^1 we can also define the Banach spaces L^p for $p \in [1, \infty]$.

Suppose now additionally that Ω is a Hausdorff locally compact space and $\Sigma = \mathcal{B}(\Omega)$. Furthermore, assume μ to be finite on all compact sets, i.e., $|\mu|(K) < \infty$ for all compact sets $K \subseteq \Omega$. For such spaces we define analogously to the ordinary L^p-scale of Banach spaces a L^p-scale of locally convex spaces $L^p_{loc}(\Omega)$ defined by

$$L^p_{loc}(\Omega) := \{f \in L^0(\Omega) \mid \forall K \subseteq \Omega,\, K \text{ compact} :\, f \cdot \mathbb{1}_K \in L^p(K)\}$$

with $\mathbb{1}_K$ being the indicator function of the set K. Each K gives rise to a seminorm $\|\cdot\|_K$

via

$$\|f\|_K := \left(\int\limits_K |f(x)|^p \, |\mu| \, (\mathrm{d}x) \right)^{\frac{1}{p}} \qquad \big(p \in [1, \infty)\big)$$

and analogously in the case $L^\infty_{loc}(\Omega)$.

Finally, let us introduce the notation $M(\Omega)$ for the space of all Radon measures defined on $\mathcal{B}(\Omega)$. Here a measure μ will be called a Radon measure if it is finite on all compacts $K \subseteq \Omega$, inner regular on open sets, i.e., $|\mu|(O) = \sup\{|\mu|(K) \mid K \subseteq O, K \text{ compact}\}$ holds for all open sets $O \subseteq \Omega$, and outer regular on all measurable sets which means $|\mu|(B) = \inf\{|\mu|(O) \mid B \subseteq O, O \text{ open}\}$ holds even for all measurable sets $B \subseteq \Omega$. The symbol $M_b(\Omega)$ will denote the set of all bounded Radon measures.

B. Analysis in $\mathbb{R}^n$

The second part of the appendix introduces standard notations and facts about common tools from analysis in $\mathbb{R}^n$ and subsets of it. Points in $\mathbb{R}^n$ will be typically denoted by $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$. Confusion with powers seems to be unlikely. Occasionally we will extend functions from $\mathbb{R}^n$ to $\mathbb{C}^n$. In this case we shall denote points by $z = (z^1, \ldots, z^n) \in \mathbb{C}^n$. When extending to $\mathbb{C}^n$ also notions like $\operatorname{Re} z := (\operatorname{Re} z^1, \ldots, \operatorname{Re} z^n) \in \mathbb{R}^n$ and $\operatorname{Im} z := (\operatorname{Im} z^1, \ldots, \operatorname{Im} z^n) \in \mathbb{R}^n$ are useful. The symbol $\|\cdot\|$ shall, also in the case that we work in $\mathbb{C}^n$, be reserved for the norm, i.e., $\|z\| := (|z^1|^2 + \cdots + |z^n|^2)^{\frac{1}{2}}$. The standard inner product of two vectors $z, w \in \mathbb{C}^n$ will be denoted by $(z|w) := \overline{z^1} w^1 + \cdots + \overline{z^n} w^n$. Note that the definition implies linearity in the second argument.

For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, a point $x \in \mathbb{R}^n$, and a direction $v \in \mathbb{R}^n$ we shall write

$$(D_v f)(x) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

for the directional derivative along the direction v. In case $v = e_k$, the k^{th} canonical basis vector, we use the symbol $D_k := D_{e_k}$. Furthermore, if $n = 1$, we just write D instead of D_1 and occasionally f' instead of Df as well as $\frac{\mathrm{d}}{\mathrm{d}t} f(t)$ for $(Df)(t)$. Multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0$ are usually denoted by lower case greek letters. We set $|\alpha| := \alpha_1 + \cdots + \alpha_n$. With their help multiple partial derivatives are expressed via multi-index notation as in

$$D^\alpha := D_1^{\alpha_1} \ldots D_n^{\alpha_n}.$$

For subsets $\Omega \subseteq \mathbb{R}^n$ (and also in $\mathbb{C}^n$) we use standard topological notations. We write $\mathring{\Omega}$ for the interior of Ω, we use $\overline{\Omega}$ for its closure, and $\partial\Omega = \overline{\Omega} \setminus \mathring{\Omega}$ for the boundary. Let $k \in \mathbb{N}_0$ and $\Omega \subseteq \mathbb{R}^n$ such that $\overline{\mathring{\Omega}} = \overline{\Omega}$. This condition means that all boundary points of Ω can be approached from its interior. In particular open sets fulfill this condition. For such sets we write $C^k(\Omega)$ for the collection of all functions $f : \Omega \to \mathbb{C}$ which are k times continuously differentiable on the interior $\mathring{\Omega}$ and for which all partial derivatives $D^\alpha f : \mathring{\Omega} \to \mathbb{C}$, $|\alpha| \leq k$, extend to continuous functions, denoted by the same symbol, $\big(D^\alpha f : \Omega \to \mathbb{R}\big) \in C(\Omega)$. The imposed condition on Ω ensures uniqueness of the extensions. Important subsets of

$C(\Omega) := C^0(\Omega)$ are

$$C_b(\Omega) := \{f \in C(\Omega) \mid f \text{ is bounded}\},$$

the bounded continuous functions, and

$$C_0(\Omega) := \{f \in C(\Omega) \mid \forall \varepsilon > 0 \, \exists K \subseteq \Omega, \text{ compact}, \forall x \in \Omega \setminus K : |f(x)| < \varepsilon\},$$

the set of continuous functions vanishing on $\overline{\Omega} \setminus \Omega$. If Ω is open and bounded, the set $C_0(\Omega)$ are the continuous functions which vanish on the boundary $\partial\Omega$ of the set Ω. In case Ω is compact we have $C_0(\Omega) = C(\Omega)$. Furthermore,

$$C^\infty(\Omega) := \bigcap_{k=0}^{\infty} C^k(\Omega)$$

denotes the set of infinitely often continuously differentiable functions. For short we simply will refer to them as smooth functions. We shall also combine the so far introduced notions and consider spaces such as $C_b^k(\Omega)$, the set of k-times continuously differentable functions with bounded derivatives, and $C_0^k(\Omega)$, the same thing for k-times differentiable functions such that all derivatives belong to $C_0(\Omega)$. We will also make use of $C_b^\infty(\Omega)$ and $C_0^\infty(\Omega)$.

The space of functions (actually equivalence classes of functions being equal almost everywhere) integrable with respect to the Lebesgue measure over some set $\Omega \subseteq \mathbb{R}^n$ will be $L^1(\Omega)$ and similar for the other L^p-spaces. Integration with respect to the Lebesgue measure will be denoted by $\int_\Omega f(x)\mathrm{d}x$ or $\int f$ if references to the integration variables or the domain of integration are not needed. If μ is another Borel measure and absolutely continuous w.r.t. the Lebesgue measure with density g, we shall write $\mu = g\mathrm{d}x$ (or using any other variable instead of x). For common functions such as the exponential we shall even write $f(x)\mathrm{d}x$ instead of $f\mathrm{d}x$ with f given by some expression $f(x)$. For example, $\mathrm{e}^{-|x|}\mathrm{d}x$ is the Lebesgue measure weighted with the density $x \mapsto \mathrm{e}^{-|x|}$. Another important function space will be

$$\mathcal{D}(\Omega) := \{f \in C^\infty(\Omega) \mid \operatorname{supp}(f) \subseteq \Omega \text{ is compact}\}.$$

For $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$ we shall also use $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and, if f is some function defined on some set $\Omega \subseteq \mathbb{R}^n$, $x^\alpha f$ for the function $x \mapsto x^\alpha f(x)$. Especially when dealing with problems in full space, the function space of all Schwartz functions

$$\mathscr{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{N}_0^n : x^\alpha D^\beta f \in C_b(\mathbb{R}^n)\}$$

is of tremendous importance. The Fourier transformation $\mathcal{F}$, given by

$$(\mathcal{F}f)(y) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} f(x)\mathrm{e}^{-i(x|y)}\mathrm{d}x, \quad f \in \mathscr{S}(\mathbb{R}^n),$$

defines a bijection from $\mathscr{S}(\mathbb{R}^n)$ onto itself. One has $\mathscr{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ and the Fourier transformation can be extended to a unitary operator, again denoted by $\mathcal{F}$, from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$.

Let $\Omega \subset \mathbb{R}^n$ be open. Functions $f \in L^p(\Omega)$ having their distributional derivatives $D^\alpha f$ up to a certain order $k \in \mathbb{N}_0$ again in $L^p(\Omega)$ are said to be *Sobolev functions* and their set is denoted by $W^{k,p}(\Omega)$. For $p = 2$ it is common to write $W^{k,2}(\Omega) =: H^k(\Omega)$. The closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $W_0^{k,p}(\Omega)$ and again analogously for the case $p = 2$.

At some points we will also encounter functions defined on subsets of the complex plane. If f is such a function defined on an open domain $\Omega \subseteq \mathbb{C}$ and the function is moreover complex differentiable for all $z \in \Omega$, we will call f a holomorphic function and denote the set of all such functions with domain Ω by $\mathcal{H}(\Omega)$. The subalgebra of all bounded holomorphic functions on Ω will be denoted by $\mathcal{H}^\infty(\Omega)$.

Bibliography

[1] NIST Digital Library of Mathematical Functions. Version 1.0.26, visited 09.04.2020 `http://dlmf.nist.gov/`.

[2] W. Arendt, R. Chill, C. Seifert, H. Vogt, and J. Voigt. Form Methods for Evolution Equations, and Applications, 2015. Lecture Notes of the 18th Internet Seminar on Evolution Equations `https://www.mat.tuhh.de/veranstaltungen/isem18/pdf/LectureNotes.pdf`.

[3] W. Arendt, A. F. M. ter Elst, and M. Warma. Fractional powers of sectorial operators via the Dirichlet-to-Neumann operator. *Comm. Partial Differential Equations*, 43(1): 1–24, 2018.

[4] A. V. Balakrishnan. An operational calculus for infinitesimal generators of semigroups. *Trans. Amer. Math. Soc.*, 91(2):330–353, 1959.

[5] A. V. Balakrishnan. Fractional powers of closed operators and the semigroups generated by them. *Pacific J. Math.*, 10(2):419–437, 1960.

[6] C. Batty, M. Haase, and J. Mubeen. The holomorphic functional calculus approach to operator semigroups. *Acta Sci. Math. (Szeged)*, 79(1–2):289–323, 2013.

[7] K.-D. Bierstedt. Gewichtete Räume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt. II. *J. Reine Angew. Math.*, 260:133–146, 1973.

[8] K.-D. Bierstedt. *An introduction to locally convex inductive limits*. Functional analysis and its applications. Papers from the International School held in Nice, August 25–September 20, 1986, ICPAM Lecture Notes. World Scientific Publishing Co., 1988.

[9] S. Bochner. Diffusion Equation and Stochastic Processes. *Proc. Nat. Acad. Sciences*, 35(7):368–370, 1949.

[10] S. Bochner. *Harmonic Analysis and the Theory of Probability*. California Monogr. Math. Sci. University of California Press, 1955.

[11] V. I. Bogachev. *Measure Theory*, volume 1. Springer, 2007.

[12] R. C. Buck. Bounded continuous functions on a locally compact space. *Michigan Math. J.*, 5(2):95–104, 1958.

[13] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(8):1245–1260, 2007.

[14] A. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Mathematica*, 24(2):113–190, 1964.

[15] V. Calvo, C. Martinez, and M. Sanz. Fractional powers of non-negative operators in Fréchet spaces. *Int. J. Math. Math. Sci.*, 12(2):309–320, 1989.

[16] J. B. Cooper. The strict topology and spaces with mixed topologies. *Proc. Amer. Math. Soc.*, 30(3):583–592, 1971.

[17] R. deLaubenfels. Automatic extensions of functional calculi. *Studia Mathematica*, 114 (3):237–259, 1995.

[18] R. deLaubenfels, F. Yao, and S. Wang. Fractional Powers of Operators of Regularized Type. *J. Math. Anal. Appl.*, 199(3):910–933, 1996.

[19] N. Dunford. Spectral theory I, Convergence to projections. *Trans. Amer. Math. Soc.*, 54(2):185–217, 1943.

[20] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer, 2000.

[21] B. Farkas. *Perturbations of Bi-continuous Semigroups*. PhD thesis, Eötvös Loránd University, Faculty of Science Department of Applied Analysis, 2003.

[22] K. Floret and J. Wloka. *Einführung in die Theorie der lokalkonvexen Räume*, volume 56 of *Lecture Notes in Mathematics*. Springer, 1968.

[23] J. E. Galé, P. J. Miana, and P. R. Stinga. Extension problem and fractional operators: semigroups and wave equations. *J. Evol. Equ.*, 13(2):343–368, 2013.

[24] A. Grothendieck. *Produits tensoriels topologiques et espaces nucléaires*, volume 16 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1955.

[25] M. Haase. *The Functional Calculus for Sectorial Operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Basel, 2006.

[26] M. Haase. Abstract extensions of functional calculi. In Y. Tomilov and J. Zemánek, editors, *Études Opératorielles*, volume 112 of *Banach Center Publications*, pages 153–170. Institute of Mathematics, Polish Academy of Sciences, 2017.

[27] M. Haase. Lectures on Functional Calculus, 2018. Lecture Notes of the 21st Internet Seminar `https://www.math.uni-kiel.de/isem21/en/course/phase1/isem21-lectures-on-functional-calculus`.

[28] M. Heymann. The Stieltjes Convolution and a Functional Calculus for Non-negative Operators, 2002. online, `http://www.matthiasheymann.de/Download/Stieltjes.pdf`.

[29] E. Hille. *Functional analysis and semigroups*, volume 31 of *Amer. Math. Soc. Colloquium Publications*. de Gruyter, 1948.

[30] F. Hirsch. Intégrales de résolvantes et calcul symbolique. *Ann. Inst. Fourier*, 22(4): 239–264, 1972.

[31] H. Jarchow. *Locally Convex Spaces*. Mathematische Leitfäden. B.G. Teubner, 1981.

[32] W. Kaballo. *Aufbaukurs Funktionalanalysis und Operatortheorie: Distributionen - lokalkonvexe Methoden - Spektraltheorie*. Springer, 2013.

[33] T. Kato. Note on fractional powers of linear operators. *Proc. Japan Acad.*, 36(3): 94–96, 1960.

[34] H. Komatsu. Fractional powers of operators. *Pacific J. Math.*, 19(2):285–346, 1966.

[35] H. Komatsu. Fractional powers of operators, III Negative powers. *Pacific J. Math.*, 21(1):89–111, 1967.

[36] H. Komatsu. Fractional powers of operators, III Negative powers. *J. Math. Soc. Japan*, 21(2):205–220, 1969.

[37] G. Köthe. *Topological Vector Spaces II*, volume 237 of *Grundlehren der mathematischen Wissenschaften*. Springer New York, 1979.

[38] K. Kruse. Weighted vector-valued Functions and the ε-product. *Banach J. Math. Anal.*, 14(4):1509—-1531, 2020.

[39] K. Kruse, J. Meichsner, and C. Seifert. Subordination for sequentially equicontinuous equibounded C_0-semigroups, 2018. arxiv preprint `https://arxiv.org/pdf/1802.05059.pdf`.

[40] F. Kühnemund. A Hille-Yosida theorem for Bi-continuous Semigroups. *Semigroup Forum*, 67(2):205–225, 2003.

[41] J.L. Lions and J. Peetre. Sur une classe d'espaces d'interpolation. *Publications Math. Inst. Hautes Etudes Sci.*, 19(1):5–68, 1964.

[42] L. Marco, C. Martinez, and M. Sanz. Fractional powers of operators. *J. Math. Soc. Japan*, 40(2):331–347, 1988.

[43] C. Martinez and M. Sanz. An Extension of the Hirsch Symbolic Calculus. *Potential Analysis*, 9(4):301–319, 1998.

[44] C. Martinez and M. Sanz. *The Theory of Fractional Powers of Operators*, volume 187 of *North-Holland Mathematics Studies*. Elsevier Science, 1st edition, 2001.

[45] A. McIntosh. Operators which have an H_∞ functional calculus. *Miniconference on operator theory and partial differential equations*, pages 210–231, 1986.

[46] J. Meichsner and C. Seifert. Fractional powers of non-negative operators in Banach spaces via the Dirichlet-to-Neumann operator. Arxiv preprint, v3 `https://arxiv.org/pdf/1704.01876.pdf`.

[47] J. Meichsner and C. Seifert. A Note on the Harmonic Extension Approach to Fractional Powers of non-densely defined Operators. In J. Eberhardsteiner and M. Schöberl, editors, *Proceedings in Applied Mathematics and Mechanics*, volume 19, pages 1–2, 2019.

[48] J. Meichsner and C. Seifert. On the harmonic extension approach to fractional powers in Banach spaces. *Fract. Calc. Appl. Anal.*, 23(4):1054–1089, 2020.

[49] J. Meichsner and C. Seifert. On some Consequences of the Solvability of the Caffarelli–Silvestre Extension Problem. In M. A. Bastos, L. Castro, and A. Yu. Karlovich, editors, *Operator Theory, Functional Analysis and Applications*, volume 282 of *Operator Theory: Advances and Applications*. Birkhäuser Basel, 2020.

[50] R. Meise and D. Vogt. *Introduction to Functional Analysis.* Oxford Graduate Texts in Mathematics. Clarendon Press, 1997.

[51] S. A. Molchanow and E. Ostrovskii. Symmetric Stable Processes as Traces of Degenerate Diffusion Processes. *Theory Probab. Appl.*, 14(1):128–131, 1968.

[52] E. Nelson. A functional calculus using singular laplace integrals. *Trans. Amer. Math. Soc. 88*, 88(2):400–413, 1958.

[53] R. S. Phillips. On the generation of semigroups of linear operators. *Pacific J. Math.*, 2(3):343–369, 1952.

[54] R. S. Phillips. Semi-groups of operators. *Bull. Amer. Math. Soc.*, 61(1):16–33, 1955.

[55] B. Ross. The Development of Fractional Calculus 1695–1900. *Historia Math.*, 4(1): 75–89, 1977.

[56] H. H. Schaefer. *Topological Vector Spaces*, volume 3 of *Graduate Texts in Mathematics.* Springer-Verlag, 3rd printing corrected edition, 1971.

[57] R. L. Schilling, R. Song, and Z. Vondracek. *Bernstein Functions: Theory and Applications*, volume 37 of *de Gruyter Stud. Math.* de Gruyter, 2nd edition, 2012.

[58] L. Schwartz. *Lectures on Mixed Problems in Partial Differential Equations and Representations of Semi-groups*, volume 11 of *Lectures on Mathematics and Physics / Tata Institute of Fundamental Research.* Tata Institute of Fundamental Research, 1957.

[59] L. Schwartz. Théorie des distributions à valeurs vectorielles. I. *Annales de l'Institut Fourier*, 7:1–141, 1957.

[60] R. Seeley. Norms and Domains of the Complex Powers A_B^z. *American Journal of Mathematics*, 93(2):299–309, 1971.

[61] H.M. Srivastava and V.K. Tuan. A new convolution theorem for the Stieltjes transform and its application to a class of singular integral equations. *Arch. Math.*, 64(2):144–149, 1995.

[62] T.J. Stieltjes. Recherches sur les fractions continues. *Annales de la Faculté des sciences de Toulouse : Mathématiques*, 8(4):1–122, 1894.

[63] P. R. Stinga and J. L. Torrea. Extension Problem and Harnack's Inequality for Some Fractional Operators. *Comm. Partial Differential Equations*, 35(11):2092–2122, 2010.

[64] G. van Dijk. *Distribution Theory, Convolution, Fourier Transform, and Laplace Transform.* De Gruyter Textbook. De Gruyter, 2013.

[65] U. Westphal. Fractional powers of infinitesimal generators of semigroups. In R. Hilfer, editor, *Applications of fractional calculus in physics*, chapter III, pages 131–170. World Scientific, 2000.

[66] A. Wiweger. Linear spaces with mixed topology. *Studia Math.*, 20(1):47–68, 1961.

[67] I. Wood. Maximal L^p-regularity for the Laplacian on Lipschitz domains. *Math. Z.*, 255(4):855–875, 2007.

[68] A. Yagi. Coincidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'operateurs. *C.R. Acad. Sc. Paris*, 299(Série I):173–176, 1984.

[69] K. Yosida. *Functional Analysis*, volume 123 of *Grundlehren Math. Wiss.* Springer, 2nd edition, 1968.

List of symbols

X	locally convex vector space, typically abbreviated as LCS, occasionally also Y will be used to denote LCS, 5
$\|\cdot\|_p$	a continuous seminorm, typically the letters p,q and r (with possible additional subscripts) are used for indexing, 5
$\mathcal{P}_X$	system of continuous seminorms on X generating the topology, 5
$\mathcal{U}$	collection of all 0-neighbourhoods of X, 5
$B_p(x, r)$	ball around $x \in X$ with radius $r > 0$ w.r.t. the seminorm $\|\cdot\|_p \in \mathcal{P}_X$, 5
$\mathcal{C}(X, Y)$	set of all linear closed operators from X to Y, 6
$\mathcal{D}(A)$	linear subspace of a given space and domain of the linear operator A, 6
$\mathcal{L}(X, Y)$	set of all linear continuous operators from X to Y, 6
$\rho(A)$	resolvent set of the linear operator A, $\{\lambda \in \mathbb{C} \mid (\lambda - A)^{-1} \in \mathcal{L}(X)\}$, 6
$\sigma(A)$	spectrum of the linear operator A, $\mathbb{C} \setminus \rho(A)$, 6
$\mathcal{M}(X)$	set of all linear non-negative operators defined on a subspace $\mathcal{D}(A)$ of X, 6
$\arg(z)$	argument of $z \in \mathbb{C} \setminus (-\infty, 0]$, unique number in $(-\pi, \pi)$ such that the equation $z = \|z\| \mathrm{e}^{i \arg(z)}$ holds, 6
S_ω	closed sector around the positive real axis of half opening angle $\omega \in [0, \pi)$, $S_\omega = \{z \in \mathbb{C} \setminus (-\infty, 0] \mid \|\arg(z)\| \leq \omega\} \cup \{0\}$, 6
$(\mathrm{e}^{-zA})_{z \in S_\omega}$	operator semigroup, 6
1_X	identity operator on the space X, the reference to the space will be dropped if it is clear from context, 7
$\mathcal{L}_{\mathcal{B}}(X)$	set of the continuous, linear operators on a LCS X equipped with the locally convex topology induced by the bornology $\mathcal{B}$, 11
β	strong topology; the topology of uniform convergence on bounded sets, 11
X'_β	dual space of the LCS X equipped with the topology of uniform convergence on bounded sets of X, 11
X''	dual space of the strong dual X'_β, 15

X''_β	dual space of the strong dual X'_β equipped with the strong topology, i.e., the topology of uniform convergence on the bounded sets of X'_β, 15
$\sigma(X, X')$	weak topology on the LCS X induced by its dual space X', 15
$\operatorname{ind}_{\alpha\in\mathcal{A}} X_\alpha$	inductive limit of the inductive spectrum $(X_\alpha)_{\alpha\in\mathcal{A}}$, 17
$B(x, r)$	ball around x with radius $r > 0$, 17
$\operatorname{acx}(U)$	absolutely convex hull of the set $U \subset X$, 20
$\operatorname{proj}_{\alpha\in\mathcal{A}} X_\alpha$	projective limit of the projective spectrum $(X_\alpha)_{\alpha\in\mathcal{A}}$, 21
$\mathcal{S}_\omega$	set of all sectorial operators of angle ω on a given LCS X, 27
$\mathcal{E}(S_\omega)$	example of a special inductive limit, cf. Example 2.0.12 (c) and Example 3.0.2 (c), 29
κ	topology of the uniform convergence on absolutely convex compact sets in a LCS X, 30
$X \varepsilon Y$	ε-product of the LCS X and Y, set of continuous linear operators $\mathcal{L}_{ec}(Y'_{\kappa,X})$ where one uses the topology of uniform convergence on absolutely convex compact sets in Y for the dual space Y' and where the space itself is topologised by using uniform convergence on the equicontinuous subsets of Y' 30
U°	(absolute) polar of the set $U \subseteq Y$, i.e., all $y' \in Y'$ with $\lvert\langle y', y\rangle\rvert \leq 1$ for all $y \in U$; if $U \subseteq Y'$, one defines U° to be all $y \in Y$ with $\lvert\langle y', y\rangle\rvert \leq 1$ for all $y' \in U$, 30
$\mathcal{K}(Y, X)$	set of the compact operators between the Banach spaces Y and X, 31
$\sigma(Y', Y)$	weak topology on the LCS Y' induced by the space Y; readers used to Banach spaces would call it weak*-topology, 31
τ_s	shift operator; maps $f(\cdot) \mapsto f(\cdot + s)$, 36
$\mu * \nu$	either standard or Stieltjes convolution of the measures μ and ν, 37, 55
$\mathrm{LS}(\mathbb{C}_{\mathrm{Re}>0})$	the Laplace-Stieltjes algebra; the algebra of all Laplace transforms of bounded measures μ on $[0, \infty)$, 46
$\mathcal{S}$	Stieltjes algebra; the algebra of all Stieltjes transforms of elements μ of $(S^\infty)'$, 58
$L^0(\Omega, \Sigma, \mu; \Lambda, \Xi)$	vector space of all essentially measurable functions, i.e., equivalence classes of functions being μ–a.e. equal to a measurable function $f : \Omega \to \Lambda$ where (Ω, Σ, μ) is a measure space and (Λ, Ξ) is a measurable space, 112
$\mathcal{R}_{ess}(f)$	essential range of $f \in L^0(\Omega)$, all $z \in \mathbb{C}$ such that for all $\varepsilon > 0$ one has $\lvert\mu\rvert(\{\lvert f - z\rvert < \varepsilon\}) > 0$, 113

$L^p_{loc}(\Omega)$	Fréchet space of all essentially measurable functions being p-integrable or essentially bounded, respectively, on measurable subsets $K \subseteq \Omega$ of finite total variation measure, 113
$\mathbb{1}_K$	indicator function of the measurable set $K \in \Sigma$, 113
$M(\Omega)$	set of all Radon measures defined on the Borel σ-algebra $\mathcal{B})(\Omega)$ of the locally compact space Ω, $M_b(\Omega)$ denotes the subset of bounded Radon measures, 114
D^α	$D_1^{\alpha_1} \dots D_n^{\alpha_n}$ where $\alpha \in \mathbb{N}_0^n$ is a multi–index and D_k is the directional derivative in direction e_k, 115
$\mathring{\Omega}$	interior of the set $\Omega \subset \mathbb{R}^n$, $\{x \in \Omega \mid \exists \varepsilon > 0 : B(x, \varepsilon) \subseteq \Omega)\}$, 115
$\overline{\Omega}$	closure of the set $\Omega \subset \mathbb{R}^n$, $\{x \in \mathbb{R}^n \mid \exists (x_n) \text{ in } \Omega) : x_n \to x\}$, 115
$\partial\Omega$	boundary of the set $\Omega \subset \mathbb{R}^n$, $\overline{\Omega} \setminus \mathring{\Omega}$, 115
$C^k_b(\Omega)$	k times continuously differentiable functions such that $D^\alpha f$, $\alpha \in \mathbb{N}_0^n$, $\lvert\alpha\rvert \leq k$ is bounded on $\Omega \subseteq \mathbb{R}^n$, 116
$C^k_0(\Omega)$	k times continuously differentiable functions such that $D^\alpha f$, $\alpha \in \mathbb{N}_0^n$, $\lvert\alpha\rvert \leq k$ vanishes on the boundary $\partial\Omega$ of $\Omega \subseteq \mathbb{R}^n$, 116
$C^\infty_b(\Omega)$	$\bigcap_{k=0}^\infty C^k_b(\Omega)$, 116
$C^\infty_0(\Omega)$	$\bigcap_{k=0}^\infty C^k_0(\Omega)$, 116
$\mathcal{D}(\Omega)$	$\{f \in C^\infty(\Omega) \mid \mathrm{supp}(f) \subsetneq \Omega, \mathrm{supp}(f) \text{ is compact}\}$, space of test functions with support contained in $\Omega \subseteq \mathbb{R}^n$, 116
$\mathscr{S}(\mathbb{R}^n)$	$\{f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{N}_0^n : x^\alpha D^\beta f \in C_b(\mathbb{R}^n)\}$, space of Schwartz functions, 116
$\mathcal{F}$	Fourier transformation, $(\mathcal{F}f)(y) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} f(x) \mathrm{e}^{-i(x\mid y)} \mathrm{d}x$, 117
$W^{k,p}(\Omega)$	k-times weakly differentiable functions with weak derivatives in $L^p(\Omega)$, 117
$H^k(\Omega)$	$W^{k,2}$, Space of Sobolev functions for $p = 2$, 117
$W^{k,p}_0(\Omega)$	closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$, 117
$\mathcal{H}(\Omega)$	set of all holomorphic (complex differentiable) functions defined on a domain $\Omega \subseteq \mathbb{C}$, 117
$\mathcal{H}^\infty(\Omega)$	set of all bounded holomorphic functions defined on a domain $\Omega \subseteq \mathbb{C}$, 117

Index