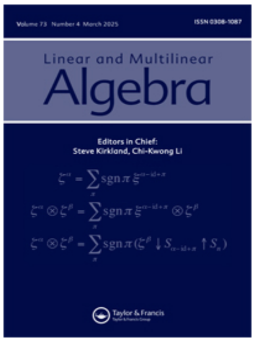

Linear and Multilinear Algebra

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/glma20

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To cite this article: Florian Bünger & Alberto Seeger (2025) Perturbation properties of the generalized spectral radius, Linear and Multilinear Algebra, 73:4, 633-648, DOI: 10.1080/03081087.2024.2366951

To link to this article: https://doi.org/10.1080/03081087.2024.2366951

Published online: 18 Jun 2024.

LINEAR AND MULTILINEAR ALGEBRA
2025, VOL. 73, NO. 4, 633–648
https://doi.org/10.1080/03081087.2024.2366951

Perturbation properties of the generalized spectral radius

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ABSTRACT

Let $n \in \mathbb{N}, \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $M_n(\mathbb{K})$ be the set of all n-by-n matrices with entries in $\mathbb{K}$. We investigate the sensitivity of the generalized spectral radius

$$\rho^{\mathbb{K}}(A) := \max\{|\lambda| : \lambda \in \mathbb{K} \text{ and } |Ax| = |\lambda x| \text{ for an } x \in \mathbb{K}^n \backslash \{0\}\}$$

of $A \in M_n(\mathbb{K})$ under perturbations, i.e. we ask how $\rho^{\mathbb{K}}(A)$ is related to $\rho^{\mathbb{K}}(A+E)$ for perturbations $E \in M_n(\mathbb{K})$. For example and somewhat surprisingly, Elsner's famous bound on the spectral variation of two complex square matrices fully translates to

$$|\rho^{\mathbb{C}}(A) - \rho^{\mathbb{C}}(B)| \leqslant \|A+B\|_2^{1-\frac{1}{n}} \|A-B\|_2^{\frac{1}{n}}$$

for all $A, B \in M_n(\mathbb{C})$, where $\|\cdot\|_2$ is the matrix 2-norm. For $\rho^{\mathbb{R}}$ this result holds true locally.

ARTICLE HISTORY
Received 3 January 2024
Accepted 7 June 2024

COMMUNICATED BY
S. Friedland

KEYWORDS
Generalized spectral radius; sign-real spectral radius; sign-complex spectral radius; perturbation of eigenvalues; Hölder continuity

2020 MATHEMATICS SUBJECT CLASSIFICATIONS
15A42; 15A18

1. Introduction

For $n \in \mathbb{N} := \{1, 2, \dots\}$ we abbreviate $\mathbb{N}_n := \{1, \dots, n\}$. The set of n-by-n matrices with entries in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is denoted by $M_n(\mathbb{K})$ and

$$\mathcal{S}_n(\mathbb{K}) := \{\text{diag}(s_1, \dots, s_n) : s_1, \dots, s_n \in \mathbb{K} \text{ and } |s_i| = 1 \text{ for all } i \in \mathbb{N}_n\}$$

is its subset of *signature matrices*. Note that $\mathcal{S}_n(\mathbb{R})$ contains all diagonal matrices with ± 1 entries on the diagonal, so that $\mathcal{S}_n(\mathbb{R})$ is a finite set with 2^n elements. Contrary, $\mathcal{S}_n(\mathbb{C})$ can be identified with the n-dimensional torus $\mathbb{T}^n$ and is therefore not discrete but still compact. Note also that the elements of $\mathcal{S}_n(\mathbb{C})$ are unitary matrices. The spectrum of $A \in M_n(\mathbb{C})$, i.e. the set of all eigenvalues of A, is denoted by $\sigma(A)$. The *spectral radius* $\rho(A)$ and the *real spectral radius* $\rho_0(A)$ are defined by

$$\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}$$
$$\rho_0(A) := \max\{|\lambda| : \lambda \in \sigma(A) \cap \mathbb{R}\}$$

with convention $\rho_0(A) := 0$ if A does not have real eigenvalues.

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In a series of papers Rump [1–6] invented for $A \in M_n(\mathbb{K})$ the so-called *generalized spectral radius*

$$\rho^{\mathbb{K}}(A) := \max\{|\lambda| : \lambda \in \mathbb{K} \text{ and } |Ax| = |\lambda x| \text{ for an } x \in \mathbb{K}^n \backslash \{0\}\}.$$

Here $|Ax| = |\lambda x|$ is meant entrywise, i.e. $|(Ax)_i| = |\lambda x_i|$ for $i \in \mathbb{N}_n$. More detailed, $\rho^{\mathbb{R}} : M_n(\mathbb{R}) \to \mathbb{R}$ and $\rho^{\mathbb{C}} : M_n(\mathbb{C}) \to \mathbb{R}$ are the *sign-real* and *sign-complex* spectral radius, respectively. It is known (cf. [6]) and not difficult to prove that

$$\rho^{\mathbb{R}}(A) = \max_{S \in \mathcal{S}_n(\mathbb{R})} \rho_0(SA) \tag{1}$$

$$\rho^{\mathbb{C}}(A) = \max_{S \in \mathcal{S}_n(\mathbb{C})} \rho(SA). \tag{2}$$

It is important to observe that $\rho^{\mathbb{R}}$ is not just the restriction of $\rho^{\mathbb{C}}$ from complex to real matrices. Both functions, although similar, cannot always be treated with a common technique. The sign-real and sign-complex spectral radii were further explored and used by Goldberger and Neumann [7], Peña [8], Radons [9], Shi and Wei [10], Varga [11], Zalar [12] and the authors [13,14].

A crucial property of $\rho^{\mathbb{K}}$ is its continuity. The continuity of $\rho^{\mathbb{R}}$ is a nontrivial result established in [1, Corollary 2.5]. In contrast, the continuity of $\rho^{\mathbb{C}}$ is quite obvious and mentioned in [6]. A short reasoning is as follows: since the spectral radius function $\rho : M_n(\mathbb{C}) \to \mathbb{R}$ is continuous and $\mathcal{S}_n(\mathbb{C})$ is compact, Berge's maximum theorem and (2) imply that $\rho^{\mathbb{C}}$ is continuous. Apart from mere continuity, more detailed infinitesimal analytic properties of $\rho^{\mathbb{K}}$ are not known. This paper addresses this gap. It is organized as follows. Section 2 introduces some notation and states known facts on generalized spectral radii which are used later on.

In Sections 3 and 4 we look at the sensitivity of $\rho^{\mathbb{K}}$ under perturbations, i.e. for given $A \in M_n(\mathbb{K})$ we bound the variation of $\rho^{\mathbb{K}}(A + E)$ for mainly but not always small perturbations $E \in M_n(\mathbb{K})$. It turns out that classical eigenvalue perturbation bounds from Elsner [15] and Chu [16] hold true for $\rho^{\mathbb{K}}$, sometimes at least locally. We shall prove in particular that $\rho^{\mathbb{K}}$ is pointwise Hölder continuous, i.e. for each $A \in M_n(\mathbb{K})$ there are a vicinity $\mathcal{U}$ of A and constants $c \geqslant 0$ and $\alpha \in (0, 1]$ such that

$$|\rho^{\mathbb{K}}(A) - \rho^{\mathbb{K}}(B)| \leqslant c\|A - B\|^{\alpha}$$

for all $B \in \mathcal{U}$, where $\|\cdot\|$ is some norm on $M_n(\mathbb{K})$. We explicitly state values for the Hölder constant c and the Hölder exponent α. The values for α are of optimal order.

2. Notation and preliminary material

Let $I = I_n$ denote the n-by-n identity matrix; if the dimension n is clear from the context, then the index n is skipped. A norm $\|\cdot\|$ on $M_n(\mathbb{K})$ is a *matrix norm* if $\|AB\| \leqslant \|A\| \, \|B\|$ for all $A, B \in M_n(\mathbb{K})$. The norm has the *signature property* if

$$\|SA\| = \|A\| \quad \text{for all } A \in M_n(\mathbb{K}),\ S \in \mathcal{S}_n(\mathbb{K})$$

and the *maximum property* if[1] $\|\operatorname{diag}(A_1, \dots, A_k)\| = \max_{1 \leqslant i \leqslant k} \|A_i\|$ for all $A_i \in M_{n_i}(\mathbb{K})$, $n_1 + \cdots + n_k = n$, $k \in \mathbb{N}_n$. Furthermore, if $\mathbb{K} = \mathbb{C}$ the norm is *unitarily invariant* if

$\|UAV\| = \|A\|$ for all $A, U, V \in M_n(\mathbb{C})$ with U, V unitary. Note that unitarily invariant norms have the signature property because signature matrices are unitary. For $p \in [1, \infty]$, the *p-norm* of a vector $x \in \mathbb{C}^n$ is defined by

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} & \text{if } p < \infty \\ \max_{i\in\mathbb{N}_n} |x_i| & \text{if } p = \infty. \end{cases}$$

The *matrix p-norm* of $A \in M_n(\mathbb{C})$ reads $\|A\|_p := \max\{\|Ax\|_p : \|x\|_p = 1\}$. Recall that the matrix 2-norm is unitarily invariant. All matrix p-norms (and, more generally, all matrix norms induced by absolute vector norms) have the signature property. Recall also that all matrix p-norms have the maximum property, cf. [17,18] for more details. For a matrix norm $\|\cdot\|$ on $M_n(\mathbb{K})$ and a regular $X \in M_n(\mathbb{K})$, $\kappa_{\|\cdot\|}(X) := \|X\|\|X^{-1}\|$ denotes the *condition number* of X with respect to $\|\cdot\|$. If the norm is clear from the context, then just $\kappa(X)$ is written. For $r > 0$ and $A \in M_n(\mathbb{K})$

$$\mathfrak{B}_r^{\mathbb{K}}(A, \|\cdot\|) := \{X \in M_n(\mathbb{K}) : \|A - X\| \leqslant r\}$$

is the closed ball with centre A and radius r. Such a ball plays the role of a vicinity of A, the parameter r being understood as a measure for the size of the vicinity. If $\mathbb{K}$ and $\|\cdot\|$ are clear from the context, then simply $\mathfrak{B}_r(A)$ is written.

For $A = (a_{i,j}) \in M_n(\mathbb{C})$, $|A| := (|a_{i,j}|)$, $\overline{A} := (\overline{a}_{i,j})$ and $A^* := \overline{A}^\top$ are the entrywise absolute value, entrywise conjugate, and conjugate transpose of A, respectively.

Lemma 2.1 collects basic known properties of the generalized spectral radius which can be found in [6].

Lemma 2.1: *For $A \in M_n(\mathbb{K})$ the following holds true:*

(a) $\rho^{\mathbb{K}}(A) = \rho^{\mathbb{K}}(A^*) = \rho^{\mathbb{K}}(\overline{A})$
(b) $\rho^{\mathbb{K}}(A) = \rho^{\mathbb{K}}(S_1AS_2)$ *for all* $S_1, S_2 \in \mathcal{S}_n(\mathbb{K})$
(c) $\rho^{\mathbb{K}}(A) = \rho^{\mathbb{K}}(P^TAP)$ *for all permutation matrices* $P \in M_n(\mathbb{R})$
(d) $\rho^{\mathbb{K}}(A) = \rho^{\mathbb{K}}(D^{-1}AD)$ *for all regular diagonal matrices* $D \in M_n(\mathbb{K})$
(e) $\rho^{\mathbb{K}}(\alpha A) = |\alpha|\rho^{\mathbb{K}}(A)$ *for all* $\alpha \in \mathbb{K}$
(f) $\rho^{\mathbb{K}}(A) \leqslant \rho(|A|)$ *with equality if A is real and nonnegative*
(g) $\rho^{\mathbb{K}}(A) \leqslant \|A\|$ *for matrix norms* $\|\cdot\|$ *having the signature property.*

Proof: The first equality of (a) and (b)–(e) are stated in [6, Lemma 2.1]. The second equality in (a) is seen likewise. Next, (f) comes from [6, Equation (9) and Lemma 2.1] for $\mathbb{K} = \mathbb{R}$. For $\mathbb{K} = \mathbb{C}$ it is the right inequality in [6, Theorem 6.3]. Finally, (g) is shown in [6, Theorem 2.6] for matrix p-norms. The stated more general case is proven as follows. Since $\|\cdot\|$ is a matrix norm it holds that $\rho(X) \leqslant \|X\|$ for all $X \in M_n(\mathbb{K})$, cf. [19, Theorem 5.6.9]. Let $A \in M_n(\mathbb{K})$. Choose $S \in \mathcal{S}_n(\mathbb{K})$ as a solution to (1) if $\mathbb{K} = \mathbb{R}$ and as a solution to (2) if $\mathbb{K} = \mathbb{C}$. In either case, we can write

$$\rho^{\mathbb{K}}(A) \leqslant \rho(SA) \leqslant \|SA\| = \|A\|,$$

where the final equality is due to the signature property of the norm. ■

3. Perturbation bounds and Hölder continuity

The next definition recalls Hölder and Lipschitz continuity in a form used later on.

Definition 3.1: Let $\|\cdot\|$ be a norm on $M_n(\mathbb{K})$. A function $f : M_n(\mathbb{K}) \to \mathbb{R}$ is:

(a) *Hölder continuous*, if there are constants $c \geqslant 0$ and $\alpha \in (0, 1]$ such that

$$|f(A) - f(B)| \leqslant c\|A - B\|^{\alpha} \tag{3}$$

for all $A, B \in M_n(\mathbb{K})$.

(b) *Hölder continuous on* $\mathcal{U} \subseteq M_n(\mathbb{K})$, if (3) holds true for all $A, B \in \mathcal{U}$. In such a case the constants c and α may depend on $\mathcal{U}$.

(c) *locally Hölder continuous*, if f is Hölder continuous on all bounded subsets.

(d) *Hölder continuous at* $A \in M_n(\mathbb{K})$, if there is a vicinity $\mathcal{U}$ of A such that (3) holds true for all $B \in \mathcal{U}$. In such a case c and α may depend on A and $\mathcal{U}$.

(e) *pointwise Hölder continuous*, if f is Hölder continuous at each $A \in M_n(\mathbb{K})$.

The constants c and α are called *Hölder constant* and *Hölder exponent*, respectively. If $\alpha = 1$, then Hölder continuity becomes *Lipschitz continuity*.

Note that local Hölder continuity implies pointwise Hölder continuity.

For $A, B \in M_n(\mathbb{C})$ the *spectral variation* $\mathrm{sv}_A(B)$ of B with respect to A and the *Hausdorff distance* between $\sigma(A)$ and $\sigma(B)$ are defined as follows (cf. [15] and [20, Definition 1.2]):

$$\mathrm{sv}_A(B) := \max_{\mu \in \sigma(B)} \min_{\nu \in \sigma(A)} |\mu - \nu|$$

$$\mathrm{hd}(A, B) := \max(\mathrm{sv}_A(B), \mathrm{sv}_B(A)).$$

The following theorem is due to Elsner [15] (see also [20, Theorem 1.3]).

Theorem 3.1 (Elsner): *For $A, B \in M_n(\mathbb{C})$ we have*

$$\mathrm{hd}(A, B) \leqslant (\|A\|_2 + \|B\|_2)^{1-\frac{1}{n}} \|A - B\|_2^{\frac{1}{n}}. \tag{4}$$

We do not know whether it was particularly noted somewhere in the literature that Elsner's bound (4) carries over to the distance between the spectral radii of two complex matrices. Moreover, it remains also true for the distance between sign-complex spectral radii. Both facts are recorded in Theorem 3.2. For the sign-real spectral radius at least a local version remains valid, cf. Theorem 3.3. For later use let us abbreviate the right-hand side of (4) as

$$\omega(A, B) := (\|A\|_2 + \|B\|_2)^{1-\frac{1}{n}} \|A - B\|_2^{\frac{1}{n}}. \tag{5}$$

Since the matrix 2-norm is unitarily invariant, we have $\omega(UA, UB) = \omega(A, B)$ for all $A, B, U \in M_n(\mathbb{C})$ with U unitary. In particular, $\omega(SA, SB) = \omega(A, B)$ for $S \in \mathcal{S}_n(\mathbb{C})$. Note also that $\omega : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to \mathbb{R}$ is absolutely homogeneous, which means that

$\omega(\lambda A, \lambda B) = |\lambda|\omega(A, B)$ for all $\lambda \in \mathbb{C}$ and all $A, B \in M_n(\mathbb{C})$. Furthermore, for $\mu \in \mathbb{C}$ we abbreviate

$$d_A(\mu) := \min_{\nu \in \sigma(A)} |\mu - \nu|$$
$$\Delta_A(\mu) := \{\nu \in \sigma(A) : |\mu - \nu| = d_A(\mu)\}.$$

The first expression corresponds to the distance from μ to the spectrum of A and the second expression is the set of all points in $\sigma(A)$ achieving this distance.

Theorem 3.2: *For $A, B \in M_n(\mathbb{C})$ it holds that*

$$|\rho(A) - \rho(B)| \leqslant \omega(A, B) \tag{6}$$
$$|\rho^{\mathbb{C}}(A) - \rho^{\mathbb{C}}(B)| \leqslant \omega(A, B). \tag{7}$$

Proof: Pick $A, B \in M_n(\mathbb{C})$. Without loss of generality we assume that $\rho(A) \leqslant \rho(B)$. Choose $\mu \in \sigma(B)$ such that $|\mu| = \rho(B)$ and $\lambda \in \Delta_A(\mu)$. Then, by Theorem 3.1, it holds that $|\mu - \lambda| \leqslant \mathrm{sv}_A(B) \leqslant \mathrm{hd}(A, B) \leqslant \omega(A, B)$. Using also $|\lambda| \leqslant \rho(A)$ gives

$$\begin{aligned}|\rho(A) - \rho(B)| &= \rho(B) - \rho(A) = |\mu| - \rho(A) \leqslant |\mu| - |\lambda| \\ &= \big||\mu| - |\lambda|\big| \leqslant |\mu - \lambda| \leqslant \omega(A, B).\end{aligned}$$

This proves (6). For proving (7) choose again $A, B \in M_n(\mathbb{C})$ and pick $S \in \mathcal{S}_n(\mathbb{C})$. By (6), we have $|\rho(SA) - \rho(SB)| \leqslant \omega(SA, SB) = \omega(A, B)$, i.e.

$$\rho(SA) \leqslant \rho(SB) + \omega(A, B) \quad \text{and} \quad \rho(SB) \leqslant \rho(SA) + \omega(A, B).$$

By building suprema over $S \in \mathcal{S}_n(\mathbb{C})$ we readily get

$$\rho^{\mathbb{C}}(A) \leqslant \rho^{\mathbb{C}}(B) + \omega(A, B) \quad \text{and} \quad \rho^{\mathbb{C}}(B) \leqslant \rho^{\mathbb{C}}(A) + \omega(A, B).$$

Thus, $|\rho^{\mathbb{C}}(A) - \rho^{\mathbb{C}}(B)| \leqslant \omega(A, B)$. ■

As a direct consequence of Theorem 3.2 we see that, while restricted to a bounded subset of $M_n(\mathbb{C})$, the functions ρ and $\rho^{\mathbb{C}}$ are Hölder continuous with Hölder exponent $1/n$. The Hölder constant, which can be taken as the same for both functions, depends of course on the bounded set under consideration. We record this for future referencing.

Corollary 3.1: *Let $\|\cdot\|$ be any norm on $M_n(\mathbb{C})$. For all bounded sets $\mathcal{U}$ in $M_n(\mathbb{C})$ there exists a constant $c \geqslant 0$ such that*

$$|\rho(A) - \rho(B)| \leqslant c\|A - B\|^{\frac{1}{n}}$$
$$|\rho^{\mathbb{C}}(A) - \rho^{\mathbb{C}}(B)| \leqslant c\|A - B\|^{\frac{1}{n}}$$

for all $A, B \in \mathcal{U}$.

Proof: We use Theorem 3.2 and the fact that $\|\cdot\|_2 \leqslant \gamma \|\cdot\|$ for some $\gamma > 0$. As constant c one may use for instance

$$c := \gamma \left[2 \sup_{X \in \mathcal{U}} \|X\| \right]^{1-\frac{1}{n}}. \tag{8}$$

■

Corollary 3.1 implies that ρ and $\rho^{\mathbb{C}}$ are locally Hölder continuous with Hölder constant c as in (8) and Hölder exponent $1/n$. Note that the continuity of $\rho^{\mathbb{C}}$ was not used to prove Theorem 3.2. Thus, Corollary 3.1 supplies an alternative proof of this fact.

We now prove pointwise Hölder continuity of $\rho^{\mathbb{R}}$. To do this, it is convenient to introduce the auxiliary set

$$\mathcal{R}(A) := \{S \in \mathcal{S}_n(\mathbb{R}) : \rho^{\mathbb{R}}(A) \in \sigma(SA)\} \tag{9}$$

which collects all real signature matrices S such that $\rho^{\mathbb{R}}(A)$ is an eigenvalue of SA. It is not hard to see that $\mathcal{R}(A)$ is always nonempty.

Theorem 3.3: *For $A \in M_n(\mathbb{R})$ there is a $\delta > 0$ such that for all $B \in \mathfrak{B}_\delta^{\mathbb{R}}(A, \|\cdot\|_2)$ the following holds true:*

(a) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$
(b) $\Delta_{SA}(\rho^{\mathbb{R}}(B)) = \{\rho^{\mathbb{R}}(A)\}$ *for all* $S \in \mathcal{R}(B)$
(c) $|\rho^{\mathbb{R}}(A) - \rho^{\mathbb{R}}(B)| \leqslant \omega(A, B)$.

Proof: Since $\Sigma := \bigcup\{\sigma(SA) : S \in \mathcal{S}_n(\mathbb{R})\}$ is a finite set, there is an $\varepsilon > 0$ such that

$$|\rho^{\mathbb{R}}(A) - \lambda| > 2\varepsilon \quad \text{for all } \lambda \in \Sigma \backslash \{\rho^{\mathbb{R}}(A)\}. \tag{10}$$

By continuity of $\rho^{\mathbb{R}}$ and by Theorem 3.1 there is a $\delta > 0$ such that for all

$$B \in \mathfrak{B}_\delta(A) := \mathfrak{B}_\delta^{\mathbb{R}}(A, \|\cdot\|_2)$$

and all $S \in \mathcal{S}_n(\mathbb{R})$ the following holds true:

$$|\rho^{\mathbb{R}}(A) - \rho^{\mathbb{R}}(B)| \leqslant \varepsilon \tag{11}$$

$$\mathrm{sv}_{SA}(SB) \leqslant \varepsilon. \tag{12}$$

The last inequality is because for all B sufficiently close to A we have

$$\mathrm{sv}_{SA}(SB) \leqslant \mathrm{hd}(SA, SB) \leqslant \omega(SA, SB) = \omega(A, B) \leqslant \varepsilon.$$

Now, take $B \in \mathfrak{B}_\delta(A)$ and $S \in \mathcal{R}(B)$ so that $\rho^{\mathbb{R}}(B) \in \sigma(SB)$. By (12) any

$$\lambda \in \Delta_{SA}(\rho^{\mathbb{R}}(B)) \tag{13}$$

fulfils $|\rho^{\mathbb{R}}(B) - \lambda| \leqslant \mathrm{sv}_{SA}(SB) \leqslant \varepsilon$. Hence (11) implies $|\rho^{\mathbb{R}}(A) - \lambda| \leqslant 2\varepsilon$ and (10) gives $\rho^{\mathbb{R}}(A) = \lambda \in \sigma(SA)$. Thus, $S \in \mathcal{R}(A)$. We have proven in this way the inclusion

$\mathcal{R}(B) \subseteq \mathcal{R}(A)$ announced in (a). Since $S \in \mathcal{R}(B)$ and λ in (13) were chosen arbitrarily, the proof also shows that $\Delta_{SA}(\rho^{\mathbb{R}}(B)) \subseteq \{\rho^{\mathbb{R}}(A)\}$ for all $S \in \mathcal{R}(B)$. This inclusion is in fact an equality as $\Delta_{SA}(\rho^{\mathbb{R}}(B))$ is nonempty. This takes care of (b). Finally, (c) follows from (b) and Theorem 3.1 applied to SA and SB:

$$|\rho^{\mathbb{R}}(A) - \rho^{\mathbb{R}}(B)| \leqslant \mathrm{sv}_{SA}(SB) \leqslant \mathrm{hd}(SA, SB) \leqslant \omega(SA, SB) = \omega(A, B).$$

■

Corollary 3.2: *The sign-real spectral radius $\rho^{\mathbb{R}}$ is pointwise Hölder continuous. In fact, for any $A \in M_n(\mathbb{R})$, there exists a positive δ such that*

$$|\rho^{\mathbb{R}}(A) - \rho^{\mathbb{R}}(B)| \leqslant (2\|A\|_2 + \delta)^{1-\frac{1}{n}} \ \|A - B\|_2^{\frac{1}{n}} \tag{14}$$

for all $B \in \mathfrak{B}_{\delta}^{\mathbb{R}}(A, \|\cdot\|_2)$.

Proof: For $B \in \mathfrak{B}_{\delta}^{\mathbb{R}}(A, \|\cdot\|_2)$ clearly $\|B\|_2 \leqslant \|A\|_2 + \delta$ so that Theorem 3.3(c) and (5) imply (14). ■

Let us have a simultaneous look at Corollaries 3.1 and 3.2. We have seen that the sign-complex spectral radius $\rho^{\mathbb{C}}$ is locally Hölder continuous. Concerning the sign-real spectral radius, we do not know whether $\rho^{\mathbb{R}}$ is locally Hölder continuous too. However, we have shown that $\rho^{\mathbb{R}}$ is pointwise Hölder continuous. Concerning the Hölder exponents of the pointwise Hölder continuous functions $\rho^{\mathbb{C}}$ and $\rho^{\mathbb{R}}$, in both cases it is $1/n$. As shown in the example below, such a Hölder exponent is optimal, which means that it cannot be replaced by some exponent $\alpha > 1/n$.

Example 3.1: Similar to [20, p. 165 et seqq.], consider a matrix of order n

$$A_{\varepsilon} := \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ \varepsilon & & & 1 \end{bmatrix},$$

depending on a nonnegative real ε placed in the lower left corner. Its characteristic polynomial $\det(xI_n - A_{\varepsilon}) = (x-1)^n - \varepsilon$ has roots $1 + \omega_j \varepsilon^{\frac{1}{n}}$ where $\omega_j := \exp(\frac{2\pi ij}{n})$, $i^2 = -1$, $j = 0, \dots, n-1$, are the nth roots of unity. Since A_{ε} is a nonnegative real matrix, we have $\rho^{\mathbb{K}}(A_{\varepsilon}) = \rho(A_{\varepsilon}) = 1 + \varepsilon^{\frac{1}{n}}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, cf. Lemma 2.1(f). Hence

$$|\rho^{\mathbb{K}}(A_{\varepsilon}) - \rho^{\mathbb{K}}(A_0)| = \varepsilon^{\frac{1}{n}}$$

behaves like $\|A_{\varepsilon} - A_0\|^{\frac{1}{n}}$ for any norm $\|\cdot\|$ on $M_n(\mathbb{K})$.

Although the Hölder exponent $1/n$ is optimal in general, there is room for improvement in a local Hölder condition of the type

$$|\rho^{\mathbb{K}}(A) - \rho^{\mathbb{K}}(B)| \leqslant c\|A - B\|^{\alpha}$$

at a given point $A \in M_n(\mathbb{K})$ and B from some vicinity of A, if one knows specific properties of A. For example, in the case of the sign-real spectral radius, if the maximum size, say m, of all Jordan blocks of all SA, $S \in \mathcal{S}_n(\mathbb{R})$, is less than n, one can use the sharper Hölder exponent $\alpha = 1/m$. The proof of this fact is fairly technical and will be treated in the next section.

4. Cases in which the Hölder exponent 1/*n* can be improved

This section focuses on the sign-real spectral radius. If we wish to improve the quality of the Hölder exponent in (14), then we must abandon Elsner's perturbation bound (4) and explore a different approach. The standard Bauer–Fike theorem, cf. [21], [19, Theorem 6.3.2], is not helpful either, so we will rely on a generalization of the Bauer–Fike theorem established by Chu, cf. [16, Theorem 3A]. Alternatively, a generalization of Bauer–Fike's theorem due to Kahan, Parlett, Jiang [22, Theorem 8] could also be considered, but we shall focus on Chu's contribution.

Theorem 4.1 (Chu): *Let $\|\cdot\|$ be either $\|\cdot\|_1$ or $\|\cdot\|_\infty$. Consider a matrix $A \in M_n(\mathbb{C})$ and its Jordan canonical form*

$$J = X^{-1}AX = \operatorname{diag}(J_1, \dots, J_k),$$

where X is regular and the ith Jordan block J_i is of order n_i. For $B \in M_n(\mathbb{C})$ and $\mu \in \sigma(B)\backslash\sigma(A)$ choose $j \in \mathbb{N}_k$ such that

$$\|(J_j - \mu I)^{-1}\| = \max_{1 \leqslant i \leqslant k} \|(J_i - \mu I)^{-1}\|. \tag{15}$$

Then, for $p := n_j$ and $\theta := p\kappa(X)\|A - B\|$, it holds that

$$d_A(\mu) \leqslant \max\{\theta, \theta^{1/p}\}. \tag{16}$$

Some comments on Theorem 4.1 are in order. Firstly, if $p = 1$, then (16) becomes the classical Bauer–Fike bound

$$d_A(\mu) \leqslant \kappa(X)\|A - B\|.$$

Secondly, Theorem 4.1 remains valid for other norms than $\|\cdot\|_1$ or $\|\cdot\|_\infty$. Thirdly, the factor $p^{1/p}$ in $\theta^{1/p}$ is bounded by

$$p^{1/p} \leqslant 3^{1/3} \approx 1.4422.$$

Fourthly, the index j in (15) may not be uniquely determined. In such a case, the optimal j, achieving the maximum in (15) and minimizing the right-hand side of (16), is the one with minimum Jordan block size n_j. However, this will not be used in what follows. Finally, note that $\|A - B\|$ in Theorem 4.1 needs not to be small. For small perturbations Chu's result reduces to the following corollary. The proof is deferred to the Appendix.

Corollary 4.1: *Let A, J, X and $\|\cdot\|$ be as in Theorem 4.1. Then, there is a $\delta > 0$ such that for all $B \in \mathfrak{B}_\delta^{\mathbb{C}}(A, \|\cdot\|)$ and all $\mu \in \sigma(B)\backslash\sigma(A)$, one has $|\Delta_A(\mu)| = 1$ and*

$$d_A(\mu) \leqslant [m\kappa(X)\|A - B\|]^{\frac{1}{m}}, \tag{17}$$

where m is the size of the largest Jordan block of J corresponding to the unique eigenvalue $\lambda \in \Delta_A(\mu)$.

We now transfer Chu's result to the sign-real spectral radius. Recall that $\mathcal{R}(A)$ is defined by (9).

Theorem 4.2: *Let $A \in M_n(\mathbb{R})$ and $\|\cdot\|$ be either $\|\cdot\|_1$ or $\|\cdot\|_\infty$. For every $S \in \mathcal{R}(A)$ let $J_S = X_S^{-1}(SA)X_S$ be the Jordan canonical form of SA with regular $X_S \in M_n(\mathbb{C})$, $m_S \in \mathbb{N}_n$ the size of the largest Jordan block of J_S corresponding to $\rho^{\mathbb{R}}(A) \in \sigma(SA)$, and $c_S := [m_S \kappa(X_S)]^{1/m_S}$. Furthermore, define*

$$m := \max_{S \in \mathcal{R}(A)} m_S \quad \text{and} \quad c := \max_{S \in \mathcal{R}(A)} c_S. \tag{18}$$

Then, there is a $\delta > 0$ such that for all $B \in \mathfrak{B}_\delta^{\mathbb{R}}(A, \|\cdot\|)$ the following holds true:

(a) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$
(b) $\Delta_{SA}(\rho^{\mathbb{R}}(B)) = \{\rho^{\mathbb{R}}(A)\}$ *for all* $S \in \mathcal{R}(B)$
(c) $|\rho^{\mathbb{R}}(A) - \rho^{\mathbb{R}}(B)| \leqslant c_S \|A - B\|^{\frac{1}{m_S}}$ *for all* $S \in \mathcal{R}(B)$
(d) $|\rho^{\mathbb{R}}(A) - \rho^{\mathbb{R}}(B)| \leqslant c \|A - B\|^{\frac{1}{m}}$.

In particular, $\rho^{\mathbb{R}}$ is Hölder continuous at A with Hölder constant c and Hölder exponent $1/m$. Thus, if $m = 1$, then $\rho^{\mathbb{R}}$ is Lipschitz continuous at A.

Proof: Since $\mathcal{R}(A)$ is finite, we can find $\delta \in (0, 1]$ such that the assertion of Corollary 4.1 holds true for SA, J_S, X_S for all $S \in \mathcal{R}(A)$. Since in finite dimension all norms are equivalent, Theorem 3.3 implies that δ can also be chosen small enough such that

$$\mathcal{R}(B) \subseteq \mathcal{R}(A) \quad \text{and} \quad \Delta_{SA}(\rho^{\mathbb{R}}(B)) = \{\rho^{\mathbb{R}}(A)\}$$

for all $B \in \mathfrak{B}_\delta^{\mathbb{R}}(A, \|\cdot\|)$ and all $S \in \mathcal{R}(B)$. This is (a) and (b). Now, pick such a matrix $B \in \mathfrak{B}_\delta^{\mathbb{R}}(A, \|\cdot\|)$ and $S \in \mathcal{R}(B)$ so that

$$\mu := \rho^{\mathbb{R}}(B) \in \sigma(SB) \quad \text{and} \quad \lambda := \rho^{\mathbb{R}}(A) \in \Delta_{SA}(\mu).$$

Since $\|SA - SB\| = \|A - B\| < \delta$, (17) gives $|\mu - \lambda| \leqslant c_S \|A - B\|^{1/m_S}$. This proves (c). Finally, (c) and (18) imply (d), whereby $\|A - B\| \leqslant \delta \leqslant 1$ is used to deduce the inequality $\|A - B\|^{1/m_S} \leqslant \|A - B\|^{1/m}$ from $m_s \leqslant m$. ∎

Identifying Lipschitz continuity of $\rho^{\mathbb{R}}$ at specific matrices is a matter of special interest. As stated in Theorem 4.2, $\rho^{\mathbb{R}}$ is Lipschitz continuous at $A \in M_n(\mathbb{R})$ if m as defined in (18) equals 1. This simple conclusion already supplies the following insight.

Corollary 4.2: *$\rho^{\mathbb{R}}$ is Lipschitz continuous at each $A \in M_n(\mathbb{R})$ satisfying the property*

$$\text{for all } S \in \mathcal{S}_n(\mathbb{R}), SA \text{ is diagonalizable.} \tag{19}$$

In particular, the set of matrices at which $\rho^{\mathbb{R}}$ is not Lipschitz continuous has Lebesgue measure zero. The same is true for the set of matrices at which $\rho^{\mathbb{R}}$ is not differentiable.

Proof: The first part of the corollary is a direct consequence of Theorem 4.2. For proving the second part it suffices to check that

$$\mathcal{X}_n := \{A \in M_n(\mathbb{R}) : A \text{ does not satisfy (19)}\}$$

is a Lebesgue null set. It is well-known that the set $\mathcal{Y}_n$ of all matrices in $M_n(\mathbb{R})$ having algebraically multiple eigenvalues is a null set.[2] Note that $\mathcal{Y}_n$ contains all non-diagonalizable matrices of $M_n(\mathbb{R})$. For $S \in \mathcal{S}_n(\mathbb{R})$, also

$$S\mathcal{Y}_n := \{SX : X \in \mathcal{Y}_n\}$$

is a null set. Since $\mathcal{S}_n(\mathbb{R})$ is finite, the finite union

$$\mathcal{Z}_n := \cup\{S\mathcal{Y}_n : S \in \mathcal{S}_n(\mathbb{R})\}$$

is a null set too. Thus, since $\mathcal{X}_n \subseteq \mathcal{Z}_n$, also $\mathcal{X}_n$ is a null set. The final statement on differentiability follows from Stepanov's theorem (see e.g. [23]). ■

Since the computation of m defined in (18) is in general of exponential cost, simpler conditions for identifying Lipschitz continuity are wanted. This is addressed in Proposition 4.1 below. In preparation, let

$$F(A) := \{x^*Ax : x \in \mathbb{C}^n, \|x\|_2 = 1\} \quad \text{and} \quad r(A) := \max\{|z| : z \in F(A)\}$$

denote the *field of values* and the *numerical radius* of $A \in M_n(\mathbb{C})$, respectively. An eigenvalue λ of A is called *normal* (cf. [24, Definition 1.6.5]) if

(a) Every eigenvector of A corresponding to λ is orthogonal to every eigenvector of A corresponding to an eigenvalue different from λ, and
(b) The geometric multiplicity of λ is equal to its algebraic multiplicity, i.e. all Jordan blocks corresponding to λ have size 1.

As recalled in the first part of Theorem 4.3, eigenvalues on the boundary $\partial F(A)$ of the field of values are always normal, cf. [24, Theorem 1.6.6].

Theorem 4.3: *If $A \in M_n(\mathbb{C})$ and $\alpha \in \partial F(A) \cap \sigma(A)$, then α is a normal eigenvalue of A. If m is the multiplicity of α, then A is unitarily similar to* $\operatorname{diag}(\alpha I_m, R)$ *with $R \in M_{n-m}(\mathbb{C})$ and $\alpha \notin \sigma(R)$.*

For the reader's convenience, we also recall Wielandt's theorem on nonnegative irreducible matrices of order $n \geqslant 2$, cf. [19, Theorem 8.4.5]. The definition of matrix irreducibility is standard and can be found for instance in [19, Def. 6.2.21].

Theorem 4.4 (Wielandt): *Let $A, B \in M_n(\mathbb{R})$, $n \geqslant 2$, such that A is irreducible and nonnegative, and $|B| \leqslant A$. Then, $\rho(B) \leqslant \rho(A)$. If $\rho(B) = \rho(A)$, then, for $\lambda \in \sigma(B)$ with $|\lambda| = \rho(B)$, there is a unitary diagonal matrix $D \in M_n(\mathbb{C})$ such that $B = \frac{\lambda}{|\lambda|}DAD^{-1}$.*

The next proposition gives simple criteria for ensuring Lipschitz continuity of $\rho^{\mathbb{R}}$ at a prescribed matrix A or, more generally, at any Zalar transform of such A. By a *Zalar transformation* (or Zalar operator) we understand any linear isomorphism $L : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ such that

$$\rho^{\mathbb{R}}(L(X)) = \rho^{\mathbb{R}}(X)$$

for all $X \in M_n(\mathbb{R})$. Such linear operators were studied in [12], where it is shown that, for $\mathbb{K} = \mathbb{R}$, the transformations (a) to (d) in Lemma 2.1 and their combinations are already all Zalar transformations.

Proposition 4.1: *If $A \in M_n(\mathbb{R})$ fulfils one of the following conditions, then $\rho^{\mathbb{R}}$ is Lipschitz continuous at A:*

(a) *$\rho^{\mathbb{R}}(A)$ is a normal eigenvalue of SA for all $S \in \mathcal{R}(A)$*
(b) *$\rho^{\mathbb{R}}(A) = \|A\|_2$*
(c) *A is irreducible and $\rho^{\mathbb{R}}(A) = \rho(|A|)$.*

In particular, $\rho^{\mathbb{R}}$ is Lipschitz continuous at $A \in M_n(\mathbb{R})$ if A is symmetric, or orthogonal, or nonnegative and irreducible. Moreover, this remains true for any Zalar transform and any scalar multiple of such a matrix.

Proof: The assertion is trivial for $n = 1$ so that we may assume $n \geqslant 2$.

(a). If $\rho^{\mathbb{R}}(A)$ is a normal eigenvalue of SA for all $S \in \mathcal{R}(A)$, then for all such S the Jordan blocks of SA corresponding to $\rho^{\mathbb{R}}(A)$ have size 1. Using the notation of Theorem 4.2, we get $m_S = 1$ for all $S \in \mathcal{R}(A)$ and therefore $m = \max_{S\in\mathcal{R}(A)} m_S = 1$. Hence, Theorem 4.2(d) yields the assertion.

(b). Pick $S \in \mathcal{R}(A)$ and choose $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$ and $SAx = \rho^{\mathbb{R}}(A)x$. Then,

$$\|A\|_2 = \rho^{\mathbb{R}}(A) = |x^\top SAx| \leqslant r(SA) \leqslant \|SA\|_2 = \|A\|_2$$

yields $r(SA) = \rho^{\mathbb{R}}(A)$ wherefore $\rho^{\mathbb{R}}(A)$ necessarily lies on the boundary of $F(SA)$. By Theorem 4.3 this implies that $\rho^{\mathbb{R}}(A)$ is a normal eigenvalue of SA and the assertion follows by (a).

(c). Irreducibility of A implies irreducibility of $|A|$ and the Perron–Frobenius theorem yields that $\rho(|A|)$ is an algebraically simple positive eigenvalue of $|A|$. Pick $S \in \mathcal{R}(A)$. Since $|SA| = |A|$, Wielandt's Theorem 4.4 applied to $(A, B, \lambda) := (|A|, SA, \rho^{\mathbb{R}}(A))$ yields $SA = D|A|D^{-1}$ for a unitary diagonal matrix D. Thus, SA is similar to $|A|$ so that $\rho^{\mathbb{R}}(A)$ is also an algebraically simple eigenvalue of SA. Again, this gives $m = \max_{S\in\mathcal{R}(A)} m_S = 1$ and Theorem 4.2(d) yields the assertion.

If A is symmetric or orthogonal, then (cf. [6, Theorem 2.6])

$$\rho^{\mathbb{R}}(A) = \rho(A) = \|A\|_2.$$

Hence (b) holds true. If A is nonnegative and irreducible, then Lemma 2.1(f) yields

$$\rho^{\mathbb{R}}(A) = \rho(|A|).$$

Hence (c) holds true. The final statement on Zalar transforms and scalar multiples is clear. ■

Although condition (b) is a particular instance of (a), we prefer to have it explicitly mentioned in the list. Checking condition (a) in practice is not always easy because the cardinality of $\mathcal{S}_n(\mathbb{R})$ grows exponentially with n. Condition (c) is independent of the others.

So far we did not strive for computing optimal Lipschitz constants. In general this is a tedious task. However, in case (a) of Proposition 4.1 it can be solved.

Proposition 4.2: *Let $A \in M_n(\mathbb{R})$ fulfil condition (a) of Proposition 4.1. Then,*

$$|\rho^{\mathbb{R}}(A) - \rho^{\mathbb{R}}(B)| \leqslant \|A - B\|_2$$

for all B in a vicinity of A, and the Lipschitz constant 1 is the smallest possible.

Since condition (b) in Proposition 4.1 is a particular instance of condition (a), Proposition 4.2 includes the case in which A is orthogonal or symmetric. The proof of Proposition 4.2 is rather technical and relies on Schur decompositions instead of Jordan decompositions. This needs quite some additional amount of extra work and is therefore differed to the Appendix.

5. Conclusion

We investigated perturbation properties of the generalized spectral radius $\rho^{\mathbb{K}}$. Our focus is on identifying Hölder or Lipschitz continuity. We proved that the sign-complex spectral radius $\rho^{\mathbb{C}}$ is locally Hölder continuous, i.e. Hölder continuous on bounded sets, cf. Corollary 3.1. Likewise, we proved that the sign-real spectral radius $\rho^{\mathbb{R}}$ is pointwise Hölder continuous, i.e. Hölder continuous at each matrix, cf. Corollary 3.2. In both cases the Hölder exponent is $1/n$, where n is the matrix order. We showed that this exponent is in general the largest possible, cf. Example 3.1. However, at specific matrices the Hölder exponent $1/n$ can be improved, cf. Section 4. It turns out that $\rho^{\mathbb{R}}$ is in fact Lipschitz continuous at almost every matrix in $M_n(\mathbb{R})$, cf. Corollary 4.2. In particular, we showed that $\rho^{\mathbb{R}}$ is Lipschitz continuous at orthogonal, or symmetric, or nonnegative irreducible matrices, cf. Proposition 4.1.

Notes

1. By writing $\|A_i\|$, we tacitly assume that $\|\cdot\|$ naturally induces a norm on $M_m(\mathbb{K})$ for all $m \leqslant n$.
2. $\mathcal{Y}_n$ is the vanishing locus of the discriminant of the characteristic polynomial and therefore an algebraic hypersurface of $M_n(\mathbb{R})$.
3. If $n_i = 1$, then N_i is understood as the zero 1-by-1 matrix.

Acknowledgments

We thank the anonymous referees for their helpful comments and suggestions.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix 1

Proof of Corollary 4.1

By Theorem 3.1, there is a sufficiently small $\delta > 0$ such that $|\Delta_A(\mu)| = 1$ for all $\mu \in \sigma(B)$, $B \in \mathfrak{B}_\delta(A)$. In other words, for each $B \in \mathfrak{B}_\delta(A)$ and each $\mu \in \sigma(B)$ there is a unique $\lambda_\mu \in \sigma(A)$ such that $d_A(\mu) = |\lambda_\mu - \mu|$. For later use let m_μ denote the size of the largest Jordan block of J corresponding to λ_μ. Furthermore, δ can be chosen small enough such that the term θ in Theorem 4.1 becomes less or equal 1 for all $B \in \mathfrak{B}_\delta(A)$. For instance, this holds true if $\delta \leqslant [\max\{n_1, \dots, n_k\}\kappa(X)]^{-1}$. For such a B and μ and p as in Theorem 4.1, inequality (16) becomes

$$d_A(\mu) \leqslant \theta^{1/p} = [p\kappa(x)\|A - B\|]^{1/p}. \tag{A1}$$

Let $\tilde{J}$ be a Jordan block of J of size p corresponding to some $\nu \in \sigma(A)$ such that $\|(\tilde{J} - \mu I)^{-1}\|$ is maximal (cf. (15)). For $z := \nu - \mu$ and $Z := \tilde{J} - \mu I$ it holds that

$$Z = \begin{bmatrix} z & 1 & & & \\ & z & 1 & & \\ & & \ddots & \ddots & \\ & & & z & 1 \\ & & & & z \end{bmatrix} \quad \text{and} \quad Z^{-1} = \begin{bmatrix} z^{-1} & -z^{-2} & z^{-3} & \cdots & -(-z)^{-p} \\ & z^{-1} & -z^{-2} & \cdots & -(-z)^{-p+1} \\ & & \ddots & \ddots & \vdots \\ & & & z^{-1} & -z^{-2} \\ & & & & z^{-1} \end{bmatrix}.$$

Recall that $\|\cdot\|$ is either $\|\cdot\|_1$ or $\|\cdot\|_\infty$. Thus, $\|Z^{-1}\| = \sum_{k=1}^{p} |z|^{-k}$, an expression that increases with p for fixed z, and also increases for fixed p and decreasing $|z| \to 0$. Therefore, by continuity of the (finitely many) eigenvalues of A with respect to the entries of A, δ can a priori also be chosen small enough such that $\nu = \lambda_\mu$ and $p = m_\mu$. Then, (A1) proves (17).

Proof of Proposition 4.2

The following generalized Bauer–Fike theorem is due to Chu [16, Theorem 3]. For the readers convenience we include a proof.

Theorem A.1 (Chu): *Let $\|\cdot\|$ be a matrix norm having the maximum property. Let $A, X \in M_n(\mathbb{C})$, X regular, such that $\tilde{A} := X^{-1}AX = \operatorname{diag}(A_1, \dots, A_k)$ is a block diagonal matrix, where each block $A_i = D_i + N_i$, of corresponding size n_i, is expressible as a sum of a diagonal matrix D_i and a strictly upper triangular N_i.*[3] *For $B \in M_n(\mathbb{C})$ and $\mu \in \sigma(B)\backslash\sigma(A)$ choose $j \in \mathbb{N}_k$ such that*

$$\|(A_j - \mu I)^{-1}\| = \max_{1 \leqslant i \leqslant k} \|(A_i - \mu I)^{-1}\|. \tag{A2}$$

Then, for $p := n_j$, $C := \sum_{i=0}^{p-1} \|N_j\|^i$ and $\theta := C\kappa(X)\|A - B\|$, it holds that

$$d_A(\mu) \leqslant \max\{\theta, \theta^{1/p}\}. \tag{A3}$$

Proof: First note that the matrices $\tilde{A} - \mu I$ and $D_j - \mu I$ are regular since $\mu \notin \sigma(A)$. Thus, $M := (\tilde{A} - \mu I)^{-1}X^{-1}(B - A)X$ is well-defined and it holds that

$$X^{-1}(B - \mu I)X = X^{-1}(A + (B - A) - \mu I)X = (\tilde{A} - \mu I)(I + M).$$

Since $B - \mu I$ is singular, so is $I + M$. Therefore, and because $\|\cdot\|$ is a matrix norm, we get $1 \leqslant \|M\| \leqslant \|(\tilde{A} - \mu I)^{-1}\| \cdot \kappa(X)\|A - B\|$. By (A2) and because $\|\cdot\|$ has the maximum property, this yields

$$\|(A_j - \mu I)^{-1}\|^{-1} = \|(\tilde{A} - \mu I)^{-1}\|^{-1} \leqslant \kappa(X)\|A - B\|. \tag{A4}$$

Furthermore,

$$A_j - \mu I = D_j - \mu I + N_j = (D_j - \mu I)(I + (D_j - \mu I)^{-1}N_j)$$

and, therefore,

$$(A_j - \mu I)^{-1} = \sum_{i=0}^{p-1} \left[-(D_j - \mu I)^{-1} N_j\right]^i (D_j - \mu I)^{-1}. \tag{A5}$$

Note that we are not taking the sum all the way to infinity, but just up to $p-1$. This is because $p = n_j$ and $(D_j - \mu I)^{-1} N_j$ is a strictly upper triangular matrix of order n_j, so that $[(D_j - \mu I)^{-1} N_j]^i = 0$ for all $i \geqslant p$. Since D_j is a diagonal matrix, we have

$$\|(D_j - \mu I)^{-1}\| = [\min_{1 \leqslant i \leqslant p} |(D_j)_{i,i} - \mu|]^{-1} \leqslant [d_A(\mu)]^{-1} := \alpha. \tag{A6}$$

From (A4), (A5) and (A6) we derive

$$\begin{aligned}[\kappa(X)\|A - B\|]^{-1} &\leqslant \|(A_j - \mu I)^{-1}\| \leqslant \sum_{i=0}^{p-1} \alpha^{i+1} \|N_j\|^i \leqslant \left[\max_{1 \leqslant i \leqslant p} \alpha^i\right] \cdot \sum_{i=0}^{p-1} \|N_j\|^i \\ &= \max\{\alpha, \alpha^p\}\, C,\end{aligned}$$

i.e.

$$\min\{\alpha^{-1}, \alpha^{-p}\} = [\max\{\alpha, \alpha^p\}]^{-1} \leqslant C\kappa(X)\|A - B\| = \theta.$$

Hence, $d_A(\mu) = \alpha^{-1} \leqslant \max\{\theta, \theta^{1/p}\}$, which is (A3). ■

We remark that the index j in (A2) may not be uniquely determined and in such a case the best choice may not be obvious. For small perturbations Theorem A.1 implies the following corollary.

Corollary A.1: *Let A, $\tilde{A}$, X and $\|\cdot\|$ be as in Theorem* A.1. *Then, there is a $\delta > 0$ such that for all $B \in \mathfrak{B}_\delta^{\mathbb{C}}(A, \|\cdot\|)$ and all $\mu \in \sigma(B) \backslash \sigma(A)$, one has $|\Delta_A(\mu)| = 1$ and*

$$d_A(\mu) \leqslant [C\kappa(X)\|A - B\|]^{\frac{1}{p}}. \tag{A7}$$

Hereby, $p := n_j$ is the size of a block A_j achieving

$$\max\{\|(A_i - \mu I)^{-1}\| : i \in \mathbb{N}_k \text{ and } \Delta_A(\mu) \subseteq \sigma(A_i)\}, \tag{A8}$$

and $C := \sum_{i=0}^{p-1} \|N_j\|^i$.

Proof: Like in the proof of Corollary 4.1, $\delta > 0$ can be chosen such that $|\Delta_A(\mu)| = 1$ for all $\mu \in \sigma(B)$, $B \in \mathfrak{B}_\delta(A)$, i.e. $\Delta_A(\mu) = \{\lambda_\mu\}$ for a unique $\lambda_\mu \in \sigma(A)$. Continuing this notation, let $p := n_j$ be the size of a block A_j achieving

$$\max\{\|(A_i - \mu I)^{-1}\| : i \in \mathbb{N}_k \text{ and } \lambda_\mu \in \sigma(A_i)\}.$$

This is an equivalent notation of (A8). For sufficiently small δ, those indices $i \in \mathbb{N}_k$ for which $\lambda_\mu \notin \sigma(A_i)$ holds true can be excluded from finding the overall maximum $\max_{1 \leqslant i \leqslant k} \|(A_i - \mu I)^{-1}\|$ considered in (A2). Additionally, we can choose

$$\delta \leqslant \left(\max_{1 \leqslant \ell \leqslant k} \sum_{i=0}^{n_\ell - 1} \|N_\ell\|^i \kappa(X) \right)^{-1}.$$

Then, $C := \sum_{i=0}^{p-1} \|N_j\|^i$ and $\theta := C\kappa(X)\|A - B\|$ as defined in Theorem A.1 fulfil $\theta \leqslant 1$ so that $\max\{\theta, \theta^{1/p}\} = \theta^{1/p}$. Hence, (A3) becomes (A7). ■

Lemma A.1: *Let $A \in M_n(\mathbb{R})$ and let $\|\cdot\|$ be a matrix norm having the signature property and the maximum property. Abbreviate $\alpha := \rho^{\mathbb{R}}(A)$ and suppose that for each $S \in \mathcal{R}(A)$ there is a block diagonal decomposition*

$$SA = X_S \mathrm{diag}(\alpha, \ldots, \alpha, R_S) X_S^{-1}, \tag{A9}$$

where R_S is upper triangular with $\alpha \notin \sigma(R_S)$ and $X_S \in M_n(\mathbb{C})$ is regular. Then, $\rho^{\mathbb{R}}$ is Lipschitz continuous at A with Lipschitz constant

$$c := \max_{S \in \mathcal{R}(A)} \kappa_{\|\cdot\|}(X_S).$$

In particular, if $\|\cdot\| = \|\cdot\|_2$ and if all X_S are unitary, then $c = 1$.

Proof: Since $\mathcal{R}(A)$ is finite, we can find $\delta > 0$ such that the assertion of Corollary A.1 holds true for $A_S := SA$, $\tilde{A}_S := \mathrm{diag}(\alpha, \ldots, \alpha, R_S)$, and X_S for all $S \in \mathcal{R}(A)$, where the leading α's are considered as separate 1-by-1 blocks. By Theorem 3.3, there is a $\delta > 0$ such that $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\Delta_{SA}(\rho^{\mathbb{R}}(B)) = \{\alpha\}$ for all $B \in \mathfrak{B}_\delta^{\mathbb{R}}(A, \|\cdot\|)$ and $S \in \mathcal{R}(B)$. Now, pick any $B \in \mathfrak{B}_\delta^{\mathbb{R}}(A, \|\cdot\|)$ and $S \in \mathcal{R}(B)$. Hence, $\mu := \rho^{\mathbb{R}}(B) \in \sigma(SB)$ and

$$|\rho^{\mathbb{R}}(A) - \rho^{\mathbb{R}}(B)| = |\alpha - \mu| = d_{SA}(\mu) \leqslant \kappa_{\|\cdot\|}(X_S)\|SA - SB\| \leqslant c\|A - B\|.$$

The first inequality is obtained by applying Corollary A.1 and the fact that all blocks of $\tilde{A}_S$ having α as an eigenvalue have size $p = 1$. Finally, if $\|\cdot\| = \|\cdot\|_2$ and if all X_S are unitary, then $\kappa_{\|\cdot\|_2}(X_S) = 1$ for all $S \in \mathcal{R}(A)$ and hence $c = 1$.

We now prove Proposition 4.2: Refining the proof of Proposition 4.1(a), Theorem 4.3 additionally asserts that (A9) holds for unitary X_S and a square block R_S for all $S \in \mathcal{R}(A)$, where $\alpha := \rho^{\mathbb{R}}(A)$ is not an eigenvalue of R_S. By taking a Schur triangularization (cf. [19, Theorem 2.3.1]) of R_S we can assume w.l.o.g. that R_S is upper triangular. Then, Lemma A.1 supplies the local Lipschitz constant 1 for the 2-norm.

Taking $A: = I$ and $B := \beta I$, $\beta \geqslant 0$, gives $|\rho^{\mathbb{R}}(A) - \rho^{\mathbb{R}}(B)| = |1 - \beta| = \|A - B\|_2$, which shows that the Lipschitz constant 1 is best possible. ■